CODEGREE CONDITIONS FOR TILING COMPLETE k-PARTITE k-GRAPHS AND LOOSE CYCLES

WEI GAO, JIE HAN, AND YI ZHAO

ABSTRACT. Given two k-graphs (k-uniform hypergraphs) F and H, a perfect F-tiling (or F-factor) in H is a set of vertex disjoint copies of F that together cover the vertex set of H. For all complete k-partite k-graphs K, Mycroft proved a minimum codegree condition that guarantees a K-factor in an n-vertex k-graph, which is tight up to an error term $o(n)$. In this paper we improve the error term in Mycroft's result to a sub-linear term that relates to the Turán number of K when the differences of the sizes of the vertex classes of K are co-prime. Furthermore, we find a construction which shows that our improved codegree condition is asymptotically tight in infinitely many cases thus disproving a conjecture of Mycroft. At last, we determine exact minimum codegree conditions for tiling $K^{(k)}(1,\dots,1,2)$ and tiling loose cycles thus generalizing the results of Czygrinow, DeBiasio, and Nagle, and of Czygrinow, respectively.

1. INTRODUCTION

Given $k \ge 2$, a k-uniform hypergraph (in short, *k-graph*) is a pair $H = (V, E)$, where V is a finite vertex set and E is a family of k-element subsets of V. Given a k-graph H and a set S of d vertices in $V(H)$, $1 \le d \le k-1$, we denote by $\deg_H(S)$ the number of edges of H containing S. The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg(S)$ over all d-subsets S of $V(H)$. Furthermore, the minimum 1-degree is usually referred as the *minimum vertex degree* and the minimum $(k-1)$-degree is referred as the *minimum collective degree (codegree)*.

As a natural extension of matching problems, (hyper)graph *tiling* (alternatively called *packing*) has received much attention in the last two decades (see [23] for a survey). Given two (hyper)graphs F and H, a *perfect F-tiling*, or an *F-factor*, of H is a spanning subgraph of H that consists of vertex disjoint copies of F. Here we are interested in minimum degree threholds that force perfect packings in hypergraphs. Given a k-graph F and an integer n divisible by $|F|$, let $\delta(n, F)$ be the smallest integer t such that every n-vertex k-graph H with $\delta_{k-1}(H) \ge t$ contains a perfect F-tiling.

Perfect tilings for graphs are well understood. In particular, extending the results of Hajnal and Szemerédi [10] and Alon and Yuster [1] (see also [21]), Kühn and Osthus [24] determined $\delta(n, F)$ for all graphs F, up to an additive constant, for sufficiently large n.

Over the last few years there has been a growing interest in obtaining degree conditions that force a perfect F-tiling in k-graphs for $k \ge 3$. In general, this appears to be much harder than the graph case (see a recent survey [34]). Let K_4^3 be the complete 3-graph on four vertices, and let K_4^{3-} be the (unique) 3-graph on four vertices with three edges. Let C_2^3 be the unique 3-graph on four vertices with two edges. Lo and Markström [26] proved that $\delta(n, K_4^3) = (1+o(1))3n/4$, and independently Keevash and Mycroft [20] determined the exact value of $\delta(n, K_4^3)$ for sufficiently large n. In [25] Lo and Markström proved that $\delta(n, K_4^{3-}) = (1+o(1))n/2$. Very recently, Han Lo, Treglown and Zhao [13] determined $\delta(n, K_4^{3-})$ exactly for large n. Kühn and Osthus [22] showed that $\delta(n, C_2^3) = (1+o(1))n/4$, and Czygrinow, DeBiasio, and Nagle [5] determined $\delta(n, C_2^3)$ exactly for large n. Han and Zhao [16] determined the exact minimum

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vertex degree threshold for perfect C_2^3-tiling for large n. With more involved arguments, Han, Zang, and Zhao [15] determined the minimum vertex degree threshold for perfect K-tiling asymptotically for all complete 3-partite 3-graphs K.

Mycroft [29] proved a general result on tiling k-partite k-graphs. To state his result, we need the following definitions. Let F be a k-graph on a vertex set U with at least one edge. A *k-partite realization* of F is a partition of U into vertex classes $U_1, \dots, U_k$ so that for any $e \in E(F)$ and $1 \le j \le k$ we have $|e \cap U_j| = 1$. We say that F is *k-partite* if it admits a k-partite realization. Define

$$\mathcal{S}(F) := \bigcup_{\chi} \{|U_1|, \dots, |U_k|\} \text{ and } \mathcal{D}(F) := \bigcup_{\chi} \{||U_i| - |U_j|| : i, j \in [k]\},$$

where in each case the union is taken over all k-partite realizations χ of F into vertex classes $U_1, \dots, U_k$ of F. Then $\gcd(F)$ is defined to be the greatest common divisor of the set $\mathcal{D}(F)$ (if $\mathcal{D}(F) = \{0\}$ then $\gcd(F)$ is undefined). We also define

$$\sigma(F) := \frac{\min_{S \in \mathcal{S}(F)} S}{|V(F)|},$$

and thus in particular, $\sigma(F) \le 1/k$. Mycroft [29] proved the following:

$$\delta(n, F) \le \begin{cases} n/2 + o(n) & \text{if } \mathcal{S}(F) = \{1\} \text{ or } \gcd(\mathcal{S}(F)) > 1; \\ \sigma(F)n + o(n) & \text{if } \gcd(F) = 1; \\ \max\{\sigma(F)n, n/p\} + o(n) & \text{if } \gcd(\mathcal{S}(F)) = 1 \text{ and } \gcd(F) = d > 1, \end{cases} \tag{1.1}$$

where p is the smallest prime factor of d. Moreover, Mycroft [29] showed that equality holds in (1.1) for all complete k-partite k-graphs F, as well as a wide class of other k-partite k-graphs. Furthermore, he conjectured that the error terms in (1.1) can be replaced by a (sufficiently large) constant that depends only on F.

Conjecture 1.1. [29] *Let F be a k-partite k-graph. Then there exists a constant C such that the error term $o(n)$ in* (1.1) *can be replaced by C.*

Let $K^{(k)}(a_1, \dots, a_k)$ denote the complete k-partite k-graph with parts of size $a_1, \dots, a_k$. In this paper we always assume that $a_1 \le \dots \le a_k$. Thus $\sigma(K^{(k)}(a_1, \dots, a_k)) = a_1/m$, where $m := a_1 + \dots + a_k$. The well-known *space-barrier* (Construction 2.1) shows that

$$\delta(n, K^{(k)}(a_1, \dots, a_k)) \ge \frac{a_1}{m} n. \tag{1.2}$$

This shows that the second line of (1.1) is asymptotically best possible when $F = K^{(k)}(a_1, \dots, a_k)$ and $\gcd(F) = 1$.

