2023 GSU Matrix Theory Workshop
In Honor of Dr. Frank J. Hall

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Georgia State University
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Titles and Abstracts of Talks

Speaker: Milica Andelić, Kuwait University , E-mail: milica.andelic@ku.edu.kw

Title: Combinatorial aspects of generalized inverse

Abstract. We provide combinatorial formulae for the Moore-Penrose inverse of various matrices associated with (signed) graphs.

Speaker: Marina Arav, Georgia State University University , E-mail: arav@gsu.edu

Title: A characterization of signed graphs with stable maximum nullity at most two

Abstract. A signed graph is a pair (G, Σ) where G is an undirected graph (we allow parallel edges but no loops) and $\Sigma \subseteq E(G)$. If (G, Σ) is a signed graph with vertex-set $V = \{1, \ldots, n\}$, $S(G, \Sigma)$ is the set of all $n \times n$ real symmetric matrices $A = [a_{i,j}]$ with $a_{i,j} > 0$ if i and j are adjacent and connected by only odd edges, $a_{i,j} < 0$ if i and j are adjacent and connected by only even edges, $a_{i,j} \in \mathbb{R}$ if i and j are adjacent and connected by both even and odd edges, $a_{i,j} = 0$ if i and j are not adjacent, and $a_{i,i} \in \mathbb{R}$ for all vertices i. The parameter $\xi(G, \Sigma)$ is defined as the largest nullity of any matrix satisfying the Strong Arnold Property. This invariant is closed under taking minors. In 2021, Arav, Hall, van der Holst, and Li gave a characterization of 2-connected signed graphs (G, Σ) with $\xi(G, \Sigma) \leq 2$. A full characterization was still open. In this talk, we discuss a full characterization of signed graphs (G, Σ) with $\xi(G, \Sigma) \leq 2$.

Speaker: Steve Butler, Iowa State University, E-mail: butler@iastate.edu

Title: Properties of coalescing sets

Abstract. Given graphs H_1, H_2 with $B_1 \subseteq V(H_1), B_2 \subseteq V(H_2)$ we way that $(H_1, B_1) \sim (H_2, B_2)$ if attaching any rooted graph G onto the vertices of B_1 in H_1 and onto the vertices of B_2 in H_2 always results in cospectral graphs (with respect to some designated matrix associated with the graphs, e.g. adjacency, Laplacian, ...). Our main result is to show that for many standard matrices (adjacency, Laplacian, signless Laplacian) that $(H_1, B_1) \sim (H_2, B_2)$ if and only if $(H_1, \overline{B_1}) \sim (H_2, \overline{B_2})$.

Speaker: Minerva Catral, Xavier University, E-mail: catralm@xavier.edu

Title: On the number of distinct eigenvalues allowed by a sign pattern

Abstract. A sign pattern (matrix) $\mathcal{A}$ has entries in $\{+, -, 0\}$. For a real matrix A, we define $q(A)$ to be the number of distinct eigenvalues of A. We say that $\mathcal{A}$ *allows k distinct eigenvalues* if there is a matrix A having the sign pattern of $\mathcal{A}$

that has $q(A) = k$. For a given sign pattern $\mathcal{A}$, we define the sequence $q_{\text{seq}}(\mathcal{A}) = \langle q_1(\mathcal{A}), q_2(\mathcal{A}), \ldots, q_n(\mathcal{A})\rangle$ as follows: For $k = 1, \ldots, n$, $q_k(\mathcal{A}) = 1$ if $\mathcal{A}$ allows k distinct eigenvalues, and $q_k(\mathcal{A}) = 0$ otherwise. We explore $q_{\text{seq}}(\mathcal{A})$ using digraph properties of the sign pattern $\mathcal{A}$, and classify patterns of small order according to their q-sequence. *This is joint work with J. Breen, C. Brouwer, M. Cavers, P. van den Driessche and K. Vander Meulen.*

Speaker: Bryan Curtis, Iowa State University, E-mail: bcurtis1@iastate.edu

Title: Orthogonal realizations of random sign patterns

Abstract. A sign pattern is an array with entries coming from the set {0,+,-}. A sign pattern $\mathcal{S}$ allows row orthogonality provided there exists a matrix Q whose rows form an orthonormal set and the signs of the entries of Q match the entries of $\mathcal{S}$. This paper utilizes a tool called the strong inner product property (SIPP) to initiate the study of conditions under which random sign patterns allow row orthogonality with high probability. A quantitative version of the Gram-Schmidt algorithm is also used to provide slightly weaker results.

