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On multipartite Hajnal–Szemerédi theorems

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ABSTRACT

Let G be a k-partite graph with n vertices in parts such that each vertex is adjacent to at least $\delta^*(G)$ vertices in each of the other parts. Magyar and Martin (2002) [18] proved that for $k = 3$, if $\delta^*(G) \ge \frac{2}{3}n + 1$ and n is sufficiently large, then G contains a K_3-factor (a spanning subgraph consisting of n vertex-disjoint copies of K_3). Martin and Szemerédi (2008) [19] proved that G contains a K_4-factor when $\delta^*(G) \ge \frac{3}{4}n$ and n is sufficiently large. Both results were proved using the Regularity Lemma. In this paper we give a proof of these two results by the absorbing method. Our absorbing lemma actually works for all $k \ge 3$ and may be utilized to prove a general and tight multipartite Hajnal–Szemerédi theorem.

1. Introduction

Let H be a graph on h vertices, and let G be a graph on n vertices. *Packing* (or *tiling*) problems in extremal graph theory are investigations of conditions under which G must contain many vertex disjoint copies of H (as subgraphs), where minimum degree conditions are studied the most. An H-matching of G is a subgraph of G which consists of vertex-disjoint copies of H. A *perfect H-matching*, or *H-factor*, of G is an H-matching consisting of $\lfloor n/h \rfloor$ copies of H. Let K_k denote the complete graph on k vertices. The celebrated theorem of Hajnal and Szemerédi [5] says that every n-vertex graph G with $\delta(G) \ge (k-1)n/k$ contains a K_k-factor (see [10] for another proof).

Using the Regularity Lemma of Szemerédi [23], researchers have generalized this theorem for packing arbitrary H [1,13, 22,14]. Results and methods for packing problems can be found in the survey of Kühn and Osthus [15].

In this paper we consider multipartite packing, which restricts G to be a k-partite graph for $k \ge 2$. A k-partite graph is called *balanced* if its partition sets have the same size. Given a k-partite graph G, it is natural to consider the minimum partite degree $\delta^*(G)$, the minimum degree from a vertex in one partition set to any other partition set. When $k = 2$, $\delta^*(G)$ is simply $\delta(G)$. In most of the rest of this paper, the minimum degree condition stands for the minimum partite degree for short.

Let $\mathcal{G}_k(n)$ denote the family of balanced k-partite graphs with n vertices in each of its partition sets. It is easy to see (e.g. using the König–Hall Theorem) that every bipartite graph $G \in \mathcal{G}_2(n)$ with $\delta^*(G) \ge n/2$ contains a 1-factor. Fischer [4] conjectured that if $G \in \mathcal{G}_k(n)$ satisfies

$$\delta^*(G) \ge \frac{k-1}{k}n, \tag{1}$$

then G contains a K_k-factor and proved the existence of an *almost* K_k-factor for $k = 3, 4$. Magyar and Martin [18] noticed that the condition (1) is not sufficient for odd k and instead proved the following theorem for $k = 3$. (They actually showed that when n is divisible by 3, there is only one graph in $\mathcal{G}_3(n)$, denoted by $\Gamma_3(n/3)$, that satisfies (1) but fails to contain a K_3-factor, and adding any new edge to $\Gamma_3(n/3)$ results in a K_3-factor.)

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Theorem 1 (*[18]*)**.** *There exists an integer n_0 such that if $n \geq n_0$ and $G \in \mathcal{G}_3(n)$ satisfies $\delta^*(G) \geq 2n/3 + 1$, then G contains a K_3-factor.*

On the other hand, Martin and Szemerédi [19] proved that the original conjecture holds for $k = 4$.

Theorem 2 (*[19]*)**.** *There exists an integer n_0 such that if $n \geq n_0$ and $G \in \mathcal{G}_4(n)$ satisfies $\delta^*(G) \geq 3n/4$, then G contains a K_4-factor.*

Recently Keevash and Mycroft [8] and independently Lo and Markström [17] proved that Fischer's conjecture is asymptotically true, namely, $\delta^*(G) \geq \frac{k-1}{k}n + o(n)$ guarantees a K_k-factor for all $k \geq 3$. Very recently, Keevash and Mycroft [9] improved this to an exact result.

In this paper we give a new proof of Theorems 1 and 2 by the absorbing method. Our approach is similar to that of [17] (in contrast, a geometric approach was employed in [8]). However, in order to prove exact results by the absorbing lemma, one needs only assume $\delta^*(G) \geq (1 - 1/k)n$, instead of $\delta^*(G) \geq (1 - 1/k + \alpha)n$ for some $\alpha > 0$ as in [17]. In fact, our absorbing lemma uses an even weaker assumption $\delta^*(G) \geq (1 - 1/k - \alpha)n$ and has a more complicated absorbing structure.

The absorbing method, initiated by Rödl, Ruciński, and Szemerédi [21], has been shown to be effective handling extremal problems in graphs and hypergraphs. One example is the re-proof of Posa's conjecture by Levitt, Sárközy, and Szemerédi [16], while the original proof of Komlós, Sárközy, and Szemerédi [11] used the Regularity Lemma. Our paper is another example of replacing the regularity method with the absorbing method. Compared with the threshold n_0 in Theorems 1 and 2 derived from the Regularity Lemma, the value of our n_0 is much smaller.

Before presenting our proof, let us first recall the approach used in [18,19]. Given a k-partite graph $G \in \mathcal{G}_k(n)$ with parts $V_1, \ldots, V_k$, the authors said that G is Δ-extremal if each V_i contains a subset A_i of size $\lfloor n/k \rfloor$ such that the density $d(A_i, A_j) \leq \Delta$ for all $i \neq j$. Using standard but involved graph theoretic arguments, they solved the extremal case for $k = 3, 4$ [18, Theorem 3.1], [19, Theorem 2.1].

Theorem 3. *Let $k = 3, 4$. There exists Δ and n_0 such that the following holds. Let $n \geq n_0$ and $G \in \mathcal{G}_k(n)$ be a k-partite graph satisfying $\delta^*(G) \geq (2/3)n + 1$ when $k = 3$ and (1) when $k = 4$. If G is Δ-extremal, then G contains a K_k-factor.*

To handle the non-extremal case, they proved the following lemma ([18, Lemma 2.2] and [19, Lemma 2.2]).

Lemma 4 (*Almost Covering Lemma*)**.** *Let $k = 3, 4$. Given $\Delta > 0$, there exists $\alpha > 0$ such that for every graph $G \in \mathcal{G}_k(n)$ with $\delta^*(G) \geq (1 - 1/k)n - \alpha n$ either G contains an almost K_k-factor that leaves at most $C = C(k)$ vertices uncovered or G is Δ-extremal.*

To improve the almost K_k-factor obtained from Lemma 4, they used the Regularity Lemma and Blow-up Lemma [12]. Here is where we need our absorbing lemma whose proof is given in Section 2. Our lemma actually gives a more detailed structure than what is needed for the extremal case when G does not satisfy the absorbing property.

We need some definitions. Given positive integers k and r, let $\Theta_{k\times r}$ denote the graph with vertices a_{ij}, $i = 1, \ldots, k$, $j = 1, \ldots, r$, and a_{ij} is adjacent to $a_{i'j'}$ if and only if $i \neq i'$ and $j \neq j'$. In addition, given a positive integer t, the graph $\Theta_{k\times r}(t)$ denotes the blow-up of $\Theta_{k\times r}$, obtained by replacing vertices a_{ij} with sets A_{ij} of size t, and edges $a_{ij}a_{i'j'}$ with complete bipartite graphs between A_{ij} and $A_{i'j'}$. Given $\epsilon, \Delta > 0$ and $t \geq 1$ (not necessarily an integer), we say that a k-partite graph G is (ϵ, Δ)-approximate to $\Theta_{k\times r}(t)$ if each of its partition sets V_i can be partitioned into $\bigcup_{i=1}^{r} V_{ij}$ such that $||V_{ij}| - t| \leq \epsilon t$ for all i, j and $d(V_{ij}, V_{i'j}) \leq \Delta$ whenever $i \neq i'$.[1]

Lemma 5 (*Absorbing Lemma*)**.** *Given $k \geq 3$ and $\Delta > 0$, there exists $\alpha = \alpha(k, \Delta) > 0$ and an integer $n_1 > 0$ such that the following holds. Let $n \geq n_1$ and $G \in \mathcal{G}_k(n)$ be a k-partite graph on $V_1 \cup \cdots \cup V_k$ such that $\delta^*(G) \geq (1 - 1/k)n - \alpha n$. Then one of the following cases holds.*

(1) *G contains a K_k-matching M of size $|M| \leq 2(k-1)\alpha^{4k-2}n$ in G such that for every $W \subset V \setminus V(M)$ with $|W \cap V_1| = \cdots = |W \cap V_k| \leq \alpha^{8k-6}n/4$, there exists a K_k-matching covering exactly the vertices in $V(M) \cup W$.*
(2) *We may remove some edges from G so that the resulting graph G' satisfies $\delta^*(G') \geq (1 - 1/k)n - \alpha n$ and is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k\times k}\left(\frac{n}{k}\right)$.*

The K_k-matching M in Lemma 5 has the so-called *absorbing* property: it can absorb *any* balanced set with a much smaller size.

