Complete tripartite subgraphs of balanced tripartite graphs with large minimum degree

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Abstract

In 1975 Bollobás, Erdős, and Szemerédi [4] asked what minimum degree guarantees an octahedral subgraph $K_3(2)$ in any tripartite graph G with n vertices in each vertex class. We show that $\delta(G) \geq n + 2n^{\frac{5}{6}}$ suffices thus improving the bound $n + (1 + o(1))n^{\frac{11}{12}}$ of Bhalkikar and Zhao [2] obtained by following the approach of [4]. Bollobás, Erdős, and Szemerédi conjectured that $n + cn^{\frac{1}{2}}$ suffices and there are many $K_3(2)$-free tripartite graphs G with $\delta(G) \geq n + cn^{\frac{1}{2}}$. We confirm this conjecture under the additional assumption that every vertex in G is adjacent to at least $(1/5 + \varepsilon)n$ vertices in any other vertex class.

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1 Introduction

As a foundation stone of extremal graph theory, the celebrated Turán's theorem in 1941 [14] determined the maximum size $\mathrm{ex}(n, K_t)$ of graphs G of order n without K_t as a subgraph (the $t = 3$ case is also known as Mantel's theorem [11]). In 1974, Bollobás, Erdős, and Straus [3] obtained a Turán-type result that determined the maximum size of an r-partite graph not containing K_t as a subgraph for any $r \geq t$. Instead of considering the size, in 1975, Bollobás, Erdős, and Szemerédi [4] investigated the following minimum degree version of this problem.

Problem 1.1. *Given integers n and $3 \leq t \leq r$, what is the largest minimum degree $\delta(G)$ among all r-partite graphs G with parts of size n and which do not contain a copy of K_t?*

The $r = t$ case of Problem 1.1 has received a lot of attention and found applications in linear arboricity, hypergraph matching, list coloring, etc. Graver (see [4]) answered Problem 1.1 for $r = t = 3$ and Jin [8] solved it for $r = t = 4, 5$. The $r = t$ case of Problem 1.1 was finally settled by Haxell and Szabó [7] and Szabó and Tardos [13]. Recently Lo, Treglown, and Zhao [10] solved many $r > t$ cases of the problem, including when $r \equiv -1 \pmod{t-1}$ and $r \geq (3t - 4)(t - 2)$. For more related results, we refer interested readers to [1, 6, 7, 8, 13].

Let $G_r(n)$ be an (arbitrary) *balanced r-partite graph* with parts of size n, and let $K_r(s)$ denote the complete r-partite graph with parts of size s. In the same paper Bollobás, Erdős,

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and Szemerédi [4] asked when $G_3(n)$ contains $K_3(2)$ (known as the *octahedral graph*) as a subgraph.

Problem 1.2. *What minimum degree $\delta(G)$ in a graph $G = G_3(n)$ ensures a subgraph $K_3(2)$?*

The authors of [4] conjectured that $\delta(G) \geq n + cn^{\frac{1}{2}}$ is sufficient for some constant c. They also stated that for $n \geq 2^8$, $\delta(G) \geq n + 2^{-\frac{1}{2}}n^{\frac{3}{4}}$ guarantees a $K_3(2)$, which follows from their result on $K_3(s)$ [4, Theorem 2.6]. Unfortunately, Bhalkikar and Zhao [2] found a miscalculation in the proof of [4, Theorem 2.6]. After correcting this error, they followed the approach in [4] and obtained the following result.

Theorem 1.3 ([2])**.** *Given $s \geq 2$, $\varepsilon > 0$, and sufficiently large n, every graph $G = G_3(n)$ with $\delta(G) \geq n + (1+\varepsilon)(s-1)^{1/(3s^2)}n^{1-1/(3s^2)}$ contains a copy of $K_3(s)$. In particular, $\delta(G) \geq n + (1+\varepsilon)n^{\frac{11}{12}}$ guarantees a $K_3(2)$.*

In this paper, we establish several results towards Problem 1.2. First we improved Theorem 1.3 as follows.

Theorem 1.4. *Given $s \geq 2$, $C = 2(s-1)^{\frac{1}{s+1}}$, and sufficiently large n, every graph $G = G_3(n)$ with $\delta(G) \geq n + Cn^{1-\frac{1}{s(s+1)}}$ contains a copy of $K_3(s)$. In particular, if $\delta(G) \geq n + 2n^{\frac{5}{6}}$, then G contains a $K_3(2)$.*

Note that we can easily replace the constant 2 in C by $1+o(1)$ but use the current definition of C for simplicity.

Let V_1, V_2, V_3 be the three vertex classes of a graph $G = G_3(n)$. We call $\min\{d_G(v, V_i) : v \in V(G) \setminus V_i, i \in [3]\}$ the *minimum partial degree* of G, where $d_G(v, V_i)$ denotes the number of neighbors of v in V_i. Our next result shows that $G = G_3(n)$ contains a copy of $K_3(2)$ under the conjectured condition $\delta(G) \geq n + cn^{\frac{1}{2}}$ if, in addition, the minimum partial degree is at least $(1/5+\varepsilon)n$.

Theorem 1.5. *For every $c \geq 58$, there exists $n_0 = n_0(c)$ such that every tripartite graph $G = G_3(n)$ with $n \geq n_0$, $\delta(G) \geq n + 30^5c^4n^{\frac{1}{2}}$, and the minimum partial degree at least $(1/5 + 7/c)n$ contains a $K_3(2)$.*

Bhalkikar and Zhao [2] also constructed many non-isomorphic $K_3(2)$-free tripartite graphs with minimum degree at least $n + n^{\frac{1}{2}}$. In Section 4 we construct new families of $K_3(2)$-free tripartite graphs with minimum degree at least $n + cn^{\frac{1}{2}}$ and reasonably large minimum partial degree. This shows that Problem 1.2 is difficult if $\delta(G) \geq n + cn^{\frac{1}{2}}$ suffices because it is hard to apply the stability method.

For a graph G, let $T(G)$ denote the number of triangles in G. For $n, t \in \mathbb{N}$, let $f(n,t)$ denote the minimum $T(G)$ over all $G = G_3(n)$ of minimum degree $\delta(G)$ at least $n+t$. Bollobás, Erdős, and Szemerédi [4] also studied $f(n,t)$. They showed that $f(n,1) = 4$ for $n \geq 4$, and for general t, they proved $f(n,t) \geq t^3$. For $t \leq n/5$, they constructed tripartite graphs $G = G_3(n)$ with $\delta(G) \geq n + t$ and $T(G) = 4t^3$, which implies $f(n,t) \leq 4t^3$. They asked the following question.

Problem 1.6. *For $t \leq n/5$, is it true that $f(n,t) \geq 4t^3$?*

We give a simple proof of $f(n,t) \geq n^2(3t-n)/2$, which improves $f(n,t) \geq t^3$ when $\frac{\sqrt{3}-1}{2}n \leq t \leq n$, and gives the exact value of $f(n,t)$ for even n and $t \geq n/2$.

Proposition 1.7. *For $1 \leq t \leq n$, $f(n,t) \geq n^2(3t-n)/2$ and equality holds if n is even and $t \geq n/2$.*

1.1 Notation

Recall that $G = G_3(n)$ is an (arbitrary) balanced tripartite graph with parts V_1, V_2, V_3 such that $|V_i| = n$ for $i = 1, 2, 3$. Let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G. We consider the subscript of V_i to be *modulo* 3 *with values* $1, 2, 3$, *instead of* $0, 1, 2$.

