TURÁN NUMBER OF COMPLETE MULTIPARTITE GRAPHS IN MULTIPARTITE GRAPHS

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Abstract. In this paper we study a multi-partite version of the Erdős–Stone theorem. Given integers $r < k$ and $t \geqslant 1$, let $\mathrm{ex}_k(n, K_{r+1}(t))$ be the maximum number of edges of $K_{r+1}(t)$-free k-partite graphs with n vertices in each part, where $K_{r+1}(t)$ is the complete $(r+1)$-partite graph with t vertices in each part. We determine the exact value of $\mathrm{ex}_k(n, K_{r+1}(t))$ for $t \leqslant 3$, $r < k \leqslant 2r$ and sufficiently large n. We also characterize all extremal graphs for r, k such that r divides k, analogous to a result of Erdős and Simonovits on forbidding $K_{r+1}(t)$ in general graphs.

1. Introduction

Generalizing Mantel's theorem from 1907 [18], Turán's theorem from 1941 [23] started the systemetic study of Extremal Graph Theory. Given a graph F, let $\mathrm{ex}(n, F)$ denote the largest number of edges in a graph not containing F as a subgraph (called F-free). Let K_r denote the complete graph on r vertices and $T_r(n)$ denote the complete r-partite graph on n vertices with $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ in each part (known as the Turán graph); and $t_r(n)$ be the size of $T_r(n)$. Turán's theorem [23] states that $\mathrm{ex}(n, K_{r+1}) = t_r(n)$ for all $n \geqslant r \geqslant 1$ and in addition, $T_r(n)$ is the unique *extremal graph*.

Let $K_{t_1,\ldots,t_r}$ denote the complete r-partite graph with parts of size $t_1, \ldots, t_r$ and write $K_r(t) = K_{t,\ldots,t}$ with r parts. A celebrated result of Erdős and Stone [10] determines $\mathrm{ex}(n, K_{r+1}(t))$ asymptotically:

$$\mathrm{ex}(n, K_{r+1}(t)) = t_r(n) + o(n^2) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} + o(n^2).$$

Erdős [7] and Simonovits [21] independently improved the error term above to $O(n^{2-1/t})$. Simonovits [21] also showed that any extremal graph for $K_{r+1}(t)$ can be obtained from $T_r(n)$ by adding or removing $O(n^{2-1/t})$ edges. Later Erdős and Simonovits [9] determined the structure of extremal graphs for $K_{r+1}(t)$ for $t \leqslant 3$ as follows.

Theorem 1. [9] *For $t \leqslant 3$, every extremal graph G for $K_{r+1}(t)$ has a vertex partition $U_1, \ldots, U_r$ such that*

- *$G[U_i, U_j]$ is complete for all $i \neq j$,*
- *$G[U_i] = n/r + o(n)$,*
- *$G[U_1]$ is extremal for $K_{t,t}$, and*
- *$G[U_2], \ldots, G[U_r]$ are extremal for $K_{1,t}$.*

The restriction $t \leqslant 3$ in Theorem 1 comes from our knowledge on $\mathrm{ex}(n, K_{t,t})$. A well-known open problem in Extremal Graph Theory is proving $\mathrm{ex}(n, K_{t,t}) = \Omega(n^{2-1/t})$ and this is only known for $t \leqslant 3$.

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Extremal problems with multipartite graphs as host graphs have been studied since 1951, when Zarankiewicz proposed the study of the largest number of edges in a $K_{s,t}$-free bipartite graph. Let $\mathcal{G}(n_1, \dots, n_k)$ denote the family of k-partite graphs with $n_1, \dots, n_k$ vertices in its parts and write $\mathcal{G}_k(n) = \mathcal{G}(n, \dots, n)$ with k parts. Given a graph F, define $\mathrm{ex}(n_1, \dots, n_k; F)$ as the largest number of edges in F-free graphs from $\mathcal{G}(n_1, \dots, n_k)$, and let $\mathrm{ex}_k(n, F) = \mathrm{ex}(n, \dots, n; F)$ (with k parts). (Trivially $\mathrm{ex}_k(n, F) = \binom{k}{2} n^2$ if the chromatic number $\chi(F) > k$.) In 1975 Bollobás, Erdős, and Szemerédi [2] investigated several Turán-type problem for multipartite graphs. Applying a simple counting argument, they showed that, for any $n, k, r \in \mathbb{N}$ with $k > r$,

$$\mathrm{ex}_k(n, K_{r+1}) = t_r(k) n^2 \tag{1.1}$$

The main results of [2] were on the minimum degree that forces a copy of K_r in the graphs of $\mathcal{G}_k(n)$. This problem has been intensively studied and resolved when $k = r$ (frequently in its complementary form concerning *independent transversals*) [13–15, 22], and more recently for $k > r$ [17] (certain cases are still unsolved). There are several other extremal results for multipartite graphs. Bollobás, Erdős, and Straus [1] determined $\mathrm{ex}(n_1, \dots, n_r; K_r)$ for all $n_1, \dots, n_r$ and r. Let kK_r denote k vertex-disjoint copies of K_r. The problem $\mathrm{ex}(n_1, \dots, n_r; kK_r)$ were studied more recently [6, 12, 24] and settled by Chen, Lu, and Yuan [5] when $n_1, \dots, n_r$ are sufficiently large (k, r are arbitrary but fixed). The minimum pair density of multipartite graphs that forces a clique was studied by Bondy, Shen, Thomassé, and Thomassen [3] and Pfender [19].

In this paper we study $\mathrm{ex}_k(n, K_{r+1}(t))$, the multi-partite version of the Erdős–Stone theorem and Theorem 1. To give the precise value of $\mathrm{ex}_k(n, K_{r+1}(t))$, we need the following definition. Given $a, t, n_1, \dots, n_a \in \mathbb{N}$, let $z_t^{(a)}(n_1, \dots, n_a)$ be the a-partite Zarankiewicz number for $K_{t,t}$, that is, the maximum number of edges in a $K_{t,t}$-free bipartite graph with part sizes $n_1, \dots, n_a$. For simplicity, we write $z_t^{(a)}(n)$ if $n_1 = \dots = n_a$. We also write $z_t(m, n)$ for $z_t^{(2)}(m, n)$.

Theorem 1 says that any extremal graph for $K_{r+1}(t)$ is the *join* of an extremal graph for $K_{t,t}$ and $r-1$ extremal graphs for $K_{1,t}$. Inspired by this, a natural guess of an extremal graph for its r-partite analogue is as follows. Suppose $k = ar + b$ with $0 \leqslant b < r$. We start with an n-blowup of $T_r(k)$, which has r classes and each class has either a or $a+1$ parts. We add a $K_{t,t}$-free graph to one class with the most number of parts and $\{K_{1,t}, K_{2,2}\}$-free graphs to the remaining $r-1$ classes. If $r < k \leqslant 2r$, then it is easy to see (as in the proof of Theorem 2) that this graph is $K_{r+1}(t)$-free and has $t_r(k)n^2 + z_t(n, n) + (k - r - 1)(t - 1)n$ edges, and therefore

$$\mathrm{ex}_k(n, K_{r+1}(t)) \geqslant t_r(k)n^2 + z_t(n, n) + (k - r - 1)(t - 1)n.$$

In this paper we first improve this lower bound when $t \geqslant 2$ and $r < k \leqslant 2r$.

Theorem 2. *Suppose $t \geqslant 2$, $r < k \leqslant 2r$ and $n \geqslant 8t^2$. Then $\mathrm{ex}_k(n, K_{r+1}(t)) \geqslant g(n, r, k, t)$, where*

$$g(n, r, k, t) := t_r(k)n^2 + z_t(n, n) + (t - 1)(k - r - 1)n + \min\{k - r - 1, 2r - k\} \left\lfloor \frac{(t - 1)^2}{4} \right\rfloor.$$

The following is our main result, in which we prove a matching upper bound when $t \in \{2, 3\}$ and n is sufficiently large.

Theorem 3. *For any $t \in \{2, 3\}$ and integers r, there exists $n_0 = n_0(r) \in \mathbb{N}$ such that for $n \geqslant n_0$ and $r < k \leqslant 2r$, we have $\mathrm{ex}_k(n, K_{r+1}(t)) = g(n, r, k, t)$.*

In fact, we conjecture that $\mathrm{ex}_k(n, K_{r+1}(t)) = g(n, r, k, t)$ holds for all $t \geqslant 2$, $r < k \leqslant 2r$, and sufficiently large n.

A natural question is whether our result can be extended to larger values of t and k. For larger value of t, although we can use $z_t(n,n)$ without knowing its precise value, we need several properties of this function in our proof. Kővári, Sós, Turán [16] showed that $z_t(n,n) = O(n^{2-1/t})$ for $t \geqslant 2$ and proving a matching lower bound is a well-known open problem:

(Z) $z_t(n,n) = \Omega(n^{2-1/t})$ for $t \geqslant 2$.

It was shown [4, 8] that (Z) holds for $t = 2, 3$. In addition, we will need the following properties.

$(E1)$ For any $a \in \mathbb{N}$, there exists $\delta > 0$ such that for large n, $z_t^{(a+1)}(n) - z_t^{(a)}(n) \geqslant \delta n^{2-1/t}$.

$(E2)$ for any $\varepsilon \in (0,1]$, there exists $\delta > 0$ such that for large n,

$$z_t(n,n) - z_t((1-\varepsilon)n, n) \geqslant \delta n^{2-1/t}.$$

$(E3)$ $z_t(m,n) - z_t(m-1,n) \geqslant t-1$.

We can easily verify $(E1)$–$(E3)$ for $t = 2, 3$. First, a proof of $(E1)$ from (Z) for all t is given in Section 3.3. Second, when $t = 2, 3$, $(E2)$ follows from $z_t(m,n) \leqslant (1+o(1))mn^{1-1/t}$ by Füredi [11] and $z_t(n,n) \geqslant (1-o(1))n^{2-1/t}$ for $t = 2, 3$. Third, $(E3)$ holds trivially because adding a vertex with $t-1$ edges to a $K_{t,t}$-free graph will not create a copy of $K_{t,t}$.

We do not know whether similar properties hold when $k > 2r$: in $(E2)$ we must deal with the $\lceil k/r \rceil$-partite Zarankiewicz number; we also need to replace $t-1$ by $\Omega(a^2 t)$ in $(E3)$, which seems out of reach at present.

Theorems 2 and 3 show that our problem is more complex than its non-partite counterpart, Theorem 1. Finally, we show that, when r divides k, this additional complexity does not exist, and we give an analogue of Theorem 1, modulo the existence of a set of constantly many exceptional vertices.

Theorem 4. *For $r, k \in \mathbb{N}$ with $r \mid k$ and $t = 2, 3$, there exist $C_0, n_0 \in \mathbb{N}$ such that the following holds for $n \geqslant n_0$. Let G be a $K_{r+1}(t)$-free k-partite graph with n vertices in each part and $\mathrm{ex}_k(n, K_{r+1}(t))$ edges. Then there is a vertex partition of G into $U_1, \ldots, U_r$, each consisting of exactly k/r parts of G, and a vertex set $Z \subseteq V(G)$ with $|Z| \leqslant C_0$ such that*

- *$G[U_i \backslash Z, U_j \backslash Z]$ is almost complete for all $i \neq j$,*
- *$G[U_1 \backslash Z]$ is $K_{t,t}$-free, and*
- *$G[U_2 \backslash Z], \ldots, G[U_r \backslash Z]$ are $K_{1,t}$-free.*

Showing $Z = \varnothing$ in Theorem 4 requires an a-partite analogue of $(E2)$, which is unknown for $a \geqslant 3$.

For the rest of this paper we only consider $r \geqslant 2$, as it is easy to see that Theorems 2–4 hold for $r = 1$.

Notation. Given integers $n \geqslant m \geqslant 1$, let $[n] = \{1, \ldots, n\}$ and $[m,n] = \{m, m+1, \ldots, n\}$. We omit floors and ceilings unless they are crucial, e.g., we may choose a set of εn vertices even if our assumption does not guarantee that εn is an integer.

When $X, Y \subseteq V(G)$ intersect, $E_G(X,Y)$ is defined as the collection of ordered pairs in $(x,y) \in X \times Y$ such that $\{x,y\} \in E(G)$. Write $e_G(X,Y) = |E_G(X,Y)|$. For a vertex v in G, let $N(v,X) = N(v) \cap X$ and $d(v,X) = |N(v,X)|$. Moreover, given $X \subseteq V(G)$, let $e(X,G)$ be the number of edges of G incident to the vertices of X. Given two graphs G and H on a common vertex set V, $G \cap H$ denotes a graph on V with $E(G \cap H) = E(G) \cap E(H)$. Given a k-partition $\{V_1, V_2, \ldots, V_k\}$, a set S is called *crossing* if $|S \cap V_i| \leqslant 1$, $i \in [k]$.

When we choose constants $x, y > 0$, $x \ll y$ means that for any $y > 0$ there exists $x_0 > 0$ such that for any $x < x_0$ the subsequent statement holds. Hierarchies of other lengths are defined similarly. Furthermore, all constants in the hierarchy are positive and for a constant appearing in the form $1/s$, we always mean to choose s as an integer.

2. Proof of Theorem 2

In this section we prove Theorem 2, that is, $\mathrm{ex}_k(n, K_{r+1}(t)) \geqslant g(n,r,k,t)$ for $r < k \leqslant 2r$. Our proof needs a t-regular $K_{2,2}$-free bipartite graph with n vertices in each part. It is well known (see [20]) that such graph exists for infinitely many $n \in \mathbb{N}$ with $n \geqslant t^2$. The following proposition from [25, Section 2] allows n to be any integer that is at least $8t^2$.

