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Forbidding Complete Hypergraphs as Traces

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Abstract. Let $2 \leq q \leq \min\{p, t-1\}$ be fixed and $n \to \infty$. Suppose that $\mathcal{F}$ is a p-uniform hypergraph on n vertices that contains no complete q-uniform hypergraph on t vertices as a trace. We determine the asymptotic maximum size of $\mathcal{F}$ in many cases. For example, when $q = 2$ and $p \in \{t, t+1\}$, the maximum is $(\frac{n}{t-1})^{t-1} + o(n^{t-1})$, and when $p = t = 3$, it is $\lfloor \frac{(n-1)^2}{4} \rfloor$ for all $n \geq 3$. Our proofs use the Kruskal-Katona theorem, an extension of the sunflower lemma due to Füredi, and recent results on hypergraph Turán numbers.

Key words. Trace, hypergraph, turán problem, extremal problem.

1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$. Given a set X, 2^X denotes the family of all subsets of X, and $\binom{X}{q} = \{A \subseteq X : |A| = q\}$. A *hypergraph* $\mathcal{H}$ on X is a family of subsets of X; these subsets are called *edges* of $\mathcal{H}$ and X is the *vertex* set of $\mathcal{H}$. If all edges of $\mathcal{H}$ have size p, then $\mathcal{H}$ is a p-uniform hypergraph (p-graph for short).

Let G be a hypergraph on X and $S \subseteq X$. We define the *trace* of G on S as

$$G|_S := \{E \cap S : E \in G\}.$$

Note that we omit multiplicity when defining G_S.

If there exists a set S such that $G|_S$ contains a copy of F as a subhypergraph, we say that G contains F as a trace, or F is a trace of G. In this case we write $G \to F$, otherwise $G \not\to F$. Let $L^p(n, F)$ ($L(n, F)$) denote the maximum number of edges in a p-uniform (not necessarily uniform) hypergraph on $[n]$ not containing F as a trace. Extremal problems on traces started from determining $L(n, 2^{[t]})$. Sauer [16], Perles-Shelah [17], and Vapnik-Chervonenkis [19] independently found that $L(n, 2^{[t]}) = \binom{n}{0} + \cdots + \binom{n}{t-1}$. For the uniform case, Frankl and Pach [6] showed that $L^p(n, 2^{[t]}) \leq \binom{n}{t-1}$ for $t \leq p \leq n$ (see [14] for small improvement). Many

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intersecting problems and applications on traces can be found in the survey of Füredi and Pach [9].

In this paper we consider the problem of forbidding a level of the lattice $2^{[t]}$ as a trace. More precisely, given integers p, t, n with $\max\{p, t\} \leq n$, we study the value of $L^p(n, \binom{[t]}{q})$ for $1 \leq q \leq t-1$ (the $q = 0$ and $q = t$ cases are trivial). Frankl and Pach [6] studied the $q = 1$ case and obtained that $\mathrm{ex}(p+t-1, \binom{[t]}{t-1}) \leq L^p(n, \binom{[t]}{1}) \leq \binom{p+t-1}{t-1}$, where *ex* is the classical Turán number. Balogh, Keevash and Sudakov [1] investigated the trace problem of forbidding more than one non-trivial level of $2^{[t]}$.

Trivially $L^p(n, \binom{[t]}{q}) = \binom{n}{p}$ when $p < q$. Therefore throughout the paper we assume that

$$2 \leq q \leq t-1 \quad \text{and} \quad q \leq p, \tag{1}$$

and whenever we use asymptotic notation, we assume that only $n \to \infty$. Note that the $p = q$ case is exactly the Turán problem. The reason why we only consider *uniform* trace numbers is that Füredi and Quinn [10] showed that $L(n, \binom{[t]}{q}) = L(n, 2^{[t]})$ for every $0 \leq q \leq t$, in other words, forbidding a level of the lattice $2^{[t]}$ is equivalent to forbidding the whole lattice in the non-uniform case. Following graph theory language, the forbidden configuration $\binom{[t]}{q}$ is a complete q-graph on t vertices, so we denote it by K_t^q, and write $K_t = K_t^2$.

Our first result, which is little more than an observation, determines the order of magnitude of $L^p(n, K_t^q)$.

Proposition 1. $L^p(n, K_t^q) = \Theta(n^{\min\{p,t-1\}})$.

A trace problem for uniform hypergraphs is in fact a Turán problem. Given a family $\mathcal{F}$ of r-graphs, the *Turán number* $\mathrm{ex}(n, \mathcal{F})$ of $\mathcal{F}$ is the maximum number of edges in an r-graph on n vertices containing no $F \in \mathcal{F}$ (see *e.g.*, Füredi [8] for a survey). When $\mathcal{F} = \{F\}$, we write $\mathrm{ex}(n, F)$ instead of $\mathrm{ex}(n, \{F\})$. If we denote K_3 by $\{12, 23, 31\}$, then $L^3(n, K_3) = \mathrm{ex}(n, \{F_1, F_2, F_3\})$, where $F_1 = \{124, 234, 134\}$, $F_2 = \{124, 234, 135\}$, and $F_3 = \{124, 235, 136\}$. In general, for any q-graph F and $q \leq p$, we have $L^p(n, F) = \mathrm{ex}(n, \mathcal{H}^p(F))$, where $\mathcal{H}^p(F)$ is the family of all p-graphs H with $|F|$ edges such that $H \to F$.

Definition 1. Let $H_{q,t}^p$ be the member of $\mathcal{H}^p(K_t^q)$ with the maximum number of vertices. In other words, $H_{q,t}^p$ is the p-graph obtained from K_t^q by enlarging each of its $\binom{t}{q}$ edges with a (different) set of $p-q$ new vertices. Trivially $H_{p,t}^p = K_t^p$.