Speaker: Louis Deaett, Quinnipiac University, E-mail: Louis.Deaett@quinnipiac.edu

Title: Matroid theory in understanding combinatorial bounds on matrix rank

Abstract. In linear algebra and matrix theory, when we investigate combinatorial questions related to rank, it is natural to ask how much of these questions can be answered by matroid theory. The reason is that the structure of a matroid abstracts some of the essential combinatorial properties of linear independence, and hence also of rank.

In this talk, we focus on the problem of bounding the rank of a matrix from a description of its zero-nonzero structure by a graph, digraph or pattern. We show how well-studied bounds can be established within a matroid-theoretic framework. This framework allows us to give a generalization of the problem to matroids; we recover the original problem for matrices over a chosen field by considering only matroids representable over that field. Moreover, we consider the question of whether existing bounds can be improved by exploiting a deeper understanding of the relevant matroid theory, and suggest some possibilities for future work in that direction.

Speaker: Hein van der Holst, Georgia State University, E-mail: hvanderholst@gsu.edu

Title: On the strong maximum nullity of a bipartite graph

Abstract. For a bipartite graph G with bipartition U, W and $|U| = |W|$, we denote by $Q(G)$, the set of all real $U \times W$ matrices $M = [m_{i,j}]$ with $m_{i,j} \neq 0$ if i and j are connected by a single edge, $m_{i,j} \in \mathbb{R}$ if i and j are connected by parallel edges, and $m_{i,j} = 0$ if i and j are not adjacent, and we denote by $N(G)$ the set of all real matrices $X = [x_{i,j}]$ with $x_{i,j} = 0$ if i and j are adjacent. A matrix M has the ASAP if there is no nonzero matrix $X \in N(G)$ such that $X^T M = 0$ and $MX^T = 0$. There exists a matrix satisfying the ASAP if and only if G has a perfect matching. For a bipartite graph G having a perfect matching, denote by $M_S(G)$ the maximum nullity of any $M \in Q(G)$ satisfying the ASAP.

In this talk, we discuss the class of bipartite graphs G with $M_S(G) = 0$ and the class of bipartite graph G with $M_S(G) \leq 1$. We do this in terms of forbidden substructures (minors).

Speaker: Selcuk Koyuncu, University of North Georgia, E-mail: skoyuncu@ung.edu

Title: A note on Multilevel Toeplitz Matrices

Abstract. Chien, Liu, Nakazato and Tam proved that all $n \times n$ classical Toeplitz matrices (one-level Toeplitz matrices) are unitarily similar to complex symmetric matrices via two types of unitary matrices and the type of the unitary matrices only depends on the parity of n. In this paper we extend their result to multilevel Toeplitz matrices that any multilevel Toeplitz matrix is unitarily similar to a complex symmetric matrix. We provide a method to construct the unitary matrices that uniformly turn any multilevel Toeplitz matrix to a complex symmetric matrix by taking tensor products of these two types of unitary matrices for one-level Toeplitz matrices according to the parity of each level of the multilevel Toeplitz matrices. In addition, we introduce a class of complex symmetric matrices that are unitarily similar to some p-level Toeplitz matrices.