Proof of Theorems 1 and 2. Let $k = 3, 4$. Let $\alpha \ll \Delta$, where Δ is given by Theorem 3 and α satisfies both Lemmas 4 and 5. Suppose that n is sufficiently large. Let $G \in \mathcal{G}_k(n)$ be a k-partite graph satisfying $\delta^*(G) \geq (2/3)n + 1$ when $k = 3$ and (1) when $k = 4$. By Lemma 5, either G contains a subgraph which is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k\times k}\left(\frac{n}{k}\right)$ or G contains an

[1] Here we follow the definition of (ϵ, Δ)-approximation in [18,19]. It seems natural to require that $d(V_{ij}, V_{i'j'}) \geq 1 - \Delta$ whenever $i \neq i'$ and $j \neq j'$ as well. However, this follows from $d(V_{ij}, V_{i'j}) \leq \Delta$ $(i \neq i')$ when $\delta^*(G) \geq (1 - 1/r)rt$.

absorbing K_k-matching M. In the former case, for $i = 1, \dots, k$, we add or remove at most $\frac{\Delta n}{6k}$ vertices from V_{i1} to obtain a set $A_i \subset V_i$ of size $\lfloor n/k \rfloor$. For $i \neq i'$, we have

$$\begin{aligned} e(A_i, A_{i'}) &\le e(V_{i1}, V_{i'1}) + \frac{\Delta n}{6k}(|A_i| + |A_{i'}|) \\ &\le \frac{\Delta}{2}|V_{i1}||V_{i'1}| + 2\frac{\Delta n}{6k}\left\lfloor \frac{n}{k} \right\rfloor \\ &\le \frac{\Delta}{2}\left(1 + \frac{\Delta}{6}\right)^2 \left(\frac{n}{k}\right)^2 + \frac{\Delta n}{3k}\left\lfloor \frac{n}{k} \right\rfloor \\ &\le \Delta \left\lfloor \frac{n}{k} \right\rfloor \left\lfloor \frac{n}{k} \right\rfloor, \end{aligned}$$

which implies that $d(A_i, A_{i'}) \le \Delta$. Thus G is Δ-extremal. By Theorem 3, G contains a K_k-factor. In the latter case, G contains a K_k-matching M is of size $|M| \le 2(k-1)\alpha^{4k-2}n$ such that for every $W \subset V \setminus V(M)$ with $|W \cap V_1| = \cdots = |W \cap V_k| \le \alpha^{8k-6}n/4$, there exists a K_k-matching on $V(M) \cup W$. Let $G' = G \setminus V(M)$ be the induced subgraph of G on $V(G) \setminus V(M)$, and $n' = |V(G')|$. Clearly G' is balanced. As $\alpha \ll 1$, we have

$$\delta^*(G') \ge \delta^*(G) - |M| \ge \left(1 - \frac{1}{k}\right) n - 2(k-1)\alpha^{4k-2}n \ge \left(1 - \frac{1}{k} - \alpha\right) n'.$$

By Lemma 4, G' contains a K_k-matching M' such that $|V(G') \setminus V(M')| \le C$. Let $W = V(G') \setminus V(M')$. Clearly $|W \cap V_1| = \cdots = |W \cap V_k|$. Since $C/k \le \alpha^{8k-6}n/4$ for sufficiently large n, by the absorbing property of M, there is a K_k-matching M'' on $V(M) \cup W$. This gives the desired K_k-factor $M' \cup M''$ of G. □

Remarks. • Since our Lemma 5 works for all $k \ge 3$, it has the potential of proving a general multipartite Hajnal–Szemerédi theorem. To do it, one only needs to prove Theorem 3 and Lemma 4 for $k \ge 5$.

- Since our Lemma 5 gives a detailed structure of G when G does not have desired absorbing K_k-matching, it has the potential of simplifying the proof of the extremal case. Indeed, if one can refine Lemma 4 such that it concludes that G either contains an almost K_k-factor or it is approximate to $\Theta_{k\times k}\left(\frac{n}{k}\right)$ and other extremal graphs, then in Theorem 3 we may assume that G is actually approximate to these extremal graphs.
- Using the Regularity Lemma, researchers have obtained results on packing arbitrary graphs in k-partite graphs, see [24,7,3,2] for $k = 2$ and [20] for $k = 3$. With the help of the recent result of Keevash–Mycroft [8] and Lo–Markström [17], it seems not very difficult to extend these results to the $k \ge 4$ case (though exact results may be much harder). However, it seems difficult to replace the regularity method by the absorbing method for these problems.

2. Proof of the absorbing lemma

In this section we prove the Absorbing Lemma (Lemma 5). We first introduce the concepts of reachability.

Definition 6. In a graph G, a vertex x is reachable from another vertex y by a set $S \subseteq V(G)$ if both $G[x \cup S]$ and $G[y \cup S]$ contain K_k-factors. In this case, we say S connects x and y.

The following lemma plays a key role in constructing absorbing structures. We postpone its proof to the end of the section.

Lemma 7 (*Reachability Lemma*). *Given $k \ge 3$ and $\Delta > 0$, there exists $\alpha = \alpha(k, \Delta) > 0$ and an integer $n_2 > 0$ such that the following holds. Let $n \ge n_2$ and $G \in \mathcal{G}_k(n)$ be a k-partite graph on $V_1 \cup \cdots \cup V_k$ such that $\delta^*(G) \ge (1 - 1/k)n - \alpha n$. Then one of the following cases holds.*

(1) *For any x and y in V_i, $i \in [k]$, x is reachable from y by either at least $\alpha^3 n^{k-1}$ $(k-1)$-sets or at least $\alpha^3 n^{2k-1}$ $(2k-1)$-sets in G.*
(2) *We may remove some edges from G so that the resulting graph G' satisfies $\delta^*(G') \ge (1 - 1/k)n - \alpha n$ and is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k\times k}\left(\frac{n}{k}\right)$.*

With the aid of Lemma 7, the proof of Lemma 5 becomes standard counting and probabilistic arguments, as shown in [6].

Proof of Lemma 5. We assume that G does not satisfy the second property stated in the lemma.

Given a *crossing* k-tuple $T = (v_1, \dots, v_k)$, with $v_i \in V_i$, for $i = 1, \dots, k$, we call a set A an *absorbing set* for T if both $G[A]$ and $G[A \cup T]$ contain K_k-factors. Let $\mathcal{L}(T)$ denote the family of all $2k(k-1)$-sets that absorb T (the reason why our absorbing sets are of size $2k(k-1)$ can be seen from the proof of Claim 8 below).

