

MINIMUM VERTEX DEGREE THRESHOLD FOR LOOSE HAMILTON CYCLES IN 3-UNIFORM HYPERGRAPHS

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ABSTRACT. We show that for sufficiently large n , every 3-uniform hypergraph on n vertices with minimum vertex degree at least $\binom{n-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} + c$, where $c = 2$ if $n \in 4\mathbb{N}$ and $c = 1$ if $n \in 2\mathbb{N} \setminus 4\mathbb{N}$, contains a loose Hamilton cycle. This degree condition is best possible and improves on the work of Buß, Hàn and Schacht who proved the corresponding asymptotical result.

1. INTRODUCTION

The study of Hamilton cycles is an important topic in graph theory. In recent years, researchers have worked on extending the classical theorem of Dirac [7] on Hamilton cycles to hypergraphs – see recent surveys of [23, 20].

Given $k \geq 2$, a k -uniform hypergraph (in short, k -graph) consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$, where every edge is a k -element subset of V . For $1 \leq l < k$, a k -graph is called an l -cycle if its vertices can be ordered cyclically such that each of its edges consists of k consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly l vertices. (If we allow $l = 0$, then a 0-cycle is merely a matching and perfect matchings have been intensively studied recently, e.g. [1, 5, 9, 16, 15, 21, 27, 30, 31]) In k -graphs, a $(k-1)$ -cycle is often called a *tight* cycle while a 1-cycle is often called a *loose* cycle. We say that a k -graph contains a *Hamilton l -cycle* if it contains an l -cycle as a spanning subhypergraph. Note that a Hamilton l -cycle of a k -graph on n vertices contains exactly $n/(k-l)$ edges, implying that $k-l$ divides n .

Given a k -graph \mathcal{H} with a set S of d vertices (where $1 \leq d \leq k-1$) we define $\deg_{\mathcal{H}}(S)$ to be the number of edges containing S (the subscript \mathcal{H} is omitted if it is clear from the context). The *minimum d -degree* $\delta_d(\mathcal{H})$ of \mathcal{H} is the minimum of $\deg_{\mathcal{H}}(S)$ over all d -vertex sets S in \mathcal{H} . We refer to $\delta_1(\mathcal{H})$ as the *minimum vertex degree* and $\delta_{k-1}(\mathcal{H})$ the *minimum codegree* of \mathcal{H} .

1.1. Hamilton cycles in hypergraphs. Confirming a conjecture of Katona and Kierstead [13], Rödl, Ruciński and Szemerédi [25, 26] showed that for any fixed k , every k -graph \mathcal{H} on n vertices with $\delta_{k-1}(\mathcal{H}) \geq n/2 + o(n)$ contains a tight Hamilton cycle. This is best possible up to the $o(n)$ term. With long and involved arguments, the same authors [28] improved this to an exact result for $k = 3$. Loose Hamilton cycles were first studied by Kühn and Osthus [18], who proved that every 3-graph on n vertices with $\delta_2(\mathcal{H}) \geq n/4 + o(n)$ contains a loose Hamilton cycle. Czygrinow and

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Molla [6] recently improved this to an exact result. The result of Kühn and Osthus [18] was generalized for arbitrary k and arbitrary $l < k/2$ by Hàn and Schacht [10], and independently by Keevash et al. [14] for $l = 1$ and arbitrary k . Later Kühn, Mycroft, and Osthus [17] obtained an asymptotically sharp bound on codegree for Hamilton l -cycles for all $l < k$. Hence, the problem of finding Hamilton l -cycles in k -graphs with large codegree is asymptotically solved.

Much less is known under other degree conditions. Recently Rödl and Ruciński [24] gave a sufficient vertex degree condition that guarantees a tight Hamilton cycle in 3-graphs. Glebov, Person and Weps [8] gave a nontrivial vertex degree condition for tight Hamilton cycles in k -graphs for general k . Neither of these results is best possible – see more discussion in Section 4.

Recently Buß, Hàn, and Schacht [2] studied the minimum vertex degree that guarantees a loose Hamilton cycle in 3-graphs and obtained the following result.

Theorem 1.1. [2, Theorem 3] *For all $\gamma > 0$ there exists an integer n_0 such that the following holds. Suppose \mathcal{H} is a 3-graph on $n > n_0$ with $n \in 2\mathbb{N}$ and*

$$\delta_1(\mathcal{H}) > \left(\frac{7}{16} + \gamma\right) \binom{n}{2}.$$

Then \mathcal{H} contains a loose Hamilton cycle.

In this paper we improve Theorem 1.1 as follows.

Theorem 1.2 (Main Result). *There exists an $n_{1,2} \in \mathbb{N}$ such that the following holds. Suppose that \mathcal{H} is a 3-graph on $n > n_{1,2}$ with $n \in 2\mathbb{N}$ and*

$$\delta_1(\mathcal{H}) \geq \binom{n-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} + c, \quad (1.1)$$

where $c = 2$ if $n \in 4\mathbb{N}$ and $c = 1$ otherwise. Then \mathcal{H} contains a loose Hamilton cycle.

The following construction shows that Theorem 1.2 is best possible. It is slightly stronger than [2, Fact 4].

Proposition 1.3. *For every $n \in 2\mathbb{N}$ there exists a 3-graph on n vertices with minimum vertex degree $\binom{n-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} + c - 1$, where c is defined as in Theorem 1.2, and which contains no loose Hamilton cycle.*

Proof. Let $\mathcal{H}_1 = (V_1, E_1)$ be the 3-graph on $n \in 2\mathbb{N} \setminus 4\mathbb{N}$ vertices such that $V_1 = A \dot{\cup} B$ ¹ with $|A| = \lceil \frac{n}{4} \rceil - 1$ and $|B| = \lfloor \frac{3n}{4} \rfloor + 1$, and E_1 consists of all triples intersecting A . Note that $\delta_1(\mathcal{H}_1) = \binom{n-1}{2} - \binom{\lfloor \frac{3n}{4} \rfloor}{2}$. Suppose that \mathcal{H}_1 contains a loose Hamilton cycle C . There are $n/2$ edges in C and every vertex in A is contained in at most two edges in C . Since $2|A| = \frac{n-2}{2}$, there is at least one edge of C whose vertices are completely from B . This is a contradiction since B is independent. So \mathcal{H}_1 contains no loose Hamilton cycle.

Let $\mathcal{H}_2 = (V_2, E_2)$ be a 3-graph on $n \in 4\mathbb{N}$ vertices such that $V_2 = A \dot{\cup} B$ with $|A| = \frac{n}{4} - 1$ and $|B| = \frac{3}{4}n + 1$, and E_2 consists of all triples intersecting A and those containing both b_1 and b_2 , where b_1, b_2 are two fixed vertices in B . Then $\delta_1(\mathcal{H}_2) = \binom{n-1}{2} - \binom{\frac{3}{4}n}{2} + 1$. Suppose that \mathcal{H}_2 contains a loose Hamilton cycle C . There are $n/2$ edges in C and every vertex in A is contained in at most two edges

¹Throughout the paper, we write $A \dot{\cup} B$ for $A \cup B$ when sets A, B are disjoint.

in C . Thus, there are at least two edges of C whose vertices are completely from B . But due to the construction, every two edges in B share two vertices so they cannot both appear in one loose cycle. This contradiction shows that \mathcal{H}_2 contains no loose Hamilton cycle. \square

As a typical approach of obtaining exact results, we distinguish the *extremal* case from the *nonextremal* case and solve them separately.

Definition 1.4. *Given $\beta > 0$, a 3-graph \mathcal{H} on n vertices is called β -extremal if there is a set $B \subseteq V(\mathcal{H})$, such that $|B| = \lfloor 3n/4 \rfloor$ and $e(B) \leq \beta n^3$.*

Theorem 1.5 (Extremal Case). *There exist $\beta > 0$ and $n_{1.5} \in \mathbb{N}$ such that the following holds. Let $n > n_{1.5}$ be an even integer. Suppose that \mathcal{H} is a 3-graph on n vertices satisfying (1.1). If \mathcal{H} is β -extremal, then \mathcal{H} contains a loose Hamilton cycle.*

Theorem 1.6 (Nonextremal Case). *For any $\beta > 0$, there exist $\gamma > 0$ and $n_{1.6} \in \mathbb{N}$ such that the following holds. Let $n > n_{1.6}$ be an even integer. Suppose that \mathcal{H} is a 3-graph on n vertices satisfying $\delta_1(\mathcal{H}) \geq (\frac{7}{16} - \gamma) \binom{n}{2}$. If \mathcal{H} is not β -extremal, then \mathcal{H} contains a loose Hamilton cycle.*

Theorem 1.2 follows from Theorems 1.5 and 1.6 immediately by choosing β from Theorem 1.5 and letting $n_{1.2} = \max\{n_{1.5}, n_{1.6}\}$.

Let us discuss our proof ideas here. The proof of Theorem 1.5 is somewhat standard (though non-trivial). The proof of Theorem 1.6 follows the approach in the previous work [2, 10, 17, 25, 26, 28]. Roughly speaking, we use the *absorbing method* initiated by Rödl, Ruciński and Szemerédi, which reduces the task of finding a loose Hamilton cycle to finding constantly many vertex-disjoint loose paths that covers almost all the vertices of the 3-graph. More precisely, we first apply the Absorbing Lemma (Lemma 2.1) and obtain a (short) absorbing path \mathcal{P}_0 which can absorb any smaller proportion of vertices. Second, apply the Reservoir Lemma (Lemma 2.2) and find a small reservoir set R whose vertices may be used to connect any constant number of loose paths to a loose cycle. Third, apply the Path-tiling Lemma (Lemma 2.3) in the remaining 3-graph and obtain constantly many vertex-disjoint loose paths covering almost all the vertices. Fourth, connect these paths (including \mathcal{P}_0) together by the reservoir R and get a loose cycle C . Finally we absorb the vertices not in $V(C)$ to \mathcal{P}_0 and obtain the desired loose Hamilton cycle.

The Absorbing Lemma and the Reservoir Lemma are not very difficult and already proven in [2]. Thus the main step is to prove the Path-tiling Lemma, under the assumption $\delta_1(\mathcal{H}) \geq (\frac{7}{16} - \gamma) \binom{n}{2}$ and that \mathcal{H} is not β -extremal (in contrast, $\delta_1(\mathcal{H}) \geq (\frac{7}{16} + \gamma) \binom{n}{2}$ is assumed in [2]). As shown in [2, 10], after applying the (weak) Regularity Lemma, it suffices to prove that the cluster 3-graph \mathcal{K} can be tiled almost perfectly by some particular 3-graph. For example, the 3-graph \mathcal{M} given in [2] has the vertex set $[8] = \{1, 2, \dots, 8\}$ and edges $123, 345, 456, 678$.² Since it is hard to find an \mathcal{M} -tiling directly, the authors of [2] found a *fractional* \mathcal{M} -tiling instead and converted it to an (integer) \mathcal{M} -tiling by applying the Regularity Lemma again. In this paper we consider a much simpler 3-graph \mathcal{Y} with vertex set $[4]$ and edges $123, 234$, and obtain an almost perfect \mathcal{Y} -tiling in \mathcal{K} directly. Interestingly, \mathcal{Y} -tiling was studied (via the codegree condition) in the very first paper on loose Hamilton cycles [18].

²Throughout the paper, we often represent a set $\{v_1, v_2, \dots, v_k\}$ as $v_1 v_2 \dots v_k$.

Comparing with the first exact result on Hamilton cycles in hypergraphs [28], our proof is much shorter because the Absorbing and Reservoir Lemmas in [28] are much harder to prove.

The rest of the paper is organized as follows: we prove Theorem 1.6 in Section 2 and Theorem 1.5 in Section 3, and give concluding remarks in Section 4.

