Proof of a Tiling Conjecture of Komlós

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ABSTRACT: A conjecture of Komlós states that for every graph H, there is a constant K such that if G is any n-vertex graph of minimum degree at least $(1 - (1/\chi_{cr}(H)))n$, where $\chi_{cr}(H)$ denotes the critical chromatic number of H, then G contains an H-matching that covers all but at most K vertices of G. In this paper we prove that the conjecture holds for all sufficiently large values of n.
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1. INTRODUCTION

All graphs considered in this paper are finite, undirected, and simple. If H is a graph on h vertices and G is a graph on n vertices, an *H-matching* of G (or a *tiling* of G with H) is a subgraph of G consisting of vertex-disjoint copies of H. In *tiling* problems the objective is to find many vertex disjoint copies of H in G, or even a complete tiling (or H-factor) of G with $\lfloor n/h \rfloor$ copies of H. Perhaps, one of the earliest tiling results in extremal graph theory is Dirac's theorem on Hamilton paths [5] that solves the edge-factor problem. The case of triangle-factors is due to Corrádi and Hajnal [4], and finally the celebrated result of Hajnal and Szemerédi settles the K_r-factor problem for all r:

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Theorem 1.1 (Hajnal-Szemerédi 1970 [7]). *Let G be a graph on n vertices with minimum degree*

$$\delta(G) \geq \left(1 - \frac{1}{r}\right)n;$$

then G has a K_r-factor.

During the 1990's Alon and Yuster extended the Hajnal-Szemerédi theorem in various ways:

Theorem 1.2 (Alon-Yuster 1992 [2]). *For every $\varepsilon > 0$ and for every integer h there exists an $n_0 = n_0(\varepsilon, h)$ such that for every graph H on h vertices with chromatic number $\chi(H)$, any graph G with $n > n_0$ vertices and minimum degree*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right)n, \tag{1.1}$$

contains at least $(1 - \varepsilon)n/h$ vertex disjoint copies of H.

Theorem 1.3 (Alon-Yuster 1996 [3]). *For every $\varepsilon > 0$ and for every integer h there exists an $n_0 = n_0(\varepsilon, h)$ such that for every graph H on h vertices and for every $n > n_0$, any graph G with n vertices and minimum degree*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \varepsilon\right)n \tag{1.2}$$

has an H-factor.

They conjectured that the error terms εn in Theorems 1.2 and 1.3 could be relaxed to a constant. In [2] they also remarked that this is essentially best possible. These conjectures have been recently proven by Komlós, Sárközy, and Szemerédi:

Theorem 1.4 (Komlós, Sárközy, and Szemerédi 1995 [11]). *For every graph H there is a constant K such that if G is an n-graph satisfying*

$$\delta(G_n) \geq \left(1 - \frac{1}{\chi(H)}\right)n, \tag{1.3}$$

then it has an H-matching that covers all but at most K vertices.

Theorem 1.5 (Komlós, Sárközy, and Szemerédi 1995 [11]). *Given the conditions of Theorem 1.4, if*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right)n + K, \tag{1.4}$$

then G has an H-factor.

Let us use the notation

$$TT(n, H) = \min\{t : \delta(G) \geq t \text{ implies that } n\text{-graph } G \text{ has an } H\text{-factor}\}$$
$$= 1 + \max_G\{\delta(G) : G \text{ is an } n\text{-graph } G \text{ has an } H\text{-factor}\}$$

and define $TT(n, H, M)$ to be the smallest integer t such that an n-graph with minimum degree $\delta(G) \geq t$, then there is an H-matching covering at least M vertices in G.

The the sharpness of Theorem 1.4 and 1.5 would suggest that the limit of $TT(n, H)/n$ is $1 - 1/\chi(H)$; hence like for Turán-type theorems, the relevant quantity for tiling problems would also be the chromatic number $\chi(H)$. While this is true for *some* graphs H, it is false for many others: In [8], Komlós presented a much improved form of Theorem 1.4 (but not Theorem 1.5), and found that, for *any* graph H, the critical quantity for tiling problems is not the chromatic but the so-called *critical chromatic number* $\chi_{cr}(H)$.

For an r-chromatic graph H on h vertices, we write $u = u(H)$ for the smallest possible color-class size in any r-coloring of H. The critical chromatic number of H is $\chi_{cr}(H) = (r - 1)h/(h - u)$. It is easy to see that $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$, and $\chi_{cr}(H) = \chi(H) = r$ if and only if every r-coloring of H has equal color-class sizes.

Theorem 1.6 (Komlós 2000 [8]-lower bound). *Let H have parameters $\chi = \chi(H)$ and $\chi_{cr} = \chi_{cr}(H)$. Then, for all $0 < M \leq n$,*

$$TT(n, H, M) \geq M\left(1 - \frac{1}{\chi_{cr}}\right) + (n - M)\left(1 - \frac{1}{\chi - 1}\right). \tag{1.5}$$

In particular, $TT(n, H) \geq (1 - \frac{1}{\chi_{cr}})n$.

He also proved a matching upper bound:

Theorem 1.7 (Komlós [8]-upper bound). *For every graph H and $\varepsilon > 0$ there is a threshold $n_0 = n_0(H, \varepsilon)$ such that if $n \geq n_0$ and an n-graph G satisfies the degree condition*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right)n, \tag{1.6}$$

then G contains an H-matching that covers all but at most εn vertices.

He also posed the following conjecture:

Conjecture 1 [Komlós [8]]. *For every graph H there exists a constant $K = K(H)$ such that if G is an n-graph satisfying (1.6), then G contains an H-matching that covers all but at most K vertices. This is best possible for every H (by Theorem 1.6). Hence,*

$$\left(1-\frac{1}{\chi_{cr}(H)}\right)n-K\leq TT(n,H,n-K)\leq\left(1-\frac{1}{\chi_{cr}(H)}\right)n. \qquad \blacksquare$$

It should be noted that when $\chi_{cr}(H) < \chi(H)$, even if we replace $\chi_{cr}(H)$ by $\chi_{cr}(H) + C$ in (1.6), for any constant $C \leq \frac{1}{\chi_{cr}(H)} - \frac{1}{\chi(H)}$, we still cannot get an H-factor in G. For example, in [3] Alon and Yuster showed that there exists a graph G with $|V(G)| = hn$ and $\delta(G) \geq hn/2$ that does not contain an $K_{a,b}$-factor for $a > b$ and $a + b = h$.

In [13] we proved the correctness of this conjecture when H is a 3-chromatic graph, for sufficiently large values of n. In this paper we will generalize our result to arbitrary fixed graphs H:

Theorem 1.8. *For any k-chromatic graph H on h vertices with smallest color-class of order u, there exists an n_0 such that, for all $n \geq n_0$, if G is any n vertex graph with*

$$\delta(G)\geq\left(1-\frac{1}{\chi_{cr}(H)}\right)n, \tag{1.7}$$

then G contains an H-matching that covers all but at most $\frac{5(k-2)(h-u)^2}{u(k-1)}$ vertices of G.

In our proof we will use the concept of *bottle-graphs.* A bottle-graph of chromatic number r is a complete r-partite graph with color-class sizes $(u, w, w, \ldots, w)$, where $u = \alpha w$ for some $\alpha \leq 1$. Clearly, the critical chromatic number of this graph is $r - 1 + \alpha$. The vector $(\frac{\alpha}{r-1+\alpha}, \frac{1}{r-1+\alpha}, \ldots, \frac{1}{r-1+\alpha})$ is called the *color-vector* of the bottle-graph. The parameters u and w will be referred to as the *neck* and the *width* of the bottle-graph, respectively.

Given an r-chromatic graph H of order h with smallest color-class size $u = u(H)$, we say that a graph $\mathcal{B} = \mathcal{B}(H)$ is the bottle-graph of H if $\mathcal{B}$ is the smallest bottle-graph that contains an H-factor, with the color-vector $\underline{\beta} = (s, t, \ldots, t)$, where $s = u/h$ and $t = (1 - s)/(r - 1)$. Observe that, $\chi_{cr}(\mathcal{B}) = \chi_{cr}(H) = (r - 1)/(1 - s)$. Moreover, let $\{u, u_1, u_2, \ldots, u_{r-1}\}$ denote the color-class sizes in an r-coloring of H, we can always construct a bottle-graph using $(r - 1)$ vertex disjoint copies of H, with the ith copy of H having $u, u_i, u_{i+1}, \ldots, u_{r-1}, u_1, \ldots, u_{i-1}$ vertices in color class $1, 2, \ldots, r$ of bottle-graph. Hence, the order of $\mathcal{B}(H)$ is at most $(r - 1)h$. Therefore, it is sufficient to prove Theorem 1.8 when H is a bottle-graph.

Theorem 1.9 (Main Theorem). *For every bottle-graph H with chromatic number k, neck u, and width w, there exists an n_0 such that for all $n \geq n_0$ if G is any n vertex graph with*

$$\delta(G)\geq\frac{\alpha+k-2}{\alpha+k-1}n=\left(1-\frac{1}{k-1}+\gamma\right)n, \tag{1.8}$$

where $\alpha = \frac{u}{w}$ and $\gamma = \frac{\alpha}{(k-1)(k-1+\alpha)}$, then G contains an H-matching that covers all but at most $\frac{5k}{(k-1)^2\gamma}w$ vertices.