Proposition 2.1. [25] *For $t \geqslant 1$ and $n \geqslant 8t^2$, there exists a t-regular $K_{2,2}$-free bipartite graph with n vertices in each part.*

Now we prove our lower bound on $\mathrm{ex}_k(n, K_{r+1}(t))$ stated in Theorem 2.

Proof of Theorem 2. First assume $r < k \leqslant 2r$ and $t \geqslant 2$. Let $V_1, \dots, V_k$ be disjoint sets of size n and, if $k < 2r$, let $V_{k+1}, \dots, V_{2r}$ be empty sets. Let G' be a complete r-partite graph with parts $V_1 \cup V_{r+1}, V_2 \cup V_{r+2}, \dots, V_r \cup V_{2r}$. Moreover, we add to G' a maximum $K_{t,t}$-free bipartite graph with bipartition $V_1 \cup V_{r+1}$, and a $(t-1)$-regular $K_{2,2}$-free bipartite graph on $V_i \cup V_{i+r}$ for $2 \leqslant i \leqslant k-r$ (the existence of such graph is guaranteed by Proposition 2.1). The resulting graph is $K_{r+1}(t)$-free because a copy of $K_{r+1}(t)$ has at most $2t-1$ vertices in $V_1 \cup V_{r+1}$ and at most t vertices in $V_i \cup V_{i+r}$ for $2 \leqslant i \leqslant k-r$. This graph has $t_r(k)n^2 + z_t(n) + (t-1)(k-r-1)n$ edges, and thus, $\mathrm{ex}_k(n, K_{r+1}(t)) \geqslant t_r(k)n^2 + z_t(n) + (t-1)(k-r-1)n$. This proves the theorem when $t = 2$ or $k = 2r$.

Now assume $r < k < 2r$ and $t \geqslant 2$. Let $b = k - r$. Our goal is to give a better construction that shows

$$\mathrm{ex}_k(n, K_{r+1}(t)) \geqslant t_r(k)n^2 + z_t(n,n) + (t-1)(b-1)n + \min\{b-1, r-b\} \left\lfloor \frac{(t-1)^2}{4} \right\rfloor.$$

Let $V_{i,j}$, $(i,j) \in [r] \times [2]$ be vertex sets, where $V_{b+1,2}, \dots, V_{r,2}$ are empty sets and other sets have size n. Let $G = K(V_{1,1} \cup V_{1,2}, \dots, V_{r,1} \cup V_{r,2})$ be the complete r-partite graph with parts $V_{1,1} \cup V_{1,2}, \dots, V_{r,1} \cup V_{r,2}$. Thus $e(G) = t_r(k)n^2$.

We first revise the partition as follows. Let $t' := \lceil (t-1)/2 \rceil$ and $b' := \min\{b-1, r-b\}$. Let $\{V'_{i,j}, (i,j) \in [r] \times [2]\}$ be obtained from $\bigcup V_{i,j}$ by moving a set $S_{i,1}$ of t' vertices from $V_{i,1}$ to $V_{i+b-1,1}$, and moving a set $S_{i,2}$ of t' vertices from $V_{i,2}$ to $V_{i+b-1,2}$, for every $i \in [2, b'+1]$. For $i \in [r]$, let $U_i := V'_{i,1} \cup V'_{i,2}$ and $H := K(U_1, \dots, U_r) \cap K(V_{1,1}, \dots, V_{r,1}, V_{1,2}, \dots, V_{r,2})$. Let H' be obtained from H by adding

- a $K_{t,t}$-free bipartite graph on U_1 of size $z_t(n,n)$, and
- a maximum $\{K_{1,t}, K_{2,2}\}$-free bipartite graph on U_i for $i \in [2, b]$.
- $2t'$ vertex-disjoint copies of $K_{1,t-1}$ on each U_i, $i \in [b+1, b+b']$, where the centers of stars are the $2t'$ vertices of $S_{i-b+1,1} \cup S_{i-b+1,2}$; then add a copy of $K_{t',t'}$ on $S_{i-b+1,1} \cup S_{i-b+1,2}$. See Figure 1.

For $i \in [2, b]$, Proposition 2.1 implies that each $H'[U_i]$ is $(t-1)$-regular and thus for $i \in [2, b'+1]$, $H'[U_i]$ has $(n-t')(t-1)$ edges, and for $i \in [b'+2, b]$ it has $n(t-1)$ edges; for $i \in [b+1, b+b']$, each $H'[U_i]$ has $2t'(t-1) + (t')^2$ edges. Therefore, the number of edges in $H' \backslash H$ is

$$\begin{aligned}
&z_t(n,n) + (n-t')(t-1)b' + n(t-1)(b-b'-1) + b'(2t'(t-1) + (t')^2) \\
=&z_t(n,n) + (t-1)(b-1)n + b'(t')^2 + b't'(t-1) \\
=&z_t(n,n) + (t-1)(b-1)n + 2b'(t')^2 + b't'(t-1-t') \\
=&z_t(n,n) + (t-1)(b-1)n + 2b'(t')^2 + \left\lfloor \frac{(t-1)^2}{4} \right\rfloor.
\end{aligned}$$

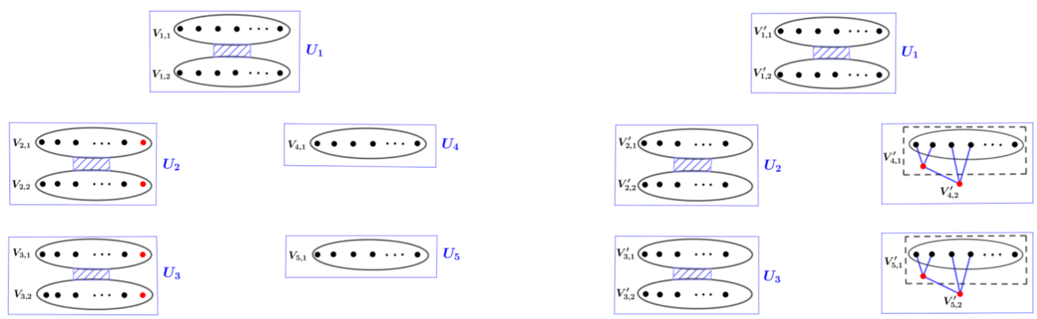

FIGURE 1. The lower bound construction for $K_{r+1}(t)$ with $r = 5$, $k = 8$ and $t = 3$. The left figure is the standard construction similar to the one given in Theorem 1; the right figure is the construction presented in our proof of Theorem 3.

We claim that H contains $t_r(k)n^2 - 2b'(t')^2$ edges. Indeed, for every $i \in [2, b'+1]$, the vertices of $S_{i,1}$ moved from $V_{i,1}$ to $V_{i+b-1,1}$ lose n edges to $V_{i+b-1,1}$ and gain $n - t'$ edges to $V_{i,2}$, thus having a net loss $(t')^2$ edges between U_i and U_{i+b-1}; the same holds for $S_{i,2}$. Thus, our claim holds after summing over all b' such rows, which implies that $e(H') = t_r(k)n^2 + z_t(n, n) + (t-1)(b-1)n + b'\lfloor\frac{(t-1)^2}{4}\rfloor$.

At last, we show that H' is $K_{r+1}(t)$-free. Recall that by construction, every U_i is triangle-free, U_1 is $K_{t,t}$-free, $U_2, \ldots, U_b$ are $\{K_{1,t}, K_{2,2}\}$-free, and each of $U_{b+1}, \ldots, U_{U_{b+b'}}$ induces vertex-disjoint copies of $K_{1,t-1}$ whose centers are joined by copies of $K_{t',t'}$. Suppose K is a copy of $K_{r+1}(t)$ in H'. By construction, K contains at most $2t-1$ vertices in U_1. We claim that $|V(K) \cap (U_i \cup U_{i+b-1})| \leqslant 2t$ for $i \in [2, b'+1]$ and $|V(K) \cap U_j| \leqslant t$ for $i \in [b'+2, b] \cup [b+b'+1, r]$, which will lead to a contradiction with $|V(K)| = (r+1)t$. Indeed, for $i \in [2, r]$, if K contains at least $t+1$ vertices in U_i, then these vertices induce either a copy of $K_{1,t}$ or $K_{2,2}$. By construction, this is only possible for $i \in [b+1, b+b']$ and in that case $V(K) \cap U_i$ must intersect both $S_{i-b+1,1}$ and $S_{i-b+1,2}$. Furthermore, there exists $v \in V(K) \cap S_{i-b+1,1}$ and $v' \in V(K) \cap S_{i-b+1,2}$ such that v and v' are in different color classes of K. Since v and v' have no common neighbor in $U_i \cup U_{i-b+1}$, K induces at most two classes on $U_i \cup U_{i-b+1}$. Thus, $|V(K) \cap (U_i \cup U_{i-b+1})| \leqslant 2t$ and we are done. $\square$

Remark. Note that for $t = 2$, both our constructions give the same value, which means that the extremal graph is not unique. Indeed, it is easy to see that one can construct $b'+1$ different ones – as we can move vertices for a subset of the b' rows.

3. STABILITY AND PROOF OUTLINE

Let us first consider extremal graph for $\mathrm{ex}_k(n, K_{r+1})$. Given $r, k \in \mathbb{N}$ with $k > r$, write $k = ar + b$ for $0 \leqslant b \leqslant r-1$. By Turán's theorem, the Turán graph $T_r(k) := K_{a,\ldots,a,a+1,\ldots,a+1}$ (with b parts of size $a+1$ and $r-b$ parts of size a) is the unique largest K_{r+1}-free graph on k vertices. The following definition shows that there are many extremal graphs for $\mathrm{ex}_k(n, K_{r+1})$.

Definition 3.1. Let $\mathcal{T}_{r,k}(n)$ be the collection of k-partite graphs with parts $V_1, \ldots, V_k$ of size n defined as follows. If $b > 0$, we arbitrarily divide $V_{ar+1}, \ldots, V_k$ into r sets $W_1, \ldots, W_r$ (some of them may be empty) such that each W_i is a subset of V_j for some j; if $b = 0$, then let $W_1, \ldots, W_r$ be empty sets. Now let T be the r-partite graph with parts $U_1, \ldots, U_r$ such that

$$U_i = W_i \cup Z_i, \text{ where } Z_i := V_{(i-1)a+1} \cup \cdots \cup V_{ia},$$

obtained from the complete r-partite graph $K(U_1, \ldots, U_r)$ by removing edges between W_i and $W_{i'}$, $i \neq i'$, whenever $W_i, W_{i'} \subseteq V_j$ for some j (in other words, $T = K(U_1, \ldots, U_r) \cap K(V_1, \ldots, V_k)$).

Since T is r-partite, it is K_{r+1}-free. Let $U = \bigcup_{i \in [r]} U_i$ and $W = \bigcup_{i \in [r]} W_i$. Note that $e_T(U) = \binom{r}{2} a^2 n^2$ while the number of edges of T incident to W is equal to $|W|(k-a-1)n = b(k-a-1)n^2$. Since $t_r(k) = \binom{r}{2} a^2 + b(k-a-1)$, it follows that $e(T) = t_r(k)n^2$. By (1.1), T is an extremal graph for K_{r+1}.[1]

3.1. **A stability theorem.** We need the following stability result for $\mathrm{ex}_k(n, K_{r+1}(t))$. Given two graphs $G, H \in \mathcal{G}_k(n)$ on the same parts $V_1, \ldots, V_k$, we say that G and H are γ-close if $|E(G) \triangle E(H)| \leqslant \gamma n^2$.

Theorem 5. *For any $k, r, t \in \mathbb{N}$ and any $\gamma > 0$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for every integer $n \geqslant n_0$. Suppose $G \in \mathcal{G}_k(n)$ is $K_{r+1}(t)$-free and $e(G) \geqslant (t_r(k) - \varepsilon)n^2$. Then G is γ-close to a member of $\mathcal{T}_{r,k}(n)$. In particular, we have $e(G) \leqslant (t_r(k) + \gamma)n^2$.*

In the earlier version of this paper, we gave a self-contained proof of Theorems 5. Here we derive it from a stronger result of Chen, Lu and Yuan [5, Theorem 1.5], in which they provide more structural information. Their definition and result are more general by allowing the parts V_i of G to have different sizes. Here we only state their definition and result that we need for the balanced case.

Definition 3.2 (Stable partition)**.** Let $k \geqslant r \geqslant 2$ be integers. Let $\mathcal{P} = \{P_1, \ldots, P_r\}$ and $\mathcal{V} = \{V_1, \ldots, V_k\}$ where $|V_i| = n$ for all $i \in [k]$ be partitions of a common vertex set. For $i \in [r], j \in [k]$, a set $W = P_i \cap V_j$ is called an integral part if $W = V_j$, and called a partial part otherwise. We say that $\mathcal{P}$ is stable to $\mathcal{V}$, if each of $P_1, \ldots, P_r$ has the same number of integral parts and at most one partial part.

Definition 3.3 ((X, ε)-stable)**.** Let n, r, k be integers with $n \geqslant k \geqslant r \geqslant 2$. For a given spanning subgraph G of $K_k(n)$, let $\mathcal{P} = \{P_1, \ldots, P_r\}$ be a partition of $V(G)$ and $\mathcal{V} = \{V_1, \ldots, V_k\}$ be the natural k-partition. Given $\varepsilon \in (0, 1)$ and a set $X \subseteq V(G)$ of size at most εn, we say that $\mathcal{P}$ is an (X, ε)-stable partition if

- $G - X$ is ε-close to $K(P_1, \ldots, P_r) - X$,
- $\{P_1 \backslash X, \ldots, P_r \backslash X\}$ is stable to $\{V_1 \backslash X, \ldots, V_k \backslash X\}$.