We first give a simple construction (Construction 2.2) that strengthens the space-barrier. Applying this construction, we obtain the following proposition, whose Part (1) shows that Conjecture 1.1 is false for all complete k-partite k-graphs K with $\gcd(K) = 1$ and $a_{k-1} \ge 2$. Given two k-graphs F and H, we call H *F-free* if H does not contain F as a subgraph. The well-known *Turán number* $\mathrm{ex}(n, F)$ is the maximum number of edges in an F-free k-graph on n vertices. Correspondingly, the *codegree Turán number* $\mathrm{coex}(n, F)$ is the maximum of the minimum codegree of an F-free k-graph on n vertices. Note that $\mathrm{coex}(n, F)\binom{n}{k-1}/k \le \mathrm{ex}(n, F)$ because an n-vertex k-graph H with $\delta_{k-1}(H) \ge \mathrm{coex}(n, F)$ has at least $\mathrm{coex}(n, F)\binom{n}{k-1}/k$ edges.

Proposition 1.2. *Let $K := K^{(k)}(a_1, \dots, a_k)$ such that $a_1 \le \dots \le a_k$ and $m = a_1 + \dots + a_k$.*

(1) If $a_{k-1} \ge 2$, then $\delta(n, K) \ge a_1 n/m + (1 - o(1))\sqrt{(m - a_1)n/m}$.

(2) If $a_1 = 1$, then

$$\delta(n, K) \ge \frac{n}{m} + \mathrm{coex}(\tfrac{m-1}{m} n + 1, K).$$

Our main result sharpens the second case of (1.1) by using the Turán number and the Frobenius number. Given integers $0 \le b_1 \le \cdots \le b_k$ such that $\gcd(b_1, \dots, b_k) = 1$, the *Frobenius number* $g(b_1, \dots, b_k)$ is the largest integer that cannot be expressed as $\ell_1 b_1 + \cdots + \ell_k b_k$ for any nonnegative integers $\ell_1, \dots, \ell_k$. [1] By definition, $g(b_1, \dots, b_k) = -1$ if some $b_i = 1$; otherwise $g(b_1, \dots, b_k) > 0$. No general formula of $g(b_1, \dots, b_k)$ is known but it is known [7, 33] that $g(b_1, \dots, b_k) \le (b_k - 1)^2$.

Theorem 1.3. *Let $k \ge 3$ and $K := K^{(k)}(a_1, \dots, a_k)$ such that $a_1 \le \cdots \le a_k$, $m = a_1 + \cdots + a_k$ and $\gcd(K) = 1$. Let $n \in m\mathbb{N}$ be sufficiently large. Suppose H is an n-vertex k-graph such that*

$$\delta_{k-1}(H) \ge \frac{a_1}{m} n + f(n) + C, \tag{1.3}$$

where

$$f(n) := \max_{1-C \le i \le 1} \mathrm{ex}\left(\tfrac{m-a_1}{m} n + i, K\right) k \binom{\frac{m-a_1}{m} n + i}{k-1}^{-1}$$

and $C = g(a_2 - a_1, \dots, a_k - a_1) + 1$. Then H contains a K-factor.

A classical result of Erdős [6] states that given integers $k \ge 2$ and $1 \le a_1 \le \cdots \le a_k$, there exists c such that for all sufficiently large n,

$$\mathrm{ex}(n, K^{(k)}(a_1, \dots, a_k)) \le c n^{k - 1/a_1 \cdots a_{k-1}}. \tag{1.4}$$

This implies that $f(n)$ in Theorem 1.3 is at most $O(n^{1-1/a_1 \cdots a_{k-1}})$, which is smaller than the error term $o(n)$ in (1.1). Due to Proposition 1.2 (2), the term $f(n)$ in Theorem 1.3 would be asymptotically tight if $a_1 = 1$ and $\mathrm{coex}(n, K) = (1 - o(1))\, \mathrm{ex}(n, K) k / \binom{n}{k-1}$ (i.e., the extremal k-graph of K is almost regular in terms of codegree). Mubayi [28] determined $\mathrm{ex}(n, K^{(k)}(1, \dots, 1, 2, t))$ asymptotically for all $t \ge 2$. Since the extremal k-graphs in this case is almost regular in terms of codegree, we obtain sharpened value of $\delta(n, K^{(k)}(1, \dots, 1, 2, t))$. Moreover, Mubayi [28] also determined the order of magnitude of $\mathrm{ex}(n, K^{(k)}(1, \dots, 1, s, t))$ for $s \ge 3$ and $t \ge (s-1)! + 1$. This gives the correct order of magnitude of the second term of $\delta(n, K^{(k)}(1, \dots, 1, s, t))$ for $s \ge 3$ and $t \ge (s-1)! + 1$ such that $\gcd(s - 1, t - s) = 1$.

Corollary 1.4. *Let $k \ge 3$.*

(1) *For any $t \ge 2$,*

$$\delta(n, K^{(k)}(1, \dots, 1, 2, t)) = \frac{n}{k+t} + (1 + o(1)) \sqrt{\frac{(t-1)(k+t-1)}{k+t} n};$$

(2) *For any $s \ge 3$ and $t \ge (s-1)! + 1$ such that* $\gcd(s-1, t-s) = 1$,

$$\delta(n, K^{(k)}(1, \dots, 1, s, t)) = \frac{n}{k+s+t-2} + \Theta(n^{1-1/s}).$$

If $K := K^{(k)}(a_1, \dots, a_k)$ satisfies $\gcd(K) = 1$ and $a_{k-1} = 1$, then $a_k = 2$ and consequently, $K = K^{(k)}(1, \dots, 1, 2)$. In this case $\mathrm{ex}(n, K) \le \binom{n}{k-1}/k$ because in a K-free k-graph, every $(k-1)$-set has degree at most 1. Moreover, $C = g(0, \dots, 0, 1) + 1 = 0$ in this case. Theorem 1.3 thus gives that $\delta(n, K) \le n/(k+1) + 1$. By a more careful analysis on the proof of Theorem 1.3, we are able to determine $\delta(n, K^{(k)}(1, \dots, 1, 2))$ exactly (for sufficiently large n).

Theorem 1.5. *Given $k \ge 3$, let $n \in (k+1)\mathbb{Z}$ be sufficiently large. Then*

$$\delta(n, K^{(k)}(1, \dots, 1, 2)) = \begin{cases} \frac{n}{k+1} + 1 & \text{if } k - i \mid \binom{n'-i}{k-1-i} \text{ for all } 0 \le i \le k-2; \\ \frac{n}{k+1} & \text{otherwise,} \end{cases}$$

where $n' = \frac{kn}{k+1} + 1$.