Claim 8. *For every crossing k-tuple T, we have $|\mathcal{L}(T)| > \alpha^{4k-3}n^{2k(k-1)}$.*

Proof. Fix a crossing k-tuple T. First we try to find a copy of K_k containing v_1 and avoiding $v_2, \dots, v_k$. By the minimum degree condition, there are at least

$$\prod_{i=2}^{k}\left(n-1-(i-1)\left(\frac{1}{k}+\alpha\right)n\right) \geq \prod_{i=2}^{k}\left(n-(i-1)\frac{n}{k}-((k-1)\alpha n+1)\right)$$

such copies of K_k. When $n \geq 3k^2$ and $\frac{1}{\alpha} \geq 3k^2$, we have $(k-1)\alpha n + 1 \leq n/(3k)$ and thus the number above is at least

$$\prod_{i=2}^{k}\left(n-(i-1)\frac{n}{k}-\frac{n}{3k}\right) \geq \left(\frac{n}{k}\right)^{k-1}, \quad \text{when } k \geq 3.$$

Fix such a copy of K_k on $\{v_1, u_2, u_3, \dots, u_k\}$. Consider u_2 and v_2. By Lemma 7 and the assumption that G does not satisfy the second property of the lemma, we can find at least $\alpha^3 n^{k-1}$ $(k-1)$-sets or $\alpha^3 n^{2k-1}$ $(2k-1)$-sets to connect u_2 and v_2. If S is a $(k-1)$-set that connects u_2 and v_2, then $S \cup K$ also connects u_2 and v_2 for any k-set K such that $G[K] \cong K_k$ and $K \cap S = \emptyset$. There are at least

$$(n-2)\prod_{i=2}^{k}\left(n-1-(i-1)\left(\frac{1}{k}+\alpha\right)n\right) \geq \frac{n}{2}\left(\frac{n}{k}\right)^{k-1}$$

copies of K_k in G avoiding u_2, v_2 and S. If there are at least $\alpha^3 n^{k-1}$ $(k-1)$-sets that connect u_2 and v_2, then at least

$$\alpha^3 n^{k-1} \cdot \frac{n}{2}\left(\frac{n}{k}\right)^{k-1} \frac{1}{\binom{2k-1}{k-1}} \geq 2\alpha^4 n^{2k-1}$$

$(2k-1)$-sets connect u_2 and v_2 because a $(2k-1)$-set can be counted at most $\binom{2k-1}{k-1}$ times. Since $2\alpha^4 < \alpha^3$, we can assume that there are always at least $2\alpha^4 n^{2k-1}$ $(2k-1)$-sets connecting u_2 and v_2. We inductively choose disjoint $(2k-1)$-sets that connects v_i and u_i for $i = 2, \dots, k$. For each i, we must avoid $T, u_2, \dots, u_k$, and $i-2$ previously selected $(2k-1)$-sets. Hence there are at least $2\alpha^4 n^{2k-1} - (2k-1)(i-1)n^{2k-2} > \alpha^4 n^{2k-1}$ choices of such $(2k-1)$-sets for each $i \geq 2$. Putting all these together, and using the assumption that α is sufficiently small, we have

$$|\mathcal{L}(T)| \geq \left(\frac{n}{k}\right)^{k-1} \cdot (\alpha^4 n^{2k-1})^{k-1} > \alpha^{4k-3} n^{2k(k-1)}. \quad \square$$

Every set $S \in \mathcal{L}(T)$ is *balanced* because $G[S]$ contains a K_k-factor and thus $|S \cap V_1| = \cdots = |S \cap V_k| = 2(k-1)$. Note that there are $\binom{n}{2(k-1)}^k$ balanced $2k(k-1)$-sets in G. Let $\mathcal{F}$ be the random family of $2k(k-1)$-sets obtained by selecting each balanced $2k(k-1)$-set from $V(G)$ independently with probability $p := \alpha^{4k-3} n^{1-2k(k-1)}$. Then by Chernoff's bound, since n is sufficiently large, with probability $1 - o(1)$, the family $\mathcal{F}$ satisfies the following properties:

$$|\mathcal{F}| \leq 2\mathbb{E}(|\mathcal{F}|) \leq 2p\binom{n}{2(k-1)}^k \leq \alpha^{4k-2} n, \tag{2}$$

$$|\mathcal{L}(T) \cap \mathcal{F}| \geq \frac{1}{2}\mathbb{E}(|\mathcal{L}(T) \cap \mathcal{F}|) \geq \frac{1}{2}p|\mathcal{L}(T)| \geq \frac{\alpha^{8k-6} n}{2} \quad \text{for every crossing } k\text{-tuple } T. \tag{3}$$

Let Y be the number of intersecting pairs of members of $\mathcal{F}$. Since each fixed balanced $2k(k-1)$-set intersects at most $2k(k-1)\binom{n-1}{2(k-1)-1}\binom{n}{2(k-1)}^{k-1}$ other balanced $2k(k-1)$-sets in G,

$$\mathbb{E}(Y) \leq p^2 \binom{n}{2(k-1)}^k 2k(k-1)\binom{n-1}{2k-3}\binom{n}{2(k-1)}^{k-1} \leq \frac{1}{8}\alpha^{8k-6} n.$$

By Markov's bound, with probability at least $\frac{1}{2}$, $Y \leq \alpha^{8k-6} n/4$. Therefore, we can find a family $\mathcal{F}$ satisfying (2), (3) and having at most $\alpha^{8k-6} n/4$ intersecting pairs. Remove one set from each of the intersecting pairs and the sets that have no K_k-factor from $\mathcal{F}$, we get a subfamily $\mathcal{F}'$ consisting of pairwise disjoint absorbing $2k(k-1)$-sets which satisfies $|\mathcal{F}'| \leq |\mathcal{F}| \leq \alpha^{4k-2} n$ and for all crossing T,

$$|\mathcal{L}(T) \cap \mathcal{F}'| \geq \frac{\alpha^{8k-6} n}{2} - \frac{\alpha^{8k-6} n}{4} \geq \frac{\alpha^{8k-6} n}{4}.$$

Since $\mathcal{F}'$ consists of disjoint absorbing sets and each absorbing set is covered by a K_k-matching, $V(\mathcal{F}')$ is covered by some K_k-matching M. Since $|\mathcal{F}'| \leq \alpha^{4k-2} n$, we have $|M| \leq 2k(k-1)\alpha^{4k-2} n/k = 2(k-1)\alpha^{4k-2} n$. Now consider a balanced set

$W \subseteq V(G) \setminus V(\mathcal{F}')$ such that $|W \cap V_1| = \cdots = |W \cap V_k| \le \alpha^{8k-6}n/4$. Arbitrarily partition W into at most $\alpha^{8k-6}n/4$ crossing k-tuples. We absorb each of the k-tuples with a different $2k(k-1)$-set from $\mathcal{L}(T) \cap \mathcal{F}'$. As a result, $V(\mathcal{F}') \cup W$ is covered by a K_k-matching, as desired. □

The rest of the paper is devoted to proving Lemma 7. First we prove a useful lemma. A weaker version of it appears in [19, Proposition 1.4] with a brief proof sketch.

Lemma 9. *Let $k \ge 2$ be an integer, $t \ge 1$ and $\epsilon \ll 1$. Let H be a k-partite graph on $V_1 \cup \cdots \cup V_k$ such that $|V_i| \ge (k-1)(1-\epsilon)t$ for all i and each vertex is nonadjacent to at most $(1+\epsilon)t$ vertices in each of the other color classes. Then either H contains at least $\epsilon^2 t^k$ copies of K_k, or H is $(16k^4\epsilon^{1/2^{k-2}}, 16k^4\epsilon^{1/2^{k-2}})$-approximate to $\Theta_{k\times(k-1)}(t)$.*

Proof. First we derive an upper bound for $|V_i|$, $i \in [k]$. Suppose for example, that $|V_k| \ge (k-1)(1+\epsilon)t + \epsilon t$. Then if we greedily construct copies of K_k while choosing the last vertex from V_k, by the minimum degree condition and $\epsilon \ll 1$, there are at least

$$\begin{aligned}&|V_1| \cdot (|V_2| - (1+\epsilon)t) \cdots (|V_{k-1}| - (k-2)(1+\epsilon)t) \cdot (|V_k| - (k-1)(1+\epsilon)t)\\ &\quad \ge (k-1)(1-\epsilon)t \cdot (k-2-k\epsilon)t \cdots (1-(2k-3)\epsilon)t \cdot \epsilon t\\ &\quad \ge \left(k-1-\frac{1}{2}\right)\left(k-2-\frac{1}{2}\right)\cdots\left(1-\frac{1}{2}\right)\epsilon t^k \ge \frac{\epsilon}{2}t^k\end{aligned}$$

copies of K_k in H, so we are done. We thus assume that for all i,

$$|V_i| \le (k-1)(1+\epsilon)t + \epsilon t < (k-1)(1+2\epsilon)t. \tag{4}$$

Now we proceed by induction on k. The base case is $k = 2$. If H has at least $\epsilon^2 t^2$ edges, then we are done. Otherwise $e(H) < \epsilon^2 t^2$. Using the lower bound for $|V_i|$, we obtain that

$$d(V_1, V_2) < \frac{\epsilon^2 t^2}{|V_1| \cdot |V_2|} \le \frac{\epsilon^2}{(1-\epsilon)^2} < \epsilon.$$

Hence H is $(2\epsilon, \epsilon)$-approximate to $\Theta_{2\times 1}(t)$. When $k = 2$, $16k^4\epsilon^{1/2^{k-2}} = 256\epsilon$, so we are done.