Given $v \in V(G)$ we write $N(v)$ for the neighborhood of v and define $d_G(v) = |N(v)|$ as the degree of v in G. Let $\delta(G)$ and $e(G)$ denote the minimum degree and the number of edges of G respectively. We often view G as an oriented graph with edges directed from V_i to V_{i+1}. For $v \in V_i$ let $N_G^+(v)$ (resp. $N_G^-(v)$) be the set of vertices in V_{i+1} (resp. V_{i-1}) that are joined to v. Let $d_G^+(v) = |N_G^+(v)|$ and $\delta^+(G) = \min_{v \in V(G)} |N_G^+(v)|$. We define $d_G^-(v)$ and $\delta^-(G)$ analogously. For $W \subseteq V(G)$ we define $d_G(v, W)$ as the number of edges between v and W.

For graphs G, H, $G - H$ is the subgraph of G obtained by deleting edges in $E(G) \cap E(H)$. For $W \subseteq V(G)$, $G[W]$ and $G \setminus W$ are the induced subgraphs of G on W and $V(G) \setminus W$, respectively. We write $G[A, B]$ for the induced bipartite subgraph of G with parts A and B. We often write $\{x, y\}$ as xy; for $xy \in E(G)$, let $T_G(xy)$ denote the number of triangles in G that contain x and y.

Finally, let $[n]$ denote the set $\{1, \ldots, n\}$ for $n \in \mathbb{N}$. Given a set X and an integer s, we write $\binom{X}{s}$ for the set of all s-element subsets of X. We omit floors and ceilings whenever this does not affect the argument.

1.2 Organization of paper

In Sections 2 we prove Theorem 1.4 and Proposition 1.7. In Section 3 we present the proof of Theorem 1.5. In Section 4 we give multiple constructions of $K_3(2)$-free tripartite graphs with minimum degree at least $n + cn^{\frac{1}{2}}$ and large minimum partial degree.

2 Proofs of Theorem 1.4 and Proposition 1.7

Let us denote by $z(m, n; s, s)$ the Zarankiewicz number, which is the maximum number of edges in a bipartite graph $G = (U, V; E)$ with $|U| = m$ and $|V| = n$ containing no copy of $K_{s,s}$. The following well-known result on $z(m, n; s, s)$ was proved by Kövári, Sós and Turán [9].

Lemma 2.1 ([9])**.** *For $s, m, n \in \mathbb{N}$, $z(m, n; s, s) \le (s-1)^{\frac{1}{s}} m n^{1-\frac{1}{s}} + (s-1)n$.*

Next we prove Theorem 1.4 by double counting and Lemma 2.1. This approach is similar to the one used in [4, 2] and our improvement on $\delta(G)$ is due to the counting of triangles xyz with $x \in T_1$ and $y \in T_x$ instead of $x \in V_1$ and $y \in V_3$ as in [4, 2].

Proof of Theorem 1.4. Let $t = Cn^{1-\frac{1}{s(s+1)}}$, where $C = 2(s-1)^{\frac{1}{s+1}}$. Let $d_G^+(S) = \sum_{x \in S} d_G^+(x)$ for any set $S \subseteq V(G)$. Without loss of generality, suppose that there is a subset $T_1 \subseteq V_1$ of size t satisfying

$$d_G^+(T_1) = \max\{d_G^+(T) \colon |T| = t,\, T \subseteq V_i,\, i \in [3]\}.$$

For $x \in V_1$, let $T_x \subset V_3$ be an arbitrary t-element subset of $N_G^-(x)$. Note that $\delta(G) \ge n + t$ guarantees that T_x exists. For an edge xy, let $T(xy)$ be the number of triangles containing xy. Since $d^+(y) + d^-(y) \ge n + t$ for any $y \in V(G)$, we have

$$\sum_{x \in T_1, y \in T_x} T(xy) \ge \sum_{x \in T_1} \sum_{y \in T_x} \left(d_G^+(x) + d_G^-(y) - n\right) \ge \sum_{x \in T_1} \sum_{y \in T_x} \left(t + d_G^+(x) - d_G^+(y)\right)$$

$$= \sum_{x\in T_1} \left(t^2 + td_G^+(x) - d_G^+(T_x)\right) = t^3 + td_G^+(T_1) - \sum_{x\in T_1} d_G^+(T_x).$$

Using the maximality of $d_G^+(T_1)$, we derive that

$$\sum_{x\in T_1, y\in T_x} T(xy) \geq t^3 + td_G^+(T_1) - td_G^+(T_1) = t^3.$$

For any s distinct vertices $z_1, \dots, z_s \in V_2$, let

$$\mathcal{T}(z_1, \dots, z_s) = \{xy \in E(G) : x \in T_1, y \in T_x, G[\{x, y, z_i\}] \text{ is an triangle for all } i \in [s]\}.$$

By double counting and convexity, we have

$$\sum_{\{z_1,\dots,z_s\}\in\binom{V_2}{s}} |\mathcal{T}(z_1, \dots, z_s)| = \sum_{x\in T_1, y\in T_x} \binom{T(xy)}{s} \geq t^2 \binom{\frac{1}{t^2}\sum_{x\in T_1, y\in T_x} T(xy)}{s} \geq t^2\binom{t}{s}.$$

By averaging, there exist s distinct vertices $z_1, \dots, z_s \in V_2$ such that

$$\begin{aligned} |\mathcal{T}(z_1, \dots, z_s)| &\geq \frac{t^2\binom{t}{s}}{\binom{n}{s}} \geq \frac{t^{s+2}}{2^s n^s} = \frac{C^{s+1}}{2^s} tn^{1-\frac{1}{s}} > \left(\frac{C}{2}\right)^{s+1} \left(tn^{1-\frac{1}{s}} + n\right) \\ &\geq (s-1)tn^{1-\frac{1}{s}} + (s-1)n > z(t, n; s, s) \end{aligned}$$

by Lemma 2.1. Thus, the bipartite graph on $T_1 \cup V_3$ with the edge set $\mathcal{T}(z_1, \dots, z_s)$ contains a copy of $K_{s,s}$. Together with $z_1, \dots, z_s$, this gives the desired copy of $K_3(s)$ in G. □

We next prove Proposition 1.7.

Proof of Proposition 1.7. Let $G = G_3(n)$ with $\delta(G) \geq n + t$. Let $\overline{G}$ be the tripartite complement graph of G, that is $K_3(n) - G$, where $K_3(n)$ has the same vertex classes as G. Note that $\Delta(\overline{G}) \leq n - t$ and $e(\overline{G}) \leq 3n(n - t)/2$. Since each edge is in at most n triangles, so

$$T(G) \geq T(K_3(n)) - ne(\overline{G}) \geq n^3 - \frac{3n^2(n-t)}{2} = \frac{1}{2}n^2(3t - n).$$

Thus we have $f(n, t) \geq n^2(3t - n)/2$.

Suppose that n is even and $t \geq n/2$. Let $A_1, B_1, A_2, B_2, A_3, B_3$ be disjoint vertex sets each of size $n/2$. For $i \in [3]$, let $V_i = A_i \cup B_i$. Let H be a tripartite graph with vertex classes V_1, V_2, V_3 such that $H = \bigcup_{i\in[3]} H[A_{i+1}, B_i]$ and each $H[A_{i+1}, B_i]$ is $(n - t)$-regular. Let G be the tripartite complement graph of H, so G is $(n + t)$-regular. Since no triangle in $K_3(n)$ contains two edges of H, by the calculation above we have $T(G) = n^2(3t - n)/2$. □

3 Proof of Theorem 1.5

We now sketch the proof of Theorem 1.5. Suppose that G is $K_3(2)$-free with minimum partial degree $\beta n \geq (1/5 + 7/c)n$. Using the fact that $\delta^+(G) \geq \beta n$, we show that G contains a blow-up of C_6 with parts of size $\beta n + o(n)$ (Lemma 3.2). Moreover, for each vertex v in this C_6-blow-up, we have $d^+(v) \leq \beta n + o(n)$ and $d^-(v) \geq n - \beta n + o(n)$. Similarly, by $\delta^-(G) \geq \beta n$, we obtain another C_6-blow-up with similar properties. If these two C_6-blow-ups intersect, then it leads to a contradiction immediately. Otherwise, we use Lemma 3.3 to deduce that G contains a $K_3(2)$ and thus complete the proof.