[1] The usual definition of Frobenius numbers requires that all $b_1, \dots, b_k$ are positive and distinct.

A *Steiner system* $S(t,k,n)$ is an n-vertex k-uniform hypergraph in which every set of t vertices has degree exactly 1. The divisibility conditions in Theorem 1.5 are necessary for the existence of $S(k-1,k,n')$. Our proof of Theorem 1.5 applies a recent breakthrough of Keevash [18], who showed that these divisibility conditions are also sufficient for the existence of a Steiner system $S(k-1,k,n')$ for sufficiently large n'.

When $k=3$, the divisibility conditions in Theorem 1.5 reduce to $8 \mid n$. Since $K^{(3)}(1,1,2)=C_2^3$, Theorem 1.5 gives the aforementioned result of Czygrinow, DeBiasio and Nagle [5]. When k is even, the divisibility conditions in Theorem 1.5 always fail and consequently, $\delta(n,K)=n/(k+1)$. To see this, letting $i=k-2$, we have $k-i=2$ and $\binom{n'-i}{k-1-i}=n'-k+2=\frac{kn}{k+1}-k+3$. When k is even, $\frac{kn}{k+1}-k+3$ is odd and thus $k-i\nmid\binom{n'-i}{k-1-i}$.

Our last result is on tiling loose cycles. For $k\ge 3$ and $s>1$, a loose cycle of length s, denoted C_s^k, is a k-graph with $s(k-1)$ vertices $1,\dots,s(k-1)$ and s edges $\{j(k-1)+1,\dots,j(k-1)+k\}$ for $0\le j<s$, where we regard $s(k-1)+1$ as 1. It is easy to see that $\gcd(C_s^k)=1$ unless $s=k=3$ (see Proposition 6.4). Rödl and Ruciński [30, Problem 3.15] asked for the value of $\delta(n,C_s^3)$. Mycroft [29] determined $\delta(n,C_s^k)$ asymptotically for all $s\ge 2$ and $k\ge 3$. Recently, Gao and Han [9] show that $\delta(n,C_3^3)=n/6$ and independently Czygrinow [4] determined $\delta(n,C_s^3)$ for all $s\ge 3$. By modifying the proof of Theorem 1.3, we determine the exact value of $\delta(n,C_s^k)$ for $k\ge 4$ and $s\ge 2$.

Theorem 1.6. *Given $k\ge 4$ and $s\ge 2$, let $n\in s(k-1)\mathbb{N}$ be sufficiently large. Suppose H is an n-vertex k-graph such that $\delta_{k-1}(H)\ge \frac{\lceil s/2\rceil}{s(k-1)}n$. Then H contains a C_s^k-factor.*

Construction 2.1 shows that the codegree condition in Theorem 1.6 is sharp.

The rest of the paper is organized as follows. We prove Proposition 1.2 and Corollary 1.4 in Section 2. Next we discuss proof ideas and give auxiliary lemmas and use them to prove Theorems 1.3, 1.5, and 1.6 in Section 3. We prove the auxiliary lemmas in Sections 4–6.

2. Proof of Theorem 1.4

The following well-known construction is often called the *space barrier* (for tiling problems). Given a k-graph F, let $\tau(F)$ be the smallest size of a *vertex cover* of F, namely, a set that meets each edge of F. Trivially $\tau(K^{(k)}(a_1,\dots,a_k))=a_1$. We also have $\tau(C_s^k)\ge\lceil s/2\rceil$ because C_s^k has s edges and every vertex of C_s^k has degree at most two.[2]

Construction 2.1. *Fix a k-graph F of m vertices. Let $H_0=(V,E)$ be an n-vertex k-graph such that $V=A\cup B$ with $|A|=\tau(F)n/m-1$ and $|B|=n-|A|$, and E consists of all k-sets that intersect A. We have $\delta_{k-1}(H_0)=|A|=\tau(F)n/m-1$.*

Since each copy of F in H_0 contains at least $\tau(F)$ vertices in A, H_0 does not contain a perfect F-tiling. We slightly strengthen Construction 2.1 as follows.

Construction 2.2. *Let $K:=K^{(k)}(a_1,\dots,a_k)$ such that $a_1\le\cdots\le a_k$ and $m=a_1+\cdots+a_k$. Let $H_1=(V,E)$ be an n-vertex k-graph as follows. Let $V=A\cup B$ such that $|A|=a_1n/m-1$ and $|B|=n-|A|$. Let G be a k-graph on B which is $K^{(k)}(b_1,b_2,\dots,b_k)$-free for all $1\le b_1\le\cdots\le b_k$ such that $\sum_{i\in[k]}b_i=m-a_1+1$ and $b_i\le a_i$ for $i\in[k]$. Let E be the union of $E(G)$ and the set of all k-tuples that intersect A, and thus $\delta_{k-1}(H_0)=|A|+\delta_{k-1}(G)=a_1n/m+\delta_{k-1}(G)-1$.*

In Construction 2.2, no m-set with at most a_1-1 vertices in A spans a copy of K. Therefore each copy of K in H_0 contains at least a_1 vertices in A and consequently, H_0 does not contain a perfect K-tiling.

[2]We also know $\tau(C_s^k)\le\lceil s/2\rceil$ from Proposition 6.4.

Now we give a construction of Mubayi [28]. Given $t \geq 2$, suppose that q is a prime number such that $q \equiv 1 \bmod t-1$. Let $n_0 = (q-1)^2/(t-1)$. Let $\mathbf{F}$ be the q-element finite field, and let S be a (multiplicative) subgroup of $\mathbf{F} \setminus \{0\}$ of order $t-1$. We define a k-graph G_0 whose vertex set consists of all equivalence classes in $(\mathbf{F} \setminus \{0\}) \times (\mathbf{F} \setminus \{0\})$, where $(a,b) \sim (x,y)$ if there exists $s \in S$ such that $a = sx$ and $b = sy$. The class represented by (a,b) is denoted by $\langle a, b \rangle$. A set of k distinct classes $\langle a_i, b_i \rangle$ $(1 \leq i \leq k)$ forms an edge in G_0 if

$$\prod_{i=1}^{k} a_i + \prod_{i=1}^{k} b_i \in S.$$

It is easily observed that this relation is well-defined, and $\delta_{k-1}(G_0) \geq q-k$. Moreover, as shown in [28], G_0 is $K^{(k)}(1,\dots,1,2,t)$-free.