Now assume that $k \ge 3$ and the conclusion holds for $k-1$. Let H be a k-partite graph satisfying the assumptions and assume that H contains less than $\epsilon^2 t^k$ copies of K_k.

For simplicity, write $N_i(v) = N(v) \cap V_i$ for any vertex v. Let $V_1' \subset V_1$ be the vertices which are in at least ϵt^{k-1} copies of K_k in H, and let $\tilde{V}_1 = V_1 \setminus V_1'$. Note that $|V_1'| < \epsilon t$ otherwise we get at least $\epsilon^2 t^k$ copies of K_k in H. Fix $v_0 \in \tilde{V}_1$. For $2 \le i \le k$, by the minimum degree condition and $k \ge 3$,

$$|N_i(v_0)| \ge (k-1)(1-\epsilon)t - (1+\epsilon)t = (k-2)\left(1 - \frac{k}{k-2}\epsilon\right)t \ge (k-2)(1-3\epsilon)t.$$

On the other hand, following the same arguments as we used for (4), we derive that

$$|N_i(v_0)| \le (k-2)(1+2\epsilon t). \tag{5}$$

The minimum degree condition implies that a vertex in $N(v_0)$ misses at most $(1+\epsilon)t$ vertices in each $N_i(v_0)$. We now apply induction with $k-1$, t and 3ϵ on $H[N(v_0)]$. Because of the definition of V_1', we conclude that $N(v_0)$ is (ϵ', ϵ')-approximate to $\Theta_{(k-1)\times(k-2)}(t)$, where

$$\epsilon' := 16(k-1)^4(3\epsilon)^{1/2^{k-3}}.$$

This means that we can partition $N_i(v_0)$ into $A_{i1} \cup \cdots A_{i(k-2)}$ for $2 \le i \le k$ such that

$$\forall\, 2 \le i \le k,\ 1 \le j \le k-2, \quad (1-\epsilon')t \le |A_{ij}| \le (1+\epsilon')t \quad \text{and} \tag{6}$$

$$\forall\, 2 \le i < i' \le k,\ 1 \le j \le k-2, \quad d(A_{ij}, A_{i'j}) \le \epsilon'. \tag{7}$$

Furthermore, let $A_{i(k-1)} := V_i \setminus N(v_0)$ for $i = 2, \ldots, k$. By (5) and the minimum degree condition, we get that

$$(1-(3k-5)\epsilon)t \le |A_{i(k-1)}| \le (1+\epsilon)t, \tag{8}$$

for $i = 2, \ldots, k$.

Let $A_{ij}^c = V_i \setminus A_{ij}$ denote the complement of A_{ij}. Let $\bar{e}(A, B) = |A||B| - e(A, B)$ denote the number of non-edges between two disjoint sets A and B, and $\bar{d}(A, B) = \bar{e}(A, B)/(|A||B|) = 1 - d(A, B)$. Given two disjoint sets A and B (with density close to one) and $\alpha > 0$, we call a vertex $a \in A$ is *α-typical* to B if $\deg_B(a) \ge (1-\alpha)|B|$.

Claim 10. *Let $2 \le i \ne i' \le k$, $1 \le j \ne j' \le k-1$.*

(1) $d(A_{ij}, A_{i'j'}) \ge 1 - 3\epsilon'$ *and* $d(A_{ij}, A^c_{i'j}) \ge 1 - 3\epsilon'$.

(2) *All but at most* $\sqrt{3\epsilon'}$ *vertices in* A_{ij} *are* $\sqrt{3\epsilon'}$*-typical to* $A_{i'j'}$*; at most* $\sqrt{3\epsilon'}$ *vertices in* A_{ij} *are* $\sqrt{3\epsilon'}$*-typical to* $A^c_{i'j}$.

Proof. (1). Since $A^c_{i'j} = \bigcup_{j' \ne j} A_{i'j'}$, the second assertion $d(A_{ij}, A^c_{i'j}) \ge 1 - 3\epsilon'$ immediately follows from the first assertion $d(A_{ij}, A_{i'j'}) \ge 1 - 3\epsilon'$. Thus it suffices to show that $d(A_{ij}, A_{i'j'}) \ge 1 - 3\epsilon'$, or equivalently that $\bar{d}(A_{ij}, A_{i'j'}) \le 3\epsilon'$.

Assume $j \ge 2$. By (7), we have $e(A_{ij}, A_{i'j}) \le \epsilon'|A_{ij}||A_{i'j}|$. So $\bar{e}(A_{ij}, A_{i'j}) \ge (1-\epsilon')|A_{ij}||A_{i'j}|$. By the minimum degree condition and (6),

$$\begin{aligned}\bar{e}(A_{ij}, A^c_{i'j}) &\le [(1+\epsilon)t - (1-\epsilon')|A_{i'j}|]|A_{ij}| \\ &\le [(1+\epsilon)t - (1-\epsilon')(1-\epsilon')t]|A_{ij}| \\ &< (\epsilon + 2\epsilon')t|A_{ij}|,\end{aligned}$$

which implies that $\bar{e}(A_{ij}, A_{i'j'}) \le (\epsilon + 2\epsilon')t|A_{ij}|$ for any $j' \ne j$ and $1 \le j' \le k-1$. By (6) and (8), we have $|A_{i'j'}| \ge (1-\epsilon')t$. Hence

$$\bar{d}(A_{ij}, A_{i'j'}) \le (\epsilon + 2\epsilon')\frac{t}{|A_{i'j'}|} \le (\epsilon + 2\epsilon')\frac{t}{(1-\epsilon')t} \le 3\epsilon',$$

where the last inequality holds because $\epsilon \ll \epsilon' \ll 1$.

(2) Given two disjoint sets A and B, if $\bar{d}(A, B) \le \alpha$ for some $\alpha > 0$, then at most $\sqrt{\alpha}|A|$ vertices $a \in A$ satisfy $\deg_B(a) < \left(1 - \sqrt{\alpha}\right)|B|$. Hence Part (2) immediately follows from Part (1). $\square$

We need a lower bound for the number of copies of K_k in a dense k-partite graph.

Proposition 11. *Let G be a k-partite graph with vertex class $V_1, \ldots, V_k$. Suppose for every two vertex classes, the pairwise density $d(V_i, V_j) \ge 1 - \alpha$ for some $\alpha \le (k+1)^{-4}$, then there are at least $\frac{1}{2}\prod_i |V_i|$ copies of K_k in G.*

Proof. Given two disjoint sets V_i and V_j, if $\bar{d}(V_i, V_j) \le \alpha$ for some $\alpha > 0$, then at most $\sqrt{\alpha}|V_i|$ vertices $v \in V_i$ satisfy $\deg_{V_j}(v) < \left(1 - \sqrt{\alpha}\right)|V_j|$. Thus, by choosing typical vertices greedily and the assumption $\alpha \le (k+1)^{-4}$, there are at least

$$\left(1 - \sqrt{\alpha}\right)|V_1|\left(1 - 2\sqrt{\alpha}\right)|V_2| \cdots \left(1 - k\sqrt{\alpha}\right)|V_k| > \left(1 - (1 + \cdots + k)\sqrt{\alpha}\right)\prod_i |V_i| > \frac{1}{2}\prod_i |V_i|$$

copies of K_k in G. $\square$

Let $\epsilon'' = 2k\sqrt{\epsilon'}$. Now we want to study the structure of $\tilde{V}_1$.