We begin with a definition. Recall that $G = G_3(n)$ is viewed as an oriented graph with edges from V_i to V_{i+1} for $i \in [3]$. For $v \in V(G)$ and $\alpha > 0$, let

$$\widetilde{D}^+_{G,\alpha}(v) = \left\{w \in N^+_G(v) : T(vw) \geq \alpha n\right\},$$
$$\widetilde{D}^-_{G,\alpha}(v) = \left\{w \in N^-_G(v) : T(vw) \geq \alpha n\right\}.$$

The following lemma shows that only a small number of vertices $w \in V(G)$ can have large $\widetilde{D}^+_{G,\alpha}(w)$ or $\widetilde{D}^-_{G,\alpha}(w)$.

Lemma 3.1. *Let $G = G_3(n)$ and $k > 1$. Suppose that for some $i \in [3]$, there exists a vertex subset $W_i \subseteq V_i$ of size $|W_i| \geq k^2 n^{\frac{1}{2}}$ such that either $|\widetilde{D}^+_{G,2/k}(w)| \geq k^2 n^{\frac{1}{2}}$ for all $w \in W_i$ or $|\widetilde{D}^-_{G,2/k}(w)| \geq k^2 n^{\frac{1}{2}}$ for all $w \in W_i$. Then G contains a $K_3(2)$.*

Proof. We only prove the $|\widetilde{D}^+_{G,2/k}(w)| \geq k^2 n^{\frac{1}{2}}$ case because the proof of the $|\widetilde{D}^-_{G,2/k}(w)| \geq k^2 n^{\frac{1}{2}}$ case is similar. Without loss of generality, we can assume $|W_i| = k^2 n^{\frac{1}{2}}$. For each $w \in W_i$, let $W(w) \subseteq \widetilde{D}^+_{G,2/k}(w)$ be of size $k^2 n^{\frac{1}{2}}$.

Define an auxiliary bipartite graph H with parts W_i and $V_{i+1} \times V_{i+2}$ such that for $w \in W_i$ and $(v_{i+1}, v_{i+2}) \in V_{i+1} \times V_{i+2}$, $w(v_{i+1}, v_{i+2})$ is an edge of H if $v_{i+1} \in W(w)$ and $wv_{i+1}v_{i+2}$ forms a triangle in G. For all $w \in W_i$, $d_H(w) \geq k^2 n^{\frac{1}{2}} \cdot (2/k)n = 2kn^{\frac{3}{2}}$, so $e(H) \geq 2k^3 n^2$. Thus

$$\begin{aligned}\sum_{\{w_i,w_i'\}\in\binom{W_i}{2}} |N_H(w_i) \cap N_H(w_i')| &= \sum_{(v_{i+1},v_{i+2})\in V_{i+1}\times V_{i+2}} \binom{d_H\left((v_{i+1}, v_{i+2})\right)}{2} \\ &\geq |V_{i+1} \times V_{i+2}| \binom{e(H)/|V_{i+1} \times V_{i+2}|}{2} \geq n^2 \binom{2k^3}{2} \geq k^6 n^2.\end{aligned}$$

By averaging, there exists $\{w_i, w_i'\} \in \binom{W_i}{2}$ with $|N_H(w_i) \cap N_H(w_i')| > 2k^2 n$. Note that $N_H(w_i) \cap N_H(w_i')$ can be viewed as a subgraph of $G[W(w_i), V_{i+2}]$. By Lemma 2.1, we have

$$z\left(k^2 n^{\frac{1}{2}}, n; 2, 2\right) \leq k^2 n + n < |N_H(w_i) \cap N_H(w_i')|,$$

implying $K_3(2) \subseteq G$. □

Now we introduce two lemmas and postpone their proofs to the next two subsections. The first lemma gives the structure of $K_3(2)$-free tripartite graph with $\delta(G) \geq n$ and linear partial degree.

Lemma 3.2. *Let $\varepsilon > 0$ and n be sufficiently large. Suppose $G = G_3(n)$ is a tripartite graph with $\delta(G) \geq n$ and $\delta^+(G) \geq 2\varepsilon n$. Further, assume that $T(uv) \leq \left(\frac{\varepsilon}{30}\right)^2 n$ for all edges uv in G. Then either G contains a $K_3(2)$ or there exists a partition $\mathcal{P} = \{W_1, \ldots, W_6, U_1, U_2, U_3\}$ of $V(G)$ such that for $i \in [6]$ and $j \in [3]$,*

(a) *$W_j, W_{j+3}, U_j \subseteq V_j$;*
(b) *$\delta^+(G) - \varepsilon n \leq |W_i| \leq \delta^+(G) + \varepsilon n$;*
(c) *for all $w \in W_i$, $S \in \{W_{i-1}, W_{i+1}, U_{i-1}\}$ and $S' \in \mathcal{P} \setminus \{W_{i-1}, W_{i+1}, U_{i-1}\}$ (the subscript of W_i is modulo 6 with values $1, \ldots, 6$ while the subscript of U_i is modulo 3 with values $1, 2, 3$), we have $d_G(w, S) \geq |S| - \varepsilon n$, and $d_G(w, S') \leq \varepsilon n$;*

(d) *for all* $w \in W_i$, $d^+_G(w) \le \delta^+(G) + \varepsilon n$ *and* $d^-_G(w) \ge n - \delta^+(G) - \varepsilon n$.

The second lemma deals with the case when G contains two C_6 blow-ups.

Lemma 3.3. *Let* $1 < c \le n^{\frac{1}{6}}$ *and* $n \in \mathbb{N}$. *Let* $G = G_3(n)$ *be a tripartite graph with* $\delta(G) \ge n + 28c^2 n^{\frac{1}{2}}$. *Let* $d = d(c,n)$ *be a non-negative integer such that* $\delta^+(G) \ge d$ *and* $3\delta^+(G) + 2d \ge n + 26c^{-1}n$. *Suppose that there exist disjoint vertex sets* $W_1, \dots, W_6, X_1, \dots, X_6$ *(where* X_i *can be empty for* $i \in [6]$ *and the subscripts of* X_i *are modulo* 6*) such that*

(i) *for all* $j \in [3]$, $W_j, W_{j+3}, X_j, X_{j+3} \subseteq V_j$*;*

(ii) *for all* $i \in [6]$, $\delta^+(G) - c^{-1}n \le |W_i| \le \delta^+(G) + c^{-1}n$ *and* $d - c^{-1}n \le |X_i| \le d + c^{-1}n$*;*

(iii) *for all* $i \in [6]$, $w \in W_i$ *and* $x \in X_i$, *we have*

$$d(w,S) \ge |S| - c^{-1}n \qquad \text{if } S \in \{W_{i-1}, W_{i+1}, X_{i-1}, X_{i-4}\},$$
$$d(x,S) \ge |S| - c^{-1}n \qquad \text{if } S \in \{X_{i-1}, X_{i+1}, W_{i+1}, W_{i+4}\}.$$

Then G *contains a* $K_3(2)$.