2. NOTATIONS

$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G, and we write $v(G) = |V(G)|$ (order of G) and $e(G) = |E(G)|$ (size of G). G_n denotes n-graphs. $N(v, X)$ is the set of neighbors of $v \in V$ in the set $X \subset V$. We use $N(v)$ to denote $N(v, V)$. Hence $|N(v, X)| = \deg(v, X) = \deg_G(v, X)$ is the degree of v in set X, and $\deg(v) = \deg(v, V)$. In a directed graph D, we use $N_{out}(v)$ to denote $\{u \in V(D) \mid (v, u) \in E(D)\}$ (the *out* neighborhood of v), and $\deg_{out}(v) = |N_{out}(v)|$. $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G. $\nu_i(G)$ denotes the size of a maximum set of vertex disjoint i-stars (stars with i leaves) in G. In particular, $\nu_1(G) = \nu(G)$ is the size of a maximum matching in G. We write $\chi(G)$ and $\chi_{cr}(G)$ for the chromatic number and critical chromatic number of G, respectively. For an r-chromatic graph H of order h, we write $u = u(H)$ for the smallest possible color-class size in any r-coloring of H. When A and B are disjoint subsets of $V(G)$, we use $\deg(A, B)$ to denote the number of edges in $E(G)$ with one endpoint in A and the other in B. A bipartite graph G with color classes A and B and edge set E will be denoted by $G = (A, B, E)$, with $E \subseteq A \times B$. $K(n_1, n_2, \ldots, n_r)$ is the complete r-partite graph with color classes of sizes $n_1, n_2, \ldots, n_r$. The *density* between disjoint sets X and Y is defined as

$$d(X, Y) = \frac{\deg(X, Y)}{|X||Y|}.$$

In the proof of the Main Theorem, Szemerédi's *Regularity Lemma* [14] plays a pivotal role. We will need the following definition to state the regularity lemma:

Definition [Regularity Condition]. *Let $\varepsilon > 0$. A pair (A, B) of disjoint vertex-sets in G is* ε-regular *if for every $X \subset A$ and $Y \subset B$, satisfying*

$$|X| > \varepsilon|A|, \quad |Y| > \varepsilon|B|,$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

This definition implies that regular pairs are highly uniform bipartite graphs; namely, the density of any reasonably large subgraph is almost the same as the density of a regular pair. We will use the following form of the Regularity Lemma:

Lemma 2.1 (Regularity Lemma-Degree Form). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- $\ell \leq M$,
- $|V_0| \leq \varepsilon|V|$,
- *all clusters V_i, $i \geq 1$, are of the same size $L \leq \lceil \varepsilon|V| \rceil$,*
- $\deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)|V|$ *for all $v \in V$,*

- $G'|_{V_i} = \emptyset$ *(V_i is an independent set in G'), for all i,*
- *all pairs (V_i, V_j), $1 \le i < j \le \ell$, are ε-regular, each with density either* 0 *or greater than d, in G'.*

A stronger one-sided property of regular pairs is super regularity:

Definition [Super-Regularity condition]. *Given a graph G and two disjoint subsets of its vertices A and B, the pair (A, B) is (ε, d)*-super-regular, *if it is ε-regular and*

$$\deg(a) > d|B| \quad \forall\, a \in A, \quad \deg(b) > d|A| \quad \forall\, b \in B.$$

We also use the Blow-up Lemma (see [9, 10]):

Lemma 2.2. *Given a graph R of order r and positive parameters δ and Δ, there exists an $\varepsilon = \varepsilon(\delta, \Delta, r) > 0$ such that the following holds. Let $n_1, n_2, \ldots, n_r$ be arbitrary positive integers, and let us replace the vertices $v_1, v_2, \ldots, v_r$ of R with pairwise disjoint sets V_1, $V_2, \ldots, V_r$ of sizes $n_1, n_2, \ldots, n_r$ (blowing up). We construct two graphs on the same vertex-set $V = \cup_i V_i$. The first graph R_b is obtained by replacing each edge $\{v_i, v_j\}$ of R with the complete bipartite graph between the corresponding vertex-sets V_i and V_j. A sparser G is constructed by replacing each edge $\{v_i, v_j\}$ of R arbitrarily with some (ε, δ)-super-regular pair between V_i and V_j. If a graph H with $\Delta(H) \le \Delta$ is embeddable into R_b, then it is already embeddable into G.*

3. NUMBER OF LEFT-OVER VERTICES

In Theorem 1.8 the number of left-over vertices, $K(H)$, is a constant that only depends on the graph H. A conjecture of El-Zahar [6] (recently proved by Abbasi [1] for large n) states that $K(H) = 0$ when H is a union of cycles. But the following example shows that this is not always the case. In fact, $K(H)$ in Theorem 1.9 is only far from the number of leftovers in the following tight example by a constant factor.

Let G be a on n vertices with $V(G) = A_0 \cup A_1 \cup \cdots \cup A_{k-2}$, where $|A_0| = (\frac{1}{k-1} + (k-2)\gamma)n$ and $|A_j| = (\frac{1}{k-1} - \gamma)k$ for all $j \ge 1$, with $\deg(A_i, A_j) = |A_i|\,|A_j|$, for $0 \le i < j \le k-2$, i.e., G is complete between sets $A_0, A_1, \ldots, A_{k-2}$. Moreover, assume each A_j for $j \ge 1$ forms an independent set, while A_0 consists of s connected components with $s=\lceil\frac{|A_0|}{2(k-1)\gamma n + 2w - 2}\rceil$. Each of the components in A_0 is a complete bipartite graph $\{(L_i, R_i)\}$, with $|L_i| = |R_i| = (k-1)\gamma n + w - 1$ for $i \in \{1, \ldots, s-1\}$. Without loss of generality, we will assume $w + u$ divides $2\gamma n$. It is easy to see that the graph G satisfies the degree condition in (1.8).

Let $\mathcal{H}$ denote an H-matching of G. For every copy of H in $\mathcal{H}$, two of its color classes must come from some component (L_i, R_i) in A_0, and the other $(k-2)$ color classes will reside in $A_1, \ldots, A_{k-2}$. Let us first assume that in each copy of H, the u-vertex color class comes from A_0, i.e., the restriction of $\mathcal{H}$ on A_0 is a union of $K(w, u)$ graphs. There must be at least $2w - 2$ vertices left uncovered in each (L_i, R_i) of A_0, for $1 \le i \le s - 1$. In fact, let $t = \frac{2\gamma n}{w+u}$; then we can find $2t$ copies of $K(w, u)$ in each (L, R) component by placing t copies of w-sets and u-sets in each side (thus $2w - 2$ vertices will be left uncovered). Assume to the contrary that $2t + 1$ copies of $K(w, u)$ could be placed in an

(L, R) component of A_0, i.e., either L or R contains $(t + i)$ w-sets and $(t - i + 1)$ u-sets, for $i \geq 1$. But this is impossible, because

$$(t + i)w + (t - i + 1)u = tw + tu + i(w - u) + u \geq t(w + u) + w > 2\gamma n + w - 1.$$

Finally, our assumption that all copies of H have u-sets in A_0 is necessary for achieving the maximum number of copies of H in $\mathcal{H}$. In fact, embedding $K(w, w)$ in (L_i, R_i) of A_0 can only result in less than $2t$ copies of H in $\{L_i, R_i, A_1, \ldots, A_{k-2}\}$. Therefore, in any H-matching of G, there will be at least $2w - 2$ uncovered vertices for each $(L_i, R_i) \in A_0$, $1 \leq i \leq s - 1$, which leaves a total of at least $(s - 1)(2w - 2)$ vertices uncovered in A_0. Since an H-matching of G contains at most $(|A_0| - (s - 1)(2w - 2))/(w + u)$ copies of H, the number of uncovered vertices is at least

$$(s - 1)(2w - 2)\,\frac{h}{w + u} \approx \left(\frac{|A_0|}{2(k - 1)\gamma n}\right)\frac{h(2w - 2)}{w + u} \approx \frac{w}{(k - 1)\gamma}.$$

4. OUTLINE OF THE PROOF

In our proof we will assume $u < w$ (that is, $\chi_{cr} < r$), since otherwise Theorem 1.4 has already covered the case. Throughout the paper, we will also assume that n is sufficiently large and will use the following main parameters:

$$\varepsilon \ll d \ll \mu \ll \min(\alpha, 1 - \alpha). \tag{4.1}$$

We will start by applying Lemma 2.1 to G, with ε and d as in (4.1), and will get a partition of $V(G)$ into clusters $V_0, V_1, \ldots, V_\ell$. We assume that $L = |V_i|$, for $i \geq 1$, is divisible by a few integers that will be determined later in the proof; otherwise we can move a constant number of vertices from each V_i, for $i \geq 1$, to V_0 to satisfy this condition. We define the following *reduced graph* G_r:

The vertices of G_r are the clusters V_i for $i \geq 1$, and we have an edge between two clusters if they form an ε-regular pair in G' with density exceeding d. Since in G', $\delta(G') \geq (1 - \frac{1}{k-1} + \gamma)n - (d + \varepsilon)n$, it is easy to see that

$$\delta(G_r) \geq \left(1 - \frac{1}{k - 1} + \gamma - 2d\right)\ell. \tag{4.2}$$

There is a one-to-one correspondence between every vertex c in G_r and a cluster of vertices in G. We will use V_c to denote those G vertices that correspond to a vertex c in G_r. Similar to a complete r-partite graph $K(n_1, n_2, \ldots, n_r)$, an ε-regular k-clique $\mathcal{R}_\varepsilon(n_1, n_2, \ldots, n_r)$ is an r-partite graph with color-class sizes $n_1, \ldots, n_r$, such that every pair of color-classes form an ε-regular pair. For $0 < a \leq 1$, $\mathcal{R}^a_{\varepsilon,r}(t)$ denotes an ε-regular r-clique with $n_i = t$ for all $1 \leq i \leq r - 1$, and $n_r = at$. In particular, $\mathcal{R}^1_{\varepsilon,r}(t)$ will be called *balanced* ε-regular r-clique, while $\mathcal{R}^a_{\varepsilon,r}(t)$ is referred to as *unbalanced* when $a < 1$. The parameters ε, k, or t may be omitted if they are clear from the context.