Theorem 6. [5, Theorem 1.5] *Let F be a graph with chromatic number $r + 1 \geqslant 3$. For every $\varepsilon > 0$ there exists $\delta > 0$ and integer $n_0 > 0$ such that the following holds for $n \geqslant n_0$. Let G be an F-free subgraph of $K_k(n)$ with $k > r$ such that $e(G) \geqslant \mathrm{ex}_k(n, F) - \delta n^2$. Then, G has an (X, ε)-stable partition $\{P_1, \ldots, P_{t-1}\}$ for some set $X \subseteq V(G)$ of size at most εn.*

Now Theorem 5 follows from Theorem 6 by setting $F = K_{r+1}(t)$ and noticing that i) a partition is stable if and only if it is a partition required in Definition 3.1 and ii) as $|X| \leqslant \varepsilon n$, X is incident to at most εn^2 edges that we have no control on.

We remark that although Theorem 6 is stronger, the additional structural information is not immediately useful to us. For example, the set X introduced in Theorem 6 is the set of atypical vertices of G (see also the proof outline in next section). However, we need a stronger control and indeed we need to distinguish two kinds of atypical vertices. Identifying them from X is almost equivalent to identifying them from $V(G)$. So we choose to present and use the easier and more classical version, Theorem 5, in this paper.

[1]Indeed, Chen, Lu, and Yuan [5] showed that $\mathcal{T}_{r,k}(n)$ are *all* the extremal graphs for $\mathrm{ex}_k(n, K_{r+1})$ and described all extremal graphs for $\mathrm{ex}(n_1, \ldots, n_k; K_{r+1})$.

3.2. **Outline of the proofs.** Now we give an outline of our proofs. Let $G \in \mathcal{G}_k(n)$ be $K_{r+1}(t)$-free and has the maximum number of edges. Since $e(G) > t_r(k)n^2$, we can assume that G is γ-close to some $T \in \mathcal{T}_{r,k}(n)$. Since $e(G)$ is maximum, we can easily derive a minimum degree condition by symmetrization arguments.

Next we define *atypical* vertices. Roughly speaking, there are two types of *atypical* vertices: the first type of vertices, denoted by $Z'' \cup W''$, are the "wrong" ones that do not exist in $\mathcal{T}_{r,k}(n)$; the second type of vertices, denoted by $(W_i' \backslash W_i) \cup \bigcup_{j\neq i} Z_j^i$ for $i \in [r]$, are the vertices that are not in U_i but *behave* like the vertices of U_i, in other words, they are in the wrong place. We temporarily ignore the first type of atypical vertices because there are only a constant number of them (see $(P1)$ and $(P3)$) and they contribute only $O(n)$ to $e(G)$. For the second type of atypical vertices, there are only $o(n)$ of them (see $(P1)$ and $(P3)$) and we move them to appropriate rows and redefine our partition as $\tilde{U}_1, \ldots, \tilde{U}_r$ (see (4.2)). A key observation is that $Z_j^i \neq \varnothing$ (namely, there is a vertex in Z_j but behaves as a vertex in Z_i) only if $|W_j| \geqslant (1 - o(1))n$.

Now we estimate $e(G)$. We split $E(G)$ into $E_G(\tilde{U}_1), \ldots, E_G(\tilde{U}_r)$, and $E(G')$, where $G' := G \cap K(\tilde{U}_1, \ldots, \tilde{U}_r)$. We have a relatively good estimate of $e(G')$ (see Claim 4.3) taking into account that the partition is no longer balanced. In contrast, due to the second type of atypical vertices, we can only show that each $G[\tilde{U}_i]$ is "almost" $K_{t,t}$-free (see Claim 4.4). Similarly, we show that all but at most one rows are "almost" $K_{1,t}$-free (see Claim 4.6). Assuming that $e_G(\tilde{U}_1)$ is the largest among all $e_G(\tilde{U}_i)$, $i \in [r]$, we can use these properties and give an upper bound of $e(G)$. Next we show that $\tilde{U}_1$ has no atypical vertices (Claim 4.8), and thus $e_G(\tilde{U}_1) \leqslant z_t^{(a+1)}(n)$. We further refine our estimate on $\tilde{U}_i$, $i > 1$, and show that each second type atypical vertex contributes at most a constant number of edges to $E(G)\backslash E(G')$ (Claim 4.9). In summary, $\tilde{U}_1$ is indeed $K_{t,t}$-free, $e_G(\tilde{U}_i) = O(n)$ for $i > 1$, and $|Z'' \cup W''| = O(1)$, from which we conclude the proof of Theorem 4.

To prove Theorem 3, we refine earlier estimates as follows. We first show that $|W_1'| = (1 - o(1))n$ and $Z'' \cup W'' = \varnothing$, where we use $(E2)$. The rest of the proofs are further refinements of our estimates. In particular, we show that if $|W_i'|$ is not too small, then $\tilde{U}_i$ essentially contains no vertex from other rows.

3.3. **Two quick proofs.** We first derive $(E1)$ from (Z).

Proof of $(Z) \Rightarrow (E1)$. Given $a + 1$ sets $V_1, \ldots, V_{a+1}$ of size n, we define an $(a + 1)$-partite graph G on $V_1, \ldots, V_{a+1}$ as follows. Let V_2' be a set of n vertices consisting of $\lfloor n/2 \rfloor$ vertices from V_1 and $\lceil n/2 \rceil$ vertices from V_2. We place an extremal graph G' for $z_t^{(a)}(n)$ on $V_2', V_3, \ldots, V_{a+1}$, in other words, G' is an a-partite $K_{t,t}$-free graph with $z_t^{(a)}(n)$ edges. Next we add a maximum bipartite $K_{t,t}$-free graph G'' on the remaining vertices of V_1 and V_2. By (Z), $e(G'') \geqslant z_t^{(2)}(\lfloor n/2 \rfloor) \geqslant \delta n^{2-1/t}$ for some $\delta > 0$. Thus $G = G' \cup G''$ is $K_{t,t}$-free and $e(G) = e(G') + e(G'') \geqslant z_t^{(a)}(n) + \delta n^{2-1/t}$. This gives $(E1)$. ☐

We need the following simple proposition.

Proposition 3.4. *Given $r, t \in \mathbb{N}$ and reals $\gamma, \varepsilon > 0$ such that $\varepsilon^2 > 3r^2t^2\gamma$, and let n be sufficiently large. Suppose G is a $K_{r+1}(t)$-free graph with vertex partition $V = U_1 \cup \cdots \cup U_r$ such that $|U_i| \geqslant n$ for $i \in [r]$ and $d(U_i, U_j) \geqslant 1 - \gamma$, $i, j \in [r]$, $i \neq j$. Let $X \subseteq V$ be the set of vertices v satisfies that $d(v, U_i) \geqslant \varepsilon|U_i|$ for all $i \in [r]$. Then $|X| \leqslant 2(t - 1)\varepsilon^{-rt}$.*

Proof. We call a copy of $K_r(t)$ in G *useful* if it consists of exactly t vertices from each of $U_1, \ldots, U_r$. We first show that for every $v \in X$, $N(v)$ contains many useful copies of $K_r(t)$. Indeed, $d(U_i, U_j) \geqslant 1 - \gamma$ for every $i, j \in [r]$, $i \neq j$ implies that $G[U_i, U_j]$ has at most $\gamma|U_i||U_j|$ non-edges. Since

$d(v, U_i) \geqslant \varepsilon|U_i|$ for all $i \in [r]$, take $W_i \subseteq N(v) \cap U_i$ of size exactly $\varepsilon|U_i|$. We can find $\prod_{i\in[r]} \binom{\varepsilon|U_i|}{t}$ rt-sets which consists of t vertices from each U_i, amongst which, at most

$$\sum_{i,j\in[r],i\neq j} \gamma|U_i||U_j| \cdot \left(\prod_{i'\in[r]} \binom{\varepsilon|U_{i'}|}{t}\right) \cdot \frac{t}{\varepsilon|U_i|}\frac{t}{\varepsilon|U_j|} \leqslant \frac{r^2t^2\gamma}{\varepsilon^2} \prod_{i'\in[r]} \binom{\varepsilon|U_{i'}|}{t}$$

of them contain crossing non-edges. Therefore, $N(v)$ contains at least

$$\left(1 - \frac{r^2t^2\gamma}{\varepsilon^2}\right) \prod_{i'\in[r]} \binom{\varepsilon|U_{i'}|}{t} \geqslant \frac{\varepsilon^{rt}}{2} \prod_{i'\in[r]} \binom{|U_{i'}|}{t}$$

useful copies of $K_r(t)$, where we used that $r^2t^2\gamma\varepsilon^{-2} < 1/3$. Since G is $K_{r+1}(t)$-free, each useful copy K of $K_r(t)$ is in $N(v)$ for at most $t-1$ choices of $v \in X$. Double counting on the number of pairs (v, K) such that $K \subseteq N(v)$ is useful, we obtain that

$$|X|\frac{\varepsilon^{rt}}{2} \prod_{i'\in[r]} \binom{|U_{i'}|}{t} \leqslant (t-1) \prod_{i'\in[r]} \binom{|U_{i'}|}{t},$$

which gives $|X| \leqslant 2(t-1)\varepsilon^{-rt}$. □

4. Main Proofs

Given integers $1 \leqslant s \leqslant t$ and sufficiently large m, n, Kővári, Sós, Turán [16] showed that $z(m, n, s, t) \leqslant Cmn^{1-1/s}$ for some $C = C(t) > 0$, that is, a bipartite graph G with parts of size m and n has at most $Cmn^{1-1/s}$ edges if G has no copy of $K_{s,t}$ where the part of size s is in the part of G of size m. We start the proof with the general setting, that is, for $k = ar + b$ with $0 \leqslant b < r$. After we conclude the proof of Theorem 4, we focus on the case $k \leqslant 2r$.

Proofs of Theorems 3 and 4. Suppose $t = 2, 3$ and thus *(Z)* holds, that is, $z_t^{(2)}(n) \geqslant cn^{2-1/t}$ for some $c > 0$. Take $C = C(t)$ as in the Kővári–Sós–Turán result in the previous paragraph. We choose constants

$$1/n \ll \gamma \ll \varepsilon \ll \varepsilon' \ll 1/k, 1/t, c, C.$$

Suppose G is $K_{r+1}(t)$-free and has the maximum number of edges, that is, $e(G) = \mathrm{ex}_k(n, K_{r+1}(t))$. Suppose further that $e(G) > g(n, r, k, t) > t_r(k)n^2$. By Theorem 5, G is γ-close to some $T \in \mathcal{T}_{r,k}(n)$ such that $T = K(U_1, \dots, U_r) \cap K(V_1, \dots, V_k)$, where $U_1, \dots, U_r$ is a partition of $V(T)$ satisfying the following properties:

- $U_i = W_i \cup Z_i$ such that $Z_i = V_{(i-1)a+1} \cup \cdots \cup V_{ia}$, and
- $W_i = \varnothing$ if $b = 0$ and W_i is a subset of V_{q_i} for some q_i with $ar < q_i \leqslant k$ otherwise.

For simplicity, we write $z_t(n) = z_t^{(\lceil k/r\rceil)}(n)$.

The fact that G is γ-close to T gives the following observation.

(D0) for any $i \in [r]$, there exists $B_i \subseteq U_i$ of size at most $2\sqrt{\gamma}n$ such that for any $v \in U_i \backslash B_i$ and $A \subseteq \bigcup_{j\in[r]\backslash\{i\}} U_j$ satisfying that none of the vertices of A is in the same cluster as v is, we have $\overline{d}(v, A) \leqslant \sqrt{\gamma}n$.

To see it, fix $i \in [r]$ and write $U^* := \bigcup_{j\neq i} U_j$. Since G is γ-close to T, we have

$$e_G(Z_i, U^*) \geqslant |Z_i||U^*| - \gamma n^2, \quad \text{and} \quad e_G(W_i, U^*\backslash V_{q_i}) \geqslant |W_i||U^*\backslash V_{q_i}| - \gamma n^2.$$

Let $B_i' \subseteq Z_i$ be the set of vertices v such that $\overline{d}(v, U^*) > \sqrt{\gamma}n$, and $B_i'' \subseteq W_i$ be the set of vertices w such that $\overline{d}(w, U^* \backslash V_{q_i}) > \sqrt{\gamma}n$. The displayed line above implies that $|B_i'| \leqslant \sqrt{\gamma}n$ and $|B_i''| \leqslant \sqrt{\gamma}n$. Now (D0) holds by setting $B_i = B_i' \cup B_i''$.

