

TRIPARTITE ZARANKIEWICZ NUMBERS AND NORM GRAPHS

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ABSTRACT. For fixed integers $s \geq t \geq 2$, let $\text{ex}(n, n, n, K_{s,t})$ denote the maximum number of edges in a tripartite $K_{s,t}$ -free graph with n vertices in each part. When $s \geq (t-1)! + 1$, let r be the largest integer satisfying $s \geq (t-1)!r^{t-1} + 1$. Using the quotient norm graphs of Alon, Rónyai and Szabó, we prove that

$$\text{ex}(n, n, n, K_{s,t}) \geq \left(\frac{3}{2^{1/t}} r^{1-1/t} + o(1) \right) n^{2-1/t}.$$

Improving an upper bound of Tait and Timmons, we prove that, for all $s \geq t \geq 2$,

$$\text{ex}(n, n, n, K_{s,t}) \leq \left(\frac{3}{2^{1/t}} (s-t+1)^{1/t} + o(1) \right) n^{2-1/t}.$$

Together, these bounds recover the results of [13, 14, 17] for $t = 2$, and give the new asymptotic formula

$$\text{ex}(n, n, n, K_{3,3}) = \left(\frac{3}{\sqrt[3]{2}} + o(1) \right) n^{5/3}.$$

Analogous results extend to k -partite graphs containing no $K_{s,t}$ whose s -vertex or t -vertex side lies in a single part. As an application of our tripartite construction, we determine the tripartite multicolor Ramsey number of $K_{3,3}$ asymptotically.

1. INTRODUCTION

Given graphs G and F , we say that G is F -free if no subgraph of G is isomorphic to F . The *Turán number of F in G* is

$$\text{ex}(G, F) = \max\{e(G') : G' \subseteq G \text{ and } G' \text{ is } F\text{-free}\}.$$

In particular, $\text{ex}(n, F) = \text{ex}(K_n, F)$ is the classical Turán number, while

$$\text{ex}(n, n, F) = \text{ex}(K_{n,n}, F), \quad \text{ex}(n, n, n, F) = \text{ex}(K_{n,n,n}, F)$$

are the (*balanced*) *bipartite and tripartite Turán numbers*, respectively.

The classical Zarankiewicz problem concerns $\text{ex}(n, n, K_{s,t})$. We use the convention $s \geq t \geq 2$ throughout the paper. Kővári, Sós and Turán [12] proved

$$\text{ex}(n, n, K_{s,t}) \leq (s-1)^{1/t} n^{2-1/t} + (t-1)n. \quad (1)$$

For integers $m \geq s$ and $n \geq t$, let $z(m, n, s, t)$ be the maximum number of edges in a bipartite graph with parts of sizes m and n containing no copy of $K_{s,t}$ whose s -vertex side lies in the m -vertex part. Füredi [8] sharpened the leading constant in (1) as follows:

$$z(m, n, s, t) \leq (s-t+1)^{1/t} nm^{1-1/t} + tm^{2-2/t} + tn. \quad (2)$$

Nikiforov [16] subsequently improved the lower-order terms.

Determining whether $\text{ex}(n, K_{s,t}) = \Omega(n^{2-1/t})$ for all fixed $s \geq t \geq 2$ is a major open problem; see [9] for a survey. For $K_{2,2} = C_4$, Erdős, Rényi and Sós [6] and, independently, Brown [2]

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proved that $\text{ex}(n, K_{2,2}) = \left(\frac{1}{2} + o(1)\right) n^{3/2}$. More generally, Füredi [7] proved that, for every $s \geq 2$, $\text{ex}(n, K_{s,2}) = \left(\frac{\sqrt{s-1}}{2} + o(1)\right) n^{3/2}$. Brown [2] also constructed $K_{3,3}$ -free graphs showing that $\text{ex}(n, K_{3,3}) \geq (1/2 - o(1)) n^{5/3}$. Together with Füredi's bound (2) and the fact $2\text{ex}(n, K_{s,t}) \leq \text{ex}(n, n, K_{s,t})$, it follows that $\text{ex}(n, K_{3,3}) = (1/2 + o(1)) n^{5/3}$. Using norm graphs, Kollár, Rónyai and Szabó [11] proved that $\text{ex}(n, K_{s,t}) = \Omega(n^{2-1/t})$ for $t \geq 4$ and $s \geq t! + 1$. Alon, Rónyai and Szabó [1] extended this result to $t \geq 2$ and $s \geq (t-1)! + 1$ by modifying the construction of [11]. In particular, the quotient variant of the constructions of [1] gives, for every fixed $t \geq 2$, $r \geq 1$ and $s \geq (t-1)!r^{t-1} + 1$,