Since forbidding a family of hypergraphs (as a subgraph) is not easier than forbidding any member of the family,

$$L^p(n, K_t^q) = \mathrm{ex}(n, \mathcal{H}^p(K_t^q)) \leq \mathrm{ex}(n, H_{q,t}^p). \tag{2}$$

Our second result, which is also not hard to prove, shows that the inequality in (2) is asymptotically an equality when $p < t$.

Proposition 2. *Let $p < t$. Then $L^p(n, K_t^q) = \mathrm{ex}(n, H_{q,t}^p) + o(n^p)$.*

Our main result reduces $L^p(n, K_t^q)$ when $p \geq t$ to Turán numbers in many cases.

Theorem 1. *Fix $2 \leq q < t \leq p$. Suppose that $q \in \{t-2, t-1\}$ or $p \in \{t, t+1\}$. Then*

$$L^p(n, K_t^q) = L^{t-1}(n, K_t^q) + o(n^{t-1}) = \mathrm{ex}(n, H_{q,t}^{t-1}) + o(n^{t-1}). \tag{3}$$

This suggests that determining $L^p(n, K_t^q)$ could be as difficult as a hypergraph Turán problem. For example, (3) implies that $L^4(n, K_4^3) = \mathrm{ex}(n, K_4^3) + o(n^3)$, and determining $\mathrm{ex}(n, K_4^3)$ is a well-known open problem of Turán [18]. Together with Mantel's Theorem on $\mathrm{ex}(n, K_3)$ [12], Theorem 1 gives

$$L^p(n, K_3) = \mathrm{ex}(n, K_3) + o(n^2) = \left(\frac{n}{2}\right)^2 + o(n^2). \tag{4}$$

Determining $\mathrm{ex}(n, H_{q,t}^p)$ in general seems hopeless. However, the $q = 2$ case was recently solved by the first author [13] and Pikhurko [15]. Given $2 \leq p \leq \ell$, a p-graph is *ℓ-partite* if its vertices can be partitioned into ℓ classes, such that every edge has at most one vertex from each class. An ℓ-partite p-graph is called *complete* if it contains all allowable edges. We denote by $T_\ell^p(n)$ the complete ℓ-partite p-graph (a *generalized* Turán graph) on n vertices with no two class sizes differ more than one. Let $p < t$. Clearly $T_{t-1}^p(n)$ contains no $H_{2,t}^p$ as a subgraph and

$$\left|T_\ell^p(n)\right| = \sum_{S \in \binom{[\ell]}{p}} \prod_{i \in S} \left\lfloor \frac{n+i-1}{\ell} \right\rfloor = \binom{\ell}{p}\left(\frac{n}{\ell}\right)^p + o(n^p).$$

The first author [13] showed that $\mathrm{ex}(n, H_{2,t}^p) = |T_{t-1}^p(n)| + o(n^p)$ as $n \to \infty$. Pikhurko [15] improved this to $\mathrm{ex}(n, H_{2,t}^p) = |T_{t-1}^p(n)|$ for sufficiently large n. Applying (2), we thus have $L^p(n, K_t) \leq |T_{t-1}^p(n)|$ for sufficiently large n. On the other hand, it is easy to see that $T_{t-1}^p(n)$ contains no K_t^q for any $q \geq 2$ as a trace. In fact, every t-vertex set S of $T_{t-1}^p(n)$ must contain two vertices a, b from the same vertex class, but no edge of $T_{t-1}^p(n)$ contains both a and b. Thus for $q \geq 2$, every q-subset of S containing a and b is absent from $T_{t-1}^p(n)|_S$. Consequently $T_{t-1}^p(n) \not\to K_t^q$, in particular, $L^p(n, K_t) \geq |T_{t-1}^p(n)|$. Putting the upper and lower bounds together, for $2 \leq p < t$ and sufficiently large n,

$$L^p(n, K_t) = |T_{t-1}^p(n)| = \mathrm{ex}(n, H_{2,t}^p). \tag{5}$$

By combining (5) with Theorem 1, we obtain the following result.

Corollary 1. *Suppose that $t = 4$ or $p \in \{t, t+1\}$. Then*

$$L^p(n, K_t) = |T_{t-1}^{t-1}(n)| + o(n^{t-1}) = \left(\frac{n}{t-1}\right)^{t-1} + o(n^{t-1}). \tag{6}$$

We conjecture the values of $L^p(n, K_t^q)$ as follows.

Conjecture 1. Fix p, q, t, n with $2 \le q < \min\{t, p\}$. For $n > n_0$,

$$L^p(n, K_t^q) = \begin{cases} \mathrm{ex}(n, H_{q,t}^p) & \text{if } p < t, \\ L^{t-1}(n-p+t-1, K_t^q) = \mathrm{ex}(n-p+t-1, H_{q,t}^{t-1}) & \text{if } p \ge t. \end{cases}$$

The equation (5) confirms the conjecture for the case of $q = 2$, $p < t$, and sufficiently large n. As further evidence of Conjecture 1, we prove its smallest non-trivial case: $(p, q, t) = (3, 2, 3)$. Note that this sharpens the $p = 3$ case of (4).

Theorem 2. *Let $n \ge 3$. Then*

$$L^3(n, K_3) = \mathrm{ex}(n-1, K_3) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor.$$

2. Preliminary Results

In this section we prove Proposition 1, Proposition 2 and the supersaturation property for trace problems.

We first observe that $L^p(n, K_t^q)$ is close to a monotone function of n.

Proposition 3. $L^p(n, K_t^q) \ge L^{p-i}(n-i, K_t^q)$ *for* $1 \le i \le p-q$.