Minimum degree. For $i \in [k]$, let $N_i := N_T(u_i)$ for some $u_i \in V_i$. Note that this is well-defined as the vertices of V_i share the same neighborhood in T. Using the maximality of $e(G)$, we derive that for every $u \in V_i$, $i \in [k]$

$$d_G(u) \geqslant d_T(u) - 2t\gamma n.$$

Indeed, since G is γ-close to T, that is, $|E(G) \triangle E(T)| \leqslant \gamma n^2$, for each $i \in [k]$, we can greedily pick distinct $u_1, \ldots, u_t \in V_i$, such that $|N_G(u_j) \triangle N_i| \leqslant \gamma n^2/(n-j) \leqslant 2\gamma n$, for $j \in [t]$. Let $N_i' := \bigcap_{j \in [t]} N_G(u_j)$ and note that $|N_i' \triangle N_i| \leqslant 2t\gamma n$. In particular, $|N_i'| \geqslant |N_i| - 2t\gamma n$. Now for a contradiction suppose there is $u \in V_i$ such that $d_G(u) < d_T(u) - 2t\gamma n = |N_i| - 2t\gamma n$. Then we replace $N_G(u)$ by N_i', that is, we disconnect all the edges of u in G and connect u to the vertices of N_i'. Thus, we obtain a k-partite graph on the same vertex set as G and has more edges than G. Therefore, by the maximality of G, this new graph contains a copy of $K_{r+1}(t)$, denoted by K. Clearly, K must contain the vertex u, as G is $K_{r+1}(t)$-free. Moreover, K must miss at least one vertex from $u_1, \ldots, u_t$, say u_j, because the set $\{u, u_1, \ldots, u_t\}$ is independent in G and K has independence number t. However, as the neighborhood of u N_i' is a subset of $N_G(u_j)$, we can replace u by u_j and still get a copy of $K_{r+1}(t)$, which is in G, a contradiction.

Therefore, comparing with the degrees in T, we derive that for any vertex u,

$$d_G(u) \geqslant \begin{cases} (k-a)n - |W_i| - 2t\gamma n, & \text{if } u \in Z_i \text{ for } i \in [r], \\ (k-1-a)n - 2t\gamma n, & \text{if } u \in W. \end{cases} \tag{4.1}$$

Atypical vertices. In this step we identify a set of atypical vertices, that is, those behave differently from the majority of the vertices. Let $W := \bigcup_{i \in [r]} W_i = V_{ar+1} \cup \cdots \cup V_k$. We define $W'' := \{v \in W : d(v, Z_j) \geqslant \varepsilon n, \text{ for all } j \in [r]\}$ and $W_i' := \{v \in W : d(v, Z_i) < \varepsilon n\}$. Then we have $W = W'' \cup W_1' \cup \cdots \cup W_r'$. Next, for $i \in [r]$, let $Z'' := \bigcup_{i \in [r]} Z_i''$, where

$$Z_i'' := \{v \in Z_i : d(v, Z_j) \geqslant \varepsilon n, \text{ for all } j \in [r] \backslash \{i\} \quad \text{and} \quad d(v, U_i) \geqslant \varepsilon n\}.$$

Furthermore, let $Z_i' := Z_i \backslash Z_i''$ and write Z_i' as $\bigcup_{j \in [r]} Z_i^j$, where Z_i^j, $j \neq i$, consists of the vertices $v \in Z_i$ such that $d(v, Z_j) < \varepsilon n$, and Z_i^i consists of the vertices v such that $d(v, U_i) < \varepsilon n$. The following are some useful properties of these sets.

Claim 4.1. *The following properties hold for all $i \in [r]$.*

- $(P1)$ $|W_i' \backslash W_i| \leqslant 2\gamma n$ *and* $|W''| \leqslant C_0 := 2t\varepsilon^{-rt}$.
- $(P2)$ $W = W'' \cup W_1' \cup \cdots \cup W_r'$ *is a partition of* W.
- $(P3)$ $|Z_i''| \leqslant C_0$, $|Z_i^j| \leqslant \sqrt{\gamma}n$ *for* $j \neq i$, *and* $|Z_i^i| \geqslant (1 - \sqrt{\gamma})an$.
- $(P4)$ $\bigcup_{j \in [r]} Z_i^j$ *is a partition of* Z_i'.

Proof. Recall the definition of W'' and that $d(Z_i, Z_j) \geqslant 1 - \gamma$ for distinct $i, j \in [r]$. Applying Proposition 3.4 to the graph $G[W'' \cup Z]$ with vertex partition $(U_1, \ldots, U_r)$, we obtain that $|W''| \leqslant C_0 := 2t\varepsilon^{-rt}$. We next show that $|W_i' \backslash W_i| \leqslant 2\gamma n$ for each $i \in [r]$. Indeed, because G is γ-close to T, we have $e_G(Z_i, W_i' \backslash W_i) \geqslant an|W_i' \backslash W_i| - \gamma n^2$. On the other hand, by definition, $e_G(Z_i, W_i' \backslash W_i) < |W_i' \backslash W_i| \cdot \varepsilon n$. Thus, we get $|W_i' \backslash W_i| < \gamma n/(a - \varepsilon) < 2\gamma n$, verifying $(P1)$.

To see $(P2)$, suppose there is a vertex $v \in W_i' \cap W_j'$. By definition, $d(v) \leqslant (k-1)n - 2(a-\varepsilon)n < (k-1-a)n - \sqrt{\gamma}n$, contradicting (4.1).

Next we show (*P*3). Fix $i \in [r]$. Since G is γ-close to T, we have $d(Z_j, Z_{j'}) \geqslant 1-\gamma$ and $d(U_i, Z_j) \geqslant 1-\gamma$ for distinct $j, j' \in [r]\backslash\{i\}$. Thus, we can apply Proposition 3.4 on $G[U_i \cup \bigcup_{j\neq i} Z_j]$ (with the obvious r-partition) and obtain $|Z_i''| \leqslant C_0$. Moreover, for $i \neq j$, from $d(Z_i, Z_j) \geqslant 1-\gamma$ we infer $|Z_i^j| \leqslant (\gamma/\varepsilon)n \leqslant \sqrt{\gamma}n$, as $\gamma \ll \varepsilon$. Therefore, we also get $|Z_i^i| \geqslant |Z_i| - |Z_i''| - \sum_{j\neq i}|Z_i^j| \geqslant an - C_0 - (r-1)\gamma n/\varepsilon \geqslant (1-\sqrt{\gamma})an$.

Now we show (*P*4). By definition, if $v \in Z_i^i$, then $d(v, U_i) < \varepsilon n$; if $v \in Z_i^j$ for $j \neq i$, then $d(v, Z_j) < \varepsilon n$. Thus, we have $Z_i' \subseteq \bigcup_{j\in[r]} Z_i^j$ by definition. A vertex $v \in Z_i^i \cap Z_i^j$, $j \neq i$, satisfies that $d(v) < kn - (|U_i| - \varepsilon n) - (a-\varepsilon)n \leqslant (k-a)n - |W_i| - (1-2\varepsilon)n$, contradicting (4.1). A vertex $v \in Z_i^j \cap Z_i^{j'}$ for distinct $j, j' \in [r]\backslash\{i\}$ satisfies that $d(v) < (k-1)n - 2(a-\varepsilon)n \leqslant (k-a-2)n + 2\varepsilon n$, contradicting (4.1) as well. Thus, $\bigcup_{j\in[r]} Z_i^j$ is a partition of Z_i'. □

For $i \in [r]$, our refined partition is defined by

$$\tilde{U}_i := \tilde{Z}_i \cup W_i', \text{ where } \tilde{Z}_i := \bigcup_{j\in[r]} Z_j^i. \tag{4.2}$$

Then $V(G) = Z'' \cup W'' \cup \bigcup_{i\in[r]} \tilde{U}_i$. Note that for any $v \in \tilde{U}_i$, we have $d(v, Z_i^i) \leqslant d(v, Z_i) \leqslant \varepsilon n$, and thus $d(v, \tilde{Z}_i) \leqslant \varepsilon n + (r-1)\sqrt{\gamma}n$ by (*P*3).

For every $i \in [r]$, note that (*P*1) implies that $|W_i \backslash W_i'| \leqslant C_0 + (r-1)2\gamma n \leqslant 2r\gamma n$, and similarly (*P*3) implies that $|Z_i \backslash \tilde{Z}_i| \leqslant C_0 + (r-1)\sqrt{\gamma}n \leqslant r\sqrt{\gamma}n$.

We now derive a more handy minimum degree condition. For convenience, define $\overline{d}(v, A) = |A| - d(v, A)$. For $v \in Z_i^i$, we have $\overline{d}(v, \tilde{U}_i) \geqslant \overline{d}(v, U_i) - |U_i \backslash \tilde{U}_i|$. Since $\overline{d}(v, U_i) > an + |W_i| - \varepsilon n$ and $|U_i \backslash \tilde{U}_i| \leqslant |Z_i \backslash \tilde{Z}_i| + |W_i \backslash W_i'| \leqslant \varepsilon n/2$, we have $\overline{d}(v, \tilde{U}_i) \geqslant an + |W_i| - \varepsilon n - \varepsilon n/2$. By (4.1), $\overline{d}(v) \leqslant an + |W_i| + \sqrt{\gamma}n$. It follows that $\overline{d}(v, V\backslash\tilde{U}_i) \leqslant 2\varepsilon n$. Now consider $v \in \tilde{U}_i \backslash Z_i^i$. The definition of $\tilde{U}_i$ implies that $d(v, Z_i) < \varepsilon n$ and $\overline{d}(v, Z_i) > an - \varepsilon n$. Assume $v \in V_j$. Then $V_j \cap Z_i = \varnothing$ and trivially $\overline{d}(v, V_j) = n$. It follows that $\overline{d}(v, Z_i \cup V_j) > (a+1)n - \varepsilon n$. Hence $\overline{d}(v, \tilde{Z}_i \cup V_j) \geqslant \overline{d}(v, Z_i \cup V_j) - |Z_i \backslash \tilde{Z}_i| > (a+1)n - \frac{3}{2}\varepsilon n$. On the other hand, either case of (4.1) implies that $\overline{d}(v) \leqslant (a+1)n + \sqrt{\gamma}n$. Consequently, $\overline{d}(v, V\backslash(\tilde{Z}_i \cup V_j)) \leqslant 2\varepsilon n$. In summary, for $i \in [r]$ and $j \in [k]$,

(Deg) If $v \in Z_i^i$, then $\overline{d}(v, V\backslash\tilde{U}_i) \leqslant 2\varepsilon n$; if $v \in (\tilde{U}_i \backslash Z_i^i) \cap V_j$, then $\overline{d}(v, V\backslash(\tilde{Z}_i \cup V_j)) \leqslant 2\varepsilon n$.

Next we prove further properties on Z_i^j and $\tilde{Z}_j$.

Claim 4.2. *If $Z_i^j \neq \varnothing$ for some $i \neq j$, then the following holds.*

- (*Q*1) *For $v \in Z_i^j$ and $A \subseteq V(G)\backslash(Z_i \cup Z_j)$, we have $d(v, A) \geqslant |A| - \varepsilon n - \sqrt{\gamma}n$.*
- (*Q*2) *$|W_i| \geqslant (1-\varepsilon-\sqrt{\gamma})n$.*
- (*Q*3) *If $|\tilde{Z}_j \backslash Z_j| \geqslant t$, then $|W_j| \leqslant 2t\varepsilon n$.*

Proof. Note that $d(v, Z_j) \leqslant \varepsilon n$ and $d(v, Z_i) \leqslant (a-1)n$, that is, v has at least $n + (an - \varepsilon n) = (a+1)n - \varepsilon n$ non-neighbors in $Z_i \cup Z_j$. On the other hand, (4.1) says that v has at most $an + |W_i| + \sqrt{\gamma}n$ non-neighbors in G. Combining these two we get that v has at most $|W_i| - n + \varepsilon n + \sqrt{\gamma}n \leqslant \varepsilon n + \sqrt{\gamma}n$ non-neighbors outside $Z_i \cup Z_j$, and thus (*Q*1) holds. The fact that $|W_i| - n + \varepsilon n + \sqrt{\gamma}n \geqslant 0$ implies (*Q*2).

For (*Q*3), suppose to the contrary, $|\tilde{Z}_j \backslash Z_j| \geqslant t$ and $|W_j| > 2t\varepsilon n$. By (*Q*1) with $A = W_j$, arbitrary t vertices in $\tilde{Z}_j \backslash Z_j$ have at least $|W_j| - t(\varepsilon + \sqrt{\gamma})n \geqslant t$ common neighbors in W_j. We thus obtain a copy of $K_{t,t}$ with one part in $\tilde{Z}_j \backslash Z_j$ and the other part in W_j – denote its vertex set by B. For any $i' \in [r]\backslash\{j\}$ such that $B \cap Z_{i'}^j \neq \varnothing$, we have $|W_{i'}| \geqslant (1-\varepsilon-\sqrt{\gamma})n$ by (*Q*2). Since $|W_j| > 2t\varepsilon n$, $W_{i'}$ and

W_j do not belong to the same cluster, and thus no vertex of B is in the same cluster that contains $W_{i'}$, which implies that the vertices of B have at least $|W_{i'}| - 2t(2\varepsilon n) \geqslant n/2$ common neighbors in $W_{i'}$ by (Deg). For any $i'' \in [r]\backslash\{j\}$ such that $B \cap Z^j_{i''} = \varnothing$ (and thus $B \cap Z_{i''} = \varnothing$), by (Deg) we have that the vertices of B have at least $n/2$ common neighbors in $Z_{i''}$. Because G is γ-close to T, these common neighborhoods, each of size at least $n/2$, have densities close to one between each pair, and thus contain a copy of $K_{r-1}(t)$. Together with B, they form a copy of $K_{r+1}(t)$ in G, a contradiction. □

In particular, when $b = 0$ (and thus $W_i = \varnothing$ for all i), (*Q*2) implies that $Z^j_i = \varnothing$ whenever $i \neq j$. Consequently,

$$\tilde{U}_i = Z^i_i = Z_i \backslash Z'' \quad \text{for all } i \in [r] \text{ when } b = 0. \tag{4.3}$$

Let $L \subseteq [r]$ be the set of indices i such that $|W_i| \geqslant (1 - \varepsilon - \sqrt{\gamma})n$. (*Q*2) and (*Q*3) imply that

- for $i \in [r]\backslash L$, we have $Z^j_i = \varnothing$ for $j \neq i$.
- for $i \in L$, $|\tilde{Z}_i \backslash Z_i| \leqslant t - 1$ and thus $|\tilde{Z}_i| \leqslant an + t - 1$.