$$\text{ex}(n, K_{s,t}) \geq \left(\frac{1}{2}r^{1-1/t} - o(1)\right) n^{2-1/t}.$$

More recently, Bukh [3] proved that $\text{ex}(n, K_{s,t}) = \Omega(n^{2-1/t})$ for $s > 9^t t^{4t^{2/3}}$, improving the factorial function on t to an exponential one.

Tait and Timmons [17] initiated the systematic study of the *multipartite Zarankiewicz problem*, that is, the problem of forbidding complete bipartite graphs in multipartite graphs. Multipartite Turán problems had been studied earlier in the hypergraph setting, e.g., [15], and more recently, e.g., [10]. For an integer $k \geq 1$, let $\text{ex}_{\chi \leq k}(N, F)$ be the maximum number of edges in an N -vertex F -free graph with chromatic number at most k . Trivially, for any graph F , $\text{ex}(n, \dots, n, F) \leq \text{ex}_{\chi \leq k}(kn, F) \leq \text{ex}(kn, F)$, where n occurs k times in the first term. Under our convention $s \geq t$, Tait and Timmons [17] proved that

$$\text{ex}_{\chi \leq 3}(N, K_{s,t}) \leq \left(\frac{(s-1)^{1/t}}{2^{1/t} 3^{1-1/t}} + o(1)\right) N^{2-1/t}. \quad (3)$$

They proved sharper estimates for $k \geq 3$:

$$\text{ex}_{\chi \leq k}(N, K_{s,2}) \leq \left(\frac{\sqrt{s-1}}{2} \left(1 - \frac{1}{k}\right)^{1/2} + o(1)\right) N^{3/2}, \quad (4)$$

$$\text{ex}_{\chi \leq k}(N, K_{3,3}) \leq \left(\frac{1}{2} \left(1 - \frac{1}{k}\right)^{2/3} + o(1)\right) N^{5/3}. \quad (5)$$

For $k = 3$, matching lower bounds for $K_{s,2}$ were given in [17] when s is odd. Lv, Lu and Fang [13] solved the case for $s = 2$, and Luo [14] recently extended it to all even s . Together, these results imply that, for every $s \geq 2$,

$$\text{ex}_{\chi \leq 3}(N, K_{s,2}) = \left(\sqrt{\frac{s-1}{6}} + o(1)\right) N^{3/2}, \quad \text{ex}(n, n, n, K_{s,2}) = \left(3\sqrt{\frac{s-1}{2}} + o(1)\right) n^{3/2}. \quad (6)$$

Our first theorem improves (3) by replacing $s-1$ with $s-t+1$, while also recovering (4) and (5). The proof follows a standard approach: we apply Füredi's estimate (2) to the partitions determined by a proper coloring and then optimize the part sizes using concavity.

Theorem 1.1. *Fix integers $k \geq 2$ and $s \geq t \geq 2$. Then*

$$\text{ex}_{\chi \leq k}(N, K_{s,t}) \leq \left(\frac{1}{2}(s-t+1)^{1/t} \left(1 - \frac{1}{k}\right)^{1-1/t} + o(1)\right) N^{2-1/t}.$$

Our next theorem gives a lower bound for $\text{ex}(n, n, n, K_{s,t})$. It follows from the more general Theorem 1.5 presented later.