Proof. Suppose that $G \subseteq \binom{[n-i]}{p-i}$ satisfies $G \not\to K_t^q$. We extend G to a p-graph G' by adding a set C of i new vertices and replacing each $E \in G$ by $E \cup C$. We claim that $G' \not\to K_t^q$. Consider a t-set S of vertices. If S contains a vertex $x \in C$, then all edges of G' contain x, and consequently all q-subsets of $E \backslash \{x\}$ are absent from $G'|_E$. Otherwise $E \cap C = \emptyset$, and we have $G'|_E = G|_E \not\to K_t^q$. □

Proof of Proposition 1. We need to show that $L^p(n, K_t^q) = \Theta(n^{\min\{p,t-1\}})$. When $p \ge t$, Frankl and Pach [6] showed that $L^p(n, K_t^q) \le \binom{n}{t-1}$. When $p < t$, trivially $L^p(n, K_t^q) \le \binom{n}{p}$. We now consider lower bounds. When $p \le t-1$, Since $T_{t-1}^p(n) \not\to K_t^q$ (for $q \ge 2$), we have $L^p(n, K_t^q) \ge |T_{t-1}^p(n)| = \Omega(n^p)$. Now let $p \ge t$. Since $T_{t-1}^{t-1}(n-p+t-1) \not\to K_t^q$ and $|T_{t-1}^{t-1}(n-p+t-1)| = \Omega(n^{t-1})$, we have $L^{t-1}(n-p+t-1, K_t^q) = \Omega(n^{t-1})$. Proposition 3 thus implies that $L^p(n, K_t^q) \ge L^{t-1}(n-p+t-1, K_t^q) = \Omega(n^{t-1})$. □

Definition 2. Let F be a p-graph ($p \ge 2$) on $[\ell]$ and $\vec{m} = \langle m_1, \ldots, m_\ell \rangle$ be a vector of positive integers. The *blow-up* $F(\vec{m})$ of F is obtained by replacing each vertex i by a vertex class V_i of size m_i, and each edge $\{i_1, \ldots, i_p\}$ by the family of all p-sets $\{w_1, \ldots, w_p\}$, where $w_j \in V_{i_j}$. We simply write $F(m)$ if all $m_i = m$.

A phenomenon discovered by Brown, Erdős and Simonovits [5], usually called *supersaturation*, implies that $\text{ex}(n, F(\vec{m})) = \text{ex}(n, F) + o(n^r)$ for every r-graph F and its blow-up $F(\vec{m})$. To prove Proposition 2, we need a lemma from [13], which is a simple consequence of supersaturation.

Lemma 1 (Lemma 4 in [13]). *Let m, p be positive integers with $p \geq 2$, and let $\mathcal{F}$ be a finite family of p-graphs. If H is a p-graph satisfying $H \subseteq F(m)$ for all $F \in \mathcal{F}$, then* $\text{ex}(n, H) \leq \text{ex}(n, \mathcal{F}) + o(n^p)$. □

Proof of Proposition 2. Here $p < t$ and we must show that $L^p(n, K_t^q) = \text{ex}(n, H_{q,t}^p) + o(n^p)$. Because of (2), we only need to show that $L^p(n, K_t^q) \geq \text{ex}(n, H_{q,t}^p) + o(n^p)$ when $n \to \infty$. For each $F \in \mathcal{H}^p(K_t^q))$, it is easy to see that $H_{q,t}^p \subseteq F(\binom{t}{q})$. Lemma 1 implies that

$$\text{ex}(n, H_{q,t}^p) \leq \text{ex}(n, \mathcal{H}^p(K_t^q)) + o(n^p) = L^p(n, K_t^q) + o(n^p). \tag{7}$$

□

Next we prove the supersaturation phenomenon for trace problems.

Lemma 2. $L^p(n, K_t^q(m)) \leq L^p(n, K_t^q) + o(n^p)$. *In particular, when $p < t$,* $L^p(n, K_t^q(m)) = (1 + o(1))L^p(n, K_t^q)$.

Proof. The second assertion follows from the first by realizing that $L^p(n, K_t^q) = \Theta(n^p)$ for $p < t$ from Proposition 1. To prove the first claim, recall that $\mathcal{H}^p(K_t^q(m))$ is the family of p-graphs whose $|K_t^q(m)|$ edges contain $K_t^q(m)$ as a trace, and $H_{q,t}^p$ is obtained from K_t^q by enlarging each of its $\binom{t}{q}$ edges with a different set of $p-q$ new vertices. Let $\tilde{H} = H_{q,t}^p(\vec{m})$, where $m_i = m$ for all the vertices v_i in the original K_t^q, and $m_i = 1$ for the other vertices. It is easy to see that $\tilde{H}$ is a member of $\mathcal{H}^p(K_t^q(m))$. We thus have

$$\begin{aligned} L^p(n, K_t^q(m)) &= \text{ex}(n, \mathcal{H}^p(K_t^q(m))) \\ &\leq \text{ex}(n, \tilde{H}) \\ &\leq \text{ex}(n, H_{q,t}^p) + o(n^p) \\ &\leq L^p(n, K_t^q) + o(n^p), \end{aligned}$$

where the first inequality holds because $\tilde{H} \in \mathcal{H}^p(K_t^q(m))$, the second inequality holds because of supersaturation for the Turán problems, and the last one holds because of (7). □