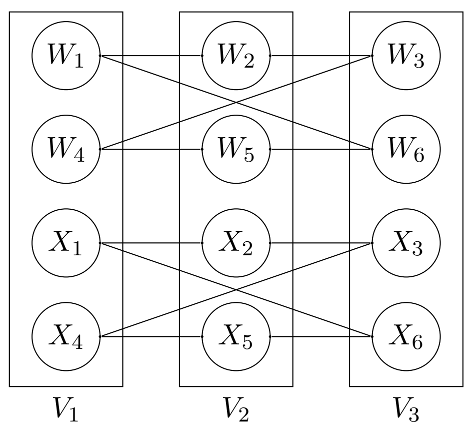

Figure 1: Graph of Lemma 3.3.

Proof of Theorem 1.5. This proof of the theorem actually proves the following slightly stronger statement: *For every* $c \ge 58$, *there exists* $n_0 = n_0(c)$ *such that every tripartite graph* $G = G_3(n)$ *with* $n \ge n_0$, $\delta(G) \ge n + 30^5 c^4 n^{\frac{1}{2}}$ *and*

$$2\delta^+(G) + 2\delta^-(G) + \max\{\delta^+(G), \delta^-(G)\} \ge \left(1 + 35c^{-1}\right) n \tag{3.1}$$

contains a $K_3(2)$.

Suppose to the contrary that there exists a $K_3(2)$-free tripartite graph $G = G_3(n)$ satisfies the conditions above. Let $\alpha = (35c)^{-2}$ and $\widetilde{G}$ be the spanning subgraph of G with $E(\widetilde{G}) = \{uv \in E(G) : T(uv) \le \alpha n\}$. For $i \in [3]$, let S_i^+ (and S_i^-) be a subset $S \subseteq V_i$ of size $4\alpha^{-2}n^{1/2}$ with $\sum_{v \in S} d^+_{G - \widetilde{G}}(v)$ (and $\sum_{v \in S} d^-_{G-\widetilde{G}}(v)$, respectivley) maximal. By Lemma 3.1 with $k = 2/\alpha$, we deduce that, for any $v \in V_i \setminus (S_i^+ \cup S_i^-)$, we have

$$d^+_{\widetilde{G}}(v) \ge d^+_G(v) - 4\alpha^{-2}n^{1/2} \quad \text{and} \quad d^-_{\widetilde{G}}(v) \ge d^-_G(v) - 4\alpha^{-2}n^{1/2}. \tag{3.2}$$

Let $G' = \widetilde{G} \setminus \bigcup_{i\in[3]}(S_i^+ \cup S_i^-)$. Clearly, G' is a tripartite graph with parts of size $n' = n - 8\alpha^{-2}n^{1/2}$. By (3.2),

$$\delta^+(G') \geq \delta^+(\widetilde{G}) - 8\alpha^{-2}n^{1/2} > \delta^+(G) - 12\alpha^{-2}n^{1/2}$$

and analogously $\delta^-(G') \geq \delta^-(G) - 12\alpha^{-2}n^{1/2}$. Using (3.2) and $\alpha = (35c)^{-2}$, we obtain that

$$\delta(G') \geq \delta(G) - 24\alpha^{-2}n^{\frac{1}{2}} \geq n + 30^5c^4n^{\frac{1}{2}} - 24(35c)^4n^{\frac{1}{2}} \geq n' + 28c^2n'^{\frac{1}{2}}.$$

Without loss of generality, we assume $\delta^+(G') \geq \delta^-(G')$ (otherwise we reverse the direction of G'). We claim that

$$3\delta^+(G') + 2\delta^-(G') \geq n' + 30c^{-1}n'. \tag{3.3}$$

Indeed, if $\delta^+(G) \geq \delta^-(G)$, then we have

$$\begin{aligned} 3\delta^+(G') + 2\delta^-(G') &\geq 3\delta^+(G) + 2\delta^-(G) - 60\alpha^{-2}n^{\frac{1}{2}} \\ &\overset{(3.1)}{\geq} \left(1 + 35c^{-1}\right)n - 60\alpha^{-2}n^{\frac{1}{2}} \geq \left(1 + 30c^{-1}\right)n' \end{aligned}$$

as n is sufficiently large. If $\delta^-(G) \geq \delta^+(G)$, then, since $\delta^+(G') \geq \delta^-(G')$, we have

$$\begin{aligned} 3\delta^+(G') + 2\delta^-(G') \geq 3\delta^-(G') + 2\delta^+(G') &\geq 3\delta^-(G) + 2\delta^+(G) - 60\alpha^{-2}n^{\frac{1}{2}} \\ &\overset{(3.1)}{\geq} \left(1 + 30c^{-1}\right)n'. \end{aligned}$$

Hence (3.3) holds.

Now we prove that G' contains a $K_3(2)$, which contradicts our assumption that G is $K_3(2)$-free. Let V_1', V_2', V_3' be the three vertex classes of G'. Note that $T_{G'}(uv) \leq T_{\widetilde{G}}(uv) \leq \alpha n = n/(35c)^2 \leq n'/(30c)^2$ for all $uv \in E(G')$. By (3.3), $\delta^+(G') \geq n'/5$. By Lemma 3.2 with $\varepsilon = c^{-1}$, $V(G')$ can be partitioned into $W_1, \dots, W_6, U_1, U_2, U_3$ of such that, for $i \in [6]$,

(a) $W_i, U_i \subseteq V'_{i \pmod 3}$;
(b) $\delta^+(G') - c^{-1}n' \leq |W_i| \leq \delta^+(G') + c^{-1}n'$;
(c) for all $w \in W_i$ and $S \in \{W_{i-1}, W_{i+1}, U_{i-1}\}$, $d_{G'}(w, S) \geq |S| - c^{-1}n'$;
(d) for all $w \in W_i$, $d^+_{G'}(w) \leq \delta^+(G') + c^{-1}n'$ and $d^-_{G'}(w) \geq n' - \delta^+(G') - c^{-1}n'$.

If $\delta^+(G') \geq (n' + 26c^{-1}n')/3$, then we apply Lemma 3.3 with $d = 0$ and $X_i = \emptyset$ for all $i \in [6]$ and obtain a $K_3(2)$ in G', a contradiction.

If $\delta^-(G') \leq \delta^+(G') < (n' + 26c^{-1}n')/3$, then by (3.3), we have

$$\delta^-(G') \geq \frac{1}{2}(n' + 30c^{-1}n' - 3\delta^+(G')) \geq \frac{1}{2}(n' + 30c^{-1}n' - (n' + 26c^{-1}n')) \geq 2c^{-1}n'.$$

By reversing the direction of G' and applying Lemma 3.2 with $\varepsilon = c^{-1}$ and noting $\delta^-(G') \geq 2c^{-1}n'$, we obtain a partition $W_1', \dots, W_6', U_1', U_2', U_3'$ of $V(G')$ such that, for $i \in [6]$,

(e) $W_i', U_i' \subseteq V'_{5-i \pmod 3}$;
(f) $\delta^-(G') - c^{-1}n' \leq |W_i'| \leq \delta^-(G') + c^{-1}n'$;
(g) for all $w' \in W_i'$ and $S \in \{W'_{i-1}, W'_{i+1}, U'_{i-1}\}$, $d_{G'}(w', S) \geq |S| - c^{-1}n'$;
(h) for all $w' \in W_i'$, $d^-_{G'}(w') \leq \delta^-(G') + c^{-1}n'$ and $d^+_{G'}(w') \geq n' - \delta^-(G') - c^{-1}n'$.