To extend this construction, we use the fact that for any $\epsilon > 0$ and sufficiently large n, there exists a prime q such that $q \equiv 1 \bmod t-1$ and $n \leq (q-1)^2/(t-1) \leq (1+\epsilon/3)n$ (see [17]). Let G_0 be the k-graph on $(q-1)^2/(t-1)$ vertices defined above. To obtain a $K^{(k)}(1,\dots,1,2,t)$-free k-graph G on n vertices, we delete a random set T of order $(q-1)^2/(t-1) - n$ from G_0 and let $G := G_0 \setminus T$. Since the expected value of the codegree survived is at least $(q-k)/(1+\epsilon/3)$, standard concentration results (e.g., Chernoff's bound) show that $\delta_{k-1}(G) \geq (1-\epsilon)\sqrt{(t-1)n}$ with positive probability. We summarize this construction together with the result on $\mathrm{ex}(n, K^{(k)}(1,\dots,1,2,t))$ from [28] in the following proposition.

Proposition 2.3. [28] *For any $t \geq 2$, we have $\mathrm{coex}(n, K^{(k)}(1,\dots,1,2,t)) = (1+o(1))\sqrt{(t-1)n}$, and $\mathrm{ex}(n, K^{(k)}(1,\dots,1,2,t)) = (1+o(1))\frac{\sqrt{t-1}}{k!}n^{k-1/2}$.*

For integers $s \geq 3$ and $t \geq (s-1)!+1$, a more involved construction in [28] shows there exists a $K^{(k)}(1,\dots,1,s,t)$-free k-graph of order $q^s - q^{s-1}$ for some prime number q with the desired minimum codegree. We omit the detail of this construction and note that the construction can be extended to all sufficiently large n as above.

Proposition 2.4. [28] *Given $s \geq 3$ and $t \geq (s-1)!+1$, we have $\mathrm{coex}(n, K^{(k)}(1,\dots,1,s,t)) = \Theta(n^{1-1/s})$, and $\mathrm{ex}(n, K^{(k)}(1,\dots,1,s,t)) = \Theta(n^{k-1/s})$.*

Proof of Proposition 1.2. Assume $K := K^{(k)}(a_1,\dots,a_k)$ such that $a_1 \leq \cdots \leq a_k$, $m = a_1 + \cdots + a_k$ and $a_{k-1} \geq 2$. We will show that for any choice of $b_1, b_2, \dots, b_k$ such that $\sum_{i\in[k]} b_i = m - a_1 + 1$ and $b_i \leq a_i$ for $i \in [k]$, we have $b_{k-1} \geq 2$ (thus $K^{(k)}(b_1, b_2, \dots, b_k)$ contains $K^{(k)}(1,\dots,1,2,2)$ as a subgraph). Then Proposition 1.2 (1) follows from putting the k-graph G given by Proposition 2.3 with $t = 2$ into Construction 2.2. To see why $b_{k-1} \geq 2$, first assume that $a_1 = 1$. In this case $b_i = a_i$ for all $i \in [k]$. Since $a_{k-1} \geq 2$, we have $b_{k-1} \geq 2$. Second assume that $a_1 \geq 2$. If $b_{k-1} = 1$, then $b_1 = \cdots = b_{k-1} = 1$ and consequently, $\sum_{i\in[k]} a_i - \sum_{i\in[k]} b_i \geq (k-1)(a_1-1) > a_1 - 1$, a contradiction. Thus $b_{k-1} \geq 2$.

Proposition 1.2 (2) follows from Construction 2.2 immediately because $a_1 = 1$ implies that $a_i = b_i$ for $i \in [k]$. □

Proof of Corollary 1.4. The upper bounds in Corollary 1.4 (1) and (2) follow from Theorem 1.3 and the results on the Turán numbers from Propositions 2.3 and 2.4. The lower bounds follow from Proposition 1.2 (2) and the results on the codegree Turán numbers from Propositions 2.3 and 2.4. □

3. Proof ideas and lemmas

Mycroft's proofs [29] use the newly developed Hypergraph Blow-up Lemma by Keevash [19]. Instead, our proofs include several new ingredients, which allow us to obtain a better bound by a much shorter proof. First, to obtain *exact* results, we separate the proof into a *non-extremal* case and an *extremal* case and deal with them separately. The proof of the non-extremal case utilizes the *lattice-based absorbing* *method* developed recently by the second author [12], which builds on the *absorbing method* initiated by Rödl, Ruciński and Szemerédi [31]. In order to find an almost perfect K-tiling, we use the so-called *fractional homomorphic tiling*, which was used by Buß, Hàn and Schacht in [2], together with the weak regularity lemma for hypergraphs. At last, we deal with the extremal case by careful analysis.

Now we give our lemmas. Throughout the paper, we write $\alpha \ll \beta \ll \gamma$ to mean that we can choose the positive constants α, β, γ from right to left. More precisely, there are increasing functions f and g such that, given γ, whenever we choose some $\beta \leq f(\gamma)$ and $\alpha \leq g(\beta)$, the subsequent statement holds. Hierarchies of other lengths are defined similarly.

Lemma 3.1 (Absorbing Lemma). *Let $k \geq 3$ and $K := K^{(k)}(a_1, \dots, a_k)$ such that $a_1 \leq \cdots \leq a_k$, $m = a_1 + \cdots + a_k$ and $\gcd(K) = 1$. Suppose $\gamma' \ll \gamma \ll \rho, 1/m$ and n is sufficiently large. If H is an n-vertex k-graph such that $\delta_{k-1}(H) \geq \rho n$, then there exists a vertex set $W \subseteq V(H)$ with $|W| \leq \gamma n$ such that for any vertex set $U \subseteq V(H) \setminus W$ with $|U| \leq \gamma' n$ and $|U| \in m\mathbb{Z}$, both $H[W]$ and $H[U \cup W]$ contain K-factors.*

We say H is ξ-*extremal* if there exists a set $B \subseteq V(H)$ of $\lfloor (1-\sigma(K))n \rfloor$ vertices such that $e(B) \leq \xi \binom{|B|}{k}$. In the following lemma we do not need the assumption $\gcd(K) = 1$, instead we assume that $a_1 < a_k$ (which is necessary for $\gcd(K) = 1$). Note that the $a_1 = a_k$ (i.e., $a_1 = \cdots = a_k$) case has been solved in [9, Lemma 2.4].

Lemma 3.2 (K-tiling Lemma). *Let $k \geq 3$ and $K := K^{(k)}(a_1, \dots, a_k)$ such that $a_1 \leq \cdots \leq a_k$, $a_1 < a_k$, $m = a_1 + \cdots + a_k$. For any $\alpha, \gamma, \xi > 0$ such that $\gamma \ll 1/m$ and $\xi \geq 5bk^2\gamma$, there exists an integer n_0 such that the following holds. If H is a k-graph on $n > n_0$ vertices with $\delta_{k-1}(H) \geq (a_1/m - \gamma)n$, then H has a K-tiling that covers all but at most αn vertices unless H is ξ-extremal.*

Finally we give the extremal cases for Theorems 1.3, 1.5 and 1.6, respectively.