1.2. Notations. Given a vertex v and disjoint vertex sets S, T in a 3-graph \mathcal{H} , we denote by $\deg_{\mathcal{H}}(v, S)$ the number of edges that contain v and two vertices from S , and by $\deg_{\mathcal{H}}(v, ST)$ the number of edges that contain v , one vertex from S and one vertex from T . Furthermore, let $\overline{\deg}_{\mathcal{H}}(v, S) = \binom{|S|}{2} - \deg_{\mathcal{H}}(v, S)$ and $\overline{\deg}_{\mathcal{H}}(v, ST) = |S| \cdot |T| - \deg_{\mathcal{H}}(v, ST)$. Given not necessarily disjoint sets $X, Y, Z \subseteq V(\mathcal{H})$, we define

$$E_{\mathcal{H}}(XYZ) = \{xyz \in E(\mathcal{H}) : x \in X, y \in Y, z \in Z\},$$

$$\overline{E}_{\mathcal{H}}(XYZ) = \left\{ xyz \in \binom{V(\mathcal{H})}{3} \setminus E(\mathcal{H}) : x \in X, y \in Y, z \in Z \right\},$$

$e_{\mathcal{H}}(XYZ) = |E_{\mathcal{H}}(XYZ)|$, and $\overline{e}_{\mathcal{H}}(XYZ) = |\overline{E}_{\mathcal{H}}(XYZ)|$. The subscript \mathcal{H} is often omitted when it is clear from the context.

A *loose path* $\mathcal{P} = v_1 v_2 \cdots v_{2k+1}$ is a 3-graph on $\{v_1, v_2, \dots, v_{2k+1}\}$ with edges $v_{2i-1} v_{2i} v_{2i+1}$ for all $i \in [k]$. The vertices v_1 and v_{2k+1} are called the *ends* of \mathcal{P} .

2. PROOF OF THEOREM 1.6

In this section we prove Theorem 1.6 by following the same approach as in [2].

2.1. Auxiliary lemmas and Proof of Theorem 1.6. For convenience, we rephrase the Absorbing Lemma [2, Lemma 7] as follows.³

Lemma 2.1 (Absorbing Lemma). *For any $0 < \gamma_1 \leq 10^{-14}$ there exists an integer $n_{2.1}$ such that the following holds. Let \mathcal{H} be a 3-graph on $n > n_{2.1}$ vertices with $\delta_1(\mathcal{H}) \geq \frac{13}{32} \binom{n}{2}$. Then there is a loose path \mathcal{P} with $|V(\mathcal{P})| \leq \gamma_1 n$ such that for every subset $U \subseteq V \setminus V(\mathcal{P})$ with $|U| \leq \gamma_1^3 n$ and $|U| \in 2\mathbb{N}$ there exists a loose path \mathcal{Q} with $V(\mathcal{Q}) = V(\mathcal{P}) \cup U$ such that \mathcal{P} and \mathcal{Q} have the same ends.*

We also need the Reservoir Lemma [2, Lemma 6].

Lemma 2.2 (Reservoir Lemma). *For any $0 < \gamma_2 < 1/4$ there exists an integer $n_{2.2}$ such that for every 3-graph \mathcal{H} on $n > n_{2.2}$ vertices satisfying*

$$\delta_1(\mathcal{H}) \geq (1/4 + \gamma_2) \binom{n}{2},$$

there is a set R of size at most $\gamma_2 n$ with the following property: for every $k \leq \gamma_2^3 n / 12$ mutually disjoint pairs $\{a_i, b_i\}_{i \in [k]}$ of vertices from $V(\mathcal{H})$ there are $3k$ vertices $u_i, v_i, w_i, i \in [k]$ from R such that $a_i u_i v_i, v_i w_i b_i \in \mathcal{H}$ for all $i \in [k]$.

The main step in our proof of Theorem 1.6 is the following lemma, which is stronger than [2, Lemma 10].

³Lemma 7 in [2] assumes that $\delta_1(\mathcal{H}) \geq (\frac{5}{8} + \gamma)^2 \binom{n}{2}$ and returns $|V(\mathcal{P})| \leq \gamma^7 n$ with $|U| \leq \frac{\gamma^{14}}{14336} n$. We simply take their γ^7 as our γ_1 and thus $\gamma_1 \leq \left(\sqrt{\frac{13}{32}} - \frac{5}{8}\right)^7 \approx 10^{-14}$.

Lemma 2.3 (Path-tiling lemma). *For any $0 < \gamma_3, \alpha < 1$ there exist integers p and $n_{2.3}$ such that the following holds for $n > n_{2.3}$. Suppose \mathcal{H} is a 3-graph on n vertices with minimum vertex degree*

$$\delta_1(\mathcal{H}) \geq \left(\frac{7}{16} - \gamma_3\right) \binom{n}{2},$$

then there are at most p vertex disjoint loose paths in \mathcal{H} that together cover all but at most αn vertices of \mathcal{H} unless \mathcal{H} is $2050\gamma_3$ -extremal.

Proof of Theorem 1.6. Given $\beta > 0$, let $\gamma = \min\{\frac{\beta}{4101}, 10^{-14}\}$. We choose $n_{1.6} = \max\{n_{2.1}, 2n_{2.2}, 2n_{2.3}, 192(p+1)/(\gamma/3)^9\}$, where p is the constant returned from Lemma 2.3 with $\gamma_3 = 2\gamma$ and $\alpha = (\gamma/3)^3/2$. Let $n > n_{1.6}$ be an even integer.

Suppose that $\mathcal{H} = (V, E)$ is a 3-graph on n vertices with $\delta_1(\mathcal{H}) \geq (\frac{7}{16} - \gamma) \binom{n}{2}$. Since $\frac{7}{16} - \gamma > \frac{13}{32}$, we can apply Lemma 2.1 with $\gamma_1 = \gamma/3$ and obtain an absorbing path \mathcal{P}_0 with ends a_0, b_0 . We next apply Lemma 2.2 with $\gamma_2 = (\gamma/3)^3/2$ to $\mathcal{H}[(V \setminus V(\mathcal{P}_0)) \cup \{a_0, b_0\}]$ and obtain a reservoir R . Let $V' = V \setminus (V(\mathcal{P}_0) \cup R)$ and $n' = |V'|$. Note that $n - n' \leq \gamma_1 n + \gamma_2 n < \gamma n/2$. The induced subhypergraph $\mathcal{H}' = \mathcal{H}[V']$ satisfies

$$\delta_1(\mathcal{H}') \geq \left(\frac{7}{16} - \gamma\right) \binom{n}{2} - \frac{\gamma}{2} n \cdot (n-2) > \left(\frac{7}{16} - 2\gamma\right) \binom{n'}{2}.$$

Applying Lemma 2.3 to \mathcal{H}' with $\gamma_3 = 2\gamma$ and $\alpha = (\gamma/3)^3/2$, we obtain at most p vertex disjoint loose paths that cover all but at most $\alpha n'$ vertices of \mathcal{H}' , unless \mathcal{H}' is $2050\gamma_3$ -extremal. In the latter case, there exists $B' \subseteq V'$ such that $|B'| = \lfloor \frac{3}{4}n' \rfloor$ and $e(B') \leq 4100\gamma(n')^3$. Then we add $\lfloor \frac{3}{4}n \rfloor - \lfloor \frac{3}{4}n' \rfloor < \gamma n/2$ arbitrary vertices from $V \setminus B'$ to B' to get a vertex set B such that $|B| = \lfloor \frac{3}{4}n \rfloor$ and

$$e(B) \leq 4100\gamma(n')^3 + \frac{\gamma n}{2} \binom{n-1}{2} < 4101\gamma n^3 \leq \beta n^3,$$

which means that \mathcal{H} is β -extremal, a contradiction. In the former case, denote these loose paths by $\{\mathcal{P}_i\}_{i \in [p']}$ for some $p' \leq p$, and their ends by $\{a_i, b_i\}_{i \in [p']}$. The choice of $n_{1.6}$ guarantees that $p' + 1 \leq p + 1 \leq \gamma_2^3 n/24$. We can thus connect $\{a_i, b_{i+1}\}_{0 \leq i \leq p'-1} \cup \{a_{p'}, b_0\}$ by using vertices from R obtaining a loose cycle C . Since $|V \setminus C| \leq |R| + \alpha n' \leq \gamma_2 n + \gamma_2 n' \leq \gamma_1^3 n$, we can use \mathcal{P}_0 to absorb all unused vertices in R and uncovered vertices in V' . \square

The rest of this section is devoted to the proof of Lemma 2.3.

2.2. Proof of Lemma 2.3. Following the approach in [2], we will use the weak regularity lemma which is a straightforward extension of Szemerédi's regularity lemma for graphs [29]. Below we only state this lemma for 3-graphs.

Let $\mathcal{H} = (V, E)$ be a 3-graph and let A_1, A_2, A_3 be mutually disjoint non-empty subsets of V . We define $e(A_1, A_2, A_3)$ to be the number of edges with one vertex in each A_i , $i \in [3]$, and the density of \mathcal{H} with respect to (A_1, A_2, A_3) as

$$d(A_1, A_2, A_3) = \frac{e(A_1, A_2, A_3)}{|A_1||A_2||A_3|}.$$

Given $\epsilon > 0$, the triple (V_1, V_2, V_3) of mutually disjoint subsets $V_1, V_2, V_3 \subseteq V$ is called ϵ -regular if

$$|d(A_1, A_2, A_3) - d(V_1, V_2, V_3)| \leq \epsilon$$

for all triple of subsets of $A_i \subseteq V_i$, $i \in [3]$, satisfying $|A_i| \geq \epsilon|V_i|$. We say (V_1, V_2, V_3) is (ϵ, d) -regular if it is ϵ -regular and $d(V_1, V_2, V_3) \geq d$ for some $d \geq 0$. It is immediate from the definition that in an (ϵ, d) -regular triple (V_1, V_2, V_3) , if $V'_i \subseteq V_i$ has size $|V'_i| \geq c|V_i|$ for some $c \geq \epsilon$, then (V'_1, V'_2, V'_3) is $(\max\{\epsilon/c, 2\epsilon\}, d - \epsilon)$ -regular.

Theorem 2.4. [2, Theorem 14] *For any $t_0 \geq 0$ and $\epsilon > 0$, there exist T_0 and n_0 so that for every 3-graph $\mathcal{H} = (V, E)$ on $n > n_0$ vertices, there exists a partition $V = V_0 \cup V_1 \cup \dots \cup V_t$ such that*

- (i) $t_0 \leq t \leq T_0$,
- (ii) $|V_1| = |V_2| = \dots = |V_t|$ and $|V_0| \leq \epsilon n$,
- (iii) for all but at most $\epsilon \binom{t}{3}$ sets $i_1 i_2 i_3 \in \binom{[t]}{3}$, the triple $(V_{i_1}, V_{i_2}, V_{i_3})$ is ϵ -regular.

A partition as given in Theorem 2.4 is called an (ϵ, t) -regular partition of \mathcal{H} . For an (ϵ, t) -regular partition of \mathcal{H} and $d \geq 0$ we refer to $\mathcal{Q} = (V_i)_{i \in [t]}$ as the family of clusters and define the cluster hypergraph $\mathcal{K} = \mathcal{K}(\epsilon, d, \mathcal{Q})$ with vertex set $[t]$ and $i_1 i_2 i_3 \in \binom{[t]}{3}$ is an edge if and only if $(V_{i_1}, V_{i_2}, V_{i_3})$ is (ϵ, d) -regular.

The following corollary shows that the cluster hypergraph inherits the minimum degree of the original hypergraph. Its proof is the same as that of [2, Proposition 15] after we replace $7/16 + \gamma$ by c (we thus omit the proof).

Corollary 2.5. *For $c > d > \epsilon > 0$ and $t_0 \geq 0$ there exist T_0 and n_0 such that the following holds. Suppose \mathcal{H} is a 3-graph on $n > n_0$ vertices which has minimum vertex degree $\delta_1(\mathcal{H}) \geq c \binom{n}{2}$. Then there exists an (ϵ, t) -regular partition \mathcal{Q} with $t_0 < t < T_0$ such that the cluster hypergraph $\mathcal{K} = \mathcal{K}(\epsilon, d, \mathcal{Q})$ has minimum vertex degree $\delta_1(\mathcal{K}) \geq (c - \epsilon - d) \binom{t}{2}$.*

In 3-graphs, a loose path is 3-partite with partition sizes about $m, m, 2m$ for some integer m . Proposition 2.6 below shows that every regular triple with partition sizes $m, m, 2m$ contains an almost spanning loose path as a subhypergraph. In contrast, [2, Proposition 25] (more generally [10, Lemma 20]) shows that every regular triple with partition sizes $3m, 3m, 2m$ contains constant many vertex disjoint loose paths. The proof of Proposition 2.6 uses the standard approach of handling regularity.