Claim 12. *Given $v \in \tilde{V}_1$ and $2 \le i \le k$, there exists $j \in [k-1]$, such that $|N_{A_{ij}}(v)| < \epsilon'' t$.*

Proof. Suppose instead, that there exist $v \in \tilde{V}_1$ and some $2 \le i_0 \le k$, such that $|N_{A_{i_0j}}(v)| \ge \epsilon'' t$ for all $j \in [k-1]$. By the minimum degree condition, for each $2 \le i \le k$, there is at most one $j \in [k-1]$ such that $|N_{A_{ij}}(v)| < t/3$. Therefore we can greedily choose $k-2$ distinct j_i for $i \ne i_0$, such that $|N_{A_{ij_i}}(v)| \ge t/3$. Let j_{i_0} be the (unique) unused index. Note that

$$\forall\, i \ne i_0, \quad \frac{|A_{ij_i}|}{|N_{A_{ij_i}}(v)|} \le \frac{(1+\epsilon')t}{t/3} < 4, \quad \text{and} \quad \frac{|A_{i_0j_{i_0}}|}{|N_{A_{i_0j_{i_0}}}(v)|} \le \frac{(1+\epsilon')t}{\epsilon'' t} < \frac{2}{\epsilon''}.$$

So for any $i \ne i'$, by Claim 10 and the definition of ϵ'', we have

$$\bar{d}(N_{A_{ij_i}}(v), N_{A_{i'j_{i'}}}(v)) \le \frac{3\epsilon'|A_{ij_i}||A_{i'j_{i'}}|}{|N_{A_{ij_i}}(v)||N_{A_{i'j_{i'}}}(v)|} \le 3\epsilon' \cdot 4 \cdot \frac{2}{\epsilon''} = \frac{6}{k^2}\epsilon''. \tag{9}$$

Since $\epsilon \ll \epsilon'' \ll 1$, by Proposition 11, there are at least

$$\frac{1}{2}\prod_i N_{A_{ij_i}}(v) \ge \frac{1}{2} \cdot \epsilon'' t \left(\frac{t}{3}\right)^{k-2} = \frac{\epsilon''}{2 \cdot 3^{k-2}} t^{k-1} > \epsilon t^{k-1}$$

copies of K_{k-1} in $N(v)$, contradicting the assumption $v \in \tilde{V}_1$. $\square$

Note that if $\deg_{A_{ij}}(v) < \epsilon'' t$, at least $|A_{ij}| - \epsilon'' t$ vertices of A_{ij} are not in $N(v)$. By the minimum degree condition, (6) and (8), it follows that

$$|A_{ij}^c \setminus N(v)| \le (1+\epsilon)t - (|A_{ij}| - \epsilon'' t) \le (1+\epsilon)t - (1-\epsilon')t + \epsilon'' t \le 2\epsilon'' t. \quad (10)$$

Fix a vertex $v \in \tilde{V}_1$. Given $2 \le i \le k$, let ℓ_i denote the (unique) index such that $|N_{A_{i\ell_i}}(v)| < \epsilon'' t$ (the existence of ℓ_i follows from Claim 12).

Claim 13. *We have $\ell_2 = \ell_3 = \cdots = \ell_k$.*

Proof. Otherwise, say $\ell_2 \ne \ell_3$, then we set $j_2 = \ell_3$ and for $3 \le i \le k$, greedily choose distinct $j_k, j_{k-1}, \dots, j_3 \in [k-1] \setminus \{\ell_3\}$ such that $j_i \ne \ell_i$ (this is possible as j_3 is chosen at last). Let us bound the number of copies of K_{k-1} in $\bigcup_{i=2}^k N_{A_{ij_i}}(v)$. By (10), we get $|N_{A_{ij_i}}(v)| \ge |A_{ij_i}| - 2\epsilon'' t \ge t/2$ for all i. As in (9), for any $i \ne i'$, we derive that $\bar{d}(N_{A_{ij_i}}(v), N_{A_{i'j_{i'}}}(v)) \le 3\epsilon'' \cdot 4 \cdot 4 = 48\epsilon''$. Since $\epsilon'' \ll 1$, by Proposition 11, we get at least $\frac{1}{2}\left(\frac{t}{2}\right)^{k-1} > \epsilon t^{k-1}$ copies of K_{k-1} in $N(v)$, a contradiction. □

We define $A_{1j} := \{v \in \tilde{V}_1 : |N_{A_{2j}}(v)| < \epsilon'' t\}$ for $j \in [k-1]$. By Claims 12 and 13, this yields a partition of $\tilde{V}_1 = \bigcup_{j=1}^{k-1} A_{1j}$ such that

$$d(A_{1j}, A_{ij}) < \frac{\epsilon'' t |A_{1j}|}{|A_{1j}||A_{ij}|} \le \frac{\epsilon'' t}{(1-\epsilon')t} < (1+2\epsilon')\epsilon'' \quad \text{for } i \ge 2 \text{ and } j \ge 1. \quad (11)$$

By (6), (8) and (10), as $(3k-5)\epsilon \le \epsilon'$, we have

$$\bar{d}(A_{1j}, A_{ij'}) < \frac{|A_{1j}| 2\epsilon'' t}{|A_{1j}||A_{ij'}|} \le \frac{2\epsilon'' t}{(1-\epsilon')t} < 3\epsilon'' \quad \text{for } i \ge 2 \text{ and } j \ne j'. \quad (12)$$

We claim $|A_{1j}| \le (1+\epsilon)t + (1+2\epsilon')\epsilon''|A_{1j}|$ for all j. Otherwise, by the minimum degree condition, we have $\deg_{A_{1j}}(v) > (1+2\epsilon')\epsilon''|A_{1j}|$ for all $v \in A_{ij}$, and consequently $d(A_{1j}, A_{ij}) > (1+2\epsilon')\epsilon''$, contradicting (11). We thus conclude that

$$|A_{1j}| \le \frac{1+\epsilon}{1-(1+2\epsilon')\epsilon''} t < (1+2\epsilon'')t. \quad (13)$$

Since $|V_1'| \le \epsilon t$, we have $|\bigcup_{j=1}^{k-1} A_{1j}| = |V_1 \setminus V_1'| \ge |V_1| - \epsilon t$. Using (13), we now obtain a lower bound for $|A_{1j}|$, $j \in [k-1]$:

$$|A_{1j}| \ge (k-1)(1-\epsilon)t - (k-2)(1+2\epsilon'')t - \epsilon t \ge (1-2k\epsilon'')t. \quad (14)$$

It remains to show that for $2 \le i \ne i' \le k$, $d(A_{i(k-1)}, A_{i'(k-1)})$ is small. Write $N(v_1 \cdots v_m) = \bigcap_{1 \le i \le m} N(v_i)$.

Claim 14. *$d(A_{i(k-1)}, A_{i'(k-1)}) \le 6\epsilon''$ for $2 \le i, i' \le k$.*

Proof. Suppose to the contrary, that say $d(A_{(k-1)(k-1)}, A_{k(k-1)}) > 6\epsilon''$. We first select $k-2$ sets A_{ij} with $1 \le i \le k-2$ and $1 \le j \le k-2$ such that no two of them are on the same row or column – there are $(k-2)!$ choices. Fix one of them, say $A_{11}, A_{22}, \dots, A_{(k-2)(k-2)}$. We construct copies of K_{k-2} in $A_{11} \cup A_{22} \cup \cdots \cup A_{(k-2)(k-2)}$ as follows. Pick arbitrary $v_1 \in A_{11}$. For $2 \le i \le k-2$, we select $v_i \in N_{A_{ii}}(v_1 \cdots v_{i-1})$ such that v_i is $\sqrt{3\epsilon'}$-typical to $A_{(k-1)(k-1)}, A_{k(k-1)}$ and all A_{jj}, $i < j \le k-2$. By Claim 10 and (10), there are at least $(1-(k-2)\sqrt{3\epsilon'})|A_{ii}| - 2\epsilon'' t \ge t/2$ choices for each v_i. After selecting $v_1, \dots, v_{k-2}$, we select adjacent vertices $v_{k-1} \in A_{(k-1)(k-1)}$ and $v_k \in A_{k(k-1)}$ such that $v_{k-1}, v_k \in N(v_1 \cdots v_{k-2})$. For $j \in \{k-1, k\}$, we know that $N(v_1)$ misses at most $2\epsilon'' t$ vertices in $A_{j(k-1)}$, and at most $(k-3)\sqrt{3\epsilon'}|A_{j(k-1)}|$ vertices of $A_{j(k-1)}$ are not in $N(v_2 \cdots v_{k-2})$. Since $d(A_{(k-1)1}, A_{k1}) > 6\epsilon''$ and $\epsilon'' = 2k\sqrt{\epsilon'}$, there are at least