3. Proof of Theorem 1

Throughout this section we will assume that $p \geq t$. Our goal is to prove that if $q \in \{t-2, t-1\}$ or $p \in \{t, t+1\}$, then

$$L^p(n, K_t^q) = L^{t-1}(n, K_t^q) + o(n^{t-1}).$$

In fact, the second equality of (3) in Theorem 1, $L^{t-1}(n, K_t^q) = \mathrm{ex}(n, H_{q,t}^{t-1}) + o(n^{t-1})$, follows from Proposition 2 (note that the second condition in (1) still holds because $t-1 \geq q$). Furthermore, we claim that

$$L^p(n, K_t^q) \geq L^{t-1}(n, K_t^q) + o(n^{t-1}). \tag{8}$$

To see this, first observe that Proposition 3 implies that $L^p(n, K_t^q) \geq L^{t-1}(n-p+t-1, K_t^q)$. Proposition 2 further gives that $L^p(n, K_t^q) \geq \mathrm{ex}(n-p+t-1, H_{q,t}^{t-1}) + o(n^{t-1})$. Now we recall a fact on the Turán number, which immediately follows from the existence of $\lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{p}$. Given a family $\mathcal{F}$ of r-graphs and an integer $c > 0$,

$$\mathrm{ex}(n, \mathcal{F}) - \mathrm{ex}(n-c, \mathcal{F}) = o(n^r). \tag{9}$$

Therefore $L^p(n, K_t^q) \geq \mathrm{ex}(n, H_{q,t}^{t-1}) + o(n^{t-1})$ and (8) follows after applying Proposition 2 again.

Therefore the main task is to verify

$$L^p(n, K_t^q) \leq L^{t-1}(n, K_t^q) + o(n^{t-1}). \tag{10}$$

for $q \in \{t-2, t-1\}$ or $p \in \{t, t+1\}$. The $q = t-1$ case (Section 3.1) is the easiest: its main idea is to find a one-to-one function from a p-graph G with $G \not\to K_t^q$ to a $(t-1)$-graph G' such that $G' \not\supseteq K_t^{t-1}$. The remaining cases are harder: we present two lemmas in Section 3.2, and complete the proofs in Section 3.3. The main tools include the Erdős-Ko-Rado theorem, the Kruskal-Katona theorem and a lemma on sunflowers due to Füredi.

3.1. $q = t-1$

Let G be a hypergraph and S be a subset of its vertex set. The *degree* of S in G, $\deg_G(S)$, or $\deg(S)$ if the underlying hypergraph is clear from the context, is the number of edges in G containing S (frequently called codegree when $|S| \geq 2$). Given a p-graph G, if every edge $E \in G$ contains at least one p'-subset E' with $\deg_G(E') = 1$, then $\phi(E) = E'$ defines a one-to-one function from G to $G' = \{E' : E \in G\}$ (if more than one p'-subsets are of degree 1, then arbitrarily pick one of them to be $\phi(E)$).

Proposition 4. *Let G be a p-graph such that $G \not\to K_t^q$. If there exists a function ϕ mapping every edge $E \in G$ to a q-set $E' \subset E$ such that $\deg_G(E') = 1$, then $\phi(G) = \{\phi(E) : E \in G\}$ contains no K_t^q as a subgraph.*

Proof. Suppose instead, that G' contains a subgraph G'_1 on a t-set T such that $G_1 \cong K_t^q$. Clearly ϕ is one-to-one. Let ϕ^{-1} be the inverse function. We claim that each edge $E \in G$ with $\phi(E) \in G'_1$ satisfies that $E \cap T = \phi(E)$ and therefore $G|_T \supseteq K_t^q$, contradicting the assumption that $G \not\to K_t^q$. In fact, if $E \cap T \supset \phi(E)$, then $E \cap T$ contains another q-set $Q \in G'_1$. Clearly $E \neq \phi^{-1}(Q)$ because ϕ is a function. The fact that both E and $\phi^{-1}(Q)$ contain Q implies that $\deg_G(Q) \geq 2$, a contradiction. □

The following lemma is the key observation for proving the $q = t - 1$ case of (10).

Lemma 3. *Let $2 \le t \le p$. Suppose that S is a p-set and H is a family of proper subsets of S. If every $(t-1)$-subset of S is contained in some member of H, then $H \to K_t^{t-1}$.*

Proof. We do induction on p for fixed $t \ge 2$. The base case $p = t$ is trivial, since every $(t-1)$-subset of S is a member of H, or $\binom{S}{p-1} \subseteq H$. For the induction step, let $p > t$ and consider two cases. If $\binom{S}{p-1} \subseteq H$, then for a fixed t-set $T \subset S$, we have $H|_T \supseteq \binom{T}{t-1}$ because each $(t-1)$-subset T' of T is contained in $T' \cup (S \backslash T)$, which is a member of H. Otherwise $\binom{S}{p-1} \not\subseteq H$, and there exists an $(p-1)$-set $S' \notin H$. It is easy to see that $H|_{S'}$ satisfies the assumption of the lemma with $p-1$ instead of p. We then apply the induction hypothesis to S' and $H|_{S'}$ obtaining that $H|_{S'} \to K_t^{t-1}$ and consequently $H \to K_t^{t-1}$. □

Proof of (10) for $q = t - 1$. Let G be an n-vertex p-graph not having K_t^{t-1} as a trace. Each edge $E \in G$ must contain a $(t-1)$-subset E' with $\deg_G(E') = 1$, otherwise we apply Lemma 3 with $S = E$ and $H = G|_E - \{E\}$ to conclude that $G \to K_t^{t-1}$. We thus define $\phi(E) = E'$ and ϕ is a one-to-one function from G to $\binom{[n]}{t-1}$. By Proposition 4, the resulting $(t-1)$-graph G' contains no K_t^{t-1} as a subgraph, thus $|G| = |G'| \le \mathrm{ex}(n, K_t^{t-1}) = L^{t-1}(n, K_t^{t-1})$. □

3.2. Two Lemmas

Fix $G \subseteq \binom{[n]}{p}$ with $|G| \ge 2$. The following partition of G will be needed in our proofs. Define a function $f : G \to [p]$ such that for $E \in G$,

$$f(E) = \min\{|D| : D \subseteq E, \deg(D) = 1 \text{ and } \forall S \subset D, \deg(S) \ge 2\}.$$

(Throughout this subsection $\deg = \deg_G$.) Since $\deg(E) = 1$ and $\deg(\emptyset) = |G| \ge 2$, there always exists a subset $D \subset E$ such that $\deg(D) = 1$ but $\deg(S) \ge 2$ for all $S \subset D$. Hence f is well defined. For $1 \le i \le p$, let $G_i = \{E \in G : f(E) = i\}$. Clearly, $G_p + G_{p-1} + \cdots + G_1$ is a partition of G.