Proposition 2.6. *Fix any $\epsilon > 0$, $d > 2\epsilon$, and an integer $m \geq \frac{d}{\epsilon(d-2\epsilon)}$. Suppose that $V(\mathcal{H}) = V_1 \cup V_2 \cup V_3$ and (V_1, V_2, V_3) is (ϵ, d) -regular with $|V_i| = m$ for $i = 1, 3$ and $|V_2| = 2m$. Then there is a loose path P omitting at most $8\epsilon m/d + 3$ vertices of \mathcal{H} .*

Proof. We will greedily construct the loose path $P = v_1 v_2 \dots v_{2k+1}$ such that $v_{2i} \in V_2$, $v_{4i+1} \in V_1$ and $v_{4i+3} \in V_3$ until $|V_i \setminus V(P)| < \frac{2\epsilon}{d}|V_i|$ for some $i \in [3]$. For $j \in [3]$, let $U_j^0 = V_j$ and $U_j^i = V_j \setminus \{v_1, \dots, v_{2i-1}\}$ for $i \in [k]$. In addition, we require that for $i = 0, \dots, k$,

$$\deg(v_{2i+1}, U_2^i U_r^i) \geq (d - \epsilon)|U_2^i||U_r^i|, \quad (2.1)$$

where $r \equiv 2i - 1 \pmod{4}$. We proceed by induction on i . First we pick a vertex $v_1 \in V_1$ such that $\deg(v_1, V_2 V_3) \geq (d - \epsilon)|V_2||V_3|$ (thus (2.1) holds for $i = 0$). By regularity, all but at most $\epsilon|V_1|$ vertices can be chosen as v_1 . Suppose that we have selected v_1, \dots, v_{2i-1} . Without loss of generality, assume that $v_{2i-1} \in V_1$. Our goal is to choose $v_{2i} \in U_2^i, v_{2i+1} \in U_3^i$ such that

- (i) $v_{2i-1} v_{2i} v_{2i+1} \in E(\mathcal{H})$,
- (ii) $\deg(v_{2i+1}, U_1^i U_2^i) \geq (d - \epsilon)|U_1^i||U_2^i|$.

In fact, the induction hypothesis implies that $\deg(v_{2i-1}, U_2^{i-1}U_3^{i-1}) \geq (d - \epsilon)|U_2^{i-1}||U_3^{i-1}|$. Since $U_2^i = U_2^{i-1} \setminus \{v_{2i-2}\}$ and $U_3^i = U_3^{i-1}$, we have

$$\deg(v_{2i-1}, U_2^iU_3^i) \geq (d - \epsilon)|U_2^{i-1}||U_3^{i-1}| - |U_3^{i-1}| = ((d - \epsilon)|U_2^{i-1}| - 1)|U_3^{i-1}|.$$

By regularity, at most $\epsilon|V_3|$ vertices in V_3 does not satisfy (ii). So, at least

$$\deg(v_{2i-1}, U_2^iU_3^i) - \epsilon|V_3| \cdot |U_2^{i-1}| \geq ((d - \epsilon)|U_2^{i-1}| - 1)|U_3^{i-1}| - \epsilon|V_3| \cdot |U_2^{i-1}| \quad (2.2)$$

pairs of vertices can be chosen as v_{2i}, v_{2i+1} . Since $|U_3^{i-1}| \geq \frac{2\epsilon}{d}|V_3|$ and $|U_2^{i-1}| \geq \frac{2\epsilon}{d}|V_2| \geq \frac{4}{d-2\epsilon}$ (using $m \geq \frac{d}{\epsilon(d-2\epsilon)}$), the right side of (2.2) is at least

$$\left((d - \epsilon)|U_2^{i-1}| - 1 \right) \frac{2\epsilon}{d}|V_3| - \epsilon|V_3| \cdot |U_2^{i-1}| = \left((d - 2\epsilon)|U_2^{i-1}| - 2 \right) \frac{\epsilon}{d}|V_3| > 0,$$

thus the selection of v_{2i}, v_{2i+1} satisfying (i) and (ii) is guaranteed.

To calculate the number of the vertices omitted by $P = v_1v_2 \cdots v_{2k+1}$, note that $|V_1 \cap V(P)| = \lceil \frac{k+1}{2} \rceil$, $|V_2 \cap V(P)| = k$, and $|V_3 \cap V(P)| = \lfloor \frac{k+1}{2} \rfloor$. Our greedy construction of P stops as soon as $|V_i \setminus V(P)| < \frac{2\epsilon}{d}|V_i|$ for some $i \in [3]$. As $|V_1| = |V_3| = m = |V_2|/2$, one of the following three inequalities holds:

$$m - \left\lfloor \frac{k+1}{2} \right\rfloor < \frac{2\epsilon}{d}m, \quad 2m - k < \frac{2\epsilon}{d}2m, \quad m - \left\lceil \frac{k+1}{2} \right\rceil < \frac{2\epsilon}{d}m.$$

Thus we always have $m - \lceil \frac{k+1}{2} \rceil < \frac{2\epsilon}{d}m$, which implies that $\frac{k+2}{2} > (1 - \frac{2\epsilon}{d})m$ or $k > 2(1 - \frac{2\epsilon}{d})m - 2$. Consequently,

$$|V(\mathcal{H}) \setminus V(P)| = 4m - (2k + 1) < 4m - \left(4 \left(1 - \frac{2\epsilon}{d} \right) m - 4 + 1 \right) = \frac{8\epsilon}{d}m + 3. \quad \square$$

Let \mathcal{Y} be the 3-graph on the vertex set [4] with edges 123, 234 (the unique 3-graph with four vertices and two edges). The following lemma is the main step in our proof of Lemma 2.3. In general, given two (hyper)graphs \mathcal{F} and \mathcal{G} , an \mathcal{F} -tiling is a sub(hyper)graph of \mathcal{G} that consists of vertex disjoint copies of \mathcal{F} . The \mathcal{F} -tiling is *perfect* if it is a spanning sub(hyper)graph of \mathcal{G} .

Lemma 2.7 (\mathcal{Y} -tiling Lemma). *For any $\gamma > 0$, there exists an integer $n_{2.7}$ such that the following holds. Suppose \mathcal{H} is a 3-graph on $n > n_{2.7}$ vertices with*

$$\delta_1(\mathcal{H}) \geq \left(\frac{7}{16} - \gamma \right) \binom{n}{2},$$

then there is a \mathcal{Y} -tiling covering all but at most $2^{19}/\gamma$ vertices of \mathcal{H} unless \mathcal{H} is $2^{10}\gamma$ -extremal.

Now we are ready to prove Lemma 2.3 using the same approach as in [2].

Proof of Lemma 2.3. Given $0 < \gamma_3, \alpha < 1$, let $n_{2.3} = \max\{n_0, 4T_0/\epsilon\}$ and $p = T_0/2$, where T_0 and n_0 are the constants returned from Corollary 2.5 with $c = \frac{7}{16} - \gamma_3$, $d = \gamma_3/2$, $\epsilon = \frac{\alpha d}{8+\alpha}$, and $t_0 = \max\{n_{2.7}, \frac{2^{20}}{\gamma_3\alpha}\}$.

Suppose that \mathcal{H} is a 3-graph on $n > n_{2.3}$ vertices with $\delta_1(\mathcal{H}) \geq (\frac{7}{16} - \gamma_3) \binom{n}{2}$. By applying Corollary 2.5 with the constants chosen above, we obtain an (ϵ, t) -regular partition \mathcal{Q} . The cluster hypergraph $\mathcal{K} = \mathcal{K}(\epsilon, d, \mathcal{Q})$ satisfies $\delta_1(\mathcal{K}) \geq (\frac{7}{16} - 2\gamma_3) \binom{t}{2}$. Let m be the size of each cluster except V_0 , then $(1 - \epsilon)\frac{n}{t} \leq m \leq \frac{n}{t}$. By Lemma 2.7, either \mathcal{K} is $2^{10}(2\gamma_3)$ -extremal, or there is a \mathcal{Y} -tiling \mathcal{S} of \mathcal{K} that covers all but at most $2^{19}/(2\gamma_3)$ vertices of \mathcal{K} . In the first case, there exists a set $B \subseteq V(\mathcal{K})$ such

that $|B| = \lfloor \frac{3t}{4} \rfloor$ and $e(B) \leq 2^{11}\gamma_3 t^3$. Let $B' \subseteq V(\mathcal{H})$ be the union of the clusters in B . By regularity,

$$e(B') \leq e(B) \cdot m^3 + \binom{t}{3} \cdot d \cdot m^3 + \epsilon \cdot \binom{t}{3} \cdot m^3 + \binom{m}{2} n,$$

where the right-hand side bounds the number of edges from regular triples with high density, edges from regular triples with low density, edges from irregular triples and edges that are from at most two clusters. Since $m \leq \frac{n}{t}$, $\epsilon < d < \gamma_3$, and $t^{-2} < t_0^{-2} < \gamma_3$, we get

$$e(B') \leq 2^{11}\gamma_3 t^3 \left(\frac{n}{t}\right)^3 + d \binom{t}{3} \left(\frac{n}{t}\right)^3 + \epsilon \binom{t}{3} \left(\frac{n}{t}\right)^3 + \binom{n/t}{2} n < 2049\gamma_3 n^3.$$

Note that $|B'| = \lfloor \frac{3t}{4} \rfloor m \leq \frac{3t}{4} \cdot \frac{n}{t} = \frac{3n}{4}$ implies that $|B'| \leq \lfloor \frac{3n}{4} \rfloor$. On the other hand,

$$|B'| = \left\lfloor \frac{3t}{4} \right\rfloor m \geq \left(\frac{3t}{4} - 1\right) (1 - \epsilon) \frac{n}{t} \geq \left(\frac{3t}{4} - \epsilon t\right) \frac{n}{t} = \frac{3n}{4} - \epsilon n,$$

by adding at most ϵn vertices from $V \setminus B'$ to B' , we get a set $B'' \subseteq V(\mathcal{H})$ of size exactly $\lfloor 3n/4 \rfloor$, with $e(B'') \leq e(B') + \epsilon n \cdot n^2 < 2050\gamma_3 n^3$. Hence \mathcal{H} is $2050\gamma_3$ -extremal.

In the second case, the union of the clusters covered by \mathcal{Y} contains all but at most $\frac{2^{19}}{2\gamma_3} m + |V_0| \leq \alpha n/4 + \epsilon n < 3\alpha n/8$ vertices (here we use $t \geq \frac{2^{20}}{\gamma_3 \alpha}$). We will apply Proposition 2.6 to each member $\mathcal{Y}' \in \mathcal{Y}$. Suppose that \mathcal{Y}' has the vertex set $[4]$ with edges $123, 234$. For $i \in [4]$, let V_i denote the corresponding cluster in \mathcal{H} . We split V_i , $i = 2, 3$, into two disjoint sets V_i^1 and V_i^2 of equal sizes. Then the triples (V_1, V_2^1, V_3^1) and (V_4, V_2^2, V_3^2) are $(2\epsilon, d - \epsilon)$ -regular and of sizes $m, \frac{m}{2}, \frac{m}{2}$. Applying Proposition 2.6 to these two triples with $m' = \frac{m}{2}$, we find a loose path in each triple covering all but at most $\frac{8(2\epsilon)}{d-\epsilon} m' + 3 = \alpha m + 3$ vertices (here we need $\epsilon = \frac{\alpha d}{8+\alpha}$).