Theorem 1.2. *Let $s \geq t \geq 2$ satisfy $s \geq (t-1)! + 1$, and let r be the largest integer such that $s \geq (t-1)!r^{t-1} + 1$. Then*

$$\text{ex}(n, n, n, K_{s,t}) \geq \left(\frac{3}{2^{1/t}} r^{1-1/t} - o(1) \right) n^{2-1/t}.$$

When $t = 2$, Theorems 1.1 and 1.2 together recover the formula (6) established in [13, 14, 17]. We obtain the following corollary by setting $t = 3$ in Theorem 1.2 and applying Theorem 1.1 with $k = 3$ and $N = 3n$.

Corollary 1.3. *Fix an integer $s \geq 3$, and let r be the largest integer such that $s \geq 2r^2 + 1$. Then*

$$\left(\frac{3}{\sqrt[3]{2}} r^{2/3} - o(1) \right) n^{5/3} \leq \text{ex}(n, n, n, K_{s,3}) \leq \left(\frac{3}{\sqrt[3]{2}} (s-2)^{1/3} + o(1) \right) n^{5/3}.$$

In particular, when $s = 3$, we have $r = 1$ and hence,

$$\text{ex}(n, n, n, K_{3,3}) = \left(\frac{3}{\sqrt[3]{2}} + o(1) \right) n^{5/3}. \quad (7)$$

A natural question is whether Theorem 1.2 can be extended to the k -partite Zarankiewicz number $\text{ex}(n, \dots, n, K_{s,t})$. At present, we do not even know how to do so for $\text{ex}(n, n, n, n, K_{2,2})$ – the main difficulty is that a copy of $K_{2,2}$ may have four vertices from four different parts of the host graph.

This motivates a one-sided version of the multipartite problem. We call a copy of $K_{s,t}$ in a multipartite graph *one-sided* if its s -vertex or t -vertex side lies entirely in a single part. Let $\text{ex}_k^*(n, K_{s,t})$ denote the maximum number of edges in a k -partite graph with n vertices in each part that contains no one-sided copy of $K_{s,t}$. We obtain the following upper bound from Füredi's bound (2).

Proposition 1.4. *Fix integers $k \geq 2$ and $s \geq t \geq 2$. Then*

$$\text{ex}_k^*(n, K_{s,t}) \leq \left(\frac{k}{2} (s-t+1)^{1/t} (k-1)^{1-1/t} + o(1) \right) n^{2-1/t}.$$

If only one-sided copies are forbidden, then Theorem 1.2 extends to an arbitrary number of parts.

Theorem 1.5. *Fix integers $k \geq 2$ and $s \geq t \geq 2$ satisfying $s \geq (t-1)! + 1$, and let r be the largest integer such that $s \geq (t-1)!r^{t-1} + 1$. Then*

$$\text{ex}_k^*(n, K_{s,t}) \geq \left(\frac{k}{2} (k-1)^{1-1/t} r^{1-1/t} - o(1) \right) n^{2-1/t}.$$

The proof uses quotient norm graphs of Alon, Rónyai and Szabó [1]. Let H be the subgraph of the quotient norm graph after removing a small portion of vertices. We construct an automorphism h of H and a set $U \subseteq V(H)$ so that the $k-1$ sets $h^{2-k}U, h^{4-k}U, \dots, h^{k-2}U$ partition $V(H)$. Using k copies of U as the parts, we define edges of the k -partite graph so that, for any set of vertices in one part, its common neighbors in the other $k-1$ parts correspond to the common neighborhoods of these vertices in $h^{2-k}U, h^{4-k}U, \dots, h^{k-2}U$.

Every copy of $K_{s,t}$ in a tripartite graph is one-sided: one part of the tripartite graph cannot meet both sides of $K_{s,t}$, and if neither side were contained in one part, then the copy of $K_{s,t}$ would meet at least four parts, which is impossible. Thus, $\text{ex}_3^*(n, K_{s,t}) = \text{ex}(n, n, n, K_{s,t})$, and Theorem 1.5 with $k = 3$ implies Theorem 1.2 because $(3/2)2^{1-1/t} = 3/2^{1/t}$.

Combining Proposition 1.4 and Theorem 1.5 gives the following corollary.