In Sections 5 and 6, we will show that, except for an extreme case, the vertices in G'' can be tiled by copies of $\mathcal{R}^{\alpha'}_k(L_1)$ for α' satisfying $\alpha < \alpha' \leq 1$, and $L_1 = CL$ with $0 <$

$C \le 1$. Given one such H-matching, the vertices of V_0 will be inserted into some clusters of appropriate ε-regular k-cliques such that after we remove copies of H containing new vertices, the remaining parts of the cliques still contain almost perfect H-matchings. We further use the connections among different ε-regular k-cliques to reduce the total number of uncovered vertices to a constant depending only on H.

In the extremal case (Section 7), we will show that $V(G)$ can be partitioned to two sets A and B, where A is the union of several almost-independent vertex sets $\{U_1, \dots, U_t\}$ with $|U_i| \approx \frac{n}{(k-1)+\alpha}$, while there are two possible cases for B:

(1) B can be partitioned into almost-independent vertex sets $\{U_{t+1}, \dots, U_k\}$ with $|U_i| \approx \frac{n}{(k-1)+\alpha}$ for $i < k$, and $|U_k| \approx \frac{\alpha n}{k-1+\alpha}$, or
(2) B can be almost tiled by copies of $K(w, \dots, w, u)$.

We then show that in either case, all but a constant number of vertices can be covered by an H-matching.

5. THE MAXIMAL CLIQUE COVER OF THE REDUCED GRAPH

Given a graph $\mathcal{G}$, a k-clique-cover (or -decomposition) $\Phi = \{\Phi_k, \Phi_{k-1}, \dots, \Phi_1\}$ is a collection of disjoint cliques, in which Φ_i corresponds to a family of cliques of order i, for $1 \le i \le k$, and $V(\mathcal{G}) = \cup_{i=1}^{k} V(\Phi_i)$. A k-clique-cover $\Phi = \{\Phi_k, \Phi_{k-1}, \dots, \Phi_1\}$ is *maximal* if, for any other k-clique-cover $\Phi' = \{\Phi'_k, \Phi'_{k-1}, \dots, \Phi'_1\}$, suppose $|\Phi'_i| > |\Phi_i|$ for some $1 \le i \le k$; then there is a $i < j \le k$ such that $|\Phi_j| > |\Phi'_j|$. We will use K^i to denote a clique of order i (in family Φ_i) in any maximal clique-decomposition.

Consider a maximal clique-cover $\Phi = \{\Phi_k, \Phi_{k-1}, \dots, \Phi_1\}$ of $\mathcal{G}$. Let K^i, K^j, K^k denote cliques in Φ_i, Φ_j, Φ_k, respectively, for $1 \le i \le j \le k$. We will use the following notation to represent the connectivity among these cliques:

- *Well-connected* (or $K^i \hookrightarrow K^j$), if $\deg(v, K^j) = j - 1$, $\forall v \in K^i$
- *Over-connected* (or $K^i \overset{>}{\hookrightarrow} K^j$), if $\deg(K^i, K^j) \ge i(j-1)$ and $K^i \not\hookrightarrow K^j$.
- *Under-connected* (or $K^i \overset{<}{\hookrightarrow} K^j$), if $\deg(K^i, K^j) < i(j-1)$.

The following proposition is a direct consequence of Φ being maximal:

Proposition 5.1.

1. For $i \le j < k$, $\deg(K^i, K^j) \le i(j-1)$. *In other words, there is no K^i and K^j such that $K^i \overset{>}{\hookrightarrow} K^j$, since, otherwise, there exists a $v_0 \in K^i$ such that* $\deg(v_0, K^j) = j$ *and $\{v_0\} \cup K^j$ forms a $(j+1)$-clique, contradicting that Φ is maximal. In particular, if* $\deg(K^i, K^j) = i(j-1)$, *then $K^i \hookrightarrow K^j$.*
2. If $K^i \overset{>}{\hookrightarrow} K^k$, then there exists a $v_0 \in K^i$, with $\deg(v_0, K^k) = k$. *Moreover, for any other clique K^j,* $\deg(K^j, K^k) \le k(j-1)$. *In other words, any k-clique that is over-connected to some K^i will be under-connected to all other cliques of order smaller than k and at least i. Assume to the contrary* $\deg(K^j, K^k) > k(j-1)$; *then there exists at least one vertex $v_1 \in K^k$ with* $\deg(v_1, K^j) = j$. *As a result $\{v_1\} \cup K^j$ will form a $(j+1)$-clique, while $\{v_0\} \cup K^k \backslash \{v_1\}$ forms a k-clique, that is a contradiction.*

Consequently, for every K^k *and any arbitrary set of* k *cliques* $K_1^i, K_2^i, \ldots, K_k^i \in \Phi_i$ *we must have*

$$\deg\left(\bigcup_{l=1}^{k} K_l^i, K^k\right) \leq ik(k-1).$$

Indeed, if none of K_l^i, $1 \leq l \leq k$ *satisfies* $\deg(K_l^i, K^k) > i(k-1)$, *we can derive the conclusion immediately. When one of the i-cliques, say* K_1^i, *is over-connected to* K^k, *it will follow that*

$$\deg\left(\bigcup_{l=2}^{k} K_l^i, K^k\right) \leq k(i-1)(k-1).$$

Together

$$\deg\left(\bigcup_{l=1}^{t} K_l^i, K^k\right) \leq ki + k(i-1)(k-1) \leq ik(k-1).$$

3. *Combining Items* **1** *and* **2**, *if* $K^j \in \Phi_j$ *and two cliques* K^i, $\bar{K}^i \in \Phi_i$ *satisfy*

$$\deg(K^i, K^j) = \deg(\bar{K}^i, K^j) = i(j-1),$$

we will have $K^i \hookrightarrow K^j$, *and* $\bar{K}^i \hookrightarrow K^j$. *Moreover, we can write* $K^j = A_i(K^j) \cup B_i(K^j)$ *such that, for all* $v \in A_i(K^j)$, $\deg(v, K^i) = \deg(v, \bar{K}^i) = i - 1$. *Moreover, for all* $u \in B_i(K^j)$ *we will have* $\deg(u, K^i) = \deg(u, \bar{K}^i) = i$, $|A_i(K^j)| = i$, *and* $|B_i(K^j)| = j - i$. *Finally, let*

$$S_i^j = \{K^j \in \Phi_j : \deg(K^i, K^j) = \deg(\bar{K}^i, K^j) = i(j-1)\}$$

and

$$A_i^j = \{A_i(K^j) : K^j \in S_i^j\}.$$

$\mathcal{G}(A_i^j)$ *is an i-partite subgraph with* $|S_i^j|$ *vertices in each color-class. Moreover,* $\mathcal{G}(A_i^i \cup A_i^{i+1} \cup \cdots \cup A_i^k)$ *is an i-partite graph with* $\sum_{j=i}^{k} |S_i^j|$ *vertices in each color-class.* ∎

Let us assume $v(\mathcal{G}) = \ell$ and φ_i denote the normalized size of the set Φ_i, i.e., $\varphi_i = \frac{|\Phi_i|}{\ell}$, for $i \geq 1$. Then

$$k\varphi_k + (k-1)\varphi_{k-1} + \cdots + \varphi_1 = 1. \tag{5.1}$$

Let $i_0 = \min\{1 \leq i \leq k : |\Phi_i| \geq k\}$. We further assume that $i_0 < k$ and $\varphi_i = 0$ for

$i < i_0$. The following proposition presents a lower-bound on φ_k when $\mathcal{G}$ satisfies the degree condition of the reduced graph in (4.2):

Proposition 5.2. *If* $\delta(\mathcal{G}) \geq (1 - \frac{1}{k-1} + \gamma - 2d)\ell$, *then*

$$\varphi_k \geq \sum_{i=2}^{k-i_0} (i-1)\varphi_{k-i} + (k-1)\gamma - 2(k-1)d. \tag{5.2}$$

Proof. Pick any k cliques $K_1, K_2, \ldots, K_k \in \Phi_{i_0}$; using Items **1** and **2** of Proposition 5.1, we will obtain

$$\deg\left(\bigcap_{l=1}^{k} K_l, \Phi_j\right) \leq ki_0(j-1)\varphi_j\ell, \quad \forall\, i_0 \leq j \leq k-1,$$

$$\deg\left(\bigcup_{l=1}^{k} K_l, \Phi_k\right) \leq ki_0(k-1)\varphi_k\ell,$$

and, using the minimum degree condition in (4.2), we have

$$\begin{aligned} ki_0\left(1 - \frac{1}{k-1} + \gamma - 2d\right)\ell &\leq \deg\left(\bigcup_{l=1}^{k} K_l, \mathcal{G}\right) \\ &\leq ki_0(k-1)\varphi_k\ell + ki_0(k-2)\varphi_{k-1}\ell + \cdots + ki_0(i_0-1)\varphi_{i_0}\ell, \end{aligned}$$

or

$$1 - \frac{1}{k-1} + \gamma - 2d \leq (k-1)\varphi_k + (k-2)\varphi_{k-1} + \cdots + (i_0-1)\varphi_{i_0}. \tag{5.3}$$

In fact, the right side of (5.3) can be rewritten as

$$\begin{aligned} &\left(\frac{1}{k-1} + \frac{k-2}{k-1}k\right)\varphi_k + \frac{k-2}{k-1}(k-1)\varphi_{k-1} + \left(\frac{k-2}{k-1}(k-2) - \frac{1}{k-1}\right)\varphi_{k-2} + \cdots \\ &\qquad + \left(\frac{k-2}{k-1}i_0 - \frac{k-1-i_0}{k-1}\right)\varphi i_0 \\ &= \frac{1}{k-1}\varphi_k + \frac{k-2}{k-1}(k\varphi_k + (k-1)\varphi_{k-1} + \cdots + i_0\varphi_{i_0}) - \frac{1}{k-1}(\varphi_{k-2} + 2\varphi_{k-3} \\ &\qquad + \cdots + (k-1-i_0)\varphi_{i_0}) \\ &= \frac{1}{k-1}\varphi_k + \frac{k-2}{k-1} - \frac{1}{k-1}\sum_{i=2}^{k-i_0}\varphi_{k-i}(i-1). \end{aligned}$$

That is,

$$(k-1)\varphi_k + \cdots + (i_0-1)\varphi i_0 = \frac{1}{k-1}\left(\varphi_k + k - 2 - \sum_{i=2}^{k-i_0} \varphi_{k-i}(i-1)\right). \quad (5.4)$$

If we replace (5.4) in (5.3), then (5.2) will follow. ∎

Consequently, we can let

$$\varphi_k = \sum_{i=2}^{k-i_0} (i-1)\varphi_{k-i} + (k-1)\gamma + s, \quad (5.5)$$

where $s \geq -2(k-1)d$.