Furthermore, for $k \le p$, let $\partial^k G$ denote the shadow of G at level k, namely, $\partial^k G = \{D : |D| = k, D \subseteq E \text{ for some } E \in G\}$. In particular, $\partial G = \partial^{p-1} G$. Let

$$G^i = \{D \in \partial^i G : \deg(D) = 1 \text{ and } \forall S \subset D, \deg(S) \ge 2\}.$$

If we map each $D \in G^i$ to the unique $E \in G$ such that $D \subseteq E$, then we obtain an onto function from G^i to G_i. Hence $|G_i| \le |G^i|$ for $1 \le i \le p$. We are ready to state two lemmas, which are the key ingredients in our proofs.

Lemma 4. *Let $t \le k \le p$ and $G \subseteq \binom{[n]}{p}$. If $G \not\to K_t^q$, then $|\partial^t(G^k)| = O(n^{t-2})$.*

Lemma 5. *Let $t \le i \le k \le p$ and $G \subseteq \binom{[n]}{p}$. If $G \not\to K_t^{t-2}$, then $|\partial^i(G^k)| = O(n^{t-2})$.*

In order to prove Lemma 4. We need the following lemma on sunflowers, which is an easy corollary of a result of Füredi [7] and the Erdős-Ko-Rado Theorem [4]. A *sunflower* (or Δ-*system*) with k petals and a core Y is a collection of distinct sets $S_1, \ldots, S_k$ such that $S_i \cap S_j = Y$ for all $i \neq j$.

Lemma 6. *Given k and r, there exists $C = C(k, r)$ such that every $F \subseteq \binom{[n]}{k}$ with $|F| \geq Cn^{k-i}$ contains an r-petal sunflower with a core of size less than i.*

Proof. Füredi [7] extended the well-known Sunflower Lemma of Erdős and Rado [3] as follows: *given k and r, there exists $c = c(k, r)$ such that every $F \subseteq \binom{[n]}{k}$ contains a subfamily F' such that $|F'| > c|F|$ and for all distinct $E_1, E_2 \in F'$, F' contains an r-petal sunflower with core $E_1 \cap E_2$.* (The original statement in [7] is actually stronger.) Let $C = 1/c$. We apply this result to $F \subseteq \binom{[n]}{k}$ with $|F| \geq n^{k-i}/c$. Since $|F'| \geq n^{k-i} > \binom{n-i}{k-i}$, by the Erdős-Ko-Rado Theorem [4], F' contains E_1, E_2 such that $|E_1 \cap E_2| < i$. Then F' contains an r-petal sunflower with core $E_1 \cap E_2$ of size less than i. □

Fix $i \in [p]$. We say that a hypergraph $H \subseteq \partial^i G$ satisfies the property ($\diamond$) if

$$\text{for all } D \in H \text{ and } x \in D, \text{ there exists } E \in G \text{ s.t. } D \backslash \{x\} \subset E, x \notin E.$$

We claim that $\partial^i G^k$ satisfies ($\diamond$) for all $t \leq i \leq k$. First we show that G^k satisfies ($\diamond$). Pick $D \in G^k$ and $x \in D$. Since $D \in G^k$, there exists a unique $E_1 \in G$ such that $D \subseteq E_1$. Since $\deg(D\backslash\{x\}) \geq 2$, there exists $E \in G$, $E \neq E_1$ such that $D\backslash\{x\} \subset E$. In addition, $x \notin E$, otherwise $D \subseteq E$, contradicting $\deg(D) = 1$. We next observe that if H satisfies ($\diamond$), then ∂H also satisfies ($\diamond$). In fact, let $S \in \partial H$ and $x \in S$. Suppose that $S \subset D \in H$. Then there exists $E \in G$ such that $D\backslash\{x\} \subset E, x \notin E$, in particular, $S\backslash\{x\} \subset E$.

Given a function $\phi : A \to B$ and $y \in B$, let $\phi^{-1}(y) = \{x \in A : \phi(x) = y\}$.

Proof of Lemma 4. Let $H = \partial^t(G^k)$. Since $G \not\to K_t^q$, each $D \in H$ contains at least one q-element subset Q such that $Q \notin G_D$. We denote such a Q by $\psi(D)$ (arbitrarily pick one if more than one set can be chosen). In order to show that $|H| = O(n^{t-2})$, it suffices to show that for each set $Q \in \binom{[n]}{q}$, we have $|\psi^{-1}(Q)| = O(n^{t-2-q})$. Define a $(t-q)$-graph $F = \{D - Q : D \in \psi^{-1}(Q)\}$. Suppose to the contrary, that $|\psi^{-1}(Q)| = |F| > Cn^{t-q-2}$ for the constant $C = C(t-q, p-t+3)$ from Lemma 6. By Lemma 6, F contains a sunflower $S_1, \ldots, S_{p-t+3}$ with core Y of size at most 1. For all i, let $D_i = S_i \cup Q \in H$.