Since $|\mathcal{Y}| \leq t/4$, we obtain a path tiling that consists of at most $2t/4 \leq T_0/2 = p$ paths and covers all but at most

$$2(\alpha m + 3) \frac{t}{4} + \frac{3\alpha}{8} n \leq \frac{\alpha}{2} n + \frac{3t}{2} + \frac{3\alpha}{8} n < \alpha n$$

vertices. This completes the proof. \square

2.3. Proof of \mathcal{Y} -tiling Lemma (Lemma 2.7).

Fact 2.8. *Let \mathcal{H} be a 3-graph on m vertices which contains no copy of \mathcal{Y} , then $e(\mathcal{H}) \leq \frac{1}{3} \binom{m}{2}$.*

Proof. Since there is no copy of \mathcal{Y} , then given any $u, v \in V(\mathcal{H})$, we have that $\deg(uv) \leq 1$, which implies $e(\mathcal{H}) \leq \frac{1}{3} \binom{m}{2} \cdot 1 = \frac{1}{3} \binom{m}{2}$. \square

Proof of Lemma 2.7. Fix $\gamma > 0$ and let $n \in \mathbb{N}$ be sufficiently large. Let \mathcal{H} be a 3-graph on n vertices that satisfies $\delta_1(\mathcal{H}) \geq (\frac{7}{16} - \gamma) \binom{n}{2}$. Fix a largest \mathcal{Y} -tiling $\mathcal{Y} = \{\mathcal{Y}_1, \dots, \mathcal{Y}_m\}$ and let $V_i = V(\mathcal{Y}_i)$ for $i \in [m]$. Let $V' = \bigcup_{i \in [m]} V_i$ and $U = V(\mathcal{H}) \setminus V'$. Assume that $|U| > 2^{19}/\gamma$ – otherwise we are done.

Our goal is to find a set C of vertices in V' of size at most $n/4$ that covers almost all the edges, which implies that \mathcal{H} is extremal.

Let \mathcal{A}_i be the set of all edges with exactly i vertices in V' , for $i = 0, 1, 2, 3$. Note that $|\mathcal{A}_0| \leq \frac{1}{3} \binom{|U|}{2}$ by Fact 2.8. We may assume that $|U| < \frac{3}{4}n$ and consequently

$$m > \frac{n}{16}. \quad (2.3)$$

Indeed, if $|U| \geq \frac{3}{4}n$, then taking $U' \subseteq U$ of size $\lfloor \frac{3}{4}n \rfloor$, we get that $e(U') \leq e(U) \leq \frac{1}{3} \binom{|U|}{2} \leq \frac{1}{6}n^2 < \gamma n^3$. Thus \mathcal{H} is γ -extremal and we are done.

Claim 2.9. $|\mathcal{A}_1| \leq m \binom{|U|}{2} + 12m|U|$.

Proof. Let D be the set of vertices $v \in V'$ such that $\deg(v, U) \geq 4|U|$. First observe that every $\mathcal{Y}_i \in \mathcal{Y}$ contains at most one vertex in D . Suppose instead, two vertices $x, y \in V_i$ are both in D . Since $\deg(x, U) \geq 4|U| > |U|/2$, the link graph⁴ of x on U contains a path $u_1u_2u_3$ of length two. The link graph of y on $U \setminus \{u_1, u_2, u_3\}$ has size at least $4|U| - 3|U| > |U|/2$, so it also contains a path of length two, with vertices denoted by u_4, u_5, u_6 . Note that $xu_1u_2u_3$ and $yu_4u_5u_6$ span two vertex disjoint copies of \mathcal{Y} . Replacing \mathcal{Y}_i in \mathcal{Y} with them creates a larger \mathcal{Y} -tiling, contradicting the maximality of \mathcal{Y} . So we conclude that $|D| \leq m$. Consequently,

$$|\mathcal{A}_1| \leq |D| \cdot \binom{|U|}{2} + |V' \setminus D| \cdot 4|U| \leq m \binom{|U|}{2} + 3m \cdot 4|U| \leq m \binom{|U|}{2} + 12m|U|. \quad \square$$

Fix $u \in U$, $i \neq j \in [m]$, we denote by $L_{i,j}(u)$ the link graph of u induced on (V_i, V_j) , namely the bipartite link graph of u between V_i and V_j . Let $\mathcal{T}_{\leq 6}$ be the set of all triples uij , $u \in U$, $i, j \in [m]$ such that $e(L_{i,j}(u)) \leq 6$. Let \mathcal{T}_7^1 be the set of all triples uij , $u \in U$, $i, j \in [m]$ such that⁵ $e(L_{i,j}(u)) = 7$ and $L_{i,j}(u)$ has a vertex cover of two vertices with one from V_i and the other from V_j . Let $\mathcal{T}_{\geq 7}^2$ be the set of all triples uij , $u \in U$, $i, j \in [m]$ such that $e(L_{i,j}(u)) \geq 7$ and $L_{i,j}(u)$ has a vertex cover of two vertices both from V_i or V_j . Let $\mathcal{T}_{\geq 7}^3$ be the set of all triples uij , $u \in U$, $i, j \in [m]$ such that $e(L_{i,j}(u)) \geq 7$ and $L_{i,j}(u)$ contains a matching of size three. By the König–Egervary theorem, a bipartite graph either contains a matching of size three or a vertex cover of size two. Thus $\mathcal{T}_{\leq 6}$, \mathcal{T}_7^1 , $\mathcal{T}_{\geq 7}^2$, $\mathcal{T}_{\geq 7}^3$ form a partition of $U \times \binom{[m]}{2}$.

In order to bound the sizes of $\mathcal{T}_{\leq 6}$, \mathcal{T}_7^1 , $\mathcal{T}_{\geq 7}^2$, $\mathcal{T}_{\geq 7}^3$, we need the following fact.

Fact 2.10. (i) \mathcal{H} does not contain $i \neq j \in [m]$ and six vertices $u_1, \dots, u_6 \in U$ such that u_1, \dots, u_6 have the same (labeled) link graph on (V_i, V_j) and $u_1ij \in \mathcal{T}_{\geq 7}^3$.

(ii) \mathcal{H} does not contain distinct $i, j, k \in [m]$ and eight vertices $u_1, \dots, u_8 \in U$ such that the following holds. First, u_1, \dots, u_4 share the same link graph on (V_i, V_j) , and u_5, \dots, u_8 share the same link graph on (V_i, V_k) . Second, $u_1ij \in \mathcal{T}_{\geq 7}^2$ with the vertex cover in V_j and $u_5ik \in \mathcal{T}_{\geq 7}^2$ with the vertex cover in V_k .

Proof. To see Part (i), since there is a matching of size three in the (same) link graph of u_1, \dots, u_6 , say, a_1b_1, a_2b_2, a_3b_3 , then $u_1u_2a_1b_1$, $u_3u_4a_2b_2$ and $u_5u_6a_3b_3$

⁴Given a 3-graph \mathcal{H} with a vertex v , the link graph of v has the vertex set $V(\mathcal{H}) \setminus \{v\}$ and edge set $\{e \setminus \{v\} : e \in E(\mathcal{H})\}$.

⁵We could have $e(L_{i,j}(u)) \geq 7$ here; but since $L_{i,j}(u)$ contains a vertex cover with one vertex from V_i and one vertex from V_j , we must have $e(L_{i,j}(u)) = 7$.



FIGURE 1. The bipartite graph $L_{i,j}(u)$ when the triple is in \mathcal{T}_7^1 or $\mathcal{T}_{\geq 7}^2$, where the dotted line could be present or not.

span three copies of \mathcal{Y} . Replacing $\mathcal{Y}_i, \mathcal{Y}_j$ by them gives a \mathcal{Y} -tiling larger than \mathcal{Y} , a contradiction.

To see Part (ii), assume that $V_i = \{a, b, c, d\}$. Suppose that the vertex cover of $L_{i,j}(u_1)$ is $\{x_1, y_1\} \subseteq V_j$ and the vertex cover of $L_{i,k}(u_5)$ is $\{x_2, y_2\} \subseteq V_k$. Since $u_1 i j \in \mathcal{T}_{\geq 7}^2$, at most one pair from $\{x_1, y_1\} \times \{a, b\}$ is not in $L_{i,j}(u_1)$. Analogously at most one pair from $\{x_2, y_2\} \times \{c, d\}$ is not in $L_{i,k}(u_5)$. Thus, without loss of generality, we may assume that $x_1 a, y_1 b \in L_{i,j}(u_1)$ and $x_2 c, y_2 d \in L_{i,k}(u_5)$. Since u_1, \dots, u_4 share the same link graph on (V_i, V_j) , $u_1 u_2 x_1 a, u_3 u_4 y_1 b$ span two copies of \mathcal{Y} . Similarly, $u_5 u_6 x_2 c$ and $u_7 u_8 y_2 d$ span two copies of \mathcal{Y} . Replacing $\mathcal{Y}_i, \mathcal{Y}_j, \mathcal{Y}_k$ by these four copies of \mathcal{Y} gives a \mathcal{Y} -tiling larger than \mathcal{Y} , a contradiction. \square

We now show that most triples uij , $u \in U$, $i, j \in [m]$ are in \mathcal{T}_7^1 .

- Claim 2.11.** (i) $|\mathcal{T}_{\geq 7}^3| \leq \binom{m}{2} \cdot 2^{16} \cdot 5$,
(ii) $|\mathcal{T}_{\geq 7}^2| \leq 756 \binom{m}{2} + m \cdot |U|$,
(iii) $|\mathcal{T}_{\leq 6}| \leq \gamma \binom{n}{2} |U| + 2^{22} \binom{m}{2}$,
(iv) $|\mathcal{T}_7^1| \geq \binom{m}{2} |U| - \gamma n^2 |U|$.

Proof. To see Part (i), by Fact 2.10 (i), given $i, j \in [m]$ and a bipartite graph on (V_i, V_j) containing a matching of size three, at most five vertices in U can share this link graph on (V_i, V_j) . Since there are 2^{16} (labeled) bipartite graphs on (V_i, V_j) , we get that $|\mathcal{T}_{\geq 7}^3| \leq \binom{m}{2} \cdot 2^{16} \cdot 5$.

To see Part (ii), let \mathcal{D} denote the digraph on $[m]$ such that $(i, j) \in E(\mathcal{D})$ if and only if at least eight vertices $u_1, \dots, u_8 \in U$ share the same link graph on (V_i, V_j) such that $u_1 i j \in \mathcal{T}_{\geq 7}^2$, and the vertex cover is in V_i . We claim that $\deg_{\mathcal{D}}^-(i) \leq 1$ for every $i \in [m]$ and consequently $e(\mathcal{D}) \leq m$. Suppose instead, there are $i, j, k \in [m]$ such that $(j, i), (k, i) \in E(\mathcal{D})$, then eight vertices of U share the same link graph on (V_i, V_j) , and (not necessarily different) eight vertices of U share the same link graph on (V_i, V_k) . Thus we can pick four distinct vertices for each of (V_i, V_j) and (V_i, V_k) and obtain a structure forbidden by Fact 2.10 (ii), a contradiction. Note that there are $2 \cdot \binom{4}{2} \cdot 8 + 2 \cdot \binom{4}{2} = 108$ (labeled) bipartite graphs on (V_i, V_j) with at least seven edges and a vertex cover of two vertices both from V_i or V_j . Furthermore, fixing one of these bipartite graphs, if $(i, j) \notin E(\mathcal{D})$ and $(j, i) \notin E(\mathcal{D})$, then, by the definition of \mathcal{D} , at most seven vertices in U share this link graph. Hence

$$|\mathcal{T}_{\geq 7}^2| \leq \binom{m}{2} \cdot 108 \cdot 7 + m|U| = 756 \binom{m}{2} + m|U|.$$

To see Part (iii), recall that \mathcal{A}_i is the set of all edges of \mathcal{H} with exactly i vertices in V' . Then

$$|\mathcal{A}_2| \leq 6|\mathcal{T}_{\leq 6}| + 7|\mathcal{T}_7^1| + 8|\mathcal{T}_{\geq 7}^2| + 16|\mathcal{T}_{\geq 7}^3| + \binom{4}{2}m|U|.$$

Together with $|\mathcal{T}_{\leq 6}| + |\mathcal{T}_7^1| + |\mathcal{T}_{\geq 7}^2| + |\mathcal{T}_{\geq 7}^3| = \binom{m}{2}|U|$, we get,

$$\begin{aligned} |\mathcal{A}_2| &\leq 7\binom{m}{2}|U| - |\mathcal{T}_{\leq 6}| + |\mathcal{T}_{\geq 7}^2| + 9|\mathcal{T}_{\geq 7}^3| + 6m|U| \\ &\leq 7\binom{m}{2}|U| - |\mathcal{T}_{\leq 6}| + \binom{m}{2} \cdot (2^{16} \cdot 45 + 756) + 7m|U| \quad \text{by Parts (i), (ii)} \\ &< 7\binom{m}{2}|U| - |\mathcal{T}_{\leq 6}| + 2^{22}\binom{m}{2} + 7m|U|. \end{aligned} \quad (2.4)$$

We know that $\sum_{u \in U} \deg(u) = 3|\mathcal{A}_0| + 2|\mathcal{A}_1| + |\mathcal{A}_2|$. Thus, by $|\mathcal{A}_0| \leq \frac{1}{3}\binom{|U|}{2}$, Claim 2.9 and (2.4), we have

$$\begin{aligned} \sum_{u \in U} \deg(u) &\leq \binom{|U|}{2} + 2m\binom{|U|}{2} + 24m|U| + 7\binom{m}{2}|U| - |\mathcal{T}_{\leq 6}| + 2^{22}\binom{m}{2} + 7m|U| \\ &= \binom{|U|}{2} + m|U|^2 + 30m|U| + 7\binom{m}{2}|U| - |\mathcal{T}_{\leq 6}| + 2^{22}\binom{m}{2} \\ &< \frac{7}{16}\binom{|U|}{2}|U| + \frac{7}{4}m|U|^2 + \frac{7}{16}\binom{4m}{2}|U| - |\mathcal{T}_{\leq 6}| + 2^{22}\binom{m}{2} \quad \text{as } |U| > 40 \\ &= \frac{7}{16}\binom{n}{2}|U| - |\mathcal{T}_{\leq 6}| + 2^{22}\binom{m}{2}, \end{aligned} \quad (2.5)$$

where the last inequality is due to $\binom{|U|}{2} + 4m|U| + \binom{4m}{2} = \binom{|U|+4m}{2} = \binom{n}{2}$.