First Estimate on $e(G)$. Let $G' = G \cap K(\tilde{U}_1, \dots, \tilde{U}_r)$. We have $e(G) = e(G') + \sum_{i=1}^r e_G(\tilde{U}_i) + e(Z'' \cup W'', G)$. Since G' is r-partite, it is K_{r+1}-free. As G' is a subgraph of $G \in \mathcal{G}_k(n)$, we have $e(G') \leqslant t_r(k)n^2$ (but this is not good enough when $b > 0$). Below we give an upper bound for $e(G')$, which will be used throughout the proof. Recall that $T = K(V_1, \dots, V_k) \cap K(U_1, \dots, U_r)$ has precisely $t_r(k)n^2$ edges.

Claim 4.3. *We have $e(G') \leqslant t_r(k)n^2 + \sum_{i \in [r]}(\beta_i - \alpha_i)$, where*

$$\beta_i := \sum_{j \in L\backslash\{i\}} |Z^i_j| \left(|\tilde{Z}_j \backslash Z_j| + |W'_j| - n + |Z_i \backslash \tilde{Z}_i|\right) \text{ and}$$
$$\alpha_i := |\tilde{Z}_i \backslash Z_i||W'_i| + e_T(W'_i) + e_T(\tilde{Z}_i \backslash Z_i).$$

Proof. We first obtain $G^{(0)} := K(Z_1 \cup W'_1, \dots, Z_r \cup W'_r) \cap K(V_1, \dots, V_k)$ from T. During this process, we lose the edges of T between W_i and W_j, $j \neq i$, if both ends of the edges are placed in W'_i. Thus

$$e(G^{(0)}) = t_r(k)n^2 - \sum_{i \in [r]} e_T(W'_i). \tag{4.4}$$

We imagine a dynamic process of obtaining G' from $G^{(0)}$ by recursively moving vertices. To estimate $e(G')$, we track the changes of the edges with respect to complete r-partite graphs (but also respecting the k-partition of G). More precisely, for $l > 0$, let

$$G^{(l)} := K(Z^{(l)}_1 \cup W'_1, \dots, Z^{(l)}_r \cup W'_r) \cap K(V_1, \dots, V_k)$$

such that the r-partition of $G^{(l)}$ can be obtained by moving exactly one vertex from the partition of $G^{(l-1)}$. The process terminates after $m := \sum_{i \in [r]} |\tilde{Z}_i \backslash Z_i|$ steps and thus G' is a subgraph of $G^{(m)}$. Furthermore, throughout the process, we only move vertices from the color classes in L to other color classes. Therefore, we can give a linear ordering to the members of L, and for $i \in L$ we move vertices from Z_i only after we have moved the vertices in color classes j prior to i (denoted by $j <_L i$). Now, in the l-th step, suppose we move v from $Z^{(l-1)}_j$ to $Z^{(l-1)}_i$, namely, $v \in Z^i_j$, then the change is

$$e(G^{(l)}) - e(G^{(l-1)}) = |Z^{(l-1)}_j \backslash V_p| + |W'_j| - |\tilde{Z}^{(l-1)}_i| - |W'_i|,$$

where $V_p \ni v$ and $\tilde{Z}^{(l-1)}_i = Z^{(l-1)}_i \backslash V_p$.

Note that we have $|Z_j^{(l-1)}\backslash V_p| \leqslant (a-1)n+|\tilde{Z}_j\backslash Z_j|$. Moreover for any $j' <_L j$, we have $Z_{j'}^i \subseteq Z_i^{(l-1)}$. Therefore, we have $|\tilde{Z}_i^{(l-1)}| \geqslant an - |Z_i\backslash\tilde{Z}_i| + \sum_{j'<_L j}|Z_{j'}^i|$. Putting all these together, we get

$$e(G^{(l)}) - e(G^{(l-1)}) \leqslant |\tilde{Z}_j\backslash Z_j| + |W_j'| - n + |Z_i\backslash\tilde{Z}_i| - \sum_{j'<_L j}|Z_{j'}^i| - |W_i'|.$$

Recalling that we moved v from $Z_j^{(l-1)}$ to $Z_i^{(l-1)}$ at the l-th step, we obtain

$$e(G') - e(G^{(0)}) \leqslant \sum_{l=1}^{m}\left(|\tilde{Z}_j\backslash Z_j| + |W_j'| - n + |Z_i\backslash\tilde{Z}_i| - \sum_{j'<_L j}|Z_{j'}^i| - |W_i'|\right),$$

where i, j depends on l. Since $m = \sum_{i\in[r]}|\tilde{Z}_i\backslash Z_i|$, we have

$$\begin{aligned}
&\sum_{l=1}^{m}(|\tilde{Z}_j\backslash Z_j| + |W_j'| - n + |Z_i\backslash\tilde{Z}_i| - |W_i'|)\\
&= \sum_{i\in[r]}\sum_{j\in L\backslash\{i\}}|Z_j^i|(|\tilde{Z}_j\backslash Z_j| + |W_j'| - n + |Z_i\backslash\tilde{Z}_i| - |W_i'|)\\
&= \sum_{i\in[r]}\sum_{j\in L\backslash\{i\}}|Z_j^i|(|\tilde{Z}_j\backslash Z_j| + |W_j'| - n + |Z_i\backslash\tilde{Z}_i|) - \sum_{i\in[r]}|\tilde{Z}_i\backslash Z_i||W_i'|.
\end{aligned}$$

Moreover, it is not hard to see that

$$\sum_{l=1}^{m}\sum_{j'<_L j}|Z_{j'}^i| = \sum_{i\in[r]}\sum_{\{j_1,j_2\}\in\binom{L\backslash\{i\}}{2}}|Z_{j_1}^i||Z_{j_2}^i| = \sum_{i\in[r]}e_T(\tilde{Z}_i\backslash Z_i).$$

Now the claim follows by combining these estimates with (4.4). ☐

What remains is to estimate the number of edges in each $\tilde{U}_i$. For $i\in[r]$, we have $e(G[\tilde{U}_i]) = e(Z_i^i, G[\tilde{U}_i]) + e_G(\tilde{U}_i\backslash Z_i^i)$. To bound $e_G(\tilde{U}_i\backslash Z_i^i) = e_G((\tilde{Z}_i\backslash Z_i)\cup W_i')$, we note that $e_G(\tilde{Z}_i\backslash Z_i, W_i') \leqslant |\tilde{Z}_i\backslash Z_i||W_i'|$ and $e_G(W_i') \leqslant e_T(W_i')$. However, we may not have $e_G(\tilde{Z}_i\backslash Z_i) \leqslant e_T(\tilde{Z}_i\backslash Z_i)$ because each Z_j^i is an independent set in T, but may not be independent in G when $a \geqslant 2$. Thus, $e_G(\tilde{Z}_i\backslash Z_i) \leqslant e_T(\tilde{Z}_i\backslash Z_i) + \sum_{j\neq i}e_G(Z_j^i)$. Putting these together, for each $i\in[r]$, we have

$$e_G(\tilde{U}_i\backslash Z_i^i) = e_G(\tilde{Z}_i\backslash Z_i, W_i') + e_G(W_i') + e_G(\tilde{Z}_i\backslash Z_i) \leqslant \alpha_i + \sum_{j\neq i}e_G(Z_j^i). \tag{4.5}$$

Let $f_i := e(Z_i^i, G[\tilde{U}_i])$. By Claim 4.3, (4.5) and $e(G) = e(G') + e(Z''\cup W'', G) + \sum_{i\in[r]}e_G(\tilde{U}_i)$, we derive that

$$e(G) \leqslant t_r(k)n^2 + e(Z''\cup W'', G) + \sum_{i\in[r]}\left(f_i + \beta_i - \alpha_i + e_G(\tilde{U}_i\backslash Z_i^i)\right) \tag{4.6}$$

$$\leqslant t_r(k)n^2 + e(Z''\cup W'', G) + \sum_{i\in[r]}\left(f_i + \beta_i + \sum_{j\neq i}e_G(Z_j^i)\right) \tag{4.7}$$

We now focus on the structure of each $\tilde{U}_i$. We first show that $G[\tilde{U}_i]$ is "almost" $K_{t,t}$-free.

Claim 4.4. *The following holds for all $i\in[r]$.*

($K1$) *Both $G[\tilde{Z}_i]$ and $G[Z_i^i\cup W_i']$ are $K_{t,t}$-free.*

($K2$) *If $|W_i'| > 2t\varepsilon n + 2\gamma n$, then $|W_i'\backslash V_{q_i}| \leqslant t-1$.*

(*K*3) *If* $|W_i'| > 2t\varepsilon n + 2\gamma n$, *then* $G[\tilde{Z}_i \cup (W_i' \cap V_{q_i})]$ *is* $K_{t,t}$*-free.*

Proof. For $(K1)$, suppose there is a copy of $K_{t,t}$ in $\tilde{U}_i$, with vertex set denoted by B, contained in $\tilde{Z}_i$ or in $Z_i^i \cup W_i'$. Let N_B be the set of common neighbors of these $2t$ vertices of B. First assume that $B \subseteq \tilde{Z}_i$. Then for any $j \in L\backslash\{i\}$, by (Deg) we have $|N_B \cap W_j'| \geqslant |W_j'| - 4t\varepsilon n$, and thus by $(P1)$ $|N_B \cap W_j \cap W_j'| \geqslant |W_j'| - 4t\varepsilon n - 2\gamma n \geqslant n/2$. For any $j \notin L \cup \{i\}$, because $B \cap Z_j = \varnothing$ by $(Q2)$, we have $|N_B \cap Z_j| \geqslant an - 4t\varepsilon n \geqslant n/2$ by (Deg). Note that every set in $\{N_B \cap Z_j : j \notin L\} \cup \{N_B \cap W_j \cap W_j' : j \in L\}$ has size at least $n/2$ and every pair of them has density at least $1 - 4\gamma$. Therefore we can find a copy of $K_{r-1}(t)$ in the union of these sets, which gives rise to a copy of $K_{r+1}(t)$ together with B, a contradiction.

Second we assume that $B \subseteq Z_i^i \cup W_i'$. In this case we note that for any $j \neq i$, we have $B \cap Z_j = \varnothing$ and thus by (Deg), we have $|N_B \cap Z_j^j| \geqslant (1 - \sqrt{\gamma})an - 4t\varepsilon n \geqslant n/2$. Then as these sets have high pairwise densities, as in the previous case, we can find a copy of $K_{r-1}(t)$ in the union of these sets, yielding a copy of $K_{r+1}(t)$ together with B, a contradiction. Now $(K1)$ is proved.

Now we turn to $(K2)$, and suppose $|W_i'| > 2t\varepsilon n + 2\gamma n$ and thus $|W_i \cap W_i'| > 2t\varepsilon n$ by $(P1)$. First, if W_i' contains at least t vertices which are not from V_{q_i} (the cluster containing W_i), then by (Deg), each of these vertices have at most $2\varepsilon n$ non-neighbors in $W_i \cap W_i'$, and thus we can find a copy of $K_{t,t}$ in W_i', contradicting $(K1)$. So we have $|W_i'\backslash V_{q_i}| \leqslant t - 1$.

For $(K3)$, suppose there is a copy of $K_{t,t}$ as stated in the claim, whose vertex set is denoted by B. As in the previous paragraph, we have $|W_i| > 2t\varepsilon n$ by $(P1)$. Now observe crucially that if $B \cap Z_j^i \neq \varnothing$, then by $(Q2)$ $|W_i| + |W_j| > n$, and thus, W_i and W_j are not from the same cluster. So by (Deg), for any $j \in [r-1]\backslash\{i\}$, if $B \cap Z_j^i = \varnothing$, then the vertices of B have large common neighborhoods in Z_j^j; if $B \cap Z_j^i \neq \varnothing$, then the vertices of B have large common neighborhoods in $W_j \cap W_j'$ (note that $|W_j| \geqslant (1 - \varepsilon - \sqrt{\gamma})n$ by $(Q2)$). Since each of these common neighborhoods have size at least $n/2$ and each pair of them has high density, we can find a copy of $K_{r-1}(t)$ in the union of these sets, yielding a copy of $K_{r+1}(t)$ together with B, a contradiction. ☐

We now derive a lower bound for $\sum f_i$ from Claims 4.3 and 4.4. For $i \in [r]$, we have $\beta_i \leqslant \sum_{j \in L\backslash\{i\}} |Z_j^i|(|\tilde{Z}_j\backslash Z_j| + |W_j'\backslash V_{q_j}| + |Z_i\backslash\tilde{Z}_i|)$ as $|W_j'| - n \leqslant |W_j'\backslash V_{q_j}|$. Fix $j \in L\backslash\{i\}$. Note that $|W_j| \geqslant (1 - 2\varepsilon)n$. We have $|\tilde{Z}_j\backslash Z_j| \leqslant t - 1$ by $(Q3)$, and $|W_j'\backslash V_{q_j}| \leqslant t - 1$ by $(K2)$. If $|W_i| > n/2$, then $|Z_j^i| \leqslant t - 1$ by $(Q3)$. Furthermore, since $|Z_i\backslash\tilde{Z}_i| \leqslant (r-1)\sqrt{\gamma}n + C_0$ by $(P3)$, it follows that

$$|Z_j^i| \left(|\tilde{Z}_j\backslash Z_j| + |W_j'\backslash V_{q_j}| + |Z_i\backslash\tilde{Z}_i|\right) \leqslant (t-1)(t-1+t-1+(r-1)\sqrt{\gamma}n + C_0) \leqslant (t-1)r\sqrt{\gamma}n.$$