Case 1. $Y = \emptyset$. Since $D_1 \in \partial^t(G^k)$, there exists $E \in G$ such that $D_1 \subset E$. At most $|E\backslash D_1| = p-t$ petals have non-empty intersection with $E\backslash D_1$. Since the total number of petals is greater than $p-t+1$, there exists $j \neq 1$ such that $S_j \cap (E\backslash D_1) = \emptyset$, or $D_j \cap E = Q$, a contradiction.

Case 2. $Y = \{x\}$. Since H satisfies ($\diamond$), there exists $E \in G$ such that $D_1\backslash\{x\} \subset E$ and $x \notin E$. At most $|E\backslash(D_1\backslash\{x\})| = p-t+1$ petals have non-empty intersection with $E\backslash(D_1\backslash\{x\})$. Since the total number of petals is $p-t+3$, there exists j, $j \neq 1$ such that $S_j \cap (E\backslash(D_1\backslash\{x\})) = \emptyset$. Since $x \notin E$ but $x \in D_j$, we have $D_j \cap E = Q$, a contradiction. □

Proof of Lemma 5. We do induction on $i \geq t$. The base case $i = t$ holds because of Lemma 4. Let $H = \partial^i G^k$. For each $D \in H$, arbitrarily pick one of its t-subsets S. Since $G \not\to K_t^{t-2}$, S contains a $(t-2)$-subset Q such that $Q \notin G|_S$. Suppose $S \backslash Q = \{x, y\}$. Let $\psi(D) = D - \{y\}$ and $\phi(D) = (\psi(D), Q, x)$. We claim that $|\psi^{-1}(D - \{y\})| \leq \binom{i-1}{t-1}(t-1)(p-i+2)$. By the pigeonhole principle, it suffices to show that $|\phi^{-1}(D - \{y\}, Q, x))| \leq p - i + 2$ (for a fixed $D - \{y\}$, there are $\binom{i-1}{t-1}(t-1)$ ways of choosing a $(t-2)$-set Q and an element $x \notin Q$). Suppose instead, that there exist $D_1, \ldots, D_{p-i+3} \in H$ forming a sunflower with core $D - \{y\}$ and petals $\{y_j\}$, $1 \leq j \leq p - i + 3$ such that $Q \notin G|_{S_j}$ for $S_j = Q \cup \{x, y_j\}$. Since H satisfies $(\diamond)$, there exists $E \in G$ such that $D_1 \backslash \{x\} \subset E$ and $x \notin E$. At most $|E \backslash (D_1 \backslash \{x\})| = p - i + 1$ petals have non-empty intersection with $E \backslash (D_1 \backslash \{x\})$. Since the total number of petals is $p - t + 3$, there exists $j \neq 1$ such that $y_j \notin E$. Since $x \notin E$ but $x \in D_j$, we have $S_j \cap E = Q$, a contradiction.

We thus have $|H| \leq C|\psi(H)|$, where $C = \binom{i-1}{t-1}(t-1)(p-i+2)$. Since $\psi(H) \subseteq \partial^{i-1} G^k$, the induction hypothesis gives $|\psi(H)| \leq |\partial^{i-1} G^k| = O(n^{t-2})$. Consequently $|H| = O(n^{t-2})$. □

3.3. *Proofs for $p \in \{t, t+1\}$ and $q = t - 2$*

We need a proposition, which can be considered as an extension of Proposition 4.

Proposition 5. *Let $q \leq p' \leq p$, and $m = \binom{t}{q}(p-q) + 1$. Suppose that G is a p-graph on $[n]$ and ϕ is a function from G to $\binom{[n]}{p'}$ such that $\phi(E) \subseteq E$ for each $E \in G$. If $G \not\to K_t^q$, then $\phi(G) \not\to K_t^q(m)$.*

Proof. Suppose instead, that $\phi(G) \to K_t^q(m)$. Then there are disjoint vertex sets $X_1, X_2, \ldots, X_t$ of size m such that the following holds. Let $\mathcal{Q}$ be the family of q-sets having non-empty intersection with exactly q of $X_1, X_2, \ldots, X_t$. For each $Q \in \mathcal{Q}$, there exists $E \in G$ such that $Q \subseteq \phi(E) \subseteq E$. Denote such E by E_Q. We say that a set $Q \in \mathcal{Q}$ is *bad* if there exists j such that $Q \cap X_j = \emptyset$ and $(E_Q \backslash Q) \cap X_j \neq \emptyset$. Given a bad $Q \in \mathcal{Q}$, a t-tuple $x_1, \ldots, x_t$ with $x_i \in X_i$ is called *bad because of* Q if $\{x_1, \ldots, x_t\}$ contains Q and at least one vertex from $E_Q \backslash Q$. A t-tuple from $X_1 \times \cdots \times X_t$ is called *bad* if it is bad because of some Q. For fixed bad $Q \in \mathcal{Q}$, the number of bad t-tuples because of Q is at most $(p-q)m^{t-q-1}$ (first select a vertex from $E_Q \backslash Q$ and then decide the remaining $t - q - 1$ coordinates). The total number of bad t-tuples is thus at most $\binom{t}{q} m^q (p-q) m^{t-q-1}$. When $m > \binom{t}{q}(p-q)$, we have $\binom{t}{q} m^q (p-q) m^{t-q-1} < m^t$, or the number of bad t-tuples is less than the total number of t-tuples in $X_1 \times \cdots \times X_t$. Hence there always exists a good t-tuple T and consequently $G|_T \supseteq K_t^q$, a contradiction. □

Proof of (10) for $p = t$. Given $G \subseteq \binom{[n]}{t}$ such that $G \not\to K_t^q$, we partition G into $G_t + \cdots + G_1$ as in the beginning of Section 3.2. By Lemma 4, $|G^t| = O(n^{t-2})$ and consequently $|G_t| \leq |G^t| = O(n^{t-2})$. Trivially $|G_i| \leq |G^i| = O(n^{t-2})$ for $i \leq t-2$. It remains to show that $|G_{t-1}| \leq L^{t-1}(n, K_t^q) + o(n^{t-1})$. In fact, for each $E \in G_{t-1}$, we define $\phi(E) = D$ where D is one of the $(t-1)$-subsets of E satisfying $\deg(D) = 1$.