On the other hand, $\delta_1(\mathcal{H}) \geq (\frac{7}{16} - \gamma)\binom{n}{2}$ implies that $\sum_{u \in U} \deg(u) \geq (\frac{7}{16} - \gamma)\binom{n}{2}|U|$. Together with (2.5), this gives the desired bound for $|\mathcal{T}_{\leq 6}|$.

To see Part (iv), note that (2.3) implies that $|U| < \frac{3}{4}n < \frac{3}{4}16m = 12m$. By Parts (i)–(iii), we have

$$\begin{aligned} |\mathcal{T}_7^1| &\geq \binom{m}{2}|U| - \left(\binom{m}{2} \cdot (2^{16} \cdot 5 + 756) + m|U| \right) - \gamma\binom{n}{2}|U| - 2^{22}\binom{m}{2} \\ &\geq \binom{m}{2}|U| - \gamma\binom{n}{2}|U| - 2^{23}\binom{m}{2} \quad \text{as } |U| < 12m \\ &\geq \binom{m}{2}|U| - \gamma\binom{n}{2}|U| - 2^{19}\binom{n}{2} \quad \text{as } m < \frac{n}{4} \\ &> \binom{m}{2}|U| - \gamma n^2|U| \quad \text{as } |U| > 2^{19}/\gamma. \quad \square \end{aligned}$$

For a triple $uij \in \mathcal{T}_7^1$, we call $v_1 \in V_i$ and $v_2 \in V_j$ *centers* for u if $\{v_1, v_2\}$ is the vertex cover of $L_{i,j}(u)$. Define G as the graph on the vertex set V' such that two vertices $v_1, v_2 \in V'$ are adjacent if and only if there are at least 16 vertices $u \in U$ such that v_1, v_2 are centers for u .

Fact 2.12. *For every $i \in [m]$, at most one vertex $v \in V_i$ satisfies $\deg_G(v) > 0$.*

Proof. Suppose to the contrary, some $V_i = \{a, b, c, d\}$ satisfies $\deg_G(a), \deg_G(b) > 0$. Let $a' \in N_G(a), b' \in N_G(b)$ and assume that $a' \in V_j, b' \in V_k$ (it is possible to have $a' = b'$). Pick $x \in V_j \setminus \{a', b'\}$ and $y \in V_k \setminus \{a', b', x\}$. By the definition of G , we can find $u_1, \dots, u_4, u'_1, \dots, u'_4 \in U$ such that a, a' are centers for u_l and b, b' are centers for u'_l for $l = 1, \dots, 4$. This gives three copies of \mathcal{Y} on $axu_1u_2, a'cu_3u_4, byu'_1u'_2$ immediately. So if $j = k$, then replacing $\mathcal{Y}_i, \mathcal{Y}_j$ by them in \mathcal{Z} gives a larger \mathcal{Y} -tiling, a contradiction. If $j \neq k$, then we have $a' \neq b'$, and we get one more copy of \mathcal{Y} on $b'du'_3u'_4$. Replacing $\mathcal{Y}_i, \mathcal{Y}_j, \mathcal{Y}_k$ by these four copies of \mathcal{Y} in \mathcal{Z} gives a larger \mathcal{Y} -tiling, a contradiction. \square

Let C be the set of vertices $v \in V'$ such that $\deg_G(v) \geq 7$ and $\deg_G(v') \geq 2$ for some $v' \in N_G(v)$, where $N_G(v)$ denotes the neighborhood of v in G .

Claim 2.13. $(1 - 2^{11}\gamma)m \leq |C| \leq m$.

Proof. The upper bound follows from Fact 2.12 immediately.

To see the lower bound, we first show that

$$e(G) \geq (1 - 2^{10}\gamma) \binom{m}{2}. \quad (2.6)$$

To see this, let M be the set of pairs $i, j \in \binom{[m]}{2}$ such that there are at most 240 vertices $u \in U$ satisfying that $uij \in \mathcal{T}_7^1$. By Claim 2.11 (iv), the number of triples $uij \notin \mathcal{T}_7^1$ ($u \in U, i \neq j \in [m]$) is at most $\gamma n^2|U|$. Thus

$$|M| \leq \frac{\gamma n^2|U|}{|U| - 240} \leq \frac{\gamma n^2|U|}{\frac{2}{3}|U|} = \frac{3\gamma n^2}{2} < \frac{3\gamma(16m)^2}{2} < 2^{10}\gamma \binom{m}{2}.$$

where the second last inequality follows from (2.3). Fix a pair $i, j \in \binom{[m]}{2} \setminus M$. There are at least $241 = 16 \cdot 15 + 1$ vertices $u \in U$ satisfying that $uij \in \mathcal{T}_7^1$. Since $V_i \times V_j$ contains 16 pairs of vertices, by the pigeonhole principle, some pair of vertices $v_1 \in V_i, v_2 \in V_j$ are centers for at least 16 vertices $u \in U$, namely, $v_1v_2 \in G$. Thus (2.6) follows.

By Fact 2.12, there are at most m vertices with positive degree in G . For convenience, define $V'' \subset V'$ as an arbitrary set of m vertices that contains all the vertices with positive degree in G . Furthermore, for any integer $t < m$, let $D_t \subseteq V''$ denote the set of vertices v such that $\deg_G(v) \leq t$. Let $F \subseteq (V'' \setminus D_1)$ denote the set of vertices v such that $N_G(v) \subseteq D_1$. We have

$$2e(G) \leq t|D_t| + (m-1)(m-|D_t|) = m(m-1) - (m-t-1)|D_t|.$$

Together with (2.6), it gives $|D_t| \leq 2^{10}\gamma \frac{m(m-1)}{m-t-1}$. By definition, each vertex $v \in F$ satisfies $\deg_G(v) \geq 2$, and its neighborhood is contained in D_1 (thus the vertices in F have disjoint neighborhoods). This implies that $|F| \leq |D_1|/2$. Recall that $C = V'' \setminus (D_6 \cup F)$. Since D_6 and D'_2 are not necessarily disjoint,

$$|C| \geq m - |D_6| - |F| \geq m - 2^{10}\gamma \frac{m(m-1)}{m-7} - 2^{10}\gamma \frac{m(m-1)}{2(m-2)} \geq (1 - 2^{11}\gamma)m.$$

as claimed. \square

Let I_C be the set of all $i \in [m]$ such that $V_i \cap C \neq \emptyset$. Fact 2.12 and Claim 2.13 together imply that $|I_C| = |C| \geq (1 - 2^{11}\gamma)m$. Let $A = (\bigcup_{i \in I_C} V_i \setminus C) \cup U$.

Claim 2.14. $\mathcal{H}[A]$ contains no copy of \mathcal{Y} , thus $e(A) \leq \frac{1}{3} \binom{n}{2}$.

Proof. The first half of the claim implies the second half by Fact 2.8. Suppose instead, $\mathcal{H}[A]$ contains a copy of \mathcal{Y} , denoted by \mathcal{Y}_0 , on V_0 . Since $\mathcal{H}[U]$ contains no copy of \mathcal{Y} , V_0 must intersect some V_i with $i \in I_C$. Without loss of generality, suppose that V_1, \dots, V_j contain the vertices of $V_0 \setminus U$ for some $1 \leq j \leq 4$ (recall that $V_i = V(\mathcal{Y}_i)$ for $i \in [m]$). Here we separate two cases.

Case 1. For any $i \in [j]$, $|V_i \cap V_0| \leq 2$.

For $i \in [j]$, let $\{c_i\} = V_i \cap C$, and suppose that $d_i \in V_i \setminus (V_0 \cup \{c_i\})$. For each $i \in [j]$, since $\deg_G(c_i) \geq 7$, we can pick distinct $v_i \in N_G(c_i) \setminus (V_1 \cup \dots \cup V_j)$. By Fact 2.12, v_1, \dots, v_j are contained in different members of \mathcal{Y} (also different from $\mathcal{Y}_1, \dots, \mathcal{Y}_j$). Let v'_1, \dots, v'_j be arbitrary vertices in these members of \mathcal{Y} , respectively, which are different from v_1, \dots, v_j . For every $i \in [j]$, since c_i, v_i are centers for at least 16 vertices of U , we find a set of four vertices $u_i^1, \dots, u_i^4 \in U \setminus V_0$ disjoint from the previous ones such that c_i, v_i are centers for them. This is possible because $|V_0 \cap U| \leq 4 - j$ and the number of available vertices in U is thus at least $16 - (4 - j) = 12 + j \geq 4j$.

Note that for $i \in [j]$, $c_i v'_i u_i^1 u_i^2, d_i v_i u_i^3 u_i^4$ span two copies of \mathcal{Y} . Together with \mathcal{Y}_0 , this gives $2j + 1$ copies of \mathcal{Y} while using vertices from $2j$ members of \mathcal{Y} , contradicting the maximality of \mathcal{Y} .

Case 2. There exists $i_0 \in [j]$, such that $|V_{i_0} \cap V_0| = 3$.

Note that $j = 1$ or 2 in this case. Without loss of generality, assume that $|V_1 \cap V_0| = 3$. First assume that $j = 1$ (then $|V_0 \cap U| = 1$). Let $\{c_1\} = V_1 \cap C$. By the definition of C , there exists $c_2 \in N_G(c_1)$ such that $\deg_G(c_2) \geq 2$. Let $c_3 \neq c_1$ be a neighbor of c_2 in G . Assume that $\mathcal{Y}_{i_2}, \mathcal{Y}_{i_3} \in \mathcal{Y}$ contains c_2, c_3 , respectively. By the definition of G , we can find $u_1, \dots, u_6 \in U \setminus V_0$ such that c_1, c_2 are centers for u_1, u_2 , and c_2, c_3 are centers for u_3, u_4, u_5, u_6 . Thus, $c_1 w_1 u_1 u_2, c_2 w_3 u_3 u_4, c_3 w_2 u_5 u_6$ span three copies of \mathcal{Y} , where w_1, w_2 are two vertices in $V_{i_2} \setminus \{c_2\}$ and $w_3 \in V_{i_3} \setminus \{c_3\}$. Together with \mathcal{Y}_0 , it gives four copies of \mathcal{Y} while using vertices from three members of \mathcal{Y} , contradicting the maximality of \mathcal{Y} .