$$\begin{aligned} &6\epsilon''|A_{(k-1)(k-1)}||A_{k(k-1)}| - 2\epsilon'' t(|A_{(k-1)(k-1)}| + |A_{k(k-1)}|) - 2(k-3)\sqrt{3\epsilon'}|A_{(k-1)(k-1)}||A_{k(k-1)}| \\ &\ge \left(6\epsilon'' - 4\epsilon'' - 4(k-3)\sqrt{\epsilon'}\right)|A_{(k-1)(k-1)}||A_{k(k-1)}| \\ &= 12\sqrt{\epsilon'}|A_{(k-1)(k-1)}||A_{k(k-1)}| \ge 6\sqrt{\epsilon'} t^2 \end{aligned}$$

such pairs v_{k-1}, v_k. Together with the choices of $v_1, \dots, v_{k-2}$, we obtain at least $(k-2)!\left(\frac{t}{2}\right)^{k-2} 6\sqrt{\epsilon'} t^2 > \epsilon t^k$ copies of K_k, a contradiction. □

In summary, by (6), (8), (13) and (14), we have $(1-2k\epsilon'')t \le |A_{ij}| \le (1+2\epsilon'')t$ for all i and j. In order to make $\bigcup_{j=1}^{k-1} A_{1j}$ a partition of V_1, we move the vertices of V_1' to A_{11}. Since $|V_1'| < \epsilon t$, we still have $||A_{ij}| - t| \le 2k\epsilon'' t$ after moving these vertices. On the other hand, by (7), (11), and Claim 14, we have $d(A_{ij}, A_{i'j}) \le 6\epsilon'' \le 2k\epsilon''$ for $i \ne i'$ and all j (we now have

$d(A_{11}, A_{i1}) \le 2\epsilon''$ for all $i \ge 2$ because $|A_{11}|$ becomes slightly larger). Therefore H is $(2k\epsilon'', 2k\epsilon'')$-approximate to $\Theta_{k\times(k-1)}(t)$. By the definitions of ϵ'' and ϵ',

$$2k\epsilon'' = 4k^2\sqrt{\epsilon'} = 4k^2\sqrt{16(k-1)^4(3\epsilon)^{1/2^{k-3}}} \le 16k^4\epsilon^{1/2^{k-2}},$$

where the last inequality is equivalent to $\left(\frac{k-1}{k}\right)^2 3^{1/2^{k-2}} \le 1$ or $3^{1/2^{k-1}} \le \frac{k}{k-1}$, which holds because $3 \le 1 + \frac{2^{k-1}}{k-1} \le \left(1+\frac{1}{k-1}\right)^{2^{k-1}}$ for $k \ge 2$.

This completes the proof of Lemma 9. □

We are ready to prove Lemma 7.

Proof of Lemma 7. First assume that $G \in \mathcal{G}_3(n)$ is minimal, namely, G satisfies the minimum partite degree condition but removing any edge of G will destroy this condition. Note that this assumption is only needed by Claim 20.

Given $0 < \Delta \le 1$, let

$$\alpha = \frac{1}{2k}\left(\frac{\Delta}{24k(k-1)\sqrt{2k}}\right)^{2^{k-1}}. \tag{15}$$

Without loss of generality, assume that $x, y \in V_1$ and y is *not* reachable by $\alpha^3 n^{k-1}$ $(k-1)$-sets or $\alpha^3 n^{2k-1}$ $(2k-1)$-sets from x.

For $2 \le i \le k$, define

$$A_{i1} = V_i \cap (N(x) \setminus N(y)), \qquad A_{ik} = V_i \cap (N(y) \setminus N(x)),$$
$$B_i = V_i \cap (N(x) \cap N(y)), \qquad A_{i0} = V_i \setminus (N(x) \cup N(y)).$$

Let $B = \bigcup_{i\ge 2} B_i$. If there are at least $\alpha^3 n^{k-1}$ copies of K_{k-1} in B, then x is reachable from y by at least $\alpha^3 n^{k-1}$ $(k-1)$-sets. We thus assume there are less than $\alpha^3 n^{k-1}$ copies of K_{k-1} in B.

Clearly, for $i \ge 2$, A_{i1}, A_{ik}, B_i and A_{i0} are pairwise disjoint. The following claim bounds the sizes of A_{ik}, B_i and A_{i0}.

Claim 15. (1) $(1-k^2\alpha)\frac{n}{k} < |A_{i1}|, |A_{ik}| \le (1+k\alpha)\frac{n}{k}$,
(2) $(k-2-2k\alpha)\frac{n}{k} \le |B_i| < (k-2+k(k-1)\alpha)\,\frac{n}{k}$,
(3) $|A_{i0}| < (k+1)\alpha n$.

Proof. For $v \in V$, and $i \in [k]$, write $N_i(v) := N(v) \cap V_i$. By the minimum degree condition, we have $|A_{i1}|, |A_{ik}| \le (1/k+\alpha)n$. Since $N_i(x) = A_{i1} \cup B_i$, it follows that

$$|B_i| \ge \left(\frac{k-1}{k} - \alpha\right)n - \left(\frac{1}{k} + \alpha\right)n. \tag{16}$$

If some B_i, say B_k, has at least $\left(\frac{k-2}{k} + (k-1)\alpha\right)n$ vertices, then there are at least $\prod_{i=2}^{k} |B_i| - (i-2)\left(\frac{1}{k}+\alpha\right)n$ copies of K_{k-1} in B. By (16) and $|B_k| \ge \left(\frac{k-2}{k} + (k-1)\alpha\right)n$, this is at least

$$\begin{aligned}
\alpha n \cdot \prod_{i=2}^{k-1}\left(\frac{k-1}{k} - \alpha\right)n - (i-1)\left(\frac{1}{k}+\alpha\right)n &= \alpha n \cdot \prod_{i=2}^{k-1}\left(\frac{k-i}{k} - i\alpha\right)n \\
&\ge \alpha n \cdot \prod_{i=2}^{k-1}\left(\frac{k-i-\frac{1}{2}}{k}\right)n \quad \text{because } 2k^2\alpha \le 1, \\
&\ge \alpha n \cdot \frac{1}{2}\left(\frac{n}{k}\right)^{k-2} \\
&\ge \alpha^2 n^{k-1} \quad \text{because } 2k^{k-2}\alpha \le 1.
\end{aligned}$$

This is a contradiction.

We may thus assume that $|B_i| < \left(\frac{k-2}{k} + (k-1)\alpha\right)n$ for $2 \le i \le k$, as required for Part (2). As $N_i(x) = A_{i1} \cup B_i$, it follows that

$$|A_{i1}| > \left(\frac{k-1}{k} - \alpha\right)n - \left(\frac{k-2}{k} + (k-1)\alpha\right)n = \left(\frac{1}{k} - k\alpha\right)n.$$

The same holds for $|A_{ik}|$ thus Part (1) follows. Finally

$$|A_{i0}| = |V_i| - |N_i(x)| - |A_{ik}| < n - \left(\frac{k-1}{k} - \alpha\right)n - \left(\frac{1}{k} - k\alpha\right)n = (k+1)\alpha n,$$

as required for Part (3). □

Let $t = n/k$ and $\epsilon = 2k\alpha$. By the minimum degree condition, every vertex $u \in B$ is nonadjacent to at most $(1 + k\alpha)n/k < (1+\epsilon)t$ vertices in other color classes of B. By Claim 15, $|B_i| \geq (k-2-2k\alpha)\frac{n}{k} = (k-2-\epsilon)t \geq (k-2)(1-\epsilon)t$. Thus $G[B]$ is a $(k-1)$-partite graph that satisfies the assumptions of Lemma 9. We assumed that B contains less than $\alpha^3 n^{k-1} < \epsilon^2 t^{k-1}$ copies of K_{k-1}, so by Lemma 9, B is (α', α')-approximate to $\Theta_{(k-1)\times(k-2)}\left(\frac{n}{k}\right)$, where

$$\alpha' := 16(k-1)^4(2k\alpha)^{1/2^{k-3}}.$$

This means that we can partition B_i, $2 \leq i \leq k$, into $A_{i2} \cup \cdots A_{i(k-1)}$ such that $(1-\alpha')\frac{n}{k} \leq |A_{ij}| \leq (1+\alpha')\frac{n}{k}$ for $2 \leq j \leq k-1$ and

$$\forall\, 2 \leq i < i' \leq k, 2 \leq j \leq k-1, \quad d(A_{ij}, A_{i'j}) \leq \alpha'. \tag{17}$$