Theorem 3.3. *Given $k \geq 3$, let $K := K^{(k)}(a_1, \dots, a_k)$ such that $a_1 \leq \cdots \leq a_k$, $m = a_1 + \cdots + a_k$ and $\gcd(K) = 1$. Suppose $1/n \ll \xi \ll 1/m$ and $n \in m\mathbb{N}$. If H is an n-vertex k-graph which is ξ-extremal and satisfies* (1.3)*, then H contains a K-factor.*

Theorem 3.4. *Given $k \geq 3$, let $1/n \ll \xi \ll 1/k$ such that $n \in (k+1)\mathbb{N}$. Suppose H is an n-vertex k-graph that is ξ-extremal. Then H contains a $K^{(k)}(1, \dots, 1, 2)$-factor if either of the following holds:*

(i) $\delta_{k-1}(H) \geq n/(k+1) + 1$;

(ii) $\delta_{k-1}(H) \geq n/(k+1)$ *and* $k - i \nmid \binom{n'-i}{k-1-i}$ *for some* $0 \leq i \leq k-2$ *and* $n' = \frac{kn}{k+1} + 1$.

Theorem 3.5. *Given $k \geq 4$ and $s \geq 2$, let $1/n \ll \xi \ll 1/s, 1/k$ such that $n \in s(k-1)\mathbb{N}$. Suppose H is an n-vertex k-graph with $\delta_{k-1}(H) \geq \frac{\lceil s/2 \rceil}{s(k-1)} n$. If H is ξ-extremal, then H contains a C_s^k-factor.*

Proofs of Theorems 1.3, 1.5 and 1.6. We first prove Theorem 1.3. Let $k \geq 3$ and $K := K^{(k)}(a_1, \dots, a_k)$ such that $a_1 \leq \cdots \leq a_k$, $m = a_1 + \cdots + a_k$ and $\gcd(K) = 1$. Suppose $1/n \ll \gamma' \ll \gamma \ll \xi \ll 1/m$ and $n \in m\mathbb{N}$. Suppose H is an n-vertex k-graph satisfying (1.3). If H is ξ-extremal, then H contains a K-factor by Theorem 3.3. Otherwise we apply Lemma 3.1 and find an absorbing set W in $V(H)$ of size at most γn which has the absorbing property. Let $H' := H \setminus W$ and $n' = |V(H')| \geq (1-\gamma)n$. Note that $m \mid n'$. If H' is $(\xi/2)$-extremal, then there exists a vertex subset B' in $V(H')$ of order $\lfloor (1-\sigma(K))n' \rfloor$ such that $e_{H'}(B') \leq \frac{\xi}{2}\binom{|B'|}{k}$. Thus by adding to B' at most $n - n' \leq \gamma n$ vertices, we get a set B of exactly $\lfloor (1-\sigma(K))n \rfloor$ vertices in $V(H)$ with

$$e_H(B) \leq e_{H'}(B') + \gamma n \cdot \binom{n-1}{k-1} \leq \frac{\xi}{2}\binom{|B'|}{k} + k\gamma\binom{n}{k} \leq \xi\binom{|B|}{k}$$

because the choice of γ. This means that H is ξ-extremal, a contradiction. We thus assume that H' is not $(\xi/2)$-extremal. By applying Lemma 3.2 on H' with $\xi/2$ and $\alpha = \gamma'$, we obtain a K-tiling M that covers all but a set U of at most $\gamma' n$ vertices. By the absorbing property of W, $H[W \cup U]$ contains a K-factor and together with the K-tiling M we obtain a K-factor of H.

The proof of Theorem 1.6 is the same except that we replace K by C_s^k and replace Theorem 3.3 by Theorem 3.5 (here we apply Lemma 3.2 with the k-partite k-graph given by Proposition 6.4). Similarly, after replacing Theorem 3.3 by Theorem 3.4, the arguments above prove the upper bounds in Theorem 1.5. To see the lower bounds, we know $\delta(n, K^{(k)}(1,\dots,1,2)) \ge n/(k+1)$ from (1.2). Let $n' = \frac{kn}{k+1}+1$. If $k-i \mid \binom{n'-i}{k-1-i}$ for all $0 \le i \le k-2$, then the result of Keevash [18] implies that the Steiner system $S(k-1,k,n')$ exists, in other words, $\mathrm{coex}(n', K^{(k)}(1,\dots,1,2)) = 1$. Then the lower bound $\delta(n, K^{(k)}(1,\dots,1,2)) \ge n/(k+1)+1$ follows from Proposition 1.2 (2). $\square$

4. Proof of the Absorbing Lemma

The following simple proposition will be useful.

Proposition 4.1. *Let H be a k-graph. If $\delta_{k-1}(H) \ge xn$ for some $0 \le x \le 1$, then $\delta_1(H) \ge x\binom{n-1}{k-1}$.*

The following concepts were introduced by Lo and Markström [26]. Given a k-graph F of order m, $\beta > 0$, $i \in \mathbb{N}$, we say that two vertices u, v in a k-graph H on n vertices are (F,β,i)*-reachable (in H)* if and only if there are at least βn^{im-1} $(im-1)$-sets W such that both $H[\{u\}\cup W]$ and $H[\{v\}\cup W]$ contain F-factors. A vertex set A is (F,β,i)*-closed in H* if every pair of vertices in A are (F,β,i)-reachable in H. For $x \in V(H)$, let $\tilde{N}_{F,\beta,i}(x)$ be the set of vertices that are (F,β,i)-reachable to x in H.

We use the following lemma in [14] which gives us a partition $\mathcal{P} = \{V_1,\dots,V_r\}$ on H such that for any $i \in [r]$, V_i is $(F,\beta,2^{c-1})$-closed.