Otherwise $|W_i| \leqslant n/2$, and by $(Q2)$, we have $Z_i^{i'} = \varnothing$ for any $i' \neq i$. This implies $|Z_i\backslash\tilde{Z}_i| = |Z_i''| \leqslant C_0$. Using $|Z_j^i| \leqslant \sqrt{\gamma}n$, $(Q3)$, and $(K2)$, we derive that

$$|Z_j^i| \left(|\tilde{Z}_j\backslash Z_j| + |W_j'\backslash V_{q_j}| + |Z_i\backslash\tilde{Z}_i|\right) \leqslant \sqrt{\gamma}n(2(t-1) + C_0) \leqslant 2C_0\sqrt{\gamma}n.$$

Summarizing these two cases for all $j \in L\backslash\{i\}$, we obtain that $\beta_i \leqslant (r-1)2C_0\sqrt{\gamma}n$, and consequently,

$$\sum_{i \in [r]} \beta_i \leqslant 2(r-1)rC_0\sqrt{\gamma}n. \tag{4.8}$$

On the other hand, for all $i \neq j$, the graph $G[Z_j^i]$ is $K_{t,t}$-free by $(K1)$ and thus, by $(P3)$, $\sum_{i,j: i \neq j} e_G(Z_j^i) \leqslant r(r-1)C\left(\sqrt{\gamma}n\right)^{2-1/t}$. Applying this with (4.7), (4.8), and the fact that $e(Z'' \cup$

$W'', G) \leqslant (r+1)C_0kn$, we obtain that

$$e(G) \leqslant t_r(k)n^2 + (r+1)C_0kn + \sum_{i\in[r]} f_i + 2(r-1)rC_0\sqrt{\gamma}n + \sum_{i,j:i\neq j} e_G(Z_j^i)$$
$$\leqslant t_r(k)n^2 + \sum_{i\in[r]} f_i + r^2C\sqrt{\gamma}n^{2-1/t},$$

as $\gamma \ll 1$. Using the assumption $e(G) \geqslant g(n,r,k,t) \geqslant t_r(k)n^2 + z_t(n)$, we infer that

$$\sum_{i\in[r]} f_i \geqslant z_t(n) - r^2C\sqrt{\gamma}n^{2-1/t} \geqslant \frac{c}{2}n^{2-1/t} \tag{4.9}$$

by using (Z), $z_t(n) \geqslant z_t^{(2)}(n) \geqslant cn^{2-1/t}$, and $\gamma \ll 1$.

We next study the existence of $K_{1,t}$ in each color class. To do so, we consider a copy of $K_3(t)$ in $G[\tilde{U}_i \cup \tilde{U}_j]$ for some $i \neq j$.

Claim 4.5. *For any $i \neq j$, if $G[\tilde{U}_i \cup \tilde{U}_j]$ contains a copy K of $K_3(t)$, then there exists $l \notin \{i,j\}$ such that $V(K)$ intersects V_{q_l} and every cluster in Z_l.*

Proof. We may assume that $r > 2$ as otherwise the claim is trivial. Suppose to the contrary that there is a copy K of $K_3(t)$ in, say, $\tilde{U}_1$ and $\tilde{U}_2$, such that for every $l \in [3,r]$, there is a cluster in U_l which does not intersect $B := V(K)$. Let V_{i_l} be a cluster in Z_l such that $B \cap V_{i_l} = \varnothing$, and if there is no such cluster in Z_l, then we choose $V_{i_l} = V_{q_l}$. Note that in the former case, we have $|\tilde{U}_l \cap V_{i_l}| = |Z_l^l \cap V_{i_l}| \geqslant (1-\sqrt{\gamma}a)n$. In the latter case, we have $Z_l^1 \neq \varnothing$ or $Z_l^2 \neq \varnothing$, which implies that $|W_l| \geqslant (1-2\varepsilon)n$ by (Q2), and thus $|\tilde{U}_l \cap V_{i_l}| = |W_l' \cap V_{i_l}| \geqslant (1-3\varepsilon)n$. Now, by (Deg), every vertex in B has at most $2\varepsilon n$ non-neighbors in $\tilde{U}_l \cap V_{i_l}$ for each $l \in [3,r]$. Since for every l we have $|\tilde{U}_l \cap V_{i_l}| \geqslant 0.9n$, one can find large common neighborhoods (e.g. of size $n/2$) of all vertices of B in each $\tilde{U}_l \cap V_{i_l}$, and then find a copy of $K_{r-2}(t)$ in these sets. Altogether we obtain a copy of $K_{r+1}(t)$, a contradiction.

Therefore, for such a copy K of $K_3(t)$, there exists $l \notin \{i,j\}$ such that K must intersect all clusters of U_l. Since $V(K) \cap Z_l \neq \varnothing$, we have $Z_l^i \neq \varnothing$ or $Z_l^j \neq \varnothing$. Then by (Q2), $|W_l| \geqslant (1-2\varepsilon)n$ and in particular, $V_{q_l} \neq \varnothing$. Therefore $V(K) \cap V_{q_l} \neq \varnothing$. □

Claim 4.6. *For all but exactly one $j \in [r]$, we have $d(v, Z_j^j) \leqslant t-1$ for all $v \in \tilde{U}_j$.*

Proof. First assume that there exists $j \in [r]$ such that $G[\tilde{U}_j]$ contains a copy of $K_{1,t}$, with vertex set denoted by $\{v, u_1, \dots, u_t\}$, $v \in \tilde{U}_j$ and $u_1, \dots, u_t \in Z_j^j$. Fix $i \in [r]\backslash\{j\}$ and let N' be the set of common neighbors of $u_1, \dots, u_t$ in $\tilde{U}_i \cap U_i$. Suppose $v \in V_p$ and let N be the set of common neighbors of these $t+1$ vertices in $\tilde{U}_i \cap U_i$. In particular, $N \subseteq N'$ and N is almost equal to the union of a or $a+1$ clusters in $\tilde{U}_i$. Suppose there is a copy of $K_{t-1,t}$ with parts S_1 of size $t-1$ and S_2 of size t such that $S_1 \subseteq N'$ and $S_2 \subseteq N$. Then by Claim 4.5, there exists $l \in [r]\backslash\{i,j\}$ such that $B \cap Z_l \neq \varnothing$ and $B \cap V_{q_l} \neq \varnothing$, where B denotes the vertex set of the copy of $K_3(t)$. This is impossible since v is the only possible vertex in $B \cap (Z_l \cup V_{q_l})$ and can not satisfy both. Therefore, letting $N^* = N \cup (N' \cap V_p)$, we infer that $e_G(N^*) = e_G(N) + e_G(N, N'\backslash N) = O(n^{2-1/(t-1)})$.

By (P1), (P3) and (Deg), we have $|\tilde{U}_i \backslash N^*| \leqslant 3(t+1)\varepsilon n$. Let E^i be the set of the edges incident to $\tilde{U}_i \backslash N^*$ and counted in f_i. We split it to $E^i \cap E_G(Z_i^i)$ and $E^i \cap E_G(\tilde{U}_i \backslash Z_i^i, Z_i^i)$. Note that by (K1), each of the terms can be split further into at most k $K_{t,t}$-free bipartite graphs, each with one part of size at most $3(t+1)\varepsilon n$ and the other part of size at most $(1+(r-2)\sqrt{\gamma})an$. Therefore, we obtain that

$$f_i = O(\varepsilon n^{2-1/t}) + O(n^{2-1/(t-1)}) = O(\varepsilon n^{2-1/t}). \tag{4.10}$$

Now assume there exist distinct $j_1, j_2 \in [r]$ such that each $G[\tilde{U}_{j_i}]$ contains a copy of $K_{1,t}$ whose part of size t is in $Z_{j_i}^{j_i}$. The arguments above imply that (4.10) holds for all $i \in [r]$, and consequently, $\sum_{i\in[r]} f_i = O(\varepsilon n^{2-1/t})$, contradicting (4.9).

On the other hand, if $d(v, Z_j^j) \leqslant t-1$ for all $j \in [r]$ and all $v \in \tilde{U}_j$, then $\sum_{j\in[r]} f_j \leqslant (t-1)kn$, again contradicting (4.9). □

By Claim 4.6, without loss of generality, we assume that,

$$\text{for } i \geqslant 2, \quad d(v, Z_i^i) \leqslant t-1 \text{ for all } v \in \tilde{U}_i, \quad \text{and thus,} \quad f_i \leqslant \begin{cases} (t-1)|\tilde{U}_i| & \text{if } a \geqslant 2, \\ (t-1)|\tilde{U}_i \backslash Z_i^i| & \text{if } a = 1. \end{cases} \tag{4.11}$$

If $b = 0$, then $\tilde{U}_i = Z_i^i = Z_i \backslash Z''$ for all i by (4.3). In this case $\tilde{U}_1$ is $K_{t,t}$-free by $(K1)$ and $\tilde{U}_i$ is $K_{1,t}$-free for all $i \geqslant 2$ by (4.11). Since G is γ-close to $K_r(an)$, $G[U_i\backslash Z'', U_j\backslash Z'']$ is almost complete for all $i \neq j$. This completes the proof of Theorem 4 with $Z := Z''$. □

By (4.9) and (4.11), we get

$$f_1 \geqslant z_t(n) - \varepsilon n^{2-1/t}. \tag{4.12}$$

In particular, we claim that

$$|W_1| > 3t\varepsilon n \quad \text{ if } b > 0 \tag{4.13}$$

(which we will refine a moment later). Indeed, the edges counted in f_1 can be covered by $G[Z_1^1]$, $G[Z_1^1, W_1 \cap W_1']$, and at most k $K_{t,t}$-free bipartite graphs, each with a part of size at most $\sqrt{\gamma}n$ and a part of size at most an. If $|W_1| \leqslant 3t\varepsilon n$, then $e_G(Z_1^1, W_1 \cap W_1') = O(\varepsilon n^{2-1/t})$. Together with $e_G(Z_1^1) \leqslant z_t^{(a)}(n)$, we have

$$f_1 \leqslant z_t^{(a)}(n) + O(\varepsilon n^{2-1/t}) < z_t^{(a+1)}(n) - \varepsilon n^{2-1/t}$$

by $(E1)$, contradicting (4.12).

Now we can give a much cleaner structure, shown in a series of claims below. A key step is to show that $Z'' \cup W'' = \varnothing$. From now on we only consider $k \leqslant 2r$.

Claim 4.7. *Suppose $v_0 \in V(G)$ and $i \in [r]$ satisfy that v_0 has at least εn neighbors in Z_j for every $j \neq i$. Then v_0 has less than εn neighbors in U_i. In particular, we have $Z'' = \varnothing$ and $W'' = \varnothing$.*

Proof. The second part of the claim follows immediately from the definitions of Z'' and W''.

Suppose to the contrary, that there exist $v_0 \in V(G)$ and $i \in [r]$ such that v_0 has at least εn neighbors in Z_j for every $j \neq i$ and at least εn neighbors in U_i. Since $|Z_j^j| \geqslant (1-\sqrt{\gamma})an$ for all $j \in [r]$, there exist sets $N_1, \ldots, N_{r-1}$ each of size $\varepsilon n - \sqrt{\gamma}n$ such that $N_j \subseteq Z_j^j \cap N(v_0)$ for $j \neq i$ and $N_i \subseteq (Z_i^i \cup W_i) \cap N(v_0)$. Recall that $W_1' = W_1' \cap V_{q_1}$. By averaging, there exists $N_1' \subseteq N_1$ with $|N_1'| \geqslant (\varepsilon n - \sqrt{\gamma}n - 2r\gamma n)/2 \geqslant \varepsilon n/3$ such that all vertices of N_1' are in $Z_1^1 \cup W_1'$ and from the same cluster, that is,

$$N_1' \subseteq Q, \text{ where } Q \in \{Z_1^1, W_1'\}.$$

Note that $N_1' \subseteq W_1'$ is possible only if $i = 1$ and $a = 1$. If $i \neq 1$, then let $N_i' := N_i\backslash((W_i\backslash W_i') \cup V_{q_1})$ and for every $j \in [r]\backslash\{1, i\}$, let $N_j' := N_j$. By $(P1)$, $|W_i\backslash W_i'| \leqslant 2r\gamma n$, and by (4.16), $|W_i \cap V_{q_1}| \leqslant \gamma n$. Thus, we have $|N_j'| \geqslant \varepsilon n/3$ for all $j \in [r]$. Because the sets N_j' are small, we can not apply the degree conditions (Deg) to them and instead, we use (D0).

Recall that B_1 is given by (D0). Next we show that $G[\tilde{U}_1\backslash B_1]$ does not contain a copy of $K_{t-1,t}$ such that the part of size t is in N_1'. Suppose instead, there is such a copy of $K_{t-1,t}$, with parts denoted by A and B, such that $|A| = t$, $A \subseteq N_1'\backslash B_1$ and $B \subseteq \tilde{U}_1\backslash B_1$. Recall that $N_i' \cap V_{q_1} = \varnothing$ and for each $j \in [r]\backslash\{1, i\}$, $N_j' \subseteq Z_j^j$. Observe that for every $v \in \tilde{U}_1\backslash B_1$, we have $d(v, N_j') \geqslant |N_j'| - \sqrt{\gamma}n$. Indeed, if $j \neq i$, then $N_j' \subseteq Z_j^j$ and we have $d(v, N_j') \geqslant |N_j'| - \sqrt{\gamma}n$ by (D0); otherwise note that $N_i' \subseteq Z_i^i \cup (W_i' \cap W_i)$, and by (D0) and $N_i' \cap V_{q_1} = \varnothing$ we have $d(v, N_i') \geqslant |N_i'| - \sqrt{\gamma}n$. Therefore, we obtain that the vertices in $A \cup B$ have at least $|N_j'| - (2t-1)\sqrt{\gamma}n \geqslant (1 - \gamma^{1/3})|N_j'|$ common neighbors in each N_j', $j \in [2, r]$. Because each pair $N_j', N_{j'}'$ has a high density, we can find a copy of $K_{r-1}(t)$ in the union of these common neighborhoods, which together with $A \cup B \cup \{v_0\}$ form a copy of $K_{r+1}(t)$, a contradiction.