Corollary 1.6. *For every $k \geq 2$ and $s \geq 2$, we have*

$$\text{ex}_k^*(n, K_{s,2}) = \left(\frac{k}{2} \sqrt{(k-1)(s-1)} + o(1) \right) n^{3/2}, \quad \text{ex}_k^*(n, K_{3,3}) = \left(\frac{k}{2} (k-1)^{2/3} + o(1) \right) n^{5/3}.$$

Alon, Rónyai and Szabó [1] applied norm graph constructions to the study of multicolor Ramsey numbers. Let $r_m(H)$ be the least n such that every m -coloring of K_n contains a monochromatic copy of H . They showed that

$$r_m(K_{3,3}) = (1 + o(1))m^3. \quad (8)$$

Our construction behind Theorem 1.5 has a similar application. For integers $m \geq 1$ and $k \geq 2$, let $r_m^{(k)}(H)$ be the least n such that every m -coloring of the complete k -partite graph $K_{n,\dots,n}$ yields a monochromatic copy of H . This multipartite Ramsey parameter was introduced in [5]. In the bipartite case, it is known that $r_m^{(2)}(K_{2,s}) = (s-1 + o(1))m^2$ and $r_m^{(2)}(K_{3,3}) = (1 + o(1))m^3$ for every fixed $s \geq 2$ [18]. Our next theorem determines $r_m^{(3)}(K_{3,3})$ asymptotically.

Theorem 1.7. *We have $r_m^{(3)}(K_{3,3}) = (\frac{1}{2} + o(1))m^3$.*

Notation. All graphs are simple except for norm graphs, which have loops. Let H be a graph that may have loops. For $x \in V(H)$, let $\Gamma_H(x)$ denote the neighborhood of x in H (if there is a loop at x , then $x \in \Gamma_H(x)$). For $S \subseteq V(H)$, let $\Gamma_H(S) = \bigcap_{x \in S} \Gamma_H(x)$ denote the common neighborhood of S in H . We omit subscripts when the underlying graph is clear. We write $A \sqcup B$ for the union of two disjoint sets A and B .

For a prime power q , we use $\mathbb{F}_q$ to denote the finite field with q elements and $\mathbb{F}_q^*$ for its multiplicative group. Let $N(z) = N_{\mathbb{F}_{q^{t-1}}/\mathbb{F}_q}(z) = z^{1+q+\dots+q^{t-2}}$ be the norm of $z \in \mathbb{F}_{q^{t-1}}$ over $\mathbb{F}_q$.

Organization of the paper. Section 2 proves Theorem 1.1 and Proposition 1.4, while Section 3 gives norm graph constructions and proves Theorem 1.5. The Ramsey application appears in Section 4.

2. UPPER BOUNDS

Proof of Theorem 1.1. Let G be an N -vertex $K_{s,t}$ -free graph with a proper k -coloring of $V(G) = A_1 \sqcup \dots \sqcup A_k$, where some A_i may be empty. For $1 \leq i \leq k$, let $n_i = |A_i|$, $\delta_i = n_i/N$, $B_i = V(G) \setminus A_i$ and $e_i = e(A_i, B_i)$. Since G is $K_{s,t}$ -free, the bipartite subgraph $G[A_i, B_i]$ is also $K_{s,t}$ -free for each $1 \leq i \leq k$. If $n_i < t$ or $N - n_i < s$, then $e_i \leq n_i(N - n_i) = O(N)$. For every other i , (2) gives

$$e_i \leq z(N - n_i, n_i, s, t) \leq (s - t + 1)^{1/t} n_i (N - n_i)^{1-1/t} + t(N - n_i)^{2-2/t} + t n_i.$$

Let $f(x) = x(1-x)^{1-1/t}$ for $x \in [0, 1]$. Every edge of G is counted in exactly two of the e_i . Hence, summing the preceding bounds over i gives

$$2e(G) = \sum_{i=1}^k e_i \leq (s - t + 1)^{1/t} N^{2-1/t} \sum_{i=1}^k f(\delta_i) + o(N^{2-1/t}). \quad (9)$$