Next, consider a clique $K \in \Phi_{k-i}$, for some $i \in \{1, \ldots, k-i_0\}$ that is *not* over-connected to any of the k-cliques. Let $\Lambda(K)$ denote the set of all k-cliques that are well-connected to K, and let λ_i be the minimum value of $|\Lambda(K)|/\ell$ among all such K's in Φ_{k-i}, then we have:

Proposition 5.3.

$$\lambda_i \geq (k-1)\gamma + \frac{i-1}{k-1}s + \sum_{2\leq j\leq i}(j-1)\varphi_{k-j} + (i-1)\sum_{j>i}\varphi_{k-j} - 2(k-i)d. \quad (5.6)$$

Proof. Let $m = |\{K^k \in \Phi_k : \deg(K, K^k) < (k-1)(k-i)\}|/\ell$. We have

$$\begin{aligned}
(k-i)\left(1 - \frac{1}{k-1} + \gamma - 2d\right) &\leq \frac{\deg(K, G_r)}{\ell} \leq (k-1)(k-i)\varphi_k - m \\
&+ (k-2)(k-i)\varphi_{k-1} + \cdots + (k-i-1)(k-i)\varphi_{k-i} + \sum_{j>i}(k-i-1)(k-j)\varphi_{k-j} \\
&\leq (k-i)\left((k-1)\varphi_k + \cdots + (k-i-1)\varphi_{k-i} + \sum_{j>i}(k-j-1)\varphi_{k-j}\right) \\
&+ \sum_{j>i}(j-i)\varphi_{k-j} - m.
\end{aligned}$$

Using (5.4) and (5.5), we have

$$m \leq s\frac{k-i}{k-1} + \sum_{j>i}(j-i)\varphi_{k-j} + 2kd,$$

and (5.6) will follow. ■

For example,

$$\lambda_1 \geq (k-1)\gamma - 2kd,$$

$$\lambda_2 \geq (k-1)\gamma + \frac{s}{k-1} + \sum_{j\geq 2} \varphi_{k-j} - 2kd.$$

Now let us find a maximal clique-decomposition $\Phi = \{\Phi_k, \Phi_{k-1}, \ldots, \Phi_1\}$ of the reduced graph G_r and define i_0 as the index of first clique family in Φ of size at least k. We can assume that $i_0 < k$; otherwise, we simply remove all the vertices of G included in $\Phi_1, \ldots, \Phi_{k-1}$ to the exceptional set V_0 such that Φ becomes a $\mathcal{R}^1_{\varepsilon,k}(L)$-covering of G_r and $|V_0| = O(\varepsilon n)$. The same reason allows us assume that that $\varphi_i = 0$, for all $i < i_0$ [otherwise we can move all the vertices of G included in the clusters of these (small) cliques to V_0]. Thus, we can apply Proposition 5.2 to G_r and then obtain (5.5). For simplicity of notations we will assume $i_0 = 1$ from now on. We divide the proof into two cases, the general case, when $s \geq mu$ and the extremal case, when $s < \mu$. The proof of extremal case will be presented in Section 7.

6. THE GENERAL CASE

At present we have found a maximal clique-cover $\Phi = \{\Phi_k, \Phi_{k-1}, \ldots, \Phi_1\}$ of the reduced graph G_r. Correspondingly, this clique-cover in G_r defines a collection of disjoint ε-regular cliques, *a cluster-clique-cover* $\Psi = \{\Psi_k, \Psi_{k-1}, \ldots, \Psi_1\}$ in G''. We will use $\mathcal{K}^i$ to denote an ε-regular i-clique. All the terminologies we used in Section 5 apply to this cluster-clique-cover as well. Recall that the assumption in the general case is that $s > \mu$, where $s = \varphi_k - (k-1)\gamma - \sum_{i=2}^{k-1} \varphi_{k-i}(i-1)$.

6.1. The Decomposition Lemma

We define

$$\alpha' = \alpha + \frac{\alpha(1-\alpha)}{k^2}\mu. \tag{6.1}$$

Without loss of generality, we assume that $\mu = 1/N$ for some integer N. Recall that $\alpha = u/w$. Thus α' is a rational number, we set $\alpha' = p/q$, where p, q are two integers with no common factors. It is easy to see that $p < q \leq w^2k^2N = w^2k^2/\mu$.

For some $1 \leq i \leq k-1$, we consider an ε-regular $(k-i)$-clique $\mathcal{K} \in \Psi_{k-i}$, with corresponding $K \in \Phi_{k-i}$. An ε-regular k-clique $\mathcal{K}^k$ is *good* for $\mathcal{K}$ if its corresponding clique $K^k \in \Phi_k$ is well-connected or over-connected to K; otherwise, $\mathcal{K}^k$ is *bad* for $\mathcal{K}$. The ε-regular clique $\mathcal{K}$ is called *typical* if for a constant $c_t = \lceil \frac{k-1+\alpha'}{1-\alpha'} \rceil$, $|\{K^k \in \Phi_k : K \overset{>}{\hookrightarrow} K^k\}| < c_t$. Otherwise, we will refer to $\mathcal{K}$ as an *atypical* one. To state the decomposition algorithm for G'' we will use the following observation:

Proposition 6.1. *Let V_i and V_j, $1 \le i < j \le \ell$, denote two clusters in G'' that correspond to two endpoints of an edge e in the reduced graph G_r. Let us partition the clusters V_i and V_j into s and t subclusters $\{V_i^1, \ldots, V_i^s\}$, and $\{V_j^1, \ldots, V_j^t\}$ respectively, such that the size of the smallest subclusters is cL for some $\varepsilon \ll c \le 1$. Then the pairs (V_i^p, V_j^q) for $1 \le p \le s$ and $1 \le q \le t$, are ε'-regular pairs with $\varepsilon' = \min(\frac{\varepsilon}{2}, \frac{\varepsilon}{c})$. In particular, if $s = t$, then (V_i^t, V_j^t), $1 \le t \le s$ are s disjoint ε'-regular pairs.*

When each cluster in an ε-regular clique $\mathcal{K}$ is evenly partitioned into s parts, we obtain p new ε'-regular cliques consisting of smaller clusters. This procedure will be referred to as *s-partition* of ε-regular cliques. Although ε has changed to ε' in new regular pairs, for simplicity of the notation, we always use ε as the parameter. Naturally, we call a new clique good (or bad) if it is derived from a good (or bad) larger clique $\mathcal{K}$.

Since L has been chosen in a way that it can be divided by particular values, we assume that there is no roundoff in any division in the following algorithm.

Decomposition Algorithm of G'':

1. *$(q - p)$*-partition all cliques in Ψ_k and denote the new clique family by Ψ'_k. We use L' to denote the size of the resulting subclusters, i.e., $L' = \frac{L}{q-p}$. Because $q \le w^2k^2/\mu$, $L' > \frac{1}{w^2k^2}\mu L$.
2. For $i = 1, \ldots, k - 1$ convert each typical cliques in Ψ_{k-i} to unbalanced (ε-regular) k-cliques.
   For each typical $(k - i)$-clique $\mathcal{K} \in \Psi_{k-i}$, we arbitrarily pick a (new) well-connected k-clique $\mathcal{K}^k \in \Psi'_k$. Following the notation in Item **3** of Proposition 5.1, there are i clusters in $B(\mathcal{K}^k)$ and $k - i$ clusters in $A(\mathcal{K}^k)$. It is easy to see that a special ε-regular k-clique

$$\mathcal{R}(\underbrace{it, \ldots, it}_{i}, \underbrace{(i-1+\alpha')t, \ldots, (i-1+\alpha')t}_{k-i})$$

   is the union of i copies of $\mathcal{R}^{\alpha'}_{\varepsilon,k}(t)$ (with different ε). We divide each cluster in $B(\mathcal{K}^k)$ into two parts: the large parts containing $\frac{i-1+\alpha'}{i} L'$ vertices and the small ones containing $\frac{1-\alpha'}{i} L'$ vertices. All these large subclusters are combined with clusters in $A(\mathcal{K}^k)$ to form a $\mathcal{R}^{\alpha'}_k(\frac{L'}{i})$, while all the small subclusters are associated with clusters of $\mathcal{K}$ to make a copy of $\mathcal{R}^{\alpha'}_k(\frac{1-\alpha'}{i-1+\alpha'} \frac{L'}{i})$. Clearly when we repeat this procedure to $\frac{(i-1+\alpha')L}{(1-\alpha')L'}$ well-connected k-cliques in Ψ'_k, we eliminate $\mathcal{K}$ and obtain copies of $\mathcal{R}^{\alpha'}_k(\frac{L'}{i})$ and $\mathcal{R}^{\alpha'}_k(\frac{1-\alpha'}{i-1+\alpha'} \frac{L'}{i})$.