Now assume that $j = 2$, that is, $|V_0 \cap V_2| = 1$. We pick $c_2, c_3, u_1, \dots, u_6$ in the same way as in the $j = 1$ case. If $c_2 \in V_2$, then this gives four copies of \mathcal{Y} by using vertices from three members of \mathcal{Y} , a contradiction. Otherwise, let $\{c_4\} = V_2 \cap C$ and pick $c_5 \in N_G(c_4) \setminus \{c_1, c_2, c_3\}$ (this is possible because $\deg_G(c_4) \geq 7$). Suppose that \mathcal{Y}_{i_5} contains c_5 . We pick four new vertices $u_7, \dots, u_{10} \in U$ for whom c_4, c_5 are centers. Thus, we can form two copies of \mathcal{Y} by using vertices from $\mathcal{Y}_2, \mathcal{Y}_{i_5}$ and u_7, \dots, u_{10} . Together with the four copies of \mathcal{Y} given in the previous case, we obtain six copies of \mathcal{Y} while using vertices from five members of \mathcal{Y} , a contradiction. \square

Note that the edges not incident to C are either contained in A or incident to some $V_i, i \notin I_C$. By Claim 2.14, C is incident to all but at most

$$\begin{aligned} e(A) + 4 \cdot 2^{11} \gamma m \binom{n-1}{2} &< \frac{1}{3} \binom{n}{2} + 2^{10} \gamma (4m) n^2 \\ &< 2^{10} \gamma n^2 \left(\frac{1}{2^{10} \gamma} + 4m \right) < 2^{10} \gamma n^3, \end{aligned}$$

edges, where the last inequality holds because $|U| > \frac{1}{2^{10} \gamma}$. Since $|C| \leq m \leq n/4$, we can pick a set $B \subseteq V \setminus C$ of order $\lfloor \frac{3}{4} n \rfloor$. Then $e(B) < 2^{10} \gamma n^3$, which implies that \mathcal{H} is $2^{10} \gamma$ -extremal. \square

In Claim 2.14 we proved that $\mathcal{H}[A]$ contains no copy of \mathcal{Y} , where, by Claim 2.13,

$$|A| = n - m - 3(m - |C|) \geq n - \frac{n}{4} - 3 \cdot 2^{11}\gamma m \geq (1 - 2^{11}\gamma)\frac{3}{4}n.$$

We summarize this in a lemma and will use it in our forthcoming paper [12].

Lemma 2.15. *For any $\gamma > 0$, there exists an integer n_0 such that the following holds. Suppose \mathcal{H} is a 3-graph on $n > n_0$ vertices with*

$$\delta_1(\mathcal{H}) \geq \left(\frac{7}{16} - \gamma\right) \binom{n}{2},$$

then there is a \mathcal{Y} -tiling covering all but at most $2^{19}/\gamma$ vertices of \mathcal{H} unless \mathcal{H} contains a set of order at least $(1 - 2^{11}\gamma)\frac{3}{4}n$ that contains no copy of \mathcal{Y} .

3. THE EXTREMAL THEOREM

In this section we prove Theorem 1.5. Let n be sufficiently large and \mathcal{H} be a 3-graph on n vertices satisfying (1.1). Assume that \mathcal{H} is β -extremal, namely, there is a set $B \subseteq V(\mathcal{H})$, such that $|B| = \lfloor \frac{3}{4}n \rfloor$ and $e(B) \leq \beta n^3$. For the convenience of later calculations, we let $\epsilon_0 = 18\beta$ and derive that

$$e(B) < \epsilon_0 \binom{|B|}{3}. \quad (3.1)$$

Let us outline our proof here. We define two disjoint sets $A', B' \subseteq V(\mathcal{H})$ such that A' consists of the vertices with high degree in B and B' consists of the vertices with low degree in B . We will show that $A' \approx A$ and $B' \approx B$ (Claim 3.2). To illustrate our proof ideas, suppose that we are in an ideal case with $n \in 4\mathbb{N}$, $A' = A$, and $B' = B$. In this case we arbitrarily partition B' into three sets B_1, B_2 and B_3 of equal size, and find a labelling $B_1 = \{b_1, \dots, b_{n/4}\}$, $B_2 = \{b'_1, \dots, b'_{n/4}\}$ and $B_3 = \{b''_1, \dots, b''_{n/4}\}$ such that Γ has large minimum degree, where Γ denotes the bipartite graph on $(A, [n/4])$ in which $xi \in \Gamma$ for $x \in A$ and $i \in [n/4]$ if and only if $b_i b'_i, b'_i b''_i \in N_{\mathcal{H}}(x)$. It is easy to find a Hamilton cycle in Γ , which gives rise to a loose Hamilton cycle in \mathcal{H} . In our actual proof, we first build a short loose path P that covers all the vertices of $V_0 := V(\mathcal{H}) \setminus (A' \cup B')$ (and some vertices from A' and B') such that $|A' \setminus V(P)|/|B' \setminus V(P)|$ is at least $1/3$ (Claim 3.6). This is possible because of the minimum degree condition (1.1) and the fact that V_0 is small. We next extend P to a loose path Q such that $|A' \setminus V(Q)|/|B' \setminus V(Q)|$ is about $1/3$ and finally find a loose Hamilton path on $V(\mathcal{H}) \setminus Q$ by following the approach for the ideal case (Lemma 3.7).

3.1. Classifying vertices. Let $\epsilon_1 = 8\sqrt{\epsilon_0}$ and $A = V(\mathcal{H}) \setminus B$. Assume that the partition A and B satisfies that $|B| = \lfloor \frac{3}{4}n \rfloor$ and (3.1). In addition, assume that $e(B)$ is the smallest among all the partitions satisfying these conditions. We now define

$$\begin{aligned} A' &:= \left\{ v \in V \mid \deg(v, B) \geq (1 - \epsilon_1) \binom{|B|}{2} \right\}, \\ B' &:= \left\{ v \in V \mid \deg(v, B) \leq \epsilon_1 \binom{|B|}{2} \right\}, \\ V_0 &= V \setminus (A' \cup B'). \end{aligned}$$

Claim 3.1. *$A \cap B' \neq \emptyset$ implies that $B \subseteq B'$, and $B \cap A' \neq \emptyset$ implies that $A \subseteq A'$.*

Proof. First, assume that $A \cap B' \neq \emptyset$. Then there is some $u \in A$ which satisfies that $\deg(u, B) \leq \epsilon_1 \binom{|B|}{2}$. If there exists some $v \in B \setminus B'$, namely, $\deg(v, B) > \epsilon_1 \binom{|B|}{2}$, then we can switch u and v and form a new partition $A'' \cup B''$ such that $|B''| = |B|$ and $e(B'') < e(B)$, which contradicts the minimality of $e(B)$.

Second, assume that $B \cap A' \neq \emptyset$. Then some $u \in B$ satisfies that $\deg(u, B) \geq (1 - \epsilon_1) \binom{|B|}{2}$. Similarly, by the minimality of $e(B)$, we get that for any vertex $v \in A$, $\deg(v, B) \geq (1 - \epsilon_1) \binom{|B|}{2}$, which implies that $A \subseteq A'$. \square

Claim 3.2. $\{|A \setminus A'|, |B \setminus B'|, |A' \setminus A|, |B' \setminus B|\} \leq \frac{\epsilon_1}{64}|B|$ and $|V_0| \leq \frac{\epsilon_1}{32}|B|$.

Proof. First assume that $|B \setminus B'| > \frac{\epsilon_1}{64}|B|$. By the definition of B' and the assumption $\epsilon_1 = 8\sqrt{\epsilon_0}$, we get that

$$e(B) > \frac{1}{3}\epsilon_1 \binom{|B|}{2} \cdot \frac{\epsilon_1}{64}|B| > \frac{\epsilon_1^2}{64} \binom{|B|}{3} = \epsilon_0 \binom{|B|}{3},$$

which contradicts (3.1).

Second, assume that $|A \setminus A'| > \frac{\epsilon_1}{64}|B|$. Then by the definition of A' , for any vertex $v \notin A'$, we have that $\overline{\deg}(v, B) > \epsilon_1 \binom{|B|}{2}$. So we get

$$\overline{e}(ABB) > \frac{\epsilon_1}{64}|B| \cdot \epsilon_1 \binom{|B|}{2} = \epsilon_0|B| \binom{|B|}{2} > 3\epsilon_0 \binom{|B|}{3}.$$

Together with (3.1), this implies that

$$\sum_{b \in B} \overline{\deg}(b) \geq 3\overline{e}(B) + 2\overline{e}(ABB) > 3(1 - \epsilon_0) \binom{|B|}{3} + 6\epsilon_0 \binom{|B|}{3} = 3(1 + \epsilon_0) \binom{|B|}{3}.$$

By the pigeonhole principle, there exists $b \in B$, such that

$$\overline{\deg}(b) > (1 + \epsilon_0) \binom{|B|}{2} = (1 + \epsilon_0) \binom{\lfloor \frac{3n}{4} \rfloor - 1}{2} > \binom{\lfloor \frac{3n}{4} \rfloor}{2},$$

where the last inequality follows from the assumption that n is large enough. This contradicts (1.1).

Consequently,

$$\begin{aligned} |A' \setminus A| &= |A' \cap B| \leq |B \setminus B'| \leq \frac{\epsilon_1}{64}|B|, \\ |B' \setminus B| &= |A \cap B'| \leq |A \setminus A'| \leq \frac{\epsilon_1}{64}|B|, \\ |V_0| &= |A \setminus A'| + |B \setminus B'| \leq \frac{\epsilon_1}{64}|B| + \frac{\epsilon_1}{64}|B| = \frac{\epsilon_1}{32}|B|. \end{aligned} \quad \square$$

We next show that we can connect any two vertices of B' with a loose path of length two without using any fixed $\frac{n}{8}$ vertices of V .

Claim 3.3. *For every pair of vertices $u, v \in B'$ and every vertex set $S \subseteq V$ with $|S| \leq n/8$, there exist $a \in A' \setminus S$ and $b_1, b_2 \in B' \setminus S$ such that $ub_1a, ab_2v \in E(\mathcal{H})$.*

Proof. For any $x \in B'$, by (1.1), we have that $\overline{\deg}(x) \leq \binom{\lfloor \frac{3n}{4} \rfloor}{2} = \binom{|B|}{2}$. So by the definition of B' ,

$$\overline{\deg}(x, AB) \leq \overline{\deg}(x) - \overline{\deg}(x, B) \leq \binom{|B|}{2} - (1 - \epsilon_1) \binom{|B|}{2} = \epsilon_1 \binom{|B|}{2}.$$

By Claim 3.2, we get that

$$\begin{aligned} \overline{\deg}(x, A'B') &\leq \overline{\deg}(x, AB) + |A' \setminus A| \cdot |B'| + |B' \setminus B| \cdot |A'| \\ &\leq \epsilon_1 \binom{|B|}{2} + \frac{\epsilon_1}{64} |B|n \leq 2\epsilon_1 \binom{|B|}{2}. \end{aligned} \quad (3.2)$$

Consider a bipartite graph G on $A \setminus S$ and $B \setminus S$ with pairs $ab \in E(G)$ if and only if $uab, vab \in E(\mathcal{H})$. Since $|S| \leq \frac{n}{8}$, we have $|A \setminus S| \geq \frac{|A|}{2} \geq \frac{|B|}{6}$ and $|B \setminus S| > \frac{|B|}{2}$, so $|A \setminus S| \cdot |B \setminus S| > \frac{1}{6} \binom{|B|}{2} > 8\epsilon_1 \binom{|B|}{2}$. Consequently,

$$e(G) \geq |A \setminus S| \cdot |B \setminus S| - 4\epsilon_1 \binom{|B|}{2} \geq \frac{1}{2} |A \setminus S| \cdot |B \setminus S| > |A \setminus S|.$$

Hence there exists a vertex $a \in A \setminus S$ such that $\deg_G(a) \geq 2$. By picking $b_1, b_2 \in N_G(a)$ we finish the proof. \square

3.2. Building a short path.

Claim 3.4. *Suppose that $|A \cap B'| = q > 0$. Then there exists a family \mathcal{P}_1 of vertex disjoint loose paths in B' , where*

$$\mathcal{P}_1 \text{ consists of } \begin{cases} \text{one edge} & \text{if } q = 1 \text{ and } n \notin 4\mathbb{N} \\ \text{two edges } e_1, e_2 \text{ with } |e_1 \cap e_2| \leq 1 & \text{if } q = 1 \text{ and } n \in 4\mathbb{N} \\ 2q \text{ disjoint edges} & \text{if } q \geq 2 \end{cases}$$

Proof. Let $|A \cap B'| = q > 0$. Since $A \cap B' \neq \emptyset$, by Claim 3.1, we get $B \subseteq B'$, which implies $|B'| = \lfloor \frac{3}{4}n \rfloor + q$.