Together with Claim 15 Part (1), we obtain that (using $k^2\alpha \leq \alpha'$)

$$\forall\, 2 \leq i \leq k, 1 \leq j \leq k, \quad (1-\alpha')\frac{n}{k} \leq |A_{ij}| \leq (1+\alpha')\frac{n}{k}. \tag{18}$$

Let $A_{ij}^c = V_i \setminus A_{ij}$ denote the complement of A_{ij}. The following claim is an analog of Claim 10, and its proof is almost the same — after we replace $(1+\epsilon)t$ with $(1+k\alpha)n/k$ and ϵ' with α' (and we use $\alpha \ll \alpha'$). We thus omit the proof.

Claim 16. *Let $2 \leq i \neq i' \leq k$, $1 \leq j \neq j' \leq k$, and $\{j, j'\} \neq \{1, k\}$.*

(1) *$d(A_{ij}, A_{i'j'}) \geq 1 - 3\alpha'$ and $d(A_{ij}, A_{i'j}^c) \geq 1 - 3\alpha'$.*
(2) *All but at most $\sqrt{3\alpha'}$ vertices in A_{ij} are $\sqrt{3\alpha'}$-typical to $A_{i'j'}$; at most $\sqrt{3\alpha'}$ vertices in A_{ij} are $\sqrt{3\alpha'}$-typical to $A_{i'j}^c$.* □

Now let us study the structure of V_1. Let $\alpha'' = 2k\sqrt{\alpha'}$. Recall that $N(xv) = N(x) \cap N(v)$. Let V_1' be the set of the vertices $v \in V_1$ such that there are at least αn^{k-1} copies of K_{k-1} in each of $N(xv)$ and $N(yx)$. We claim that $|V_1'| < 2\alpha n$. Otherwise, since a $(k-1)$-set intersects at most $(k-1)n^{k-2}$ other $(k-1)$-sets, there are at least

$$2\alpha n \cdot \alpha n^{k-1}(\alpha n^{k-1} - (k-1)n^{k-2}) > \alpha^3 n^{2k-1}$$

copies of $(2k-1)$-sets connecting x and y, a contradiction.

Let $\tilde{V}_1 := V_1 \setminus V_1'$. The following claim is an analog of Claim 12 for Lemma 9 and can be proved similarly. The only difference between their proofs is that here we find at least αn^{k-1} copies of K_{k-1} in each of $N(xv)$ and $N(yv)$, which contradicts the definition of $\tilde{V}_1$.

Claim 17. *Given $v \in \tilde{V}_1$ and $2 \leq i \leq k$, there exists $j \in [k]$ such that $|N_{A_{ij}}(v)| < \alpha'' t$.* □

Fix an vertex $v \in \tilde{V}_1$. Claim 17 implies that for each $2 \leq i \leq k$, there exists ℓ_i such that $|N_{A_{i\ell_i}}(v)| < \alpha'' t$. Our next claim is an analog of Claim 13 for Lemma 9 and can be proved similarly.

Claim 18. *We have $\ell_2 = \ell_3 = \cdots = \ell_k$.* □

We now define $A_{1j} := \{v \in \tilde{V}_1 : |N_{A_{2j}}(v)| < \alpha'' t\}$ for $j \in [k]$. By Claims 17 and 18, this yields a partition of $\tilde{V}_1 = \bigcup_{j=1}^k A_{1j}$ such that

$$d(A_{1j}, A_{ij}) < \frac{\alpha'' t |A_{1j}|}{|A_{1j}||A_{ij}|} \leq \frac{\alpha'' t}{(1-\alpha')t} < (1+2\alpha')\alpha'' \quad \text{for } i \geq 2 \text{ and } j \geq 1. \tag{19}$$

For $v \in A_{1j}$, we have $|N_{A_{ij}}(v)| < \alpha'' t$ for $i \geq 2$. By the minimum degree condition and (18),

$$|A_{ij}^c \setminus N(v)| \leq \left(\frac{1}{k} + \alpha\right) n - (|A_{ij}| - \alpha'' t) < 2\alpha'' t. \tag{20}$$

By (18) and (20), we derive that

$$\bar{d}(A_{1j}, A_{ij'}) < \frac{|A_{1j}| \cdot 2\alpha'' t}{|A_{1j}||A_{ij'}|} \leq \frac{2\alpha'' t}{(1-\alpha')t} < 3\alpha'' \quad \text{for } i \geq 2 \text{ and } j \neq j'. \tag{21}$$

We claim that $|A_{1j}| \leq (1+\alpha)t + (1+2\alpha')\alpha''|A_{1j}|$ for all j. Otherwise, by the minimum degree condition, we have $\deg_{A_{1j}}(v) > (1+2\alpha')\alpha''|A_{1j}|$ for all $v \in A_{ij}$, and consequently $d(A_{1j}, A_{ij}) > (1+2\alpha')\alpha''$, contradicting (19). We thus conclude that

$$|A_{1j}| \leq \frac{1+\alpha}{1-(1+2\alpha')\alpha''} t < (1+2\alpha'')\frac{n}{k}. \tag{22}$$

Since $|V_1'| \le 2\alpha n$, we have $|\bigcup_{j=1}^k A_{1j}| = |V_1 \setminus V_1'| \ge |V_1| - 2\alpha n$. Using (22), we now obtain a lower bound for $|A_{1j}|, j \in [k]$.

$$|A_{1j}| \ge n - (k-1)(1+2\alpha'')\frac{n}{k} - 2\alpha n \ge (1-2k\alpha'')\frac{n}{k}. \tag{23}$$

It remains to show that $d(A_{i1}, A_{i'1})$ and $d(A_{ik}, A_{i'k})$, $2 \le i, i' \le k$, are small. First we show that if both densities are reasonably large then there are too many reachable $(2k-1)$-sets from x to y. The proof resembles the one of Claim 14.

Claim 19. *For $2 \le i \ne i' \le k$, either $d(A_{i1}, A_{i'1}) \le 6\alpha''$ or $d(A_{ik}, A_{i'k}) \le 6\alpha''$.*

Proof. Suppose instead, that say $d(A_{(k-1)1}, A_{k1}), d(A_{(k-1)k}, A_{kk}) > 6\alpha''$. Fix a vertex v_1 in A_{1j}, for some $2 \le j \le k-1$, say $v_1 \in A_{12}$. We construct two vertex disjoint copies of K_{k-1} in $N(xv_1)$ and $N(yv_1)$ as follows. We first select $k-3$ sets A_{ij} with $2 \le i \le k-2$ and $3 \le j \le k-1$ such that no two of them are on the same row or column – there are $(k-3)!$ choices. Fix one of them, say $A_{23}, \ldots, A_{(k-2)(k-1)}$. For $2 \le i \le k-2$, we select $v_i \in N_{A_{i(i+1)}}(v_1 \cdots v_{i-1})$ that is $\sqrt{3\alpha'}$-typical to $A_{(k-1)1}, A_{k1}$ and $A_{j(j+1)}$, $i < j \le k-2$. By Claim 16 and (20), there are at least

$$\left(1-(k-2)\sqrt{3\alpha'}\right)|A_{i(i+1)}| - (k\alpha + \alpha' + \alpha'')\frac{n}{k} \ge \frac{n}{2k}$$

such v_i. After selecting $v_2, \ldots, v_{k-2}$, we select two adjacent vertices $v_{k-1} \in A_{(k-1)1}$ and $v_k \in A_{k1}$ such that v_{k-1} and v_k are in $N(v_1 \cdots v_{k-2})$. For $j = k-1, k$, we know that $N(v_1)$ misses at most $(k\alpha + \alpha' + \alpha'')n/k$ vertices in A_{j1} and at most $(k-3)\sqrt{3\alpha'}|A_{j1}|$ vertices of A_{j1} are not in $N(v_2 \cdots v_{k-2})$. Since $d(A_{(k-1)1}, A_{k1}) > 6\alpha''$, there are at least