Let $X_i = W'_{8-i \pmod 6}$ for $i \in [6]$. This and (e) together imply that $X_i \subseteq V'_{i \pmod 3}$. We claim that $W_1, \ldots, W_6, X_1, \ldots, X_6$ are pairwise disjoint. Indeed, suppose to the contrary that there exists a vertex $v \in (W_i \cup W_{i+3}) \cap (X_i \cup X_{i+3})$ for some $i \in [3]$. Then, by (d) and (h), we have

$$n' - \delta^-(G') - c^{-1}n' \le d^+_{G'}(v) \le \delta^+(G') + c^{-1}n',$$

which implies that $\delta^+(G') + \delta^-(G') \ge n' - 2c^{-1}n'$, contradicting $\delta^-(G') \le \delta^+(G') < (n' + 26c^{-1}n')/3$ as $c^{-1} \le 58$.

Let $d = \delta^-(G')$. It is easy to see that all the assumptions of Lemma 3.3 hold, for example, (iii) holds because of (c) and (g), and fact that $W_i \subseteq U'_{i \pmod 3}$ and $W'_i \subseteq U_{i \pmod 3}$ for $i \in [6]$. We can thus obtain a $K_3(2)$ in G' by applying Lemma 3.3. This contradicts our assumption and completes the proof of Theorem 1.5. □

3.1 Proof of Lemma 3.2

We begin with a simple proposition.

Proposition 3.4. *Let $0 \le \lambda \le 1/10$ and G be a bipartite graph with vertex classes A and B. Suppose that for all $a \in A$, $d_G(a) \ge (1-\lambda)|B|$. Then there exists a subset $B' \subseteq B$ of size $|B'| \ge (1-5\lambda)|B|$ such that for all $a \in A$ and $b \in B'$, $d_G(a, B') \ge (1-2\lambda)|B'|$ and $d_G(b, A) \ge 4|A|/5$.*

Proof. Let $B' = \{b \in B : d_G(b) \ge 4|A|/5\}$. Note that

$$|A||B'| + \frac{4}{5}|A||B \setminus B'| \ge \sum_{b\in B} d_G(b) = e(G) = \sum_{a \in A} d_G(a) \ge (1-\lambda)|B||A|.$$

This implies $|B'| \ge (1-5\lambda)|B| \ge |B|/2$. Clearly $d_G(a, B') \ge |B'| - \lambda|B| \ge (1-2\lambda)|B'|$, and the result follows. □

Proof of Lemma 3.2. Suppose G is $K_3(2)$-free. Let $\alpha = (\frac{\varepsilon}{30})^2$. Without loss of generality, we will assume that there is an $a_0 \in V_3$ with $d^+(a_0) = \delta^+(G) = \beta n$. Furthermore, for all $vw \in E(G)$ with $v \in V_{i-1}$ and $w \in V_i$, we have

$$\begin{aligned} \alpha n \ge T(vw) = |N(w) \cap N(v)| &\ge d^+(w) + d^-(v) - n = d(w) - d^-(w) + d^-(v) - n \\ &\ge \delta(G) - d^-(w) + d^-(v) - n \ge d^-(v) - d^-(w), \\ d^-(w) \ge d^-(v) - \alpha n. & \end{aligned} \tag{3.4}$$

Since we can view G as an oriented graph with direction from V_i to V_{i+1} and $\delta^+(G) = \beta n$, we can find a directed path $P = a_0 a_1 \ldots a_{12}$ of length 12 in G. Let $A_0 = \{a_0\}$ and $A_i = N^+(a_{i-1})$ for $i \in [12]$. Note that $a_i \in A_i$.

Claim 3.5. *The sets A_i's satisfy the following properties.*

(i) *For $i \in [12]$, we have $|A_i| \ge \beta n$.*

(ii) *For $i \in \{0, 1, \ldots, 12\}$ and $v \in A_i$, we have $d^-(v) \ge (1 - \beta - i\alpha)n$.*

(iii) *For $i \in \{0, \ldots, 11\}$ and $v \in A_i$, we have $d(v, A_{i+1}) \ge |A_{i+1}| - (i+1)\alpha n \ge (\beta - (i+1)\alpha)n$ and $d^+(v) \le (\beta + i\alpha)n$. In particular, for $i \in [12]$, $|A_i| \le d^+(a_{i-1}) \le (\beta + (i-1)\alpha)n$.*

(iv) *For $i \in \{0, 1, \ldots, 10\}$ and $v \in A_i$, we have $d^-(v, A_{i+2}) \le (i+3)\alpha n$.*

(v) *For $i \in \{0, 1, \ldots, 9\}$, $|A_i \cap A_{i+3}| \le 5\sqrt{\alpha} n$.*

(vi) *For $i \in \{0, 1, \ldots, 10\}$ and $v \in A_i$, we have $|V_{i-1} \setminus (A_{i+2} \cup N^-(v))| \le (2i+3)\alpha n$.*

(vii) *For* $i \in \{1, \dots, 6\}$, $|A_i \cap A_{i+6}| \geq |A_i| - 8\sqrt{\alpha}n \geq (\beta - 8\sqrt{\alpha})n$.

Proof of claim. We have $|A_i| = d^+(a_{i-1}) \geq \delta^+(G) = \beta n$ giving (i).

We prove (ii) by induction on i. If $i = 0$ then using $\beta n = d^+(a_0)$, we have

$$d^-(a_0) = d(a_0) - d^+(a_0) \geq n - \beta n = (1 - \beta)n.$$

So we may assume $i \in [12]$. For $v \in A_i$, note that $a_{i-1}v \in E(G)$. Together with (3.4), we have

$$d^-(v) \geq d^-(a_{i-1}) - \alpha n \geq (1 - \beta - (i-1)\alpha)n - \alpha n = (1 - \beta - i\alpha)n.$$

Hence (ii) holds.

For $i \in \{0, \dots, 11\}$ and $v \in A_i$, we first show $d^+(v) \leq (\beta + i\alpha)n$ by using (ii). We know $d^+(a_0) = \beta n$. For $i \in [11]$, since $v \in A_i = N^+(a_{i-1})$, we have $|N^+(v) \cap N^-(a_{i-1})| = T(va_{i-1}) \leq \alpha n$. Consequently,

$$d^+(v) \leq |N^+(v) \cap N^-(a_{i-1})| + |V_{i+1} \setminus N^-(a_{i-1})| \overset{\text{(ii)}}{\leq} \alpha n + (\beta + (i-1)\alpha)n = (\beta + i\alpha)n.$$

This implies that $|A_i| \leq d^+(a_{i-1}) \leq (\beta + (i-1)\alpha)n$ for $i \in [12]$.

Next we show $d(v, A_{i+1}) \geq |A_{i+1}| - (i+1)\alpha n \geq (\beta - (i+1)\alpha)n$ for $v \in A_i$ and $i \in \{0, \dots, 11\}$. First, $d(a_0, A_1) = |A_1| > (\beta - \alpha)n$. Note that

$$\begin{aligned} |V_{i+1} \setminus (N^-(a_{i-1}) \cup N^+(v))| &= |V_{i+1} \setminus N^-(a_{i-1})| - |N^+(v) \setminus N^-(a_{i-1}))| \\ &\overset{\text{(ii)}}{\leq} (\beta + (i-1)\alpha)n - d^+(v) + T(a_{i-1}v) \\ &\leq (\beta + (i-1)\alpha)n - \delta^+(G) + \alpha n = i\alpha n. \end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned} d(v, A_{i+1}) &\geq |A_{i+1}| - |A_{i+1} \cap N^-(a_{i-1})| - |A_{i+1} \setminus (N^-(a_{i-1}) \cup N^+(v))| \\ &\geq |A_{i+1}| - T(a_i a_{i-1}) - |V_{i+1} \setminus (N^-(a_{i-1}) \cup N^+(v))| \\ &\overset{(3.5)}{\geq} |A_{i+1}| - \alpha n - i\alpha n \overset{\text{(i)}}{\geq} (\beta - (i+1)\alpha)n. \end{aligned}$$

Hence (iii) holds.