Lemma 4.2 ([14], Lemma 6.3)**.** *Let $c,k,m \ge 2$ be integers and suppose $1/n \ll \beta \ll \alpha \ll 1/c, \delta', 1/m$. Let F be an m-vertex k-graph. Assume an n-vertex k-graph H satisfies that $|\tilde{N}_{F,\alpha,1}(v)| \ge \delta' n$ for any $v \in V(H)$ and every set of $c+1$ vertices in $V(H)$ contains two vertices that are $(F,\alpha,1)$-reachable. Then we can find a partition $\mathcal{P}$ of $V(H)$ into $V_1,\dots,V_r$ with $r \le \min\{c, 1/\delta'\}$ such that for any $i \in [r]$, $|V_i| \ge (\delta'-\alpha)n$ and V_i is $(F,\beta,2^{c-1})$-closed in H.*

Let $\mathcal{P} = \{V_1,\dots,V_r\}$ be a vertex partition of H. The *index vector* $\mathbf{i}_{\mathcal{P}}(S) \in \mathbb{Z}^r$ of a subset $S \subset V$ with respect to $\mathcal{P}$ is the vector whose coordinates are the sizes of the intersections of S with each part of $\mathcal{P}$, i.e., $\mathbf{i}_{\mathcal{P}}(S)_{V_i} = |S \cap V_i|$ for $i \in [r]$. We call a vector $\mathbf{i} \in \mathbb{Z}^r$ an *s-vector* if all its coordinates are nonnegative and their sum equals to s. Given a k-graph F of order m and $\mu > 0$, an m-vector $\mathbf{v}$ is called a *μ-robust F-vector* if there are at least μn^m copies C of F in H satisfying $\mathbf{i}_{\mathcal{P}}(V(C)) = \mathbf{v}$. Let $I^{\mu}_{\mathcal{P},F}(H)$ be the set of all μ-robust F-vectors. For $j \in [r]$, let $\mathbf{u}_j \in \mathbb{Z}^r$ be the jth *unit vector*, namely, $\mathbf{u}_j$ has 1 on the jth coordinate and 0 on other coordinates. A *transferral* is a vector of form $\mathbf{u}_i - \mathbf{u}_j$ for some distinct $i, j \in [r]$. Let $L^{\mu}_{\mathcal{P},F}(H)$ be the lattice (i.e., the additive subgroup) generated by $I^{\mu}_{\mathcal{P},F}(H)$.

To prove Lemma 3.1, our main tool is Lemma 4.2 together with the following results. The next proposition is a simple counting result that follows from (1.4).

Proposition 4.3. *Given integers $k, r_0, a_1,\dots,a_k \in \mathbb{N}$, suppose that $1/n \ll \mu \ll \eta, 1/k, 1/r_0, 1/a_1,\dots,1/a_k$. Let H be a k-graph on n vertices with a vertex partition $V_1 \cup \cdots \cup V_r$ where $r \le r_0$. Suppose $i_1,\dots,i_k \in [r]$ and H contains at least ηn^k edges $e = \{v_1,\dots,v_k\}$ such that $v_1 \in V_{i_1}, \dots, v_k \in V_{i_k}$. Then H contains at least $\mu n^{a_1+\cdots+a_k}$ copies of $K^{(k)}(a_1,\dots,a_k)$ whose jth part is contained in V_{i_j} for all $j \in [k]$.*

We use the following result in [15], which says that $V(H)$ is closed when all the transferrals of $\mathcal{P}$ are present. We state it in a less general form, namely, we omit the trash set V_0 in its original form.

Lemma 4.4 ([15], Lemma 3.9). *Let $i_0, k, r_0 > 0$ be integers and let F be an m-vertex k-graph. Suppose $1/n \ll 1/i_0', \beta' \ll \epsilon, \beta, \mu$ such that $i_0' \in \mathbb{Z}$. Let H be a k-graph on n vertices with a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ such that $r \le r_0$ and for each $j \in [r]$, $|V_j| \ge \epsilon n$ and V_j is (F, β, i_0)-closed in H. If $\mathbf{u}_j - \mathbf{u}_l \in L^{\mu}_{\mathcal{P},F}(H)$ for all $1 \le j < l \le r$, then $V(H)$ is (F, β', i_0')-closed in H.*

The following lemma of Lo and Markström provides the desired absorbing set when $V(H)$ is closed.

Lemma 4.5 ([26], Lemma 1.1). *Let m and i be positive integers and let F be an m-vertex k-graph. Suppose $1/n \ll \gamma' \ll \beta, 1/m, 1/i$ and H is an (F, β, i)-closed k-graph of order n. Then there exists a vertex set $W \subseteq V(H)$ with $|W| \le \beta n$ such that for any vertex set $U \subseteq V(H) \setminus W$ with $|U| \le \gamma' n$ and $|U| \in m\mathbb{Z}$, both $H[W]$ and $H[U \cup W]$ contain F-factors.*

We need another lemma from [26].

Lemma 4.6 ([26], Lemma 4.2). *Let $k \ge 2$ be an integer and F be an m-vertex k-partite k-graph. Suppose $1/n \ll \alpha \ll \gamma$. For any k-graph H of order n, two vertices $x, y \in V(H)$ are $(F, \alpha, 1)$-reachable to each other if the number of $(k-1)$-sets $S \in N(x) \cap N(y)$ with $|N(S)| \ge \gamma n$ is at least $\gamma^2 \binom{n}{k-1}$.*

Proof of Lemma 3.1. Let $k \ge 3$ and $K := K^{(k)}(a_1, \ldots, a_k)$ such that $a_1 \le \cdots \le a_k$, $m = a_1 + \cdots + a_k$ and $\gcd(K) = 1$. Given $\rho > 0$, we select other parameters as follows. We pick $0 < \eta, \gamma \ll \rho$ such that $(\lfloor 1/\rho \rfloor + 1)\rho > 1 + (\lfloor 1/\rho \rfloor + 1)^2 \gamma$. When applying Proposition 4.3 with $r_0 = \lfloor 1/\rho \rfloor$ and η, we get $0 < \mu \ll \eta$. When applying Lemma 4.6, we get $0 < \alpha \ll \gamma$. When applying Lemma 4.2 with α, $c = \lfloor 1/\rho \rfloor$, and $\delta' = \rho - \gamma$, we get $\beta > 0$. When applying Lemma 4.4 with $i_0 = 2^{\lfloor 1/\rho \rfloor - 1}$, $r_0 = \lfloor 1/\rho \rfloor$, $\epsilon = \rho - 2\gamma$, β, and μ, we get β' and an integer i_0'. Finally, when applying Lemma 4.5 with β' and i_0', we get $\gamma' > 0$.