Proposition 5 implies that $\phi(G) \nrightarrow K_t^q(m)$ for $m = \binom{t}{q}(p-q)+1$. So

$$|G| = |\phi(G)| \le L^{t-1}(n, K_t^q(m)) \le L^{t-1}(n, K_t^q) + o(n^{t-1}),$$

where the last inequality follows from Lemma 2.

Proof of (10) for $p = t+1$. We need Lovász's version [11] of the Kruskal-Katona Theorem: *let H be a $(t+1)$-graph with $|H| = \binom{x}{t+1}$ for some real number x. Then* $\partial H \ge \binom{x}{t}$. This implies that if $|\partial H| = O(n^k)$, then $|H| = O(n^{\frac{k(t+1)}{t}})$. To see this, suppose that $|\partial H| \le Cn^k$ for some $C > 0$. Since $\left(\frac{x}{t}\right)^t \le \binom{x}{t} \le |\partial H| \le Cn^k$, we have $\frac{x}{t} \le C^{\frac{1}{t}} n^{\frac{k}{t}}$ and

$$|H| = \binom{x}{t+1} = \binom{x}{t}\frac{x-t}{t+1} \le Cn^k C^{\frac{1}{t}} n^{\frac{k}{t}} = O(n^{\frac{k(t+1)}{t}}).$$

Now given $G \subseteq \binom{[n]}{t+1}$ such that $G \nrightarrow K_t^q$, we partition G into $G_{t+1} + G_t + \cdots + G_1$. The proof of the $p = t$ case shows $\sum_{i=1}^t |G_i| \le L^{t-1}(n, K_t^q) + o(n^{t-1})$. It suffices to show that $|G_{t+1}| = o(n^{t-1})$, or $|G^{t+1}| = o(n^{t-1})$. Lemma 4 guarantees that $\partial^t(G^{t+1}) = O(n^{t-2})$ and consequently, by the result of Lovász, $|G^{t+1}| = O(n^{\frac{(t-2)(t+1)}{t}}) = o(n^{t-1})$.

Proof of (10) for $q = t-2$. Given $G \subseteq \binom{[n]}{p}$ such that $G \nrightarrow K_t^{t-2}$, we partition G into $G_p + \cdots + G_t + G_{t-1} + \cdots + G_1$. The proof of the $p = t$ case shows $\sum_{i=1}^t |G_i| \le L^{t-1}(n, K_t^q) + o(n^{t-1})$. For $t < k \le p$, we apply Lemma 5 with $i = k$ and obtain that $|G_k| \le |G^k| \le \partial^k(G^k) = O(n^{t-2})$, thus completing the proof. □

4. An Exact Result

In order to prove Theorem 2, we need the following lemma, which can be proved by following the original proof of Mantel's Theorem [12]. We use $+$ instead of $\cup$ for a disjoint union. In a graph G, given a vertex set A and a vertex x, $N(x, A)$ denotes the neighborhood of x in A, and $d(x, A) = |N(x, A)|$, in particular $d(x) = d(x, V(G))$. For disjoint vertex sets X and Y, we denote by $e(X, Y)$ the number of edges between X and Y. For simplicity we write ab instead of $\{a, b\}$.

Lemma 7. *Let $G = (V, E)$ be a triangle-free graph such that*

$$\text{for every } ab \in E, \text{ there exists } c \in V, \text{ such that } ac \notin E \text{ and } bc \notin E. \qquad (\star)$$

Then $|E| \le \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$ *with equality only when G has the following structure: $V(G) = A + B + \{z\}$, there exist $a \in A$ and a non-empty set $B_z \subseteq B$ such that $E(G) = A \times B - \{ab : b \in B_z\} + \{zb : b \in B_z\} + \{az\}$.*

Proof. Let xy be an edge. Since G is triangle-free, we have $N(x) \cap N(y) = \emptyset$. With $(\star)$, we further derive that $d(x) + d(y) \le n - 1$.

If $d(x) + d(y) \le n - 2$ for every edge xy in G, then following Mantel's proof of his theorem, we have

$$\frac{4|E|^2}{n} = \frac{\left(\sum_{x\in V} d(x)\right)^2}{n} \le \sum_{x\in V}(d(x))^2 = \sum_{xy\in E}(d(x)+d(y)) \le (n-2)|E|,$$

$$|E| \le \frac{n(n-2)}{4} < \frac{(n-1)^2}{4} < \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

Otherwise assume that $d(x) + d(y) = n - 1$ for some $e = \{x, y\}$. Let $A = N(y)$ and $B = N(x)$. We know that $A \cap B = \emptyset$ and $A \cup B = V - \{z\}$ for some vertex z. Let $d_1 = d(z, A)$ and $d_2 = d(z, B)$.

Case 1. $d_1 = 0$*, or* $d_2 = 0$.

Say, $d_1 = 0$. For each $b \in N(z, B)$, there exists $a \in A$ such that $ab \notin E$, since otherwise edge xb does not satisfy $(\star)$. This implies that

$$|E| = e(A, B) + d(z, B) \le |A||B| \le \left\lfloor \frac{(n-1)^2}{4} \right\rfloor.$$

Case 2. $d_1, d_2 > 0$.