Speaker: Zhongshan Li, Georgia State University, E-mail: zli@gsu.edu

Title: 4×4 **Irreducible sign patterns that require all distinct eigenvalues**

Abstract. A *sign pattern matrix* is a matrix whose entries are from the set $\{+, -, 0\}$. For a real matrix B, sgn(B) is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). For a sign pattern matrix A, the *qualitative class of* A, denoted $Q(A)$, is the set of all real matrices whose entries have signs given by the corresponding entries of A. An $n \times n$ sign pattern matrix A requires all distinct eigenvalues if every real matrix in $Q(A)$ has n distinct eigenvalues. In the article "Sign patterns that require all distinct eigenvalues", *JP J. Algebra Number Theory Appl.*, 2:2 (2002), 161–179, Li and Harris characterized the 2×2 and 3×3 irreducible sign pattern matrices that require all distinct eigenvalues, and established some useful general results on $n \times n$ sign patterns that require all distinct eigenvalues. In this talk, we characterize 4×4 irreducible sign patterns that require four distinct eigenvalues. This is done by characterizing 4×4 irreducible sign patterns that require four distinct real eigenvalues, or require four distinct nonreal real eigenvalues, or require two distinct real eigenvalues and a pair of conjugate nonreal eigenvalues. The last case turns out to be much more involved. The cycle structure of the signed digraph of the sign pattern plays a key role. Some interesting open problems are presented. Three important tools that are used in the paper are the following: the discriminant of a polynomial; the fact that if a square sign pattern matrix A requires all distinct eigenvalues then A requires a fixed number of real eigenvalues; and the known result that if A is an "k-cycle" sign pattern then for each $B \in Q(A)$, the k nonzero eigenvalues of B are evenly distributed on a circle in the complex plane centered at the origin.
This is joint work with Victor Bailey, Yubin Gao, Frank Hall, and Paul Kim

Speaker: Bryan Shader, University of Wyoming, E-mail: BShader@uwyo.edu

Title: The strong interlacing property for matrices with a given graph

Abstract. The (λ, μ)-problem for a graph G with n vertices asks: Given $2n - 1$ real numbers

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \text{ and } \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$$

does there exist a symmetric matrix A whose graph is G such that A has the λ_i as its eigenvalues, and the leading principal submatrix $A(n)$ of order n has the μ_j as its eigenvalues? Of course necessarily the λ and μ interlace, namely, $\lambda_i \leq \mu_i \leq \lambda_{i+1}$ for $i = 1, 2, \ldots, n-1$. Here we discuss a property of symmetric matrices A such that if A has the given property and graph G, then for each supergraph H of (λ, μ)-problem has an affirmative answer for H and the λ being the eigenvalues of A and the μ are the eigenvalues of $A(n)$.
This is joint work with Aida Abiad, Bryan A. Curtis, Mary Flagg, H. Tracy Hall, Jephian C.-H. Lin, and John Sinkovic

Speaker: Michael Stewart, Georgia State University, E-mail: mastewart@gsu.edu

Title: Numerical Ranks and Product Singular Values

Abstract. Generalization of the SVD of a single matrix to an implicit decomposition of a product of two matrices can be used to reveal information about the relation between fundamental subspaces of the two matrices. An implicit form of such a decomposition avoids numerical difficulties associated with the explicit computation of matrix products. Such decompositions have a number of applications, including to subspace identification algorithms from linear systems theory. Computing and partitioning a two matrix decomposition typically involves multiple interdependent numerical rank decisions in which small elements are truncated. The details of how this is done can have a big impact on the estimated dimensions. This talk describes some theorems related to numerical product rank, product rank decompositions, and some related computational methods.

Speaker: Tin-Yau Tam, University of Nevada, Reno, E-mail:ttam@unr.edu

Title: Extensions of Horn-Steinberg Theorem to simple complex classical group

Abstract. The polar decomposition for $A \in \mathrm{GL}_n(\mathbb{C})$ asserts that $A = U_1 P_1 = P_2 U_2$, where P_1, P_2 are $n \times n$ positive definite matrices and U_1, U_2 are unitary matrices. If A is nonsingular, then $U_1 = U_2 = U$. We call U the unitary part of A.

Let A have singular values $s_1 \geq \cdots \geq s_n > 0$, which are the square roots of the eigenvalues of P_1 (or P_2). Order the eigenvalues λ's of A so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. In 1949, Weyl established

$$|\lambda_1 \cdots \lambda_k| \leq s_1 \cdots s_k, \quad k = 1, \ldots, n, \quad |\lambda_1 \cdots \lambda_n| = s_1 \cdots s_n.$$

Such relation is called log-majorization. In 1954, A. Horn proved the converse, that is, there exists $A \in \mathrm{GL}_n(\mathbb{C})$ with prescribed singular values s's and eigenvalues λ's if they satisfy the log-majorization. In 1973, Kostant extended Wely-Horn Theorem in the context of semisimple Lie groups.