A direct calculation gives $f''(x) = (1 - 1/t)(1-x)^{-1-1/t}((2-1/t)x - 2) < 0$ for $0 < x < 1$, so f is concave on $[0, 1]$. Since $\sum_{i=1}^k \delta_i = 1$, Jensen's inequality gives $\sum_{i=1}^k f(\delta_i) \leq k f(\frac{1}{k}) = (1 - \frac{1}{k})^{1-1/t}$. Substituting this inequality into (9) yields

$$e(G) \leq \left(\frac{1}{2} (s - t + 1)^{1/t} \left(1 - \frac{1}{k} \right)^{1-1/t} + o(1) \right) N^{2-1/t}. \quad \square$$

Proof of Proposition 1.4. Let G be a k -partite graph with parts $A_1, \dots, A_k$, each of size n , and suppose that G contains no one-sided copy of $K_{s,t}$. For each i , $G[A_i, V(G) \setminus A_i]$ is $K_{s,t}$ -free. Therefore, (2) gives

$$\begin{aligned} e(A_i, V(G) \setminus A_i) &\leq z((k-1)n, n, s, t) \\ &\leq (s-t+1)^{1/t} (k-1)^{1-1/t} n^{2-1/t} + t(k-1)^{2-2/t} n^{2-2/t} + tn. \end{aligned}$$

Summing over i and noting that every edge of G is counted twice, we obtain

$$2e(G) \leq k(s-t+1)^{1/t} (k-1)^{1-1/t} n^{2-1/t} + o(n^{2-1/t}),$$

which gives the desired bound. $\square$

3. A MULTIPARTITE CONSTRUCTION FROM NORM GRAPHS

We first recall the construction of Alon, Rónyai and Szabó [1, Section 4]. Given integers $s \geq t \geq 2$ with $s \geq (t-1)! + 1$, let r be the largest integer such that $s \geq (t-1)!r^{t-1} + 1$. Let q be a prime power such that r divides $q-1$, and let Q_r be the subgroup of $\mathbb{F}_q^*$ of order r . Let $X = \mathbb{F}_{q^{t-1}} \times (\mathbb{F}_q^*/Q_r)$. Note that $|X| = q^{t-1}(q-1)/r = (q^t - q^{t-1})/r$. Construct a graph $H_r = H_r(q, t)$ with vertex set X by declaring two (not necessarily distinct) vertices $x = (A, aQ_r)$ and $y = (B, bQ_r)$ adjacent if

$$x \sim y \iff N(A+B) \in abQ_r. \quad (10)$$

(Note that $x \sim x$ if and only if $N(2A) \in a^2Q_r$.) For a vertex $x = (A, aQ_r)$, relation (10) determines a unique coset bQ_r for each $B \neq -A$. Hence $|\Gamma(x)| = q^{t-1} - 1$, including possible loops. Alon, Rónyai and Szabó [1, Section 4] showed that¹

$$|\Gamma_{H_r}(T)| \leq (t-1)!r^{t-1} < s \quad \text{for all } T \subseteq X \text{ with } |T| = t. \quad (11)$$

Our construction begins by deleting the vertices whose first coordinate is zero. Let $X^* = \mathbb{F}_{q^{t-1}}^* \times (\mathbb{F}_q^*/Q_r)$. Every vertex of X^* loses exactly one neighbor, namely the unique vertex with first coordinate zero. Therefore, every vertex of $H_r[X^*]$ has exactly $q^{t-1} - 2$ neighbors.

Fix $k \geq 2$ and suppose that $q \equiv 1 \pmod{4r(k-1)}$. Let $H = H_r[X^*]$. Choose $\omega \in \mathbb{F}_q^*$ of order $4(k-1)$, set $\iota = \omega^2$, and let $L = 1 + q + \dots + q^{t-2}$. We define a bijection $h : X^* \rightarrow X^*$ by $h(A, aQ_r) = (\iota A, \omega^L aQ_r)$. Consider $x = (A, aQ_r)$ and $y = (B, bQ_r)$ in X^* . Since $\iota \in \mathbb{F}_q^*$,

$$N(\iota(A+B)) = \iota^L N(A+B) \quad \text{and} \quad (\omega^L a)(\omega^L b) = \iota^L ab.$$

Consequently,

$$h(x) \sim h(y) \iff N(\iota(A+B)) \in (\omega^L a)(\omega^L b)Q_r \iff \iota^L N(A+B) \in \iota^L abQ_r.$$

It follows that $x \sim y$ if and only if $h(x) \sim h(y)$. Thus h is an automorphism of H .