$$6\alpha''|A_{(k-1)1}||A_{k1}| - (k\alpha + \alpha' + \alpha'')\frac{n}{k}(|A_{(k-1)1}| + |A_{k1}|) - 2(k-3)\sqrt{3\alpha'}|A_{(k-1)1}||A_{k1}| \ge 6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2$$

such pairs v_{k-1}, v_k. Hence $N(xv_1)$ contains at least

$$(k-3)!\left(\frac{n}{2k}\right)^{k-3} 6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2 \ge \sqrt{\alpha'}\left(\frac{n}{k}\right)^{k-1} \ge \sqrt{\alpha} n^{k-1}$$

copies of K_{k-1}. Let C be such a copy of K_{k-1}. Then we follow the same procedure and construct a copy of K_{k-1} on $N(yv_1) \setminus C$. After fixing $k-3$ sets A_{ij} with $2 \le i \le k-2$ and $3 \le j \le k-1$ such that no two of them are on the same row or column, there are still at least $\frac{n}{2k}$ such v_i for $2 \le i \le k-2$. Then, as $d(A_{ik}, A_{i'k}) > 6\alpha''$, there are at least $6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2$ choices of $v_{k-1} \in A_{(k-1)k}$ and $v_k \in A_{kk}$ such that v_{k-1} and v_k are in $N(v_1 \cdots v_{k-2})$. This gives at least $\sqrt{\alpha}n^{k-1}$ copies of K_{k-1} in $N(yv_1)$. Then, since there are at least $|V_1| - |A_{11}| - |A_{1k}| \ge \alpha n$ choices of v_1, totally there are at least $\alpha n\left(\sqrt{\alpha}n^{k-1}\right)^2 = \alpha^2 n^{2k-1}$ reachable $(2k-1)$-sets from x to y, a contradiction. □

Next we show that if any of $d(A_{i1}, A_{i'1})$ or $d(A_{ik}, A_{i'k})$, $2 \le i, i' \le k$, is sufficiently large, then we can remove edges from G such that the resulting graph still satisfies the minimum degree condition, which contradicts the assumption that G is minimal.

Claim 20. *For $2 \le i \ne i' \le k$, $d(A_{i1}, A_{i'1}), d(A_{ik}, A_{i'k}) \le 6k\sqrt{\alpha''}$.*

Proof. Without loss of generality, assume that $d(A_{2k}, A_{3k}) > 6k\sqrt{\alpha''}$. By Claim 19, we have $d(A_{21}, A_{31}) < 6\alpha''$. Combining this with (17), we have $d(A_{2j}, A_{3j}) < 6\alpha''$ for all $j \in [k-1]$. Now fix $j \in [k-1]$. The number of non-edges between A_{2j} and A_{3j} satisfies $\bar{e}(A_{2j}, A_{3j}) > (1-6\alpha'')|A_{2j}||A_{3j}|$. By the minimum degree condition and (18),

$$\bar{e}(A_{2k}, A_{3j}) < (1+ k\alpha)\frac{n}{k}|A_{3j}| - (1-6\alpha'')|A_{2j}||A_{3j}| \le 7\alpha''\frac{n}{k}|A_{3j}|.$$

Using (18) again, we obtain that

$$d(A_{2k}, A_{3j}) \ge 1 - \frac{7\alpha''\frac{n}{k}|A_{3j}|}{|A_{2k}||A_{3j}|} \ge 1 - 8\alpha''.$$

This implies that $d(A_{2k}, A_{3k}^c) \ge 1 - 8\alpha''$. Similarly we derive that $d(A_{3k}, A_{2k}^c) \ge 1 - 8\alpha''$. For $i = 2, 3$, define A_{ik}^T as the set of the vertices in A_{ik} that are $\sqrt{8\alpha''}$-typical to $A_{(5-i)k}^c$. Thus $|A_{ik} \setminus A_{ik}^T| \le \sqrt{8\alpha''}|A_{ik}|$.

Let $A_{ik}^{T_1} = \left\{v \in A_{ik}^T : \deg_{A_{(5-i)k}}(v) \le \sqrt{8\alpha''}|A_{jk}^c|\right\}$ and $A_{ik}^{T_2} = A_{ik}^T \setminus A_{ik}^{T_1}$. For $u \in A_{2k}^{T_2}$, we have

$$\deg_{V_3}(u) = \deg_{A_{3k}^c}(u) + \deg_{A_{3k}}(u) > \left(1-\sqrt{8\alpha''}\right)|A_{3k}^c| + \sqrt{8\alpha''}|A_{3k}^c| = |A_{3k}^c|.$$

Since $|A_{3k}^c| \ge \deg_{V_3}(x)$ and $|A_{3k}^c|$ is an integer, we conclude that $\deg_{V_3}(u) \ge \deg_{V_3}(x) + 1$. Similarly we can derive that $\deg_{V_2}(v) \ge \deg_{V_2}(x) + 1$ for every $v \in A_{3k}^{T_2}$. If there is an edge uv joining some $u \in A_{2k}^{T_2}$ and some $v \in A_{3k}^{T_2}$, then we can delete this edge and the resulting graph still satisfies the minimum degree condition. Therefore we may assume that there is no edge between $A_{2k}^{T_2}$ and $A_{3k}^{T_2}$. Then

$$\begin{aligned} e(A_{2k}, A_{3k}) &\le e(A_{2k} \setminus A_{2k}^{T}, A_{3k}) + e(A_{2k}, A_{3k} \setminus A_{3k}^{T}) + e(A_{2k}^{T_1}, A_{3k}^{T}) + e(A_{2k}^{T}, A_{3k}^{T_1}) \\ &\le 2\sqrt{8\alpha''}|A_{2k}||A_{3k}| + |A_{2k}^{T_1}|\sqrt{8\alpha''}|A_{3k}^{c}| + |A_{3k}^{T_1}|\sqrt{8\alpha''}|A_{2k}^{c}| \\ &\le \sqrt{8\alpha''}\left(2|A_{2k}||A_{3k}| + |A_{2k}||A_{3k}^{c}| + |A_{3k}||A_{2k}^{c}|\right) \\ &= \sqrt{8\alpha''}\,(|A_{2k}||V_3| + |A_{3k}||V_2|) \\ &\le 3\sqrt{\alpha''}\cdot 2k|A_{2k}||A_{3k}| \quad \text{by (18)}. \end{aligned}$$

Therefore $d(A_{2k}, A_{3k}) \le 6k\sqrt{\alpha''}$. □

In summary, by (18), (22) and (23), we have $(1 - 2k\alpha'')\frac{n}{k} \le |A_{ij}| \le (1 + 2\alpha'')\frac{n}{k}$ for all i and j. In order to make $\bigcup_{j=1}^{k} A_{ij}$ a partition of V_i, we move the vertices of V_1' to A_{11} and the vertices of A_{i0} to A_{i2} for $2 \le i \le k$. Since $|V_1'| < 2\alpha n$ and $|A_{i0}| \le (k+1)\alpha n$, we have $||A_{ij}| - \frac{n}{k}| \le 2k\alpha''\frac{n}{k}$ after moving these vertices. On the other hand, by (17), (19), and Claim 20, we have $d(A_{ij}, A_{i'j}) \le 6k\sqrt{\alpha''}$ for $i \ne i'$ and all j. (In fact, for $i \ge 2$, we now have $d(A_{11}, A_{i1}) \le 2\alpha''$ as we added at most $2\alpha n$ vertices to A_{11}. For $i' > i \ge 2$, we now have $d(A_{12}, A_{i2}) \le 2\alpha''$ and $d(A_{i2}, A_{i'2}) \le 2\alpha'$ as we moved at most $(k+1)\alpha n$ vertices to A_{i2}.) Therefore after deleting edges, G is $\left(2k\alpha'', 6k\sqrt{\alpha''}\right)$-approximate to $\Theta_{k\times k}(n/k)$. By (15), and the definitions of α'' and α', G is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k\times k}(n/k)$. □

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