The torus in $\mathbb{C}^n$ is

$$\mathbb{T}^n := \{(e^{i\theta_1}, \ldots, e^{i\theta_n}) \mid \theta_1, \ldots, \theta_n \in \mathbb{R}\} \subset \mathbb{C}^n.$$

Let α be $(\alpha_1, \ldots, \alpha_n) \in \mathbb{T}^n$. If $\alpha_j = e^{i\theta_j}$ and $\alpha_k = e^{i\theta_k}$, $j \neq k$, with $0 < \theta_j - \theta_k < \pi$ and $0 \leq \varepsilon \leq (\theta_j - \theta_k)/2$, *pinching* them means replacing $\alpha_j = e^{i\theta_j}$ and $\alpha_k = e^{i\theta_k}$ by $e^{i(\theta_j - \varepsilon)}$ and $e^{i(\theta_k + \varepsilon)}$. We denote by $\hat{\lambda} := (\lambda_1/|\lambda_1|, \ldots, \lambda_n/|\lambda_n|)$ if $\lambda_1, \ldots, \lambda_n$ are nonzero complex numbers and it is called the *polar part* of $\lambda \in \mathbb{C}^n$.

In 1959, Horn and Steinberg proved that there is $A \in \mathrm{GL}_n(\mathbb{C})$ with eigenvalues λ_j's such that its unitary part has eigenvalues α_j's if and only if α can be reduced to $\hat{\lambda}$ by a finite sequence of pinches, up to a permutation. Additionally, they further provided geometric characterizations.

In this talk, we discuss the extensions of Horn-Steinberg Theorem in the context of simple complex classical groups.

Speaker: Frank Uhlig, Auburn University, E-mail: uhligfd@auburn.edu

Title: Adapted AZNN Methods for Time-Varying and Static Matrix Problems

Abstract. We present adapted Zhang Neural Networks (AZNN) in which the parameter settings for the exponential decay constant η and the length of the start-up phase of basic ZNN are adapted to the problem at hand. Specifically we study experiments with AZNN for time-varying square matrix factorizations as a product of two time-varying symmetric matrices and for the time-varying matrix square root problem.
Differing from generally used small η values and minimal length start-up phases in ZNN, we adapt the basic ZNN method to work with large or even gigantic η settings and arbitrary length start-up phases using the Euler low accuracy and unstable finite difference formula. These adaptations improve the speed of AZNN's convergence and lower its solution error bounds for our chosen problems significantly to near machine constant levels. Parameter-varying AZNN also allows us to find full rank symmetrizers of static matrices reliably, such as for the Kahan and Frank matrices, for matrices with highly ill-conditioned eigenvalues and for matrices with complicated Jordan structures of dimensions from $n = 2$ on up where standard eigendata based symmetrizer algorithms generally have failed. AZNN helps us to find full rank static matrix symmetrizers that have never been successfully computed before.

Speaker: James Weaver, University of West Florida, E-mail: jweaver@uwf.edu

Title: Frank Hall, Georgia State and the Gang of Eight

Abstract. In recognition of Dr. Frank Hall, it is my privilege to share some past experiences and memories which we shared as colleagues and friends. This talk will focus on the time frame from 1995 until 2000. It starts with the ILAS meeting in Atlanta, GA in 1995 and goes through the year 2000 with the publication of 2 papers by the Gang of Eight. The members of the Gang of Eight are introduced with some highlights of the experiences which they shared.

Speaker: Fuzhen Zhang, Nova Southeaster University, E-mail: zhang@nova.edu

Title: Singular Value Inequalities of Matrix Sum in Log-majorizations

Abstract. We show some upper bounds for the product of arbitrarily selected singular values of the sum of two matrices. The results are additional to our previous work on the lower bound eigenvalue inequalities of the sum of two positive semidefinite matrices. Besides, we state explicitly Hoffman's minimax theorem with a proof, and as applications of our main results, we revisit and give estimates for related determinant inequalities of Hua type.