By Claim 3.2, we get that $q = |A \cap B'| \leq |A \setminus A'| \leq \frac{\epsilon_1}{64} |B|$. Hence for any vertex b in B' ,

$$\begin{aligned} \deg(b, B') &\leq \deg(b, B) + |B' \setminus B|(|B'| - 1) \\ &\leq \epsilon_1 \binom{|B|}{2} + q(|B'| - 1) < 2\epsilon_1 \binom{|B|}{2}. \end{aligned} \quad (3.3)$$

Now we assume that $q = 1$, so $|B'| - 1 = \lfloor \frac{3}{4}n \rfloor$. By (1.1), for any $b \in B'$,

$$\deg(b, B') \geq \binom{n-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} + c - \left[\binom{n-1}{2} - \binom{|B'| - 1}{2} \right] = c,$$

where $c = 1$ if $n \notin 4\mathbb{N}$ and $c = 2$ otherwise. The $n \notin 4\mathbb{N}$ case is trivial since B' actually contains at least $|B'|/3 > 1$ edges. If $n \in 4\mathbb{N}$, then we have $\deg(b, B') \geq 2$. Assume that B' does not contain the desired structure. Then any two distinct edges of B' share exactly two vertices. Fix an edge $e_0 = v_1v_2v_3$ of B' and two vertices $u, u' \in B' \setminus e_0$. Then every edge of B' containing u must have its two other vertices in e_0 . Since $\deg(u, B') \geq 2$, the link graph of u contains at least two pairs of vertices of e_0 . So does the link graph of u' . We thus find a loose path of length two from u to u' because two distinct pairs on e_0 share exactly one vertex.

Second, assume that $q > 1$. In this case we construct $2q$ disjoint edges greedily. By (1.1) and $|B'| = \lfloor \frac{3}{4}n \rfloor + q$, for any $b \in B'$,

$$\begin{aligned} \deg(b, B') &\geq \binom{n-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} + c - \left[\binom{n-1}{2} - \binom{|B'|-1}{2} \right] \\ &> \binom{|B'|-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} \\ &\geq (q-1) \left\lfloor \frac{3}{4}n \right\rfloor, \end{aligned}$$

which implies that $e(B') > \frac{1}{3}|B'|(q-1)\lfloor \frac{3}{4}n \rfloor$. Suppose we have found $i < 2q$ disjoint edges of B' . By (3.3), there are at most $3(2q-1) \cdot 2\epsilon_1 \binom{|B|}{2}$ edges of B' intersecting these i edges. Hence, there are at least

$$\begin{aligned} e(B') - 3(2q-1) \cdot 2\epsilon_1 \binom{|B|}{2} &\geq \frac{1}{3}|B'|(q-1) \left\lfloor \frac{3}{4}n \right\rfloor - 6(2q-1)\epsilon_1 \binom{|B|}{2} \\ &\geq \frac{2(q-1)}{3} \binom{|B|}{2} - 6(2q-1)\epsilon_1 \binom{|B|}{2} \\ &= \frac{2}{3} [(q-1) - 9(2q-1)\epsilon_1] \binom{|B|}{2} \end{aligned}$$

edges not intersecting the existing i edges. This quantity is positive provided that $\epsilon_1 < \frac{q-1}{9(2q-1)}$. Thus, $\epsilon_1 < \frac{1}{27}$ suffices since the minimum of $\frac{q-1}{9(2q-1)}$, $q > 1$ is $\frac{1}{27}$ attained by $q = 2$. \square

Remark 3.5. *Claim 3.4 is the only place where the constant c from (1.1) is used.*

The goal of this subsection is to prove the following claim.

Claim 3.6. *There exists a loose path P in \mathcal{H} with the following properties:*

- $V_0 \subseteq V(P)$,
- $|V(P)| \leq \frac{\epsilon_1}{4}|B|$,
- $|B' \setminus V(P)| \leq 3|A' \setminus V(P)| - 1$,
- *both ends of P are in B' .*

Proof. We split into two cases here.

Case 1. $A \cap B' \neq \emptyset$.

By Claim 3.1, $A \cap B' \neq \emptyset$ implies that $B \subseteq B'$, which implies that $V_0 \subseteq A$. Let $q = |A \cap B'|$. We first apply Claim 3.4 and find a family \mathcal{P}_1 of vertex disjoint loose paths on at most $6q$ vertices of B' . Next we put each vertex of V_0 into a loose path of length two with four vertices from B (so in B') such that these paths are pairwise vertex disjoint and also vertex disjoint from the paths in \mathcal{P}_1 . Let $V_0 = \{x_1, \dots, x_{|V_0|}\}$. Suppose that we have found loose paths for x_1, \dots, x_i with $i < |V_0|$. Since $A \setminus A' = V_0 \dot{\cup} (A \cap B')$, by Claim 3.2, we have

$$q + |V_0| = |A \setminus A'| \leq \frac{\epsilon_1}{64}|B|. \quad (3.4)$$

Thus,

$$4i + 6q < 4|V_0| + 6q \leq 6(|V_0| + q) \leq \frac{3\epsilon_1}{32}|B|$$

and consequently at most $\frac{3\epsilon_1}{32}|B|(|B|-1) = \frac{3\epsilon_1}{16} \binom{|B|}{2}$ pairs of B intersect the existing paths. By the definition of V_0 , $\deg(x_{i+1}, B) > \epsilon_1 \binom{|B|}{2}$. Since every graph on $n \geq 4$

vertices and $m \geq n$ edges contains two vertex disjoint edges, we can find two vertex disjoint pairs in the link graph of x_{i+1} in B .

Denote by \mathcal{P}_2 the family of the loose paths that we obtained so far. Now we want to glue paths of \mathcal{P}_2 together to a single loose path. For this purpose, we apply Claim 3.3 repeatedly to connect the ends of two loose paths while avoiding previously used vertices. This is possible because $|V(\mathcal{P}_2)| \leq 5|V_0| + 6q$ and at most $3(|V_0| + 2q - 1)$ vertices will be used to connect the paths in \mathcal{P}_2 . By (3.4), the resulting loose path P satisfies

$$|V(P)| \leq 8|V_0| + 12q - 3 < 12 \cdot \frac{\epsilon_1}{64}|B| < \frac{\epsilon_1}{4}|B|.$$

We next show that $|B' \setminus V(P)| \leq 3|A' \setminus V(P)| - 1$. To prove this, we split into three cases according to the structure of \mathcal{P}_1 . Note that $|B'| = \lfloor \frac{3n}{4} \rfloor + q$ and $|A'| = \lceil \frac{n}{4} \rceil - |V_0| - q$.

First, assume that $q > 1$. Our construction shows that \mathcal{P}_1 consists of $2q$ disjoint edges in B' . So $|V(P) \cap A'| = |V_0| + 2q - 1$ and $|V(P) \cap B'| = 4|V_0| + 3 \cdot 2q + 2(|V_0| + 2q - 1) = 6|V_0| + 10q - 2$. Thus,

$$\begin{aligned} |B' \setminus V(P)| &= \left\lfloor \frac{3n}{4} \right\rfloor + q - (6|V_0| + 10q - 2) \\ &\leq 3 \left(\left\lceil \frac{n}{4} \right\rceil - 2|V_0| - 3q + 1 \right) - 1 = 3|A' \setminus V(P)| - 1. \end{aligned}$$

Second, assume that $q = 1$ and $n \in 4\mathbb{N}$. Then \mathcal{P}_1 consists of a loose path of length two or two disjoint edges. For the first case, we have that $|V(P) \cap A'| = |V_0|$ and $|V(P) \cap B'| = 4|V_0| + 2|V_0| + 5 = 6|V_0| + 5$. Thus,

$$|B' \setminus V(P)| = \frac{3n}{4} + 1 - (6|V_0| + 5) = 3 \left(\frac{n}{4} - 2|V_0| - 1 \right) - 1 = 3|A' \setminus V(P)| - 1.$$

In the second case, we have that $|V(P) \cap A'| = |V_0| + 1$ and $|V(P) \cap B'| = 4|V_0| + 2(|V_0| + 1) + 6 = 6|V_0| + 8$. Thus,

$$|B' \setminus V(P)| = \frac{3n}{4} + 1 - (6|V_0| + 8) = 3 \left(\frac{n}{4} - 2|V_0| - 2 \right) - 1 = 3|A' \setminus V(P)| - 1.$$

Third, assume that $q = 1$ and $n \notin 4\mathbb{N}$, so \mathcal{P}_1 contains only one edge. We have $|V(P) \cap A'| = |V_0|$ and $|V(P) \cap B'| = 4|V_0| + 2|V_0| + 3 = 6|V_0| + 3$. Let $n = 4k + 2$ with some $k \in \mathbb{Z}$, so $|A| = k + 1$, $|B| = 3k + 1$, $|B'| = 3k + 2$ and $|A'| = k - |V_0|$. Thus,

$$|B' \setminus V(P)| = 3k + 2 - (6|V_0| + 3) = 3(k - 2|V_0|) - 1 = 3|A' \setminus V(P)| - 1.$$

Case 2. $A \cap B' = \emptyset$.

Note that $A \cap B' = \emptyset$ means that $B' \subseteq B$. The difference from the first case is that we do not need to construct \mathcal{P}_1 .

First we will put every vertex in V_0 into a loose path of length two together with four vertices from B' . By Claim 3.2, $|B \setminus B'| \leq \frac{\epsilon_1}{64}|B|$ and thus for any vertex $x \in V_0$,

$$\deg(x, B') \geq \deg(x, B) - |B \setminus B'| \cdot (|B| - 1) \geq \epsilon_1 \binom{|B|}{2} - \frac{\epsilon_1}{32} \binom{|B|}{2}. \quad (3.5)$$

Similar as in Case 1, let $V_0 = \{x_1, \dots, x_{|V_0|}\}$ and suppose that we have found loose paths for x_1, \dots, x_i with $i < |V_0|$. By Claim 3.2, $|V_0| \leq \frac{\epsilon_1}{32}|B|$. Thus, we have $4i < 4|V_0| \leq \frac{\epsilon_1}{8}|B|$ and consequently at most $\frac{\epsilon_1}{8}|B|(|B'| - 1) \leq \frac{\epsilon_1}{4} \binom{|B|}{2}$ pairs of B'

intersect the existing i loose paths. Then by (3.5), we may find two vertex disjoint pairs in the link graph of x_{i+1} in B' .

As in Case 1, we connect the paths that we obtained to a single loose path by applying Claim 3.3 repeatedly. The resulting loose path P satisfies that

$$|V(P)| = 5|V_0| + 3(|V_0| - 1) < 8 \cdot \frac{\epsilon_1}{32}|B| = \frac{\epsilon_1}{4}|B|.$$

We next show that $|B' \setminus V(P)| \leq 3|A' \setminus V(P)| - 1$. Note that $|V(P) \cap A'| = |V_0| - 1$ and $|V(P) \cap B'| = 4|V_0| + 2(|V_0| - 1) = 6|V_0| - 2$. Since $B' \subseteq B$, we have $|A'| \geq |A' \cap A| = \lceil \frac{n}{4} \rceil - |V_0|$. Thus,

$$\begin{aligned} |B' \setminus V(P)| &= |B'| - (6|V_0| - 2) \leq 3 \left\lceil \frac{n}{4} \right\rceil - 6|V_0| + 2 \\ &\leq 3(|A'| + |V_0| - 2|V_0| + 1) - 1 \\ &= 3(|A'| - |V(P) \cap A'|) - 1 = 3|A' \setminus V(P)| - 1. \quad \square \end{aligned}$$

3.3. Completing a Hamilton cycle. Let P be the loose path given by Claim 3.6. Suppose that $|B' \setminus V(P)| = 3|A' \setminus V(P)| - l$ for some integer $l \geq 1$. Since P is a loose path, $|V(P)|$ is odd. Since $V = A' \cup B' \cup V_0$ and $V_0 \subset V(P)$, we have

$$|V(P)| + |B' \setminus V(P)| + |A' \setminus V(P)| = n. \quad (3.6)$$

Since n is even, it follows that $|B' \setminus V(P)| + |A' \setminus V(P)|$ is odd, which implies that $l = 3|A' \setminus V(P)| - |B' \setminus V(P)|$ is odd.