Now given that $G[\tilde{U}_1\backslash B_1]$ does not contain a copy of $K_{t-1,t}$ such that the part of size t is in $N_1'\backslash B_1$, we give a refined estimate on f_1. Indeed, since $G[N_1'\backslash B_1, Z_1^1\backslash B_1]$ does not contain a copy of $K_{t-1,t}$ such that the part of size t is in $N_1'\backslash B_1$, we get $e_G(N_1'\backslash B_1, Z_1^1\backslash B_1) = O(n^{2-1/(t-1)})$. Similarly $e_G(N_1'\backslash B_1, W_1'\backslash B_1) = O(n^{2-1/(t-1)})$. Suppose $N_1' \subseteq V_q$ for some $q \in \{1, q_1\}$, then we have

$$E(G[\tilde{U}_1]) = E(G[\tilde{U}_1\backslash(N_1'\backslash B_1)]) \cup E(G[N_1'\backslash B_1, \tilde{U}_1\backslash(B_1 \cup V_q)]) \cup E(G[N_1'\backslash B_1, B_1 \cap \tilde{U}_1)]).$$

Recall that $|N_1'| \geqslant \varepsilon n/3$ and $|B_1| \leqslant 2\sqrt{\gamma}n$. Therefore, (regardless of $a = 1$ or $(a, b) = (2, 0)$) we can bound $f_1 \leqslant |E(G[\tilde{U}_1])|$ by

$$f_1 \leqslant z_t\left((1 - \tfrac{\varepsilon}{3})n, n\right) + O(n^{2-1/(t-1)}) + O(\sqrt{\gamma}n^{2-1/t}) < z_t(n) - 3rC_0kn,$$

where we used (E2) and $\gamma \ll \varepsilon$. This contradicts (4.12). □

When $a = 2$ and $b = 0$ (i.e., $k = 2r$), since $Z_i'' = \varnothing$ and $W_i = \varnothing$ for all $i \in [r]$, by (4.3), we get $\tilde{U}_i = Z_i$ for all $i \in [r]$. Therefore $e(G) = e(G') + \sum_{i=1}^r e_G(\tilde{U}_i) + e(Z'' \cup W'', G) \leqslant t_r(k)n^2 + z_t(n) + (r-1)(t-1)n$ by (K1) and (4.11), proving Theorem 3 for $k = 2r$.

For the remaining of the proof, we only need to consider $a = 1$ (and thus $b > 0$). Moreover, now for $i, j \in [r]$ each $Z_i^j \subseteq Z_i$ is an independent set and thus $e_G(Z_i^j) = 0$. So we can first update our bounds on $e(G)$ and f_1. Recall the bounds (4.7), (4.8) and (4.11) and we have

$$\begin{aligned} e(G) &\leqslant t_r(k)n^2 + \sum_{i\in[r]}(f_i + \beta_i) \\ &\leqslant t_r(k)n^2 + f_1 + (r-1)(t-1)(1+\sqrt{\gamma})n + 2(r-1)rC_0\sqrt{\gamma}n, \end{aligned}$$

yielding

$$f_1 \geqslant z_t(n, n) - C_0 n \tag{4.14}$$

Claim 4.8. *Suppose $b > 0$. Then $\tilde{U}_1 = Z_1^1 \cup W_1'$ and $W_1' \subseteq V_{q_1}$.*

Proof. Suppose to the contrary, there is a vertex v in $\tilde{U}_1\backslash(Z_1^1 \cup W_1')$ or $W_1'\backslash V_{q_1}$, namely, $v \in Z_i^1$ for some $2 \leqslant i \leqslant r$ or $v \in W_1'\backslash V_{q_1}$. Suppose $v \in V_l$. Then $l \neq q_1$. Moreover, if i is defined, then $W_1' \cap V_{q_1} \subseteq V\backslash(\tilde{Z}_i \cup V_l)$; otherwise, $W_1' \cap V_{q_1} \subseteq V\backslash V_l$. By (Deg), we have $\overline{d}(v, W_1' \cap V_{q_1}) \leqslant 2\varepsilon n$. Let $N := W_1' \cap V_{q_1} \cap N(v)$. We have $|(W_1' \cap V_{q_1})\backslash N| \leqslant 2\varepsilon N$. Since $|W_1'\backslash V_{q_1}| \leqslant |W_1'\backslash W_1| \leqslant 2\gamma n$, it follows that $|W_1'\backslash N| \leqslant 2\varepsilon n + 2\gamma n \leqslant 3\varepsilon n$.

Recall (4.13), $|W_1| > 3t\varepsilon n$. By (K3) (if $v \in \tilde{Z}_1\backslash Z_1$) or (K1) (if $v \in W_1'\backslash V_{q_1}$), we know that $G[Z_1^1, N]$ contains no $K_{t-1,t}$ with the part of size t in N. This implies that $e_G(Z_1^1, N) = O(n^{2-1/(t-1)})$. Furthermore, by (P3) and (K1), $G[\tilde{Z}_1\backslash Z_1^1, Z_1^1]$ is a $K_{t,t}$-free bipartite graph with one part of size at most $(r-1)\sqrt{\gamma}n$ and the other part of size at most n. Thus, $e_G(\tilde{Z}_1\backslash Z_1^1, Z_1^1) \leqslant C(r-1)\sqrt{\gamma}n^{2-1/t}$. By the similar arguments, we have $e_G(W_1'\backslash N, Z_1^1) \leqslant C(3\varepsilon n)n^{1-1/t}$.

Putting these bounds together (and note that Z_1^1 is an independent set, we get

$$\begin{aligned} f_1 &= e_G(Z_1^1, N) + e_G(\tilde{Z}_1\backslash Z_1^1, Z_1^1) + e_G(W_1'\backslash N, Z_1^1) \\ &= O(n^{2-1/(t-1)}) + O(\sqrt{\gamma}n^{2-1/t}) + O(\varepsilon n^{2-1/t}). \end{aligned}$$

By $(E1)$, this contradicts (4.14). □

Claim 4.8 shows that $\tilde{U}_1$ has no atypical vertices and is thus $K_{t,t}$-free by $(K1)$. Furthermore, since $\tilde{U}_1 = Z_1^1 \cup W_1'$ and $W_1' \subseteq V_{q_1}$, it follows that

$$\alpha_1 = \beta_1 = 0, \quad \text{and} \quad e_G(\tilde{U}_1) = f_1 \leqslant z_t(|Z_1^1|, |W_1'|). \tag{4.15}$$

Therefore, if $|W_1'| \leqslant (1-\gamma)n$, then we have $f_1 \leqslant z_t(n, |W_1'|) \leqslant z_t(n,n) - \delta n^{2-1/t}$ for some $\delta > 0$ by $(E2)$. This contradicts (4.14). So we obtain

$$\text{if } a = 1, \text{ then } |W_1'| \geqslant (1-\gamma)n \quad (\text{and thus} \quad 1 \in L). \tag{4.16}$$

Next we study $G[\tilde{U}_i]$ for $i \geqslant 2$. A key observation is that copies of $K_{1,t}$ in $G[\tilde{U}_i]$ together with copies of $K_{t-1,t}$ in $\tilde{U}_1$ may form copies of $K_3(t)$, which are restricted by Claim 4.5.

Claim 4.9. *Suppose $i \in [2, r]$.*

(1) If there is a copy of $K_{1,t}$ in $\tilde{U}_i\backslash(Z_1 \cup V_{q_1})$, then there exists $l \in [r]\backslash\{i\}$ such that the vertex set of $K_{1,t}$ intersects both V_{q_l} and Z_l.

(2) Both $\tilde{Z}_i\backslash Z_1$ and $Z_i^i \cup (W_i'\backslash V_{q_1})$ are $K_{1,t}$-free.

Proof. For Part (1), let B be the vertex set of a copy of $K_{1,t}$ in $\tilde{U}_i\backslash(Z_1 \cup V_{q_1})$. Since $B \cap (Z_1 \cup V_{q_1}) = \varnothing$ and $\tilde{U}_1 \subseteq Z_1 \cup V_{q_1}$, by (Deg), all vertices of B have at most $2\varepsilon n$ non-neighbors in $\tilde{U}_1$. Letting $N := \tilde{U}_1 \cap \bigcap_{w \in B} N(w)$, we have $|N| \geqslant |\tilde{U}_1| - (t+1)2\varepsilon n$.

First assume that N is $K_{t-1,t}$-free and thus $e_G(N) = O(n^{2-1/(t-1)})$. Note that, since $|\tilde{U}_1\backslash N| \leqslant (t+1)2\varepsilon n$, the edges in $\tilde{U}_1$ incident to $\tilde{U}_1\backslash N$ can be split into two bipartite $K_{t,t}$-free graphs each with one part of size at most $(t+1)2\varepsilon n$ and the other part of size at most n. Thus, the number of such edges is $O(\varepsilon n^{2-1/t})$. This gives $f_1 = O(n^{2-1/(t-1)}) + O(\varepsilon n^{2-1/t})$, contradicting (4.12).

We thus assume N contains a copy of $K_{t-1,t}$. Together with B, they form a copy of $K_3(t)$ in $G[\tilde{U}_1 \cup \tilde{U}_i]$ and we denote its vertex set by B'. By Claim 4.5, there exists $l \notin \{1, i\}$ such that B' intersects V_{q_l} and Z_l. By Claim 4.8, $\tilde{U}_1 \cap U_l = \varnothing$, so $B' \cap Z_l = B \cap Z_l$ and B indeed intersects Z_l. Since $\tilde{U}_i \cap Z_l \supseteq B \cap Z_l \neq \varnothing$, we infer that $|W_l| \geqslant (1-2\varepsilon)n$ from $(Q3)$, which implies that $q_l \neq q_1$ because of (4.13). It follows that $W_1 \cap V_{q_l} = \varnothing$ and thus $B \cap V_{q_l} = B' \cap V_{q_l} \neq \varnothing$, as desired.

For Part (2), let $A_i := Z_i^i \cup (W_i'\backslash V_{q_1})$ and B be the vertex set of a copy of $K_{1,t}$ in $\tilde{Z}_i\backslash Z_1$ or in A_i. Then, by the first part of the claim, there exists $l \in [r]\backslash\{i\}$ such that B intersects V_{q_l} and Z_l. This is impossible if $B \subseteq A_i$ because $A_i \cap Z_l = \varnothing$ for any $l \notin \{1, i\}$, and also impossible if $B \subseteq \tilde{Z}_i\backslash Z_1$ because in which case $B \cap W = \varnothing$ and thus $B \cap V_{q_l} = \varnothing$ for any $l \notin \{1, i\}$. □

The following claim shows a clean structure for the $\tilde{U}_i$ such that W_i' is not too small.

Claim 4.10. *For $i \in [2, r]$ such that $|W_i| \geqslant 2\varepsilon n$, we have $\tilde{U}_i \subseteq U_i \cup V_{q_i}$.*

Proof. Suppose instead, for some $i_0 \in [2, r]$ with $|W_{i_0}| \geqslant 2\varepsilon n$, there exists $v \in \tilde{U}_{i_0}\backslash(U_{i_0} \cup V_{q_{i_0}})$. By $(P4)$ and the fact that $v \in \tilde{U}_{i_0}\backslash U_{i_0}$, we infer that $d(v, Z_j) \geqslant \varepsilon n$ for all $j \neq i_0$. Then, by Claim 4.7, we have $d(v, U_{i_0}) < \varepsilon n$. Consequently, $d(v, W_{i_0}' \cap W_{i_0}) < \varepsilon n$, namely, v has at least

$2\varepsilon n - 2\gamma n - \varepsilon n \geqslant (1/2)\varepsilon n$ non-neighbors in $W'_{i_0} \cap W_{i_0}$ (in G). Note that v is adjacent to all the vertices of $W'_{i_0} \cap W_{i_0}$ in T. Since $G[W'_{i_0}] \subseteq T[W'_{i_0}]$, we infer that

$$e_G(W'_{i_0}) + e_G(\tilde{Z}_{i_0}\backslash Z_{i_0}, W'_{i_0}) \leqslant e_T(W'_{i_0}) + |\tilde{Z}_{i_0}\backslash Z_{i_0}||W'_{i_0}| - (1/2)\varepsilon n.$$

Since $e_G(\tilde{Z}_{i_0}\backslash Z_{i_0}) \leqslant e_T(\tilde{Z}_{i_0}\backslash Z_{i_0})$ and $\alpha_{i_0} = |\tilde{Z}_{i_0}\backslash Z_{i_0}||W'_i| + e_T(W'_{i_0}) + e_T(\tilde{Z}_{i_0}\backslash Z_{i_0})$, we have

$$\begin{aligned} e_G(\tilde{U}_{i_0}\backslash Z_{i_0}) &= e_G(\tilde{Z}_{i_0}\backslash Z_{i_0}) + e_G(W'_{i_0}) + e_G(\tilde{Z}_{i_0}\backslash Z_{i_0}, W'_{i_0}) \\ &\leqslant e_T(\tilde{Z}_{i_0}\backslash Z_{i_0}) + e_T(W'_{i_0}) + |\tilde{Z}_{i_0}\backslash Z_{i_0}||W'_{i_0}| - (1/2)\varepsilon n \\ &\leqslant \alpha_{i_0} - (1/2)\varepsilon n. \end{aligned} \tag{4.17}$$