Let n be a sufficiently large integer and let H be a k-graph on n vertices such that $\delta_{k-1}(H) \ge \rho n$. By Proposition 4.1, we have $\delta_1(H) \ge \rho \binom{n-1}{k-1}$. First, for every $v \in V(H)$, let us bound $|\tilde{N}_{K,\alpha,1}(v)|$ from below. Given any $(k-1)$-set S, we have $|N(S)| \ge \rho n$ because $\delta_{k-1}(H) \ge \rho n$. Then by Lemma 4.6, for any distinct $u, v \in V(H)$, $u \in \tilde{N}_{K,\alpha,1}(v)$ if $|N(u) \cap N(v)| \ge \gamma^2 \binom{n}{k-1}$. By double counting, we have

$$\sum_{S \in N(v)} |N(S)| < |\tilde{N}_{K,\alpha,1}(v)| \cdot |N(v)| + n \cdot \gamma^2 \binom{n}{k-1}.$$

Note that $|N(v)| \ge \delta_1(H) \ge \rho \binom{n-1}{k-1}$. Together with $\gamma \ll \rho$, we get

$$|\tilde{N}_{K,\alpha,1}(v)| > \rho n - \frac{\gamma^2 n^k}{|N(v)|} \ge (\rho - \gamma) n. \tag{4.1}$$

Next we claim that every set of $\lfloor 1/\rho \rfloor + 1$ vertices in $V(H)$ contains two vertices that are $(K, \alpha, 1)$-reachable. Indeed, since $\delta_1(H) \ge \rho \binom{n-1}{k-1}$, the degree sum of any $\lfloor 1/\rho \rfloor + 1$ vertices is at least $(\lfloor 1/\rho \rfloor + 1)\rho \binom{n-1}{k-1}$. By the definition of γ, we have

$$(\lfloor 1/\rho \rfloor + 1)\rho \binom{n-1}{k-1} > \left(1 + \binom{\lfloor 1/\rho \rfloor + 1}{2} \gamma\right) \binom{n}{k-1}.$$

By the Inclusion-Exclusion Principle, there exist $u, v \in V(H)$ such that $|N(u) \cap N(v)| \ge \gamma \binom{n}{k-1}$, so they are $(K, \alpha, 1)$-reachable by Lemma 4.6.

By (4.1) and the above claim, we can apply Lemma 4.2 on H with the constants chosen at the beginning of the proof. So we get a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of $V(H)$ such that $r \le \min\{\lfloor 1/\rho \rfloor, 1/(\rho - \gamma)\} = \lfloor 1/\rho \rfloor$ and for any $i \in [r]$, $|V_i| \ge (\rho - 2\gamma) n$ and V_i is $(K, \beta, 2^{c-1})$-closed in H.

Now we show that $\mathbf{u}_i - \mathbf{u}_j \in L^{\mu}_{\mathcal{P},K}(H)$ for all distinct $i, j \in [r]$. Without loss of generality, assume $i = 1$ and $j = 2$. For any $u \in V_1$ and $v \in V_2$, since $\delta_{k-1}(H) \ge \rho n$, u and v are contained in at least

$\binom{n-2}{k-3}\rho n/(k-2)$ edges. Since there are $\binom{k+r-1}{r-1}$ k-vectors, by averaging, there exists a k-vector $\mathbf{v}$ whose first two coordinates are positive and which is the index vector of at least

$$\frac{1}{\binom{k}{2}} \cdot \frac{|V_1||V_2|\binom{n-2}{k-3}\rho n/(k-2)}{\binom{k+r-1}{r-1}} > \eta n^k$$

edges (we divide a factor of $\binom{k}{2}$ because an edge may be counted at most $\binom{k}{2}$ times because of its intersections with V_1 and V_2). Note that we can write $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k$ such that all $\mathbf{v}_i \in \{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$. Without loss of generality, assume that $\mathbf{v}_1 = \mathbf{u}_1$ and $\mathbf{v}_2 = \mathbf{u}_2$.

We apply Proposition 4.3 with $a_1, a_2, \ldots, a_k$ and conclude that there are at least μn^m copies of K in H with index vector $\mathbf{v}' = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{v}_3 + \cdots + a_k\mathbf{v}_k$. We then apply Proposition 4.3 again with $a_2, a_1, \ldots, a_k$ (with a_1, a_2 exchanged) and conclude that there are at least μn^m copies of K in H with index vector $\mathbf{v}'' = a_2\mathbf{u}_1 + a_1\mathbf{u}_2 + a_3\mathbf{v}_3 + \cdots + a_k\mathbf{v}_k$. By definition, we have $\mathbf{v}', \mathbf{v}'' \in I^{\mu}_{\mathcal{P},K}(H)$ and thus $(a_1 - a_2)\mathbf{u}_1 + (a_2 - a_1)\mathbf{u}_2 = \mathbf{v}' - \mathbf{v}'' \in L^{\mu}_{\mathcal{P},K}(H)$. By repeating the arguments for other permutations of $a_1, a_2, \ldots, a_k$, we get that $\mathbf{w}_i := (a_{i+1} - a_i)\mathbf{u}_1 + (a_i - a_{i+1})\mathbf{u}_2 \in L^{\mu}_{\mathcal{P},K}(H)$ for $i = 1, 2, \ldots, k-1$. Recall that since $\gcd(K) = 1$, there exist $\ell_1, \ell_2, \ldots, \ell_{k-1} \in \mathbb{Z}$ such that $\ell_1(a_2 - a_1) + \ell_2(a_3 - a_2) + \cdots + \ell_{k-1}(a_k - a_{k-1}) = 1$. Hence $\mathbf{u}_1 - \mathbf{u}_2 = \ell_1\mathbf{w}_1 + \ell_2\mathbf{w}_2 + \cdots + \ell_{k-1}\mathbf{w}_{k-1} \in L^{\mu}_{\mathcal{P},K}(H)$ and we are done.

Since $\mathbf{u}_i - \mathbf{u}_j \in L^{\mu}_{\mathcal{P},K}(H)$ for all distinct $i, j \in [r]$, by Lemma 4.4, we conclude that $V(H)$ is (K, β', i_0')-closed. Thus the desired absorbing set is provided by Lemma 4.5. □

5. Proof of the Almost Perfect Tiling Lemma

For integers $k \geq 2$ and $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$ with $a_1 < a_k$, let $K := K^{(k)}(a_1, \ldots, a_k)$. Let $m = |V(K)| = a_1 + \cdots + a_k$ and $\sigma(K) = a_1/m$. Given an n-vertex k-graph H such that n is sufficiently large and $\delta_{k-1}(H) \geq (\sigma(K) - o(1))n$, we will show that either H has an almost perfect K-tiling, or H is ξ-extremal for some $\xi > 0$. Note that it suffices to consider the case when $a_2 = a_3 = \cdots = a_k$. In fact, let $K' := K^{(k)}((k-1)a_1, m - a_1, \ldots, m - a_1)$ and note that $\sigma(K) = \sigma(K') = a_1/m$. Moreover, since K' has a perfect K-tiling, it suffices to find an almost perfect K'-tiling in H.

We thus consider $K' := K^{(k)}(a, b, \ldots, b)$ with $a < b$ and call the vertex class of size a *small*, and those of size b *large*. Let $m = a + (k-1)b$. An *α-deficient K'-tiling* of an n-vertex k-graph H is a K'-tiling of H that covers at least $(1-\alpha)n$ vertices. The following lemma allows a small number of $(k-1)$-subsets of $V(H)$ to have low degree, and may find applications in other problems (*e.g.*, in reduced hypergraphs after we apply the regularity lemma).