Set $g = h^2$. For $(A, aQ_r) \in X^*$, it follows that $g(A, aQ_r) = (\omega^4 A, \omega^{2L} aQ_r)$. The element ω^4 has order $k-1$, and $g^{2(k-1)}$ is the identity in X^* . If $g^d(A, aQ_r) = (A, aQ_r)$, then $\omega^{4d} A = A$; since $A \neq 0$, it follows that $(k-1) \mid d$. Thus every cycle of g has length $k-1$ or $2(k-1)$. By the disjoint-cycle decomposition of a finite permutation, these cycles partition X^* . For each cycle C , choose $x_C \in C$ and define a set $U \subseteq X^*$ by

$$U \cap C = \{g^{j(k-1)}(x_C) : 0 \leq j < |C|/(k-1)\}.$$

¹Note that (11) is stronger than H_r being $K_{(t-1)!r^{t-1}+1, t}$ -free because (11) means that there are no sets $S, T \subseteq X$, not necessarily disjoint, with $|S| = (t-1)!r^{t-1} + 1$, $|T| = t$ such that $x \sim y$ for all $x \in S$ and $y \in T$.

For each C , the sets $g^a(U \cap C)$, $0 \leq a \leq k-2$, partition C . Consequently,

$$X^* = \bigsqcup_{a=0}^{k-2} h^{2a}U, \quad \text{and} \quad |U| = \frac{|X^*|}{k-1} = \frac{(q^{t-1}-1)(q-1)}{r(k-1)} =: n_q.$$

Now we construct a k -partite graph $G_q^{(k)}$. Index its parts by $\mathbb{Z}/k\mathbb{Z}$, and let $U_i = U \times \{i\}$ be the parts. We write $x_i \in U_i$ for $(x, i) \in U_i$. For distinct $i, j \in \mathbb{Z}/k\mathbb{Z}$, let $d(i, j) = j - i \pmod{k} \in \{1, \dots, k-1\}$ and set $\ell(i, j) = k - 2d(i, j)$. Since $d(j, i) = k - d(i, j)$, we have $\ell(j, i) = -\ell(i, j)$. For $x, y \in U$, join $x_i \in U_i$ and $y_j \in U_j$ if and only if $x \sim h^{\ell(i, j)}(y)$ in H . Since h is an automorphism,

$$x \sim h^{\ell(i, j)}(y) \iff y \sim h^{-\ell(i, j)}(x) \iff y \sim h^{\ell(j, i)}(x).$$

The adjacency of $G_q^{(k)}$ is thus symmetric.

For a fixed i , as j ranges over $\mathbb{Z}/k\mathbb{Z} \setminus \{i\}$, the value $d(i, j)$ ranges over $1, \dots, k-1$. Hence

$$\{\ell(i, j) : j \neq i\} = \{k-2, k-4, \dots, 2-k\} = \{2-k+2a : 0 \leq a \leq k-2\}.$$

Applying h^{2-k} to the earlier partition of X^* gives

$$X^* = \bigsqcup_{j \neq i} h^{\ell(i, j)}U. \quad (12)$$

For $T \subseteq U$, write $T_i = \{x_i : x \in T\}$. Recall that $\Gamma_H(T)$ denotes the common neighborhood of T in H .

Lemma 3.1. *For every nonempty set $T \subseteq U$ and $i \in \mathbb{Z}/k\mathbb{Z}$, we have $|\Gamma_{G_q^{(k)}}(T_i)| = |\Gamma_H(T)|$.*

Proof. Fix $j \neq i$. For $y \in U$, we have

$$\begin{aligned} y_j \in \Gamma_{G_q^{(k)}}(T_i) &\iff x \sim h^{\ell(i, j)}(y) \text{ for every } x \in T \\ &\iff h^{\ell(i, j)}(y) \in \Gamma_H(T) \cap h^{\ell(i, j)}U. \end{aligned}$$

For every $j \neq i$, the map $y_j \mapsto h^{\ell(i, j)}(y)$ is a bijection from $\Gamma_{G_q^{(k)}}(T_i) \cap U_j$ to $\Gamma_H(T) \cap h^{\ell(i, j)}U$.