If $l > 1$, then we extend P as follows. Starting from an end u of P (note that $u \in B'$), we add an edge by using one vertex from A' and one from B' . This is guaranteed by Claim 3.3, which actually provides a loose path starting from u . We repeat this $\frac{l-1}{2}$ times. The resulting loose path P' satisfies $|B' \setminus V(P')| = 3|A' \setminus V(P')| - 1$. We claim that $|V(P')| \leq \frac{3\epsilon_1}{4}|B|$ (thus Claim 3.3 can be applied repeatedly). Indeed, by (3.6) and $|V(P)| \leq \frac{\epsilon_1}{4}|B|$,

$$\begin{aligned} l &= 3|A' \setminus V(P)| - |B' \setminus V(P)| = 4|A' \setminus V(P)| - (n - |V(P)|) \\ &\leq 4|A'| - n + \frac{\epsilon_1}{4}|B|. \end{aligned}$$

Since $|A'| \leq |A| + |B \setminus B'| \leq \lceil \frac{n}{4} \rceil + \frac{\epsilon_1}{64}|B|$ from Claim 3.2, we have $l \leq \frac{\epsilon_1}{2}|B|$. Since $|V(P')| = |V(P)| + l - 1$, we derive that $|V(P')| \leq \frac{3\epsilon_1}{4}|B|$.

Finally, since both ends of P' are vertices in B' , we extend P' by one more ABB edge from each end, respectively. Denote the ends of the resulting path Q be $x_0, x_1 \in A'$. Let $A_1 = (A' \setminus V(Q)) \cup \{x_0, x_1\}$ and $B_1 = B' \setminus V(Q)$. Note that we have $|B_1| = 3(|A_1| - 1)$. By Claim 3.2, we have $|B_1 \setminus B| \leq |B' \setminus B| \leq \frac{\epsilon_1}{64}|B|$. Furthermore,

$$|B_1| \geq |B'| - \frac{3\epsilon_1}{4}|B| \geq |B| - \frac{\epsilon_1}{64}|B| - \frac{3\epsilon_1}{4}|B| - 2 > (1 - \epsilon_1)|B|. \quad (3.7)$$

For a vertex $v \in A_1$, since $\overline{\deg}(v, B) \leq \epsilon_1 \binom{|B|}{2}$, we have

$$\begin{aligned} \overline{\deg}(v, B_1) &\leq \overline{\deg}(v, B) + |B_1 \setminus B| \cdot (|B_1| - 1) \\ &\leq \epsilon_1 \binom{|B|}{2} + \frac{\epsilon_1}{64}|B| \left(1 + \frac{\epsilon_1}{64}\right)|B| \\ &< 2\epsilon_1 \binom{|B|}{2} < 3\epsilon_1 \binom{|B_1|}{2}, \end{aligned}$$

where the last inequality follows from (3.7). In addition, (3.2) and (3.7) imply that for any vertex $v \in B_1$,

$$\overline{\deg}(v, A_1 B_1) \leq \overline{\deg}(v, A' B') \leq 2\epsilon_1 \binom{|B|}{2} < \epsilon_1 |B|^2 < 4\epsilon_1 |A_1| |B_1|.$$

We finally complete the proof of Theorem 1.5 by applying the following lemma with $X = A_1$, $Z = B_1$, and $\rho = 4\epsilon_1$.

Lemma 3.7. *Let $\rho > 0$ be sufficiently small and n be sufficiently large. Suppose that \mathcal{H} be a 3-graph on n vertices with $V(\mathcal{H}) = X \dot{\cup} Z$ such that $|Z| = 3(|X| - 1)$. Further, assume that for every vertex $v \in X$, $\overline{\deg}(v, Z) \leq \rho \binom{|Z|}{2}$ and for every vertex $v \in Z$, $\overline{\deg}(v, XZ) \leq \rho |X| |Z|$. Then given any two vertices $x_0, x_1 \in X$, there is a loose Hamilton path from x_0 to x_1 .*

To prove Lemma 3.7, we follow the approach⁶ in the proof of [6, Lemma 3.4] given by Czygrinow and Molla, who applied a result of Kühn and Osthus [19]. A bipartite graph $G = (A, B, E)$ with $|A| = |B| = n$ is called (d, ϵ) -regular if for any two subsets $A' \subseteq A$, $B' \subseteq B$ with $|A'|, |B'| \geq \epsilon n$,

$$(1 - \epsilon)d \leq \frac{e(A', B')}{|A'| |B'|} \leq (1 + \epsilon)d,$$

and G is called (d, ϵ) -super-regular if in addition $(1 - \epsilon)dn \leq \deg(v) \leq (1 + \epsilon)dn$ for every $v \in A \cup B$.

Lemma 3.8. [19, Theorem 1.1] *For all positive constants $d, \nu_0, \eta \leq 1$ there is a positive $\epsilon = \epsilon(d, \nu_0, \eta)$ and an integer N_0 such that the following holds for all $n \geq N_0$ and all $\nu \geq \nu_0$. Let $G = (A, B, E)$ be a (d, ϵ) -super-regular bipartite graph whose vertex classes both have size n and let F be a subgraph of G with $|F| = \nu |E|$. Choose a perfect matching M uniformly at random in G . Then with probability at least $1 - e^{-\epsilon n}$ we have*

$$(1 - \eta)\nu n \leq |M \cap E(F)| \leq (1 + \eta)\nu n.$$

Proof of Lemma 3.7. Let $\epsilon = \epsilon(1, 7/8, 1/8)$ be given by Lemma 3.8 and $\rho = (\epsilon/2)^4$. Suppose that n is sufficiently large and \mathcal{H} is a 3-graph satisfying the assumption of the lemma. Let G be the graph of all pairs uv in Z such that $\deg(uv, X) \geq (1 - \sqrt{\rho})|X|$. We claim that for any vertex $v \in Z$,

$$\overline{\deg}_G(v) \leq \sqrt{\rho}|Z|. \tag{3.8}$$

Otherwise, some vertex $v \in Z$ satisfies $\overline{\deg}_G(v) > \sqrt{\rho}|Z|$. As each $u \notin N_G(v)$ satisfies $\overline{\deg}_{\mathcal{H}}(uv, X) > \sqrt{\rho}|X|$, we have

$$\overline{\deg}_{\mathcal{H}}(v, XZ) > \sqrt{\rho}|Z| \cdot \sqrt{\rho}|X| = \rho|Z||X|,$$

contradicting our assumption.

Let $m = |X| - 1$. Arbitrarily partition Z into three sets Z_1, Z_2, Z_3 , each of order m . By (3.8) and $|Z| = 3m$, we have $\delta(G[Z_1, Z_2]), \delta(G[Z_2, Z_3]) \geq (1 - 3\sqrt{\rho})m$. It is easy to see that both $G[Z_1, Z_2]$ and $G[Z_2, Z_3]$ are $(1, \epsilon)$ -super-regular as $\epsilon = 2\sqrt[4]{\rho}$. For any $x \in X$, let $F_x^1 := \{zz' \in E(G[Z_1, Z_2]) : xzz' \in E(\mathcal{H})\}$ and let $F_x^2 := \{zz' \in E(G[Z_2, Z_3]) : xzz' \in E(\mathcal{H})\}$. Since $\overline{\deg}(x, Z) \leq \rho \binom{|Z|}{2} \leq 5\rho m^2$,

⁶We proved this lemma by the absorbing method in the previous version of this manuscript.

we have $|F_x^1|, |F_x^2| \geq (1 - 3\sqrt{\rho})m^2 - 5\rho m^2 \geq \frac{7}{8}m^2$. Let M_1 and M_2 be perfect matchings chosen uniformly at random from $G[Z_1, Z_2]$ and $G[Z_2, Z_3]$, respectively. By applying Lemma 3.8 with $\nu_0 = 7/8$ and $\eta = 1/8$, for any $x \in X$, with probability at least $1 - e^{-\epsilon m}$, we have

$$|M_1 \cap E(F_x^1)|, |M_2 \cap E(F_x^2)| \geq (1 - \eta)\nu_0 m \geq \frac{49}{64}m. \quad (3.9)$$

Thus there exist a matching M_1 in $G[Z_1, Z_2]$ and a matching M_2 in $G[Z_2, Z_3]$ such that (3.9) holds for all $x \in X$. Label $Z_1 = \{a_1, \dots, a_m\}$, $Z_2 = \{b_1, \dots, b_m\}$ and $Z_3 = \{c_1, \dots, c_m\}$ such that $M_1 = \{a_1 b_1, \dots, a_m b_m\}$ and $M_2 = \{b_1 c_1, \dots, b_m c_m\}$. Let Γ be a bipartite graph on $(X, [m])$ such that $xi \in E(\Gamma)$ if and only if $xa_i b_i, xb_i c_i \in E(\mathcal{H})$ for $x \in X$ and $i \in [m]$. For every $i \in [m]$, since $a_i b_i, b_i c_i \in E(G)$, we have $\deg_\Gamma(i) \geq (1 - 2\sqrt{\rho})|X|$ by the definition of G . On the other hand, by (3.9), we have $\deg_\Gamma(x) \geq (1 - 2(1 - \frac{49}{64}))m = \frac{34}{64}m$ for any $x \in X$. By a result of Moon and Moser [22], Γ contains a Hamilton path $x_1 j_1 x_2 j_2 \dots x_m j_m x_0$, where $[m] = \{j_1, \dots, j_m\}$ and $X = \{x_0, x_1, \dots, x_m\}$. Since for each $i \in [m]$, $x_i j_i, x_{i+1} j_i \in E(\Gamma)$ implies that $x_i a_{j_i} b_{j_i}, b_{j_i} c_{j_i} x_{i+1} \in E(\mathcal{H})$ (with $x_{m+1} = x_0$), we get a loose Hamilton path of \mathcal{H} :

$$x_1 a_{j_1} b_{j_1} c_{j_1} x_2 a_{j_2} b_{j_2} c_{j_2} \dots x_m a_{j_m} b_{j_m} c_{j_m} x_0. \quad \square$$

4. CONCLUDING REMARKS

Let $h_d^l(k, n)$ denote the minimum integer m such that every k -uniform hypergraph \mathcal{H} on n vertices with minimum d -degree $\delta_d(\mathcal{H}) \geq m$ contains a Hamilton l -cycle (provided that $k - l$ divides n). In this paper we determine $h_1^1(3, n)$ for sufficiently large n . Can we apply the same approach to find other values $h_d^l(k, n)$? In the forthcoming paper [11], we determine $h_{k-1}^l(k, n)$ for all $l < k/2$, improving the asymptotic results in [18, 10, 14].

The authors of [2] conjectured that $h_1^1(k, n)$ is asymptotically attained by a similar construction as the one supported $h_1^1(3, n)$. At present we cannot verify this conjecture because it seems that our success on $h_1^1(3, n)$ comes from the relation $d = k - 2$ instead of the assumption $d = 1$.

It was conjectured in [23] that $h_d^{k-1}(k, n)$ approximately equals to the minimum d -degree threshold for perfect matchings in k -graphs, in particular, $h_1^{k-1}(k, n) = (1 - (1 - 1/k)^{k-1} + o(1))\binom{n}{k-1}$. This conjecture seems very hard because we do not even know the minimum d -degree threshold for perfect matchings in general.

The key lemma in our proof, Lemma 2.7, shows that every 3-graph \mathcal{H} on n vertices with $\delta_1(\mathcal{H}) \geq (7/16 - o(1))\binom{n}{2}$ either contains an almost perfect \mathcal{Y} -tiling or is in the extremal case. Naturally this raises a question: what is the minimum vertex degree threshold for a perfect \mathcal{Y} -tiling? The corresponding codegree threshold was determined in [18] (asymptotically) and [4] (exactly). We determine this minimum vertex degree threshold exactly in [12] (Czygrinow [3] independently proved a similar result).

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