3. For $i = 1, \ldots, k - 1$, convert atypical cliques in Ψ_{k-i} to unbalanced (ε-regular) k-cliques.
   We will follow the same procedure as in Step **2**. The only difference is that, when eliminating an atypical $\mathcal{K} \in \Psi_{k-i}$, its over-connected k-cliques, instead of well-connected cliques, will be used. Since any k-clique $\mathcal{K}^k$ over-connected to $\mathcal{K}$ has more than i clusters adjacent to all the clusters of $\mathcal{K}$, we can arbitrarily choose i of them to make $B(\mathcal{K}^k)$ and the rest will be $A(\mathcal{K}^k)$.

4. Partition the remaining k-cliques in Ψ'_k and all (different-size) unbalanced ε-regular k-cliques to $\mathcal{R}_k^{\alpha'}(L_1)$'s.
Here $L_1 = c_l L$, where the constant c_l is a multiple of μ.

Next, we will prove the correctness of the above algorithm:

In Step **3**, by Item **2** of Proposition 5.1, any k-clique that is over-connected to $\mathcal{K}$ does not participate in the converting operation of any other (typical or atypical) clique. While each $\mathcal{K} \in \Psi_{k-i}$ needs $\frac{(i-1+\alpha')L}{(1-\alpha')L'}$ good k-cliques in Ψ'_k, the total number of over-connected k-cliques is at least

$$c_t \frac{L}{L'} = \left\lceil \frac{k-1+\alpha'}{1-\alpha'} \right\rceil \frac{L}{L'} \geq \frac{(i-1+\alpha')L}{(1-\alpha')L'}, \quad \text{for } i < k.$$

During Step **2**, we considered all typical cliques in $\Psi_{k-1}, \ldots, \Psi_1$ sequentially. For a typical clique $\mathcal{K} \in \Psi_{k-i}$, we can ignore the existence of its over-connected k-cliques in our computation, because $c_t \ll \ell$. Thus $\lambda_i \ell$ in Proposition 5.3 gives a lower bound on the number of well-connected k-cliques (in Ψ_k). Thus in the new k-clique family Ψ'_k there are at least $\lambda_i \ell L/L'$ well-connected k-cliques for $\mathcal{K}$. We need to show that this quantity is not smaller than

$$\sum_{l=1}^{i} \frac{(l-1+\alpha')L}{(1-\alpha')L'} \varphi_{k-l}\ell.$$

Equivalently, we need to show

$$\lambda_i - \sum_{l=1}^{i} \frac{l-1+\alpha'}{1-\alpha'} \varphi_{k-l} > 0. \tag{6.2}$$

Let us first compute

$$I_i = \lambda_i - \sum_{l=1}^{i} \frac{l-1+\alpha}{1-\alpha} \varphi_{k-l}.$$

For example, we rewrite

$$I_1 = (k-1)\gamma - \frac{\alpha}{1-\alpha} \varphi_{k-1},$$

as

$$\begin{aligned} I_1 &= (k-1)\gamma - \frac{\alpha}{1-\alpha} \frac{1}{k-1} \left(1 - k\varphi_k - \sum_{j \geq 2} (k-j)\varphi_{k-j} \right) \\ &= (k-1)\gamma - \frac{\alpha}{1-\alpha} \frac{1}{k-1} \end{aligned}$$

$$\times\left(1-k((k-1)\gamma+s+\sum_{j\geq 2}(j-1)\varphi_{k-j})-\sum_{j\geq 2}(k-j)\varphi_{k-j}\right)$$

$$=(k-1)\gamma-\frac{\alpha}{1-\alpha}\left(\frac{1}{k-1}-k\gamma-\frac{k}{k-1}s-\sum_{j\geq 2}j\varphi_{k-j}\right).$$

Since

$$(k-1)\gamma-\frac{\alpha}{1-\alpha}\frac{1}{k-1}=\frac{\alpha}{k-1+\alpha}-\frac{\alpha}{(1-\alpha)(k-1)}=-\frac{\alpha}{1-\alpha}k\gamma,$$

we have

$$I_1=\frac{k}{k-1}\frac{\alpha}{1-\alpha}s+\frac{\alpha}{1-\alpha}\sum_{j\geq 2}j\varphi_{k-j}. \tag{6.3}$$

Similarly,

$$I_2=(k-1)\gamma+\frac{s}{k-1}+\sum_{j\geq 2}\varphi_{k-j}-\frac{\alpha}{1-\alpha}\varphi_{k-1}-\frac{1+\alpha}{1-\alpha}\varphi_{k-2}$$

$$=\frac{s}{k-1}+\frac{k}{k-1}\frac{\alpha}{1-\alpha}s+\sum_{j\geq 3}\varphi_{k-j}+\frac{\alpha}{1-\alpha}\sum_{j\geq 3}j\varphi_{k-j}.$$

In general,

$$I_i=\left(\frac{i-1}{k-1}+\frac{k\alpha}{(k-1)(1-\alpha)}\right)s+\sum_{j>i}\left(i-1+\frac{\alpha}{1-\alpha}j\right)\varphi_{k-j}>\mu_1, \tag{6.4}$$

where $\mu_1=\frac{k\alpha}{(1-\alpha)(k-1)}\mu$. To make (6.2) hold, we want the following inequality to be true for every $1\leq i\leq k$:

$$\mu_1\geq\sum_{l=1}^{i}\varphi_{k-l}\left(\frac{l-1+\alpha'}{1-\alpha'}-\frac{l-1+\alpha}{1-\alpha}\right).$$

It is sufficient to show

$$\frac{l-1+\alpha'}{1-\alpha'}-\frac{l-1+\alpha}{1-\alpha}\leq\mu_1,\quad\text{for every } l\in\{1,2,\ldots,k-1\}. \tag{6.5}$$

But this holds because of the definition of α'.

The following lemma summarizes the correctness of the Decomposition Algorithm:

Lemma 6.2 (Decomposition Lemma). *If the reduced graph G_r satisfies $\varphi_k \geq \sum_{i=2}^{k-1} (i-1)\varphi_{k-i} + (k-1)\gamma + \mu$, G'' can be decomposed into disjoint copies of $\mathcal{R}_k^{\alpha'}(L_1)$, for some $\alpha' = \alpha + c_\alpha \mu$, and $L_1 \geq c\mu L$.* ∎

6.2. Handling of Exceptional Vertices

The proof for the general case of the Main Theorem is immediate from the following lemma:

Lemma 6.3. *Assume $\varepsilon \leq \theta \ll \rho \ll 1$ and the n-vertex graph G satisfies the degree condition in (1.8), containing a set V_0 of exceptional vertices, with $|V_0| \leq \theta n$. If the subgraph $G'' = G \backslash V_0$ has an $\mathcal{R}_{\varepsilon,k}^{\alpha'}(L_1)$-factor with $\alpha' = \alpha + \rho$ for sufficiently large values of L_1, then G contains an H-matching that covers all but at most $\frac{5k}{(k-1)^2\gamma} w$ vertices.*

Before stating the proof of Lemma 6.3 some notation is in order. We use **R** to denote the $\mathcal{R}^{\alpha'}(L_1)$-cover of G'', and $\mathcal{R}$ represents an element of the cover, i.e., an unbalanced ε-regular k-clique $\mathcal{R}_{\varepsilon,k}^{\alpha'}(L_1)$. We refer to $\mathcal{R}$ as a *clique* if there is no confusion. Among the color classes of $\mathcal{R}$ we let U_k denote the smallest cluster, i.e., the one containing $\alpha' L_1$ vertices.

The proof of Lemma 6.3 consists of two phases. We will start by removing $|V_0|$ copies of H from G such that each copy of H contains exactly one vertex from V_0. We will also make sure none of the cluster-cliques loses many vertices in this process. Next, we will establish two directed graphs (one on clusters and another on cliques), and use them to control the number of uncovered vertices of the H-matching to meet the constant bound suggested in Lemma 6.3.

Phase I: We will use the following corollary of Key Lemma [12]:

Lemma 6.4. *Let $\mathcal{R} = \{V_1, \ldots, V_k\}$ be an ε-regular k-clique in G''. Assume $W_i \subset V_i$, and $|W_i| = d'L$, $i = 1, \ldots, k$, for some $d' \gg \varepsilon$. Let $\mathcal{R}_0 = \{W_1, \ldots, W_k\}$. Then $K_k(h) \subset \mathcal{R}_0$. In particular, $H \subset \mathcal{R}_0$, and we can put its small color-class in any W_i.*

For a vertex $v \in V_0$ and a cluster $C \in G_r$, we will write $v \sim C$ if $\deg(v, C) \geq d|C|$.

The procedure of Phase I is as follows: Consider the vertices in V_0 in any order. For each $v \in V_0$, we pick an element $\mathcal{R} = \{U_1, U_2, \ldots, U_k\}$ of the cover for which

$$v \sim U_i \quad \text{for all but at most one of the clusters } U_i \in \{U_1, \ldots, U_k\}. \tag{6.6}$$

We will create a copy of H using the vertex v and $\mathcal{R}$ as follows:

- If $v \sim U_i$ for any $i \neq k$ we will remove a copy of H consisting of vertex v, $(u-1)$ vertices of U_k, and w vertices from each U_i, $i \neq k$.
- If $\exists j \neq k$ such that $v \sim U_i$ for any $i \neq j$, we will remove u vertices from U_k, w vertices from each U_i for $i \neq j$, and $(w-1)$ vertices from U_j to form a copy of H containing v.

During this procedure in order to prevent a cluster from losing too many of its vertices, we stop considering an element of **R** if it has been selected $\theta_1 L_1$ times, for $\theta_1 = \sqrt{\theta}$. The key point in this algorithm is whether there always exists an *available* element of **R** satisfying (6.6). We claim:

Proposition 6.5. *For any vertex in V_0, regardless of the order that it will be processed, there are at least $\frac{\alpha}{2}$ proportion of elements of* **R** *satisfying* (6.6).