Suppose $i \in \{0, 1, \dots, 10\}$ and $v \in A_i$. Let $w \in N^+(v, A_{i+1})$, which exists because (iii) implies that $d^+(v, A_{i+1}) \geq |A_{i+1}| - (i+1)\alpha n > 0$. Then

$$\begin{aligned} \alpha n \geq T(vw) = |N^-(v) \cap N^+(w)| &\geq |N^-(v) \cap N^+(w) \cap A_{i+2}| \\ &\geq d^-(v, A_{i+2}) + d^+(w, A_{i+2}) - |A_{i+2}| \\ &\overset{\text{(iii)}}{\geq} d^-(v, A_{i+2}) + (|A_{i+2}| - (i+2)\alpha n) - |A_{i+2}| = d^-(v, A_{i+2}) - (i+2)\alpha n. \end{aligned}$$

This gives $d^-(v, A_{i+2}) \leq (i+3)\alpha n$, confirming (iv).

By (i) and (iii), for all $v \in A_{i+2}$, we have

$$d^+(v, A_{i+3}) > |A_{i+3}| - (i+3)\alpha n \geq (1 - \sqrt{\alpha})|A_{i+3}|,$$

where use the assumption that $\alpha = (\varepsilon/30)^2$ and $\beta \geq 30\sqrt{\alpha}$. By applying Proposition 3.4 on $G[A_{i+2}, A_{i+3}]$ we have a subset $A'_{i+3} \subseteq A_{i+3}$ with size $(1-5\sqrt{\alpha})|A_{i+3}|$ such that, for all $w \in A'_{i+3}$,

$$d(w, A_{i+2}) \geq \frac{4}{5}|A_{i+2}| > (i+3)\alpha n.$$

Since $d(v, A_{i+2}) \leq (i+3)\alpha n$ for all $v \in A_i$ by (iv), it follows that $A_i \cap A_{i+3} \subseteq A_{i+3} \setminus A'_{i+3}$. Therefore, $|A_i \cap A_{i+3}| \leq 5\sqrt{\alpha}|A_{i+3}| \leq 5\sqrt{\alpha} n$ confirming (v).

For all $v \in A_i$,

$$\begin{aligned} d^-(v, V_{i-1} \setminus A_{i+2}) = d^-(v) - d^-(v, A_{i+2}) &\overset{\text{(ii),(iv)}}{\geq} (1-\beta-i\alpha)n - (i+3)\alpha n \\ &= (1-\beta-(2i+3)\alpha)n \overset{\text{(i)}}{\geq} |V_{i-1}| - |A_{i+2}| - (2i+3)\alpha n \\ &= |V_{i-1} \setminus A_{i+2}| - (2i+3)\alpha n. \end{aligned}$$

Hence $|V_{i-1} \setminus (A_{i+2} \cup N^-(v))| \leq (2i+3)\alpha n$ and (vi) holds.

Finally, for $0 \leq i \leq 6$ and $v \in A_i$, we have

$$\begin{aligned} d^-(v, A_{i+5}) &\geq d^-(v, A_{i+5} \setminus A_{i+2}) \geq |A_{i+5} \setminus A_{i+2}| - |V_{i-1} \setminus \big(A_{i+2} \cup N^-(v)\big)| \\ &\overset{\text{(v),(vi)}}{\geq} |A_{i+5}| - 5\sqrt{\alpha}n - (2i+3)\alpha n \geq |A_{i+5}| - 6\sqrt{\alpha}n. \end{aligned}$$

Hence there exists a vertex $w \in A_{i+5}$ such that

$$d^+(w, A_i) \geq \frac{e(A_i, A_{i+5})}{|A_{i+5}|} \geq \frac{|A_i|(|A_{i+5}| - 6\sqrt{\alpha}n)}{|A_{i+5}|} = |A_i| - \frac{|A_i|}{|A_{i+5}|}6\sqrt{\alpha}n \geq |A_i| - 7\sqrt{\alpha}n,$$

where the last inequality holds because $|A_{i+5}| \geq \beta n$ and $|A_i| \leq (\beta + (i-1)\alpha)n$ by (i) and (iii), and consequently, $\frac{|A_i|}{|A_{i+1}|} \leq \frac{\beta+5\alpha}{\beta} < \frac{7}{6}$ by our assumption on α and β. Therefore,

$$\begin{aligned} |A_i \cap A_{i+6}| &\geq |A_i \cap A_{i+6} \cap N^+(w)| \geq d^+(w, A_i) + d^+(w, A_{i+6}) - d^+(w) \\ &\overset{\text{(iii)}}{>} |A_i| - 7\sqrt{\alpha}n + (\beta - (i+6)\alpha)n - (\beta + (i+5)\alpha)n \geq |A_i| - 8\sqrt{\alpha}n \end{aligned}$$

confirming (vii). ■

Now we come back to the proof of the lemma. For $i \in [6]$, let $W_i = (A_i \cap A_{i+6}) \setminus A_{i+3}$. If there exists some $i \in [3]$ such that $|W_i| + |W_{i+3}| > n$, then this contradicts the fact that $W_i, W_{i+3} \subseteq V_i$ and implies that G contains a $K_3(2)$. Otherwise, for $j \in [3]$, let $U_j = V_j \setminus (W_j \cup W_{j+3})$. Since $W_{j+3} \subseteq A_{j+3}$ and $W_j \cap A_{j+3} = \emptyset$, we have W_j, W_{j+3}, and U_j are pairwise disjoint subsets of V_j, in particular, (a) holds. Hence $\mathcal{P} = \{W_1, \ldots, W_6, U_1, U_2, U_3\}$ is a partition of $V(G)$.

By Claim 3.5 (iii), (v) and (vii), for $i \in [6]$, we have

$$\begin{aligned} (\beta - 13\sqrt{\alpha})n &\leq |A_i \cap A_{i+6}| - |A_i \cap A_{i+3}| \\ &\leq |W_i| \leq |A_i| \leq (\beta + 5\alpha)n. \end{aligned} \tag{3.6}$$

Consequently, $(1-2\beta-10\alpha)n \leq |U_j| \leq (1-2\beta+26\sqrt{\alpha})n$. Since $\varepsilon = 30\sqrt{\alpha}$, (b) holds.

Consider $i \in [6]$ and $v \in W_i$. Trivially, $d(v, W_{i+3}) = 0 = d(v, U_i)$ by (a). By Claim 3.5 (v) and (vii), we have $|W_{i+1}| \geq |A_{i+1}| - 13\sqrt{\alpha}n$ and thus

$$\begin{aligned} d(v, W_{i+1}) &\geq d(v, A_{i+1}) - 13\sqrt{\alpha}n \overset{\text{(iii)}}{>} |A_{i+1}| - (i+1)\alpha n - 13\sqrt{\alpha}n \\ &\geq |W_{i+1}| - 7\alpha n - 13\sqrt{\alpha}n \geq (\beta - 27\sqrt{\alpha})n. \end{aligned} \tag{3.7}$$

Together with Claim 3.5 (iii), this implies that

$$d(v, W_{i+4}) + d_G(v, U_{i+1}) = d^+(v) - d(v, W_{i+1}) \leq (\beta + i\alpha)n - (\beta - 27\sqrt{\alpha})n \leq 28\sqrt{\alpha}n.$$

By Claim 3.5 (iv),

$$d(v, W_{i+2}) \leq d^-(v, A_{i+2}) \leq (i+3)\alpha n \leq 9\alpha n.$$

Together with Claim 3.5 (ii), this implies that

$$\begin{aligned} d_G(v, W_{i-1}) + d_G(v, U_{i-1}) &= d_G^-(v) - d_G(v, W_{i+2}) \geq (1 - \beta - i\alpha)n - 9\alpha n \\ &\geq (1 - \beta - 15\alpha)n \overset{(3.6)}{\geq} n - |W_{i+2}| - 14\sqrt{\alpha}n \\ &= |W_{i-1}| + |U_{i-1}| - 14\sqrt{\alpha}n. \end{aligned}$$

Therefore, (c) holds. Finally, since $v \in W_i \subseteq A_i$, we have $d^+(v) \leq (\beta + i\alpha)n \leq (\beta + 6\alpha)n$ by Claim 3.5 (iii), and $d^-(v) \geq (1 - \beta - i\alpha)n \geq (1 - \beta - 6\alpha)n$ by Claim 3.5 (ii). Thus (d) holds. This completes the proof of the lemma. □

3.2 Proof of Lemma 3.3

Proof of Lemma 3.3. For $i \in [3]$, let $R_i = V_i \setminus (W_i \cup W_{i+3} \cup X_i \cup X_{i+3})$. By (ii), we have $|R_i| \leq n - 2\delta^+(G) - 2d + 4c^{-1}n$. Suppose to the contrary that G is $K_3(2)$-free. We first prove the following claim.