In this case $d_1d_2 - d_1 - d_2 + 1 = (d_1 - 1)(d_2 - 1) \ge 0$ with equality if and only if at least one of d_1, d_2 is 1. Since G is triangle-free, there is no edge between $N(z, A)$ and $N(z, B)$. Thus $e(A, B) \le |A||B| - d_1d_2$ and

$$|E| = e(A, B) + d(z, A) + d(z, B) \le |A||B| - d_1d_2 + d_1 + d_2 \le \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1,$$

where equality holds only when G has the desired structure. □

Proof of Theorem 2. To show that $L^3(n, K_3) \ge \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$, we enlarge each edge of $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ with the same new vertex.

To prove the upper bound, we consider a 3-graph H on $[n]$ such that $H \not\to K_3$. The proof of the $q = t-1$ case of Theorem 1 implies that each triple $T \in H$ contains a pair $\phi(T)$ with $\deg_H(\phi(T)) = 1$. We thus obtain a graph G on $[n]$ with edge set $E = \{\phi(T) : T \in H\}$. Clearly $|E| = |H|$, and G satisfies $(\star)$ because

$$\text{if } \phi(\{a, b, c\}) = ab, \text{ then } ac \notin E \text{ and } bc \notin E. \tag{11}$$

Next we claim that $G \ne G*$, where $G*$ is a graph causing the equality in Lemma 7. Suppose, to the contrary, that $G = G*$. Let us consider edges za and zb for any $b \in B_z$. By (11), $\phi^{-1}(za) = \{z, a, x\}$ for some $x \in A\backslash\{a\}$, and $\phi^{-1}(zb) = \{z, b, y\}$ for some $y \in B\backslash B_z$. Since a is the unique vertex which is non-adjacent to both x and b, we have $\phi^{-1}(xb) = \{a, b, x\}$. The trace of $\{z, a, x\}, \{z, b, y\}, \{a, b, x\}$ on $\{z, a, b\}$ is a K_3, contradicting $H \not\to K_3$.

Finally we apply Lemma 7 and obtain that $|H| = |E| \le \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$. □

5. Concluding Remarks and Open Problems

A less ambitious goal than proving Conjecture 1 is to verify (3), or equivalently (10), for $p \geq t+2$ and $q \leq t-3$. This will reduce the trace problem to determining $\mathrm{ex}(n, H_{q,t}^{t-1})$, which is only known for $q = 2$. To obtain the asymptotic value of $L^p(n, K_t^q)$ in other cases, one should try to verify (6) for $p \geq t+2$ and $t \geq 5$; the smallest open case is to prove that

$$L^7(n, K_5) = |T_4^4(n)| + o(n^4) = \left(\frac{n}{4}\right)^4 + o(n^4).$$

Following the ideas in Sections 3.2 and 3.3, in order to extend Theorem 1 for all $p \geq t$, one needs to show that $G^k = o(n^{t-1})$ for $t \leq k \leq p$. When $p \geq t+2$, this does not follow from Lemma 4 and the Kruskal-Katona theorem. The proof of Lemma 5 relies on the assumption $q = t-2$, and does not seem to generalize to other values of q.

A general uniform trace problem is to determine $L^p(n, F)$ for arbitrary p and F. Because of the close connection between trace problems and Turán problems, as seen in Proposition 2 and Theorem 1, it is very hard to determine $L^p(n, F)$ in general. Let us consider $L^3(n, F)$ when F is a graph. Fix $t = \chi(F)$. When $t \geq 4$, we have

$$L^3(n, F) = |T_{t-1}^3(n)| + o(n^3) = \binom{t-1}{3}\left(\frac{n}{t-1}\right)^3 + o(n^3).$$

In fact, the lower bound for $L^3(n, F)$ follows from $T_{t-1}^3(n) \not\to F$, where $T_{t-1}^3(n)$ is the generalized Turán graph defined in the introduction. The reason for $T_{t-1}^3(n) \not\to F$ is that when embedding F into a $(t-1)$-partite graph, some partition set must contain both ends of an edge of F. The upper bound follows from (5) and Lemma 2. The same arguments actually show that $L^p(n, F) = |T_{t-1}^p(n)| + o(n^{t-1})$ for every F with $t = \chi(F) > p$.

Problem 1. Determine the order of magnitude of $L^3(n, F)$ for every F with $\chi(F) \leq 3$.

This seems no easier than determining the order of magnitude of the Turán numbers for bipartite graphs. We can derive an upper bound for $L^3(n, F)$ as follows. A result of Erdős [2] implies that $\mathrm{ex}(n, K_3^3(m)) = O(n^{3-\frac{1}{m^2}})$. For a 3-graph H, it is clear that $K_3^3(m) \subseteq H$ implies that $H \to K_3(m-1)$. For each F with $\chi(F) \leq 3$, there exists m such that $F \subseteq K_3(m)$. Hence $L^3(n, F) \leq L^3(n, K_3(m)) \leq \mathrm{ex}(n, K_3^3(m+1)) = O(n^{3-c})$, where $c = 1/(m+1)^2$. However, we do not have a matching lower bound. For example, we only know $L^3(n, K_3(2)) = \Omega(n^{5/2})$, in contrast to the upper bound $O(n^{26/9})$ derived by above arguments (or $O(n^{11/4})$ by some extra ideas). This lower bound was given by Pikhurko (personal communication) following from the 3-partite 3-graph with partition sets A, B, C of size n, and the edge set $\{e \cup v : v \in C, e \in G\}$, where G is a maximum C_4-free bipartite graph on (A, B) with $\Omega(n^{3/2})$ edges.

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