Due to the restriction that no clique can be selected by more than $\theta_1 L_1$ vertices, the proportion of the off-limit elements of **R** should be at most:

$$\frac{\dfrac{\theta n}{\sqrt{\theta} L_1}}{\dfrac{n}{(k-1+\alpha')L_1}} < \sqrt{\theta} k.$$

Clearly, if Proposition 6.5 holds, since $\alpha/2 > \sqrt{\theta}k$, the procedure of Phase I can be carried out.

Proof of Proposition 6.5. Let v be the $(t+1)$st vertex of V_0 considered in the above procedure. We assume that, for the first t exceptional vertices, we have removed t (disjoint) copies of H each containing one exceptional vertex and $h-1$ vertices from some elements of **R**. Let G'' denote the (induced) subgraph of G on the remaining vertices. Observe that $|V(G'')| = n - \theta n - t(h-1)$. Let m denote the fraction of elements in **R** satisfying (6.6). We have

$$\begin{aligned}\left(1 - \frac{1}{k-1} + \gamma\right)n - \theta n - (h-1)t &\le \deg_G(v, V(G'')) \\ &\le \left((1-m)\frac{k-2+\alpha' d + d}{(k-1)+\alpha'} + m\right)(n - \theta n - (h-1)t),\end{aligned}$$

or

$$1 - \frac{1}{k-1} + \gamma - \theta h \le 1 - \frac{1+\alpha' - \alpha' d - d}{(k-1)+\alpha'} + \frac{1+\alpha' - \alpha' d - d}{(k-1)+\alpha'} m,$$

which implies

$$\frac{1+\alpha' - \alpha' d - d}{(k-1)+\alpha'} - \frac{1}{k-1+\alpha} - \theta h \le \frac{1+\alpha' - \alpha' d - d}{(k-1)+\alpha'} m.$$

Therefore,

$$m > \frac{1}{1+\alpha' - \alpha' d - d}(\alpha - 2d - \theta k h) > \frac{\alpha}{2}$$

holds since we chose $\frac{\alpha}{3} > kh\theta$. ∎

After V_0 becomes empty, we would like all regular pairs inside ε-regular cliques to satisfy the superregularity condition. Consider one such element $\mathcal{R}$ on clusters $\{U_1, \ldots, U_k\}$. We will move a vertex v from cluster U_i in $\mathcal{R}$ to V_0, if there exists $j \neq i$ with $\deg(v, U_j) < (d - \varepsilon)|U_j|$. The ε-regularity of the cluster-pairs guarantees that there are at most $(k-1)\varepsilon|U_i|$ such vertices in each cluster. Using a procedure similar to Phase I, we handle the new [at most $k(k-1)\varepsilon n$] exceptional vertices. Consequently, all cluster-pairs in the remaining graph will satisfy $(\varepsilon, \frac{d}{2})$ superregularity.

Phase II: Let $\mathbf{R}'$ denote the resulting clique-cover at the end of Phase I. Since some clusters have lost vertices during Phase I, the sizes of resulting clusters vary. Because $\mathcal{R}^{\alpha'}(L_1) \supseteq \mathcal{R}^{\alpha}(L_2) \cup \mathcal{R}^1(L_0)$, where $L_2 = \frac{1-\alpha'}{1-\alpha} L_1$, $L_0 = \frac{\alpha'-\alpha}{1-\alpha} L_1$, and $\alpha' - \alpha = \rho \gg \theta_1$, each element $\mathcal{R}'$ of $\mathbf{R}'$ at least contains an balanced ε-regular k-clique $\mathcal{R}_k^1\left(\frac{L_0}{2}\right)$. Thus $\mathcal{R}'$ contains an H-matching, in which u-vertex classes can be taken from any cluster of element $\mathcal{R}'$.

As a result, we argue that by appropriately placing u-vertex classes in the H-matching of $\mathcal{R}'$, $w - u$ vertices can be moved among the clusters of $\mathcal{R}'$. In fact, when we want to move $w - u$ vertices from U_1 to U_2 in $\mathcal{R}'$, we simply switch the color classes for one copy of H such that the u-vertex class which was supposed to come from U_1 will come from U_2. This will result in a loss of $(w - u)$ vertices from U_1 while U_2 will gain $(w - u)$ vertices. This observation will helps us to obtain the following result that will be proven later:

Proposition 6.6. *Each element of* $\mathbf{R}'$ *has a H-matching that covers all but at most* $(k-1)(2w-u) + w$ *vertices.*

If we directly apply Proposition 6.6 to all the elements of $\mathbf{R}'$, we will get an H-matching of G'' that leaves at most $((k-1)(2w-u) + w)|\mathbf{R}|$ vertices uncovered. Since $|\mathbf{R}| = O(\ell)$ and ℓ is not larger than the constant $M(\varepsilon)$ (from the Regularity Lemma), this already confirms the correctness of Conjecture 1.

To show that the number of left-over vertices is a constant independent of ε, we will use the connections among the clusters of G'' in different elements of $\mathbf{R}$, as well as the connections of elements of $\mathbf{R}$. We define two directed graphs as follows:

- Directed graph $\mathcal{D}$ on the clusters of G'' with a directed edge $U \to U_1$ in $E(\mathcal{D})$, whenever $U_1 \in \mathcal{R} = \{U_1, U_2, \ldots, U_k\}$ and $(U, U_i) \in E(G_r)$, for all $i \neq 1$.
- Directed graph $\mathcal{D}^*$ on the cliques, whose vertices are elements of $\mathbf{R}$ and a directed edge from $\mathcal{R}_1$ to $\mathcal{R}_2$ whenever $\forall U \in \mathcal{R}_1$, $U' \in \mathcal{R}_2$, $U \xrightarrow{\mathcal{D}} U'$.

Since in Phase I the cluster-sizes in G'' had insignificant changes, the adjacency of clusters in G_r, and consequently the graphs $\mathcal{D}$, $\mathcal{D}^*$ are the same with respect to both $\mathbf{R}$ and $\mathbf{R}'$.

In a directed graph D the *source set* of v is defined as $\mathcal{W}(v) = \{u \in V(D) : \exists$ a directed path from u to $v\}$. It is easy to see that $|\mathcal{W}(u)| \geq |\mathcal{W}(v)|$, if $u \in N_{out}(v)$. A set $\mathcal{S}(D) \subseteq V(D)$ will be called a *sink* set if $V(D) = \cup_{v \in \mathcal{S}(D)} \mathcal{W}(v)$. We will show that in graphs $\mathcal{D}$ and $\mathcal{D}^*$ there are sink sets $\mathcal{S}(\mathcal{D})$ and $\mathcal{S}(\mathcal{D}^*)$ of orders at most $\frac{k}{(k-1)^2\gamma}$ and $\frac{2}{k(k-1)\gamma}$. Now, we are ready to state the algorithm of Phase II:

1. For each cluster, we first compute the number of remaining vertices (called *extra vertices,* denoted by $K(C)$ for each cluster C) under the largest H-matching described in Proposition 6.6 (instead of actually performing the tiling). We will then separate the extra vertices in each cluster to two categories: (I) groups of $(w - u)$-vertices, denoted by $extra_1(\mathcal{R})$ (no matter which cluster the $(w - u)$ vertices come from); (II) less than $(w - u)$ vertices, denoted by $extra_2(C)$.
2. For each cluster C with $|extra_2(C)| > 0$ and $C \notin \mathcal{S}(\mathcal{D})$, we "move" these $extra_2(C)$ vertices to the sink set $\mathcal{S}(\mathcal{D})$ as follows:
   Let $x = |extra_2(C)|$. We find a directed (cluster) path $C, C^1, \dots, C^t$, with C^t belonging to the sink. For $i = 1, \dots, t$, let $\mathcal{R}_i$ denote the clique that contains C^i. Depending on whether C^i is a large or a small cluster, we will take either w, or v vertices from C^i, when finding a copy of H inside $\mathcal{R}_i$. Denote this number by c_i. For $i = 1, \dots, t$, x copies of H are removed by x vertices from C^{i-1} and $(h - 1)x$ vertices from $\mathcal{R}_i$ (in particular $(c_i - 1)x$ vertices from C^i). As a result, $|extra_2(C)|$ becomes zero and $|extra_2(C^t)|$ will increase by x. Note that a single cluster can be included in many paths to the sink set. Since the total number of extra vertices is a constant (much smaller than dL_1), the superregularity of the cluster pairs will not be affected.
3. Using a similar procedure we will "move" the extra vertices $extra_1(\mathcal{R}')$ form each element $\mathcal{R}'$ (not in $\mathcal{S}(\mathcal{D}^*)$) to the cliques in $\mathcal{S}(\mathcal{D}^*)$. For example, assume that $\{\mathcal{R}', \mathcal{R}'_1, \dots, \mathcal{R}'_t\}$ is a directed path. Then, there exist $U \in \mathcal{R}$ and $U_1 \in \mathcal{R}_1$, such that directed edge (U, U_1) belongs to $E(\mathcal{D})$. Since a $(w - u)$-vertex group can be moved arbitrarily from one cluster to another inside $\mathcal{R}'$, we can gather all $|extra_1(\mathcal{R}')|$ vertices in U and then "move" them to $U_1 \in \mathcal{R}'_1$, etc.
4. Apply the Blow-up Lemma to the remainder of each element of **R** to get an H-matching. Adding these copies of H to ones removed in previous two steps and Phase I, we finally get the desired H-matching of the original graph G.

Let us compute the number of uncovered vertices in this algorithm. The total number of extra vertices of category (II) is at most

$$(w - u)|\mathcal{S}(\mathcal{D})| \le (w - u)\,\frac{k}{(k-1)^2\gamma},$$

and the total number of extra vertices of category (I) can be bounded by

$$((k-1)(2w-u) + w)|\mathcal{S}(\mathcal{D}^*)| \le ((k-1)(2w-u) + w)\,\frac{2}{k(k-1)\gamma} < \frac{4w}{(k-1)\gamma}.$$

Therefore, the total number of left-over vertices is at most

$$(w-u)\,\frac{k}{(k-1)^2\gamma} + \frac{4w}{(k-1)\gamma} < \frac{5k}{(k-1)^2\gamma}\,w.$$

as mentioned in Lemma 6.3.