Using (12), we obtain that

$$|\Gamma_{G_q^{(k)}}(T_i)| = \sum_{j \neq i} |\Gamma_{G_q^{(k)}}(T_i) \cap U_j| = \sum_{j \neq i} |\Gamma_H(T) \cap h^{\ell(i, j)}U| = |\Gamma_H(T)|. \quad \square$$

Proof of Theorem 1.5. Given a prime power $q \equiv 1 \pmod{4r(k-1)}$, we construct the graph $G_q^{(k)}$ as above. Suppose that $G_q^{(k)}$ contains a one-sided copy of $K_{s,t}$. Let T_i be the side of $K_{s,t}$ contained in U_i and let $T = \{x \in U : x_i \in T_i\}$. If $|T| = t$, then Lemma 3.1 gives $|\Gamma_H(T)| = |\Gamma_{G_q^{(k)}}(T_i)| \geq s$, contradicting (11). If $|T| = s$, then Lemma 3.1 gives $|\Gamma_H(T)| = |\Gamma_{G_q^{(k)}}(T_i)| \geq t$. Choose a set $S \subseteq \Gamma_H(T)$ of size t . Then $|\Gamma_H(S)| \geq |T| = s$, again contradicting (11). Thus, $G_q^{(k)}$ contains no one-sided copy of $K_{s,t}$.

Taking $T = \{x\}$ in Lemma 3.1 shows that $G_q^{(k)}$ is $(q^{t-1}-2)$ -regular. Thus, $e(G_q^{(k)}) = \frac{k}{2}n_q(q^{t-1}-2)$. Since $n_q = (1+o(1))q^t/(r(k-1))$, we have $q^{t-1} = (1+o(1))(r(k-1)n_q)^{1-1/t}$. Consequently,

$$e(G_q^{(k)}) = \frac{k}{2}n_q(r(k-1)n_q)^{1-1/t}(1+o(1)) = \left(\frac{k}{2}(k-1)^{1-1/t}r^{1-1/t} + o(1)\right)n_q^{2-1/t},$$

which gives the desired bound when $n = n_q$.

For sufficiently large n , set $x = (r(k-1)n)^{1/t}$. By the prime number theorem in arithmetic progressions (see, e.g. [4]), there is a prime $q \equiv 1 \pmod{4r(k-1)}$ such that $q \leq x$ and $q = (1-o(1))x$.

Since $n_q < q^t/(r(k-1)) \leq n$, we may add isolated vertices to each part of $G_q^{(k)}$ to obtain a graph with n vertices in each part. Moreover, $q = (1 - o(1))x$ implies $n_q = (1 - o(1))n$, so the resulting graph has the asserted number of edges. $\square$

4. A TRIPARTITE RAMSEY APPLICATION

We prove Theorem 1.7 by following the approach used in the proof of [1, Theorem 3].

Proof of Theorem 1.7. For the upper bound, suppose that the edges of $K_{n,n,n}$ are colored with m colors and there is no monochromatic $K_{3,3}$. Let G_c be the spanning subgraph consisting of the edges of color c . Then (7) in Corollary 1.3 gives

$$3n^2 = \sum_{c=1}^m e(G_c) \leq m \left(\frac{3}{\sqrt[3]{2}} + o(1) \right) n^{5/3}.$$

This implies that $n \leq (1/2 + o(1))m^3$, which gives the upper bound for $r_m^{(3)}(K_{3,3})$.