Combining (4.5) and (4.17) gives

$$\sum_{i\in[r]} e_G(\tilde{U}_i\backslash Z_i) \leqslant \sum_{i\in[r]} \alpha_i - (1/2)\varepsilon n \tag{4.18}$$

Recall that $f_i \leqslant (t-1)|\tilde{U}_i\backslash Z_i|$ ($i \geqslant 2$) by (4.11), $|W'_1| \geqslant (1-\gamma)n$ by (4.16), and $|Z_i^i| \geqslant (1-\sqrt{\gamma})n$ by *(P3)*. Therefore, as $\sum_{i=1}^r |\tilde{U}_i\backslash Z_i| \leqslant (b-1)n + r\sqrt{\gamma}n + \gamma n$, we obtain

$$\sum_{i=2}^{r} f_i \leqslant (t-1)\left((b-1)n + r\sqrt{\gamma}n + \gamma n\right) \leqslant (t-1)(b-1)n + \sqrt[3]{\gamma}n. \tag{4.19}$$

Since $Z'' \cup W'' = \varnothing$, (4.7) becomes $e(G) \leqslant t_r(k)n^2 + \sum_{i\in[r]}(f_i + \beta_i + e_G(\tilde{U}_i\backslash Z_i) - \alpha_i)$. Recall that $\sum_{i=1}^r \beta_i \leqslant 2r^2C_0\sqrt{\gamma}n$ by (4.8). Together with (4.18) and (4.19), we derive that

$$e(G) \leqslant t_r(k)n^2 + z_t(n) + (t-1)(b-1)n + \sqrt[3]{\gamma}n + 2r^2C_0\sqrt{\gamma}n - \varepsilon n/2 < g(n,r,k,t),$$

as $\gamma \ll \varepsilon$. This is a contradiction. □

Let $L_1 \cup L_2 \cup L_3$ be a partition of $[2,r]$ such that $i \in L_1$ if and only if $|\tilde{Z}_i| < n$, $i \in L_2$ if and only if $|\tilde{Z}_i| = n$, and $i \in L_3$ if and only if $|\tilde{Z}_i| > n$. The following properties hold for L_1, L_2 and L_3.

- $(R1)$ If $i \in L_1$, then $Z_i^j \neq \varnothing$ for some $j \neq i$. By *(Q2)*, we have $i \in L$ and, by Claim 4.10, $\tilde{Z}_i = Z_i^i \subsetneq V_i$ and $W'_i \subseteq V_{q_i}$.
- $(R2)$ If $i \in L_2$, then $\tilde{Z}_i = Z_i^i = V_i$. Indeed, otherwise $\tilde{Z}_i \neq Z_i^i$, then $|Z_i^i| < n$ and $Z_i^j \neq \varnothing$ for some $j \neq i$. By *(Q2)* and Claim 4.10, we have $\tilde{Z}_i = Z_i^i$, a contradiction.
- $(R3)$ If $i \in L_3$, then $\tilde{Z}_i \nsubseteq Z_i$ (otherwise $|\tilde{Z}_i| \leqslant n$). By Claim 4.10, we have $|W_i| < 2\varepsilon n$, which implies that $Z_i^j = \varnothing$ for $j \neq i$ by *(Q2)*. Thus, $Z_i^i = Z_i = V_i \subsetneq \tilde{Z}_i$.

Now we derive our final bound on $e(G)$. Write $z_i := |\tilde{Z}_i|$ and $w_i := |W'_i|$ for $i \in [r]$.
By Claim 4.3 and the fact that $Z'' \cup W'' = \varnothing$, we have

$$e(G) = e(G') + \sum_{i\in[r]} e_G(\tilde{U}_i) \leqslant t_r(k)n^2 + \sum_{i\in[r]} \left(\beta_i - \alpha_i + e_G(\tilde{U}_i)\right).$$

Moreover, as $a = 1$, (4.5) becomes $e_G(\tilde{U}_i\backslash Z_i^i) \leqslant \alpha_i$. It follows that $e_G(\tilde{U}_i) = f_i + e_G(\tilde{U}_i\backslash Z_i) \leqslant f_i + \alpha_i$. For $i \in \{1\} \cup L_1 \cup L_2$, we simply use f_i as the upper bound and thus we get

$$e_G(\tilde{U}_i) - \alpha_i \leqslant f_i \leqslant \begin{cases} z_t(z_1, w_1) & \text{if } i = 1 \text{ by (4.15)}, \\ (t-1)\min\{z_i, w_i\} & \text{if } i \in L_1 \text{ by } (R1) \text{ and Claim 4.9 (2)}, \\ (t-1)(z_i - n + w_i) & \text{if } i \in L_2 \text{ by } (R2) \text{ and (4.11)}. \end{cases}$$

Additional work is needed for $i \in L_3$. We let $\lambda = \max\{0, |L_1| - |L_3|\}$ and for $i \in L_3$, let λ_i be the number of indices $j \in L_1$ such that $Z_j^i \neq \varnothing$. By $(R1)$–$(R3)$, we know that if $Z_j^i \neq \varnothing$, then $i \in L_3$ and $j \in L_1$. This implies that $\lambda_i \geqslant 1$ for every $i \in L_3$, and $\sum_{i\in L_3} \lambda_i \geqslant |L_1|$, yielding that

$$\sum_{i\in L_3} (\lambda_i - 1) \geqslant \lambda. \tag{4.20}$$

Recall that $G[tZ_i \backslash Z_1^i]$ is $K_{1,t}$-free by Claim 4.9 (2). Since $Z_i^i = Z_i$ is an independent set, it follows that $e_G(\tilde{Z}_i \backslash Z_1^i) \leqslant (t-1)|\tilde{Z}_i \backslash (Z_i \cup Z_1^i)|$. Together with (4.11), this gives

$$e_G(\tilde{Z}_i \backslash Z_1^i) + e_G(Z_1^i, Z_i) \leqslant (t-1)|\tilde{Z}_i \backslash (Z_i \cup Z_1^i)| + (t-1)|Z_1^i| = (t-1)(z_i - n).$$

Therefore, for $i \in L_3$, writing $\varrho_i := e_T(Z_1^i, \tilde{Z}_i \backslash (Z_i \cup Z_1^i))$, we have

$$e_G(\tilde{Z}_i) = e_G(\tilde{Z}_i \backslash Z_1^i) + e_G(Z_1^i, Z_i) + e_G(Z_1^i, \tilde{Z}_i \backslash (Z_i \cup Z_1^i) \leqslant (t-1)(z_i - n) + \varrho_i.$$

Moreover, the definition of α_i implies that

$$\begin{aligned} e_G(W_i') + e_G(\tilde{Z}_i \backslash Z_i, W_i') - \alpha_i &\leqslant e_T(W_i') + |\tilde{Z}_i \backslash Z_i||W_i'| - \alpha_i = -e_T(\tilde{Z}_i \backslash Z_i) \\ &= -\varrho_i - e_T(\tilde{Z}_i \backslash (Z_i \cup Z_1)) \\ &\leqslant -\varrho_i - \binom{\lambda_i}{2} \leqslant -\varrho_i + 1 - \lambda_i. \end{aligned}$$

Finally, by (4.11), we have $e_G(Z_i, W_i') \leqslant (t-1)w_i$ for $i \in L_3$. Combining all these inequalities together, we obtain that, for $i \in L_3$,

$$\begin{aligned} e_G(\tilde{U}_i) - \alpha_i &= e_G(\tilde{Z}_i) + e_G(Z_i, W_i') + e_G(W_i') + e_G(\tilde{Z}_i \backslash Z_i, W_i') - \alpha_i \\ &\leqslant (t-1)(z_i - n + w_i) + (1 - \lambda_i). \end{aligned}$$

It follows that $\sum_{i\in L_3}(e_G(\tilde{U}_i) - \alpha_i) \leqslant \sum_{i\in L_3}(t-1)(z_i - n + w_i) - \lambda$ by using (4.20).

Using $\sum_{i=2}^r (z_i - n) = n - z_1$ and $\sum_{i=2}^r w_i = bn - w_1$, we derive that

$$\begin{aligned} \sum_{i\in L_1} \min\{z_i, w_i\} + \sum_{i\in L_2 \cup L_3} (z_i - n + w_i) &= \sum_{i\in L_1} \min\{n - w_i, n - z_i\} + \sum_{i=2}^r (z_i - n + w_i) \\ &= \sum_{i\in L_1} \min\{n - w_i, n - z_i\} + bn - w_1 + n - z_1. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{i=2}^r (e_G(\tilde{U}_i) - \alpha_i) &\leqslant \sum_{i\in L_1} (t-1)\min\{z_i, w_i\} + \sum_{i\in L_2\cup L_3} (t-1)(z_i - n + w_i) - \lambda \\ &= (t-1)(bn - w_1 + n - z_1) + \sum_{i\in L_1} (t-1)\min\{n - w_i, n - z_i\} - \lambda. \end{aligned}$$

Finally, we work on the β_i's and recall that $\beta_i = \sum_{j\in L\backslash\{i\}} |Z_j^i|(|\tilde{Z}_j \backslash Z_j| + |W_j'| - n + |Z_i \backslash \tilde{Z}_i|)$. For $i \in \{1\} \cup L_1 \cup L_2$, as $\tilde{Z}_i \backslash Z_i = \varnothing$, $\beta_i = 0$. For $i \in L_3$, as $|W_i| < 2\varepsilon n$, we have $Z_i \backslash \tilde{Z}_i = \varnothing$; for any $j \in L \backslash \{i\}$, we have $\tilde{Z}_j \backslash Z_j = \varnothing$ and $W_j' \subseteq V_{q_j}$ again by Claim 4.10. Hence $\beta_i = \sum_{j\in L\backslash\{i\}} |Z_j^i|(|W_j'| - n) \leqslant 0$

because $|W'_j| \leqslant n$. It then follows that (noting that $L \cap L_3 = \varnothing$)

$$\sum_{i\geqslant 1}\beta_i = \sum_{i\in L_3}\beta_i = \sum_{i\in L_3}\sum_{j\in L\setminus\{i\}}|Z^i_j|(|W'_j|-n)$$
$$= \sum_{j\in L}\sum_{i\in L_3\setminus\{j\}}|Z^i_j|(|W'_j|-n) = \sum_{j\in L}(n-z_j)(w_j-n).$$

Note that $1 \in L$ by (4.16) and $(n-z_1)(w_1-n) \leqslant 0$ by Claim 4.8. Furthermore, since $n - z_j = 0$ for $j \in L_2$, it follows that $\sum_{i\geqslant 1}\beta_i = \sum_{j\in L_1}(n-z_j)(w_j-n)$.

Recall that $e_G(\tilde{U}_1) = f_1 \leqslant z_t(z_1, w_1)$. By (E3), we have $z_t(z_1,w_1)+(t-1)(n-z_1+n-w_1) \leqslant z_t(n)$. Thus, combining these estimates together, by (4.7), we get

$$e(G) \leqslant t_r(k)n^2 + \sum_{i\in[r]}(e_G(\tilde{U}_i) - \alpha_i + \beta_i) \leqslant t_r(k)n^2 + z_t(n) + (t-1)(b-1)n + y - \lambda, \tag{4.21}$$

where $y := \sum_{i\in L_1}((t-1)\min\{n-w_i, n-z_i\} - (n-z_i)(n-w_i))$. For each $i \in L_1$, let $y_i := \min\{n-w_i, n-z_i\}$ and $y'_i := \max\{n-w_i, n-z_i\}$. Then $y_i \leqslant y'_i$ and thus,

$$(t-1)\min\{n-w_i, n-z_i\} - (n-z_i)(n-w_i) = y_i(t-1-y'_i) \leqslant \lfloor (t-1)^2/4\rfloor \leqslant 1.$$

Since $L_1 \subseteq L\setminus\{1\}$, we have $|L_1| \leqslant b-1$. Moreover, by Claim 4.10, we have $w_i \leqslant |W_i| + |W'_i\setminus W_i| \leqslant 2\varepsilon n + \gamma n \leqslant 3\varepsilon n$ for each $i \in L_3$. If $|L_1 \cup L_2| \leqslant b-2$, then

$$bn = \sum_{i\in[r]}|W_i| \leqslant n + (b-2)n + (r-b+1)\cdot 3\varepsilon n < bn,$$

a contradiction. This implies $|L_1 \cup L_2| \geqslant b-1$, and $|L_3| \leqslant r-b$. Since $|L_1| - \lambda = \min\{|L_1|, |L_3|\} \leqslant |L_3| \leqslant r-b$, it follows that $|L_1| - \lambda \leqslant \min\{b-1, r-b\}$. Consequently, as $\lfloor (t-1)^2/4\rfloor \leqslant 1$, we get

$$y - \lambda \leqslant |L_1|\lfloor (t-1)^2/4\rfloor - \lambda \leqslant \min\{b-1, r-b\}\lfloor (t-1)^2/4\rfloor,$$

Together with (4.21), it gives the desired bound $e(G) \leqslant t_r(k)n^2 + z_t(n) + (t-1)(b-1)n + \min\{b-1, r-b\}\lfloor (t-1)^2/4\rfloor = g(n,r,k,t)$. This completes the proof of Theorem 3 for $k < 2r$. □

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