Claim 3.6. *There exist disjoint vertex subsets $W_1^*, \dots, W_6^*, X_1^*, \dots, X_6^*$ such that*

(i′) *for all $i \in [6]$, $W_i \subseteq W_i^* \subseteq V_i$, $X_i \subseteq X_i^* \subseteq V_i$;*

(ii′) *for all $v \in W_i^*$, $i \in [6]$ and $j \in \{i+1, i-1\}$, $d(v, W_j) \geq 3c^{-1}n$. For all $v \in X_i^*$, $i \in [6]$ and $j \in \{i+1, i-2\}$, $d(v, W_j) \geq 3c^{-1}n$;*

(iii′) $|V(G) \setminus \bigcup_{i \in [6]} (W_i^* \cup X_i^*)| \leq 24c^2 n^{\frac{1}{2}}$.

Proof of claim. Let $\mathcal{V} = \{W_i, X_i : i \in [6]\}$. For all $v \in R_1 \cup R_2 \cup R_3$, we define $I_v = \{A \in \mathcal{V} : d(v, A) \geq 3c^{-1}n\}$. Let

$$\mathcal{E} = \{\{W_i, W_{i+1}\}, \{X_i, X_{i+1}\}, \{W_{i-2}, X_i\}, \{W_{i+1}, X_i\} : i \in [6]\}$$

be a family of 24 pairs of $\mathcal{V}$. For every $\{A, B\} \in \mathcal{E}$, we claim that at most $c^2 n^{\frac{1}{2}}$ vertices $v \in R_1 \cup R_2 \cup R_3$ satisfy $\{A, B\} \subseteq I_v$ and call such vertices *bad*. Indeed, for $i \in [6]$, let Y_i be the set of vertices $v \in R_1 \cup R_2 \cup R_3$ such that $\{W_i, W_{i+1}\} \subseteq I_v$ (the argument for other pairs of $\mathcal{E}$ is similar). Note that $Y_i \subseteq V_{i+2}$. For all $v \in Y_i$, by the definition of I_v we have $d(v, W_i) \geq 3c^{-1}n$ and $d(v, W_{i+1}) \geq 3c^{-1}n$. Then at least $3c^{-1}n$ vertices $w \in N_G(v, W_i) \subseteq N_G^+(v)$ satisfy

$$T(vw) \geq |N_G(v, W_{i+1}) \cap N_G(w, W_{i+1})| \overset{\text{(iii)}}{\geq} 3c^{-1}n - c^{-1}n = 2c^{-1}n$$

implying that $|\widetilde{D}^+_{G,2c^{-1}}(v)| \geq 3c^{-1}n$. By Lemma 3.1, we have $|Y_i| \leq c^2 n^{\frac{1}{2}}$, as claimed.

Now we delete all bad vertices and denote the remaining set by $R_1' \cup R_2' \cup R_3'$ (and call their vertices *good* vertices). For $i \in [6]$, define

$$W_i^* = W_i \cup \left\{v \in R_1' \cup R_2' \cup R_3' : \{W_{i-1}, W_{i+1}\} \subseteq I_v\right\}$$

and

$$X_i^* = \begin{cases} X_i \cup \{v \in R_1' \cup R_2' \cup R_3' : \{W_{i+1}, W_{i-2}\} \subseteq I_v\}, & i \in \{1,2,3\}; \\ X_i, & i \in \{4,5,6\}. \end{cases}$$

As before, the subscripts of W_i^* and X_i^* in these definitions are modulo 6 with values $1, \ldots, 6$. Observe that for $i \in [6]$, W_i^* and X_i^* are disjoint because if $v \in W_i^* \cap X_i^*$, then $\{W_j, W_{j+1}\} \in I_v$ for some $j \in [6]$, contradicting $v \in R_1' \cup R_2' \cup R_3'$. Next we show that every vertex in $R_1' \cup R_2' \cup R_3'$ belongs to some W_i^* or X_i^*, which completes the proof.

Suppose to the contrary that there is a vertex $v \in R_1' \cup R_2' \cup R_3'$ which does not belong to any W_i^* and X_i^* for $i \in [6]$. We will show that $|I_v \cap \{W_i : i \in [6]\}| \leq 1$ and $|I_v \cap \{X_i : i \in [6]\}| \leq 2$. By (ii), it follows that

$$\begin{aligned} \delta(G) \leq d(v) &< \delta^+(G) + c^{-1}n + 2(d + c^{-1}n) + 5 \cdot 3c^{-1}n + 2\left(n - 2\delta^+(G) - 2d + 4c^{-1}n\right) \\ &= 2n + 26c^{-1}n - \left(3\delta^+(G) + 2d\right) \leq n, \end{aligned}$$

contradicting our assumption.

Without loss of generality, assume that $v \in R_2' \subseteq V_2$. Trivially $I_v \subseteq \{W_1, W_3, W_4, W_6, X_1, X_3, X_4, X_6\}$. Since v is a good vertex, if $|I_v \cap \{W_i : i \in [6]\}| \geq 2$, then I_v contains either $\{W_1, W_4\}$ or $\{W_3, W_6\}$ or $\{W_1, W_3\}$ or $\{W_4, W_6\}$. If $\{W_1, W_4\} \subseteq I_v$, then by the definition of good vertices, we have $\{W_3, W_6, X_3, X_6\} \cap I_v = \emptyset$, and thus

$$\delta^+(G) \leq d^+(v) \leq 4 \cdot 3c^{-1}n + |R_3| \leq n - 2\delta^+(G) - 2d + 16c^{-1}n,$$

implying $3\delta^+(G) + 2d < n + 16c^{-1}n$, a contradiction. If $\{W_3, W_6\}$, $\{W_1, W_3\}$ or $\{W_4, W_6\} \subseteq I_v$, then v belongs to X_2^*, W_2^* or W_5^* respectively, contradicting our assumption on v. Thus $|I_v \cap \{W_i : i \in [6]\}| \leq 1$. Furthermore, since $\{X_3, X_4\} \not\subseteq I_v$ and $\{X_1, X_6\} \not\subseteq I_v$, we have $|I_v \cap \{X_i : i \in [6]\}| \leq 2$, as claimed. ■

Now we go back to the proof of the lemma. Firstly, we have

$$\sum_{i \in [6]} |W_{i+1}^* \cup W_{i-1}^* \cup X_{i+2}^* \cup X_{i-1}^*| = 2\left(\sum_{i \in [6]} (|W_i^*| + |X_i^*|)\right) \leq 6n.$$