Let us now verify Proposition 6.6.

Proof of Proposition 6.6. Consider the element $\mathcal{R}'$ in $\mathbf{R}'$. Since all cluster-pairs in $\mathcal{R}'$ satisfy superregularity, we can apply the Blow-up Lemma to $\mathcal{R}'$. We pick the H-matching of $\mathcal{R}'$ (which leaves $a_1, a_2, \ldots, a_k$ vertices uncovered from clusters $U_1, U_2, \ldots, U_k$, respectively) satisfying the following two conditions:

- It is one of the best H-matchings, i.e., $\sum a_i$ is the minimum over all H-matchings of $\mathcal{R}'$.
- It is the most balanced, i.e., $\sum |a_i - w|$ is the smallest among the best matchings.

Assume that $a_{i_0} = \min a_i$. Naturally, $a_{i_0} < w$; otherwise a copy of H could be found in the leftover of $\mathcal{R}'$, and for some $i \neq i_0$ we have $a_i < 2w - u$; otherwise we can move $(w - u)$ vertices from U_i to U_{i_0} to get a smaller value for $\sum |a_i - w|$. Therefore, $\sum a_i < w + (k - 1)(2w - u)$. ∎

To verify that both $\mathcal{D}$ and $\mathcal{D}^*$ have small sink sets we need the following general lemma:

Lemma 6.7. *In a directed graph D, with $\delta(D) = \min_{v\in D} \deg_{out}(v)$, there is a sink set $\mathcal{S}$ of size at most $\frac{|V(D)|}{\delta(D)+1}$.*

Proof. Let x_1 be a vertex in which $|\mathcal{W}(x_1)| = \max_{v\in V(D)} |\mathcal{W}(v)|$. Since $N_{out}(x_1) \in \mathcal{W}(x_1)$ (otherwise, some out-neighbor of x_1 produces a larger source set), we have $|\mathcal{W}(x_1)| \geq \delta + 1$. Let $D' = D \backslash \mathcal{W}(x_1)$, then, for every vertex $v \in D'$, the set $N_{out}(v)$ in D' is the same as $= N_{out}(v)$ in D, since, if $v' \in N_{out}(v)$ in D and $v' \in \mathcal{W}(x_1)$, v should also be in $\mathcal{W}(x_1)$. Now, let x_2 be a vertex with the largest source set in D'. Similarly to x_1, we have $\mathcal{W}(x_2) \geq \delta + 1$ in D'. The above procedure can be repeated at most $\frac{|V(D|)}{\delta+1}$ times, and the proof will follow. ∎

The remaining questions are to compute $\delta(\mathcal{D})$ and $\delta(\mathcal{D}^*)$. We have

Proposition 6.8. *For any $U \in V(\mathcal{D})$, $\deg_{out}(U) \geq \frac{4\gamma}{3}|V(\mathcal{D})|$.*

Proof. Since these connections are not influenced by the insertion of exceptional vertices, we assume that the size of clusters are still L_1 or $\alpha' L_1$.

In fact, for every $U \in V(\mathcal{D})$, let m_1, m_2, and m_3 denote the fraction of elements $\mathcal{R} \in \mathbf{R}$ with clusters $\{U_1, U_2, \ldots, U_k\}$ for which $(U, U_i) \in E(G_r)$ for all $i < k$, $(U, U_i) \in E(G_r)$ for all $i \neq j$ with $j \neq k$, and $(U, U_i) \in E(G_r)$ for all i, respectively. By degree condition in G_r, we have

$$
\begin{aligned}
1 - \frac{1}{k-1} + \gamma - 2d &\leq \frac{k-2}{k-1+\alpha'} + \frac{1}{k-1+\alpha'} m_1 \\
&\quad + \frac{\alpha'}{k-1+\alpha'} m_2 + \frac{1+\alpha'}{k-1+\alpha'} m_3 \\
&< \frac{k-2}{k-1+\alpha'} + \frac{1}{k-1+\alpha'} m_1 + \frac{1}{k-1+\alpha'} m_2 + \frac{2}{k-1+\alpha'} m_3,
\end{aligned}
$$

or

$$
\begin{aligned}
1 - \frac{1}{k-1} + \gamma < \frac{k-2}{k-1+\alpha} + \frac{1}{k-1+\alpha}(m_1 + m_2 + 2m_3) \\
= \left(1 - \frac{1}{k-1} - (k-2)\gamma\right) + \left(\frac{1}{k-1} - \gamma\right)(m_1 + m_2 + 2m_3),
\end{aligned}
$$

that is,

$$(k-1)\gamma \le \left(\frac{1}{k-1} - \gamma\right)(m_1 + m_2 + 2m_3).$$

Therefore, in $\mathcal{D}$,

$$
\begin{aligned}
\deg_{out}(U) &= (m_1 + m_2 + km_3)|\mathbf{R}| \\
&\ge \frac{(k-1)\gamma}{\frac{1}{k-1} - \gamma} \frac{|V(\mathcal{D})|}{k} \\
&> \frac{(k-1)^2\gamma}{k} |V(\mathcal{D})|.
\end{aligned}
$$

■

Using an argument similar to Proposition 6.8, we can show

$$\delta(\mathcal{D}^*) \ge \frac{k(k-1)\gamma}{2} |V(\mathcal{D}^*)|,$$

which concludes the proof of Lemma 6.3. ■

7. THE EXTREMAL CASES

Recall that $\Phi = \{\Phi_k, \ldots, \Phi_1\}$ denotes the maximum clique-cover of G_r. In this section we will assume

$$s = \varphi_k - (k-1)\gamma - \sum_{i=2}^{k-1} \varphi_{k-i}(i-1) \le \mu. \tag{7.1}$$

Substituting

$$\varphi_k = (k-1)\gamma + \sum_{i=2}^{k-1} \varphi_{k-i}(i-1) + s$$

in (5.1), we will have

$$(k-1)\varphi_k + (k-1)\gamma + \sum_{i=2}^{k-1} \varphi_{k-i}(i-1) + s + (k-1)\varphi_{k-1} + \sum_{i=2}^{k-1} \varphi_{k-i}(k-i) = 1,$$

which implies

$$\varphi_k + \varphi_{k-1} + \cdots + \varphi_1 = \frac{1}{k-1} - \gamma - \frac{s}{k-1} \geq \frac{1}{k-1} - \gamma - \frac{\mu}{k-1}. \tag{7.2}$$

Also from (5.1), we can get

$$(k-1)k\gamma + ks + (k-1)\sum_{j\geq 1} i\varphi_{k-i} = 1,$$

Define

$$\sigma_i = \sum_{j\leq i} (k-j)\varphi_j, \text{ for } i < k,$$

we thus know,

$$\sigma_{k-1} = \frac{1}{k-1} - k\gamma - \frac{k}{k-1}s = \frac{1-\alpha}{k-1+\alpha} - \frac{k}{k-1}s \geq \alpha_0,$$

where we define $\alpha_0 = \frac{1-\alpha}{k-1+\alpha} - \frac{k}{k-1}\mu$.

Let $C = \sqrt[k]{\alpha_0/\varepsilon}$. *Since* $\sigma_{k-1} \geq \alpha_0 = C^k\varepsilon$, there is an integer t, $1 \leq t \leq k-1$, such that $\sigma_t \geq C^{j_0}\varepsilon$ and $\sigma_{t-1} \leq C^{j_0-1}\varepsilon$ for some $j_0 \leq k$. Set $\varepsilon' = k(C^{j_0-1}\varepsilon + 2\mu)$, and $\mu' = C^{j_0}\varepsilon$.

We first move all the vertices of G in Φ_j, $j < t$, to V_0. The size of the resulting exceptional set (still denoted by V_0) is less than

$$\varepsilon n + \sum_{j<t} j\varphi_j n < \varepsilon n + k\sum_{j<t}\varphi_j n < \varepsilon n + k\sigma_{t-1}n \leq kC^{j_0-1}\varepsilon n < \varepsilon' n.$$

Following the proof of Proposition 5.2, it is not hard to show that the number of k-cliques that are over-connected to some smaller clique is at most $2t\mu$. Since the size of Φ_t is not small, most t-cliques are not over-connected to any k-clique. Let us fix two such t-cliques K_1, K_2. Let $\Phi^0_{k-j} = \{K \in \Phi_{k-j} : \deg(\{K_1, K_2\}, K) < 2t(k-j-1)\}$ and $m_j = |\Phi^0_{k-j}|/\ell$ for $j = 0, 1, \ldots, k-t$. Using similar computations as in Proposition 5.3, we obtain

$$m_0 + m_1 + \cdots + m_{k-t} < \frac{2t}{k-1}s + 4dt < \frac{\varepsilon'}{k}.$$

Recall that in Item **3** of Proposition 5.1, we divide each clique K^j ($j \geq t$) which is well connected to K_1, K_2 into sets $A_t(K^j)$ and $B_t(K^j)$, where $A_t(K^j)$ consists of the clusters which are adjacent to all but one of K_1 (or K_2), $B_t(K^j)$ are those which are adjacent to all of K_1 (or K_2). When combining $A_t(K)$ for all clusters $K \notin \Phi_t^0 \cup \cdots \cup \Phi_k^0$, we obtain a cluster set A of size $t(\varphi_k + \cdots + \varphi_t - m_0 - \cdots - m_{k-t})\ell$. Using (7.2), we have $|A| \geq t(\frac{1}{k-1} - \gamma - \frac{2k}{k-1}\mu)\ell$. On one hand, A is covered by a family of t-cliques and on the other hand, by Item **3** of Proposition 5.1, A is made up of $t \leq k - 1$ independent set $U_1, \ldots, U_t$, with $|U_j| \geq (\frac{1}{k-1} - \gamma - \frac{\varepsilon'}{k-1})\ell$. The degree condition on G_r, (4.2), requires

$$\deg(c, G_r \backslash U_i) \geq |G_r \backslash U_i| - \frac{\varepsilon'}{k-1}\ell, \quad \forall\, c \in U_i \tag{7.3}$$

i.e., elements of U_i are almost adjacent to all the clusters outside U_i. Depending on the value of t, we will consider two separate cases:

7.1. Extremal Case (I): $t = k - 1$

Let $U_k = V(G_r)\backslash\{U_1, \ldots, U_{k-1}\}$, and let U_i, $1 \leq i \leq k$, refer to the underlying vertex sets in G. We will move all exceptional vertices (set V_0) to U_k. Then remove a vertex $v \in U_k$ to U_i, $i < k$ if $\deg(v, U_i) < \varepsilon_1|U_i|$, where $\varepsilon' \ll \varepsilon_1 \ll 1$. Let us still denote the resulting sets by $U_1, \ldots, U_k$. In the ideal case $|U_i| = (\frac{1}{k-1} - \gamma)n$ for all $i < k$ and $|U_k| = (k-1)\gamma n$. Because of the superregularity between every pair in $\{U_1, \ldots, U_k\}$, we can use the Blow-up Lemma to find the desired H-factor in G.