For the lower bound, we use a variant of the graph introduced at the beginning of Section 3. Suppose that $s = t = 3$, so $r = 1$. Let $q \equiv 1 \pmod{8}$ be a prime power and $X = \mathbb{F}_{q^2} \times \mathbb{F}_q^*$. For each $\lambda \in \mathbb{F}_q^*$, define a graph H_λ on X by the symmetric relation

$$(A, a) \sim_\lambda (B, b) \iff N(A + B) = \lambda ab.$$

The graph H_λ has a loop at (A, a) exactly when $N(2A) = \lambda a^2$. Alon, Rónyai and Szabó [1, Section 3] noted that

$$|\Gamma_{H_\lambda}(T)| \leq 2 \quad \text{for all } T \subseteq X \text{ with } |T| = 3. \quad (13)$$

Choose $\omega \in \mathbb{F}_q^*$ of order eight and set $\iota = \omega^2$. Let $X^* := \mathbb{F}_{q^2}^* \times \mathbb{F}_q^*$ and define $h : X^* \rightarrow X^*$ by $h(A, a) = (\iota A, \omega^{1+q}a)$. Then h is an automorphism of $H_\lambda[X^*]$. Apply the construction of Section 3 with three parts, that is, choose $U \subseteq X^*$ such that $X^* = h^{-1}U \sqcup hU$, and define the tripartite graph $G_q(\lambda)$ on $U_1 \sqcup U_2 \sqcup U_3$ by joining $x = (A, a) \in U_i$ and $y = (B, b) \in U_{i+1}$ if and only if

$$x \sim_\lambda h(y) \iff N(A + \iota B) = \lambda \omega^{1+q}ab.$$

Lemma 3.1 applies with respect to the relation $\sim_\lambda$. Using (13), the proof of Theorem 1.5 shows that $G_q(\lambda)$ contains no one-sided copy of $K_{3,3}$. Hence, $G_q(\lambda)$ is $K_{3,3}$ -free.

For each $i \in \{1, 2, 3\}$ and every $x = (A, a) \in U_i$ and $y = (B, b) \in U_{i+1}$ satisfying $A + \iota B \neq 0$, assign to xy the color

$$\chi(xy) = \frac{N(A + \iota B)}{\omega^{1+q}ab} \in \mathbb{F}_q^*.$$

In other words, an edge xy has color λ precisely when $x \sim_\lambda h(y)$. Thus none of the colors in $\mathbb{F}_q^*$ contains a copy of $K_{3,3}$.

For each $i \in \{1, 2, 3\}$ and $A \in \mathbb{F}_{q^2}^*$, let $Q_{i,A}$ be the complete bipartite graph whose two sides are

$$\{(A, a)_i : (A, a) \in U\} \quad \text{and} \quad \{(\iota A, b)_{i+1} : (\iota A, b) \in U\}.$$

These are precisely the uncolored pairs between U_i and U_{i+1} because $A + \iota(\iota A) = A + \iota^2 A = 0$. For each fixed i , the graphs $Q_{i,A}$ are vertex-disjoint as A varies, and each of their sides has size at most $q - 1$.

Set $\rho = \lceil 2q^{1/3} \rceil$. By (8), we have $r_\rho(K_{3,3}) = (1 + o(1))\rho^3 > 2(q - 1)$ for all sufficiently large q . Hence there is a ρ -coloring of the complete graph on $2(q - 1)$ vertices with no monochromatic $K_{3,3}$. For each i , color all $Q_{i,A}$ by such a coloring. Since the graphs $Q_{i,A}$, $A \in \mathbb{F}_{q^2}^*$, are vertex-disjoint, no

monochromatic $K_{3,3}$ is created. Use 3ρ new colors for the three pairs of host parts, disjoint from $\mathbb{F}_q^*$. Altogether, we use $(q-1) + 3\rho = q + o(q)$ colors on parts of size

$$n_q = \frac{(q^2-1)(q-1)}{2} = \left(\frac{1}{2} + o(1)\right) q^3.$$

Suppose m is sufficiently large. The prime number theorem in arithmetic progressions (see [4]) allows us to choose a prime $q \equiv 1 \pmod{8}$ such that $q = (1 - o(1))m$ and $q + 3\rho \leq m$. The coloring above uses at most m colors and gives $r_m^{(3)}(K_{3,3}) > n_q = \left(\frac{1}{2} - o(1)\right) m^3$. $\square$

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