Hence there exists an $i \in [6]$ such that $|W_{i+1}^* \cup W_{i-1}^* \cup X_{i+2}^* \cup X_{i-1}^*| \leq n$. Without loss of generality we assume $|W_2^* \cup W_6^* \cup X_3^* \cup X_6^*| \leq n$, and thus we have

$$\delta(G) \overset{(\text{iii}')}{\geq} |W_2^* \cup W_6^* \cup X_3^* \cup X_6^*| + |V(G) \setminus \bigcup_{i \in [6]} (W_i^* \cup X_i^*)| + 4c^2 n^{\frac{1}{2}}.$$

So for all $v \in W_1 \subseteq V_1$, we have $|N_G(v) \cap (W_3^* \cup W_5^* \cup X_2^* \cup X_5^*)| \geq 4c^2 n^{\frac{1}{2}}$. Therefore, there must exist a set $A \in \{W_3^*, W_5^*, X_2^*, X_5^*\}$ and a subset $\widetilde{W_1} \subseteq W_1$ with $|\widetilde{W_1}| \geq \frac{|W_1|}{4} \geq c^2 n^{\frac{1}{2}}$ such that for all $w \in \widetilde{W_1}$, $d(w, A) \geq c^2 n^{\frac{1}{2}}$. If A is one of W_5^*, X_2^* and X_5^*, then for all $w' \in N_G(w, A)$ we have $d(w', W_6) \geq 3c^{-1}n$ by (ii′), which implies that

$$T(ww') \geq |N_G(w', W_6) \cap N_G(w, W_6)| \overset{\text{(iii)}}{\geq} 3c^{-1}n - c^{-1}n = 2c^{-1}n.$$

Thus for all $w \in \widetilde{W_1}$, we have $|\widetilde{D}^+_{G,2c^{-1}}(w)| \geq |N_G(w, A)| \geq c^2 n^{\frac{1}{2}}$. By Lemma 3.1, G contains a $K_3(2)$, a contradiction. If $A = W_3^*$, then for all $w' \in N_G(w, A)$ we have $d(w', W_2) \geq 3c^{-1}n$. A similar argument shows that G contains a $K_3(2)$, this contradicts our assumption and completes the proof of Lemma 3.3. □

4 New constructions of $K_3(2)$-free tripartite graphs

For $n = q^2 + q + 1$ where q is a prime power, it is well known (see [12]) that there is a $K_{2,2}$-free $(q+1)$-regular bipartite graph $G_0 = G_2(n)$ (note that $q + 1 > \sqrt{n}$). Using G_0 as a building block, Bhalkikar and Zhao [2] constructed many non-isomorphic $K_3(2)$-free tripartite graphs with minimum degree at least $n + n^{\frac{1}{2}}$. However, all their constructions have minimum partial degree about $\sqrt{n}$. In this section we construct $K_3(2)$-free tripartite graphs with minimum degree $n + cn^{\frac{1}{2}}$ and *linear* minimum partial degree.

Our first construction is based on two blow-ups of C_6. It provides a $K_3(2)$-free tripartite graph $G = G_3(n)$ with $\delta(G) \geq n + \frac{1}{2}n^{\frac{1}{2}}$ and the minimum partial degree at least $\frac{n}{4} + \frac{1}{2}n^{\frac{1}{2}}$.

Construction 4.1. *Let H be the edge-colored tripartite graph such that*

- *$V(H) = \{a_j, b_j : j \in [6]\}$;*
- *$a_1a_2 \dots a_6$ and $b_1b_2 \dots b_6$ forms two blue C_6's;*
- *for $i \in [3]$, all four pairs between $\{a_i, a_{i+3}\}$ and $\{b_{i+1}, b_{i+4}\}$ are blue edges;*
- *for $j \in [6]$, a_jb_{j-1} are red edges.*

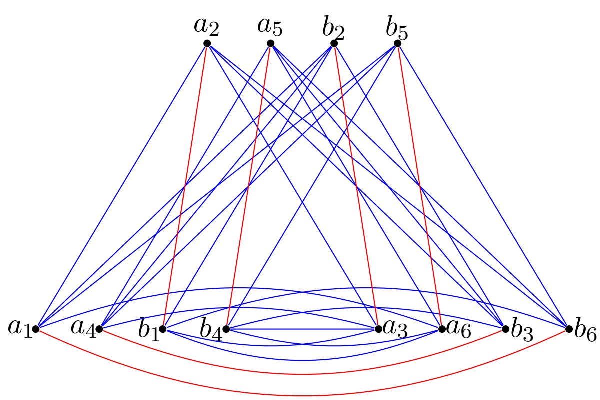

Figure 2: Graph in Construction 4.1.

Note that H is tripartite vertices classes $\{a_i, a_{i+3}, b_i, b_{i+3}\}$.

Let G be the graph with vertex subsets W_v (for $v \in V(H)$) such that $|W_v| = m$ such that the following holds. If uv is a blue edge in H, then $G[W_u, W_v]$ is a complete bipartite graph. If uv is a red edge in H, then $G[W_u, W_v]$ is isomorphic to G_0. If uv is not an edge in H, then $G[W_u, W_v]$ is empty.

Let $n = 4m$. It is easy to see that $G = G_3(n)$, $\delta(G) = 4m + m^{\frac{1}{2}} = n + \frac{1}{2}n^{\frac{1}{2}}$ and the minimum partial degree at least $m + m^{\frac{1}{2}} = \frac{n}{4} + \frac{1}{2}n^{\frac{1}{2}}$. Note that H is $K_{2,2,1}$-free, each pair of triangles do not share a blue edge and every triangle in H contains at least one red edge. Hence G is $K_3(2)$-free.

Note that in Construction 4.1, if we let $|W_{a_i}| = \alpha n$ and $|W_{b_i}| = \beta n$ for all i, where $\alpha + \beta = 1/2$, then $\delta^+(G) = \beta n + (\alpha n)^{\frac{1}{2}}$ and $\delta^-(G) = \alpha n + (\beta n)^{\frac{1}{2}}$.

One may wonder whether a linear minimum partial degree guarantees a unique extremal structure. Using the following gluing operation, we show that the answer is negative.

Construction 4.2. *Let G and G' be two $K_3(2)$-free tripartite graphs on disjoint vertex classes V_1, V_2, V_3 and V_1', V_2', V_3' of size n. Define $G \odot G'$ to be tripartite graph obtained from $G \cup G'$ by adding all edges between V_i and V_{i+1}' for all $i \in [3]$. Then $\delta(G \odot G') = n + \min\{\delta(G), \delta(G')\}$ and the minimum partial degree of $G \odot G'$ is $\min\{\delta^+(G'), \delta^-(G)\}$. Since all new edges in $G \odot G'$ does not lie in a triangle, $G \odot G'$ is $K_3(2)$-free.*

This construction allows us to create many new $K_3(2)$-free tripartite graphs. For example, let G_1 be the graph of Construction 4.1. Then $G = G_1 \odot G_1$ is a $K_3(2)$-free tripartite graph with $2n$ vertices in each part, $\delta(G) \geq 2n + \frac{1}{2}n^{\frac{1}{2}}$ and the minimum partial degree at least $\frac{n}{4} + \frac{1}{2}n^{\frac{1}{2}}$.

Recall that Bollobás, Erdős, and Szemerédi [4] conjectued there exists c such that every $G = G_3(n)$ with $\delta(G) \geq n + cn^{1/2}$ contains $K_3(2)$. If $\delta^+(G) = \beta n$, then Lemma 3.2 implies that G essentially contains a blow-up of C_6 with vertex classes of size about βn. After removing this blow-up of C_6, we believe that one should be able to reduce to the case when both $\delta^+(G)$ and $\delta^-(G)$ are sublinear in n. However, we do not know how to handle this case.

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