If $|U_i| < (\frac{1}{k-1} - \gamma)\, n$, for all $i < k$, we can create some copies of H whose w-classes are located at U_k. Then an argument similar to Proposition 6.6 shows that all but $2kw$ vertices of G will be covered by an H-matching. Consequently, we can assume that $|U_1| > (\frac{1}{k-1} - \gamma)n$, also $|U_i| < (\frac{1}{k-1} - \gamma)\, n$, for $1 < i < k$, $|U_k| < (k-1)\gamma n$ (the other cases are similar).

From U_1 we will move a vertex v for which $\deg(v, U_1) > \varepsilon_1|U_1|$ to other classes U_i with fewer vertices, without hurting the superregularity. Let $U_1, \ldots, U_k$ still denote the resulting sets. After this step, either we can achieve the ideal case, or for all $v \in U_1$, $\deg(v, U_1) \leq \varepsilon_1|U_1|$. Let us assume that the latter is the case, and let $x = |U_1| - (\frac{1}{k-1} - \gamma)n$, we will use the following fact:

Proposition 7.1.

$$\nu_i(G) \geq (\delta(G) - i + 1)\,\frac{n}{2(i+1)\Delta(G)}.$$

To see this, take a maximal set of i-stars, let m denote its size, and let **E** be the number of edges between the stars and the remaining vertices of G; we have the following chain of inequalities which in turn implies the claim:

$$(n - m(i+1))(\delta(G) - (i-1)) \leq \mathbf{E} \leq m(i+1)\Delta(G).$$

Proposition 7.1 implies that we can find x vertex disjoint w-stars in U_1 and after moving their centers to other U_i with fewer vertices, we immediately remove x copies of H. The remaining sets $U_1, \ldots, U_k$ and C have size ratio $(1, \ldots, 1, \alpha)$ and also satisfy the superregularity condition. Applying the Blow-up Lemma will complete the proof.

7.2. Extremal Case (II): $t < k - 1$

Let $B = V(G_r)\backslash A$. We use V_B to represent corresponding vertices of G in the clusters of B and V_0. V_A denotes those in the clusters of A. Let H_0 be the $(k - t)$-partite bottle graph with width w and neck u. Our goal is to find an almost perfect H_0-matching of V_B such that each copy of H_0 has a big neighborhood (defined as common neighbors of the vertices in H_0) in V_A. Thus we can apply Hall's marriage theorem to match each copy of H_0 with a copy of $K_t(w)$ from V_A. This will create the desired H-matching of G that leaves only a constant number of vertices uncovered.

Finding the H_0-matching of V_B is almost the same as the procedure of the nonextremal case: We will first tile V_B with $\mathcal{R}^{\alpha'}_{\varepsilon,k-t}(L_1)$, *unbalanced* ε*-regular* $(k - t)$*-cliques.* Then we will move the vertices of V_0 to V_B. The only difference is that we have to take special care of the vertices in V_B whose degrees in V_A are small.

Let us get into some details about the tiling of V_B with $\mathcal{R}^{\alpha'}_{(k-t)}(L_1)$: We can take advantage of the existing clique-cover. By definition, $B = \{B_0, B_1, B_2, \ldots, B_{k-t}\}$, where $B_{j-t} = \cup_{K^j\in\Phi_j\backslash\Phi_j^0} B_t(K^j)$ for $t \le j \le k$ and $B_0 = \Phi^0_k \cup \cdots \cup \Phi^0_t$, denotes all j-cliques $(j \ge t)$ which did not participate in the creation of A. We will repeat the algorithm from Section 4: As before, we use $\mathcal{K}$ to represent the cluster clique, i.e., the subgraph of G corresponding to a clique K of the reduced graph. Similarly, $\mathcal{B}_i$ denotes the family of cluster cliques corresponding to B_i for $i = 0, \ldots, k - t$. Our goal is to convert every cluster clique $\mathcal{K} \in \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{k-t-1}$ to a copy of $\mathcal{R}^{\alpha'}_{(k-t)}(L_1)$. The remaining graphs in $\mathcal{B}_{(k-t)}$ and $\mathcal{B}_0$ will naturally be divided into copies of $\mathcal{R}^{\alpha'}_{(k-t)}$. To explain why this algorithm is feasible, we will re-use the calculations in Section 4. It is easy to see that if $K' \in B_{j-t}$, $K'' \in B_{k-t}$, for $t < j < k$, came from $K^j \in \Phi_j$, and $K^k \in \Phi_k$, respectively, and $K^j \hookrightarrow K^k$, then $K' \hookrightarrow K''$. Thus, for any $K \in B_{i-t}$, the number of well-connected $(k - t)$-cliques is at least $\Lambda(K^i) \ge \lambda_i$, if K was generated from K^i. In the key expressions (6.3) and (6.4), despite the fact that s is not bigger than μ, we use $\forall i \le t$, $\sigma_i = \sum_{j\ge i} j\varphi_{k-j} \ge C^{j_0}\varepsilon = \mu'$; then μ' plays the same role as μ. That is, the right side of (6.5) was replaced by $\frac{\alpha}{1-\alpha}\mu'$. Thus, $\alpha' - \alpha = c_1\mu'$ and, eventually, we will find a $\mathcal{R}^{\alpha'}_{k-t}(L_1)$-factor of V_B.

Before inserting V_0 to V_B, we will remove (at most $2\varepsilon'|V_B|$) vertices from V_B to U_i if

$$\deg(v, U_i) < \beta_1|U_i|, \tag{7.4}$$

with $(\varepsilon' \ll \beta_1 \ll \mu')$. We might move extra vertices from $\mathcal{R}^{\alpha'}_{k-t}(L_1)$ to V_0 in order to maintain the ratio of cluster sizes as $(1, \ldots, 1, \alpha')$. If any cluster lost more than one half of its vertices due to (7.4), all the vertices in this clique will be removed to V_0. It is easy to see that the resulting exceptional set V_0 satisfies $|V_0| < 2\varepsilon' n/\alpha'$.

To finish the proof of the extremal cases, we separate cases according to sizes of U_i:

(a) The Ideal Case: $|U_i| = (\frac{1}{k-1} - \gamma)n$, for $i = 1, \ldots, t$.

Since the density between all U_i pairs is almost 1, we will first apply the Blow-up Lemma to $U_1 \cup \cdots \cup U_t$, and get a $K_t(w)$-factor. Note that the neighborhood of any copy of $K_t(w)$ in V_B is almost the entirely V_B because of (7.3). Define a subset V_{bad} of V_B, where $\forall v \in V_{bad}$,

$$\deg(v, A) \leq (1 - \beta_2)|A|, \tag{7.5}$$

with $\varepsilon' \ll \beta_2 \ll \beta_1$. It is easy to see that $|V_{bad}| < \frac{2\varepsilon'}{\beta_2} n < \beta_2 n$.

We assume that after inserting V_0 to V_B, we obtain a H_0-matching such that each copy of H_0 contains at most one vertex of V_{bad}. This assumption leads to an easy matching between H_0 copies from V_B and $K_t(w)$ from V_A: We can first use the greedy algorithm to match those H_0's that contain vertices from V_{bad}, then use Hall's marriage theorem to handle the rest of the vertices. The number of uncovered vertices by the H-matching of G will be proportional to the number of the leftovers by the H_0-matching over V_B.

To force each copy of H_{k-t} to contain at most one V_{bad} vertex, we will move all V_{bad} vertices (and some extra to maintain the size ratio of clusters in those cliques) to V_0. Since $\beta_2 \ll \mu$, we can still insert vertices in the new V_0 back to V_B.

(b) The Defected Cases: If $|U_{i_1}| < (\frac{1}{k-1} - \gamma)n$ for some $1 \leq i_1 \leq t$, instead of $K_t(w)$ we will find copies of H_t from V_A whose u-vertex class comes from U_{i_1}. To match them, we need to find the same number of $K_{k-t}(w)$'s from V_B. This is possible because $\alpha' - \alpha = c_1\mu' \gg \varepsilon'$.

When $|U_{i_1}| > (\frac{1}{k-1} - \gamma)n$ for some i_1, we will find $|U_{i_1}| - (\frac{1}{k-1} - \gamma)n$ copies of w-star in A and then move their centers to other U_i's or V_B, which has less vertices (than the ideal case). The rest is exactly the same as in the Ideal Case. ∎

This finally concludes the proof of the extremal case and also the Main Theorem.

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