Lemma 5.1. *Fix integers $k \geq 2$ and $a < b$, $0 < \gamma \ll 1/m$ and let $K' := K^{(k)}(a, b, \ldots, b)$. For any $\alpha > 0$ and $\xi \geq 5bk^2\gamma$, there exist $\epsilon > 0$ and an integer n_0 such that the following holds. Suppose H is a k-graph on $n > n_0$ vertices with $\deg(S) \geq (\sigma(K') - \gamma)n$ for all but at most ϵn^{k-1} sets $S \in \binom{V(H)}{k-1}$, then H has an α-deficient K'-tiling or H is ξ-extremal.*

We will prove Lemma 5.1 by using the Weak Regularity Lemma for hypergraphs and the so-called fractional homomorphic tilings (introduced by Buß, Hàn and Schacht [2]).

5.1. **Weak Regularity Lemma.** Let $H = (V, E)$ be a k-graph and let $A_1, \ldots, A_k$ be mutually disjoint non-empty subsets of V. We define $e(A_1, \ldots, A_k)$ to be the number of edges with one vertex in each A_i, $i \in [k]$, and the density of H with respect to $(A_1, \ldots, A_k)$ as

$$d(A_1, \ldots, A_k) = \frac{e(A_1, \ldots, A_k)}{|A_1| \cdots |A_k|}.$$

Given $\epsilon > 0$ and $d \geq 0$, we say a k-tuple $(V_1, \dots, V_k)$ of mutually disjoint subsets $V_1, \dots, V_k \subseteq V$ is *(ϵ, d)-regular* if

$$|d(A_1, \dots, A_k) - d| \leq \epsilon$$

for all k-tuples of subsets $A_i \subseteq V_i$, $i \in [k]$, satisfying $|A_i| \geq \epsilon |V_i|$. We say $(V_1, \dots, V_k)$ is *ϵ-regular* if it is (ϵ, d)-regular for some $d \geq 0$. It is immediate from the definition that in an (ϵ, d)-regular k-tuple $(V_1, \dots, V_k)$, if $V_i' \subset V_i$ has size $|V_i'| \geq c|V_i|$ for some $c \geq \epsilon$, then $(V_1', \dots, V_k')$ is $(\epsilon/c, d)$-regular.

The Weak Regularity Lemma (for hypergraphs) is a straightforward extension of Szemerédi's regularity lemma for graphs [32].

Theorem 5.2 (Weak Regularity Lemma). *Given $t_0 \geq 0$ and $\epsilon > 0$, there exist $T_0 = T_0(t_0, \epsilon)$ and $n_0 = n_0(t_0, \epsilon)$ so that for every k-graph $H = (V, E)$ on $n > n_0$ vertices, there exists a partition $V = V_0 \cup V_1 \cup \dots \cup V_t$ such that*

(i) $t_0 \leq t \leq T_0$,
(ii) $|V_1| = |V_2| = \dots = |V_t|$ *and* $|V_0| \leq \epsilon n$,
(iii) for all but at most $\epsilon\binom{t}{k}$ k-subsets $\{i_1, \dots, i_k\} \subset [t]$, the k-tuple $(V_{i_1}, \dots, V_{i_k})$ is ϵ-regular.

The partition given in Theorem 5.2 is called an *ϵ-regular partition* of H. Given an ϵ-regular partition $\mathcal{Q}$ of H and $d \geq 0$, we refer to $V_i, i \in [t]$ as *clusters* and define the *cluster hypergraph* $R = R(\epsilon, d, \mathcal{Q})$ with vertex set $[t]$ in which $\{i_1, \dots, i_k\} \subset [t]$ is an edge if and only if $(V_{i_1}, \dots, V_{i_k})$ is ϵ-regular and $d(V_{i_1}, \dots, V_{i_k}) \geq d$.

The following corollary shows that the cluster hypergraph inherits the minimum codegree of the original hypergraph. The proof is standard and very similar to that of [11, Proposition 16] so we omit the proof.

Corollary 5.3. [11] *Suppose $1/t_0 \ll \epsilon \ll c, d, 1/k$, then there exist T and n_1 such that the following holds. Let H be a k-graph on $n > n_1$ vertices such that $\deg_H(S) \geq cn$ for all but at most ϵn^{k-1} $(k-1)$-sets S. Then H has an ϵ-regular partition $\mathcal{Q} = \{V_0, V_1, \dots, V_t\}$ with $t_0 \leq t \leq T$ such that in the cluster hypergraph $R = R(\epsilon, d, \mathcal{Q})$, all but at most $\sqrt{\epsilon} t^{k-1}$ $(k-1)$-subsets S of $[t]$ satisfy $\deg_R(S) \geq (c - d - 5\sqrt{\epsilon})t$.*

5.2. **Fractional hom(K')-tiling.** Let $K' := K^{(k)}(a, b, \dots, b)$ with $a < b$. To obtain a large K'-tiling in the hypergraph H, we consider weighted homomorphisms from K' into the cluster hypergraph R. For this purpose, we extend a definition that Buß, Hàn and Schacht [2] introduced for 3-graphs.[3]

Definition 5.4. *Given a k-graph H, a function $h : V(H) \times E(H) \to [0, 1]$ is called a fractional* hom(K')-*tiling of H if*

(1) $h(v, e) = 0$ *if* $v \notin e$,
(2) $h(v) = \sum_{e \in E(H)} h(v, e) \leq 1$,
(3) for each $e \in E(H)$ there exists a labeling $e = v_1 \cdots v_k$ such that

$$h(v_1, e) \leq \dots \leq h(v_k, e) \quad \text{and} \quad \frac{h(v_1, e)}{a_1} \geq \dots \geq \frac{h(v_k, e)}{a_k}$$

By $h_{\min}$ we denote the smallest non-zero value of $h(v, e)$ (and set $h_{\min} = \infty$ if $h \equiv 0$). For $e = v_1 \cdots v_k$, we write $h(v_1, \dots, v_k) = (h(v_1, e), \dots, h(v_k, e))$ and $h(e) = \sum_{i \in [k]} h(v_i, e)$. The sum over all values is the weight $w(h)$ of h

$$w(h) = \sum_{(v,e) \in V(H) \times E(H)} h(v, e).$$

Assume that $V(K') = A_1 \cup \dots \cup A_k$ with $|A_1| = a$ and $|A_2| = \dots = |A_k| = b$, and $m = a + (k-1)b$.

Fact 5.5. *There is a fractional* hom(K')-*tiling of K' such that $w(h) = m$ and $h_{\min} = b^{1-k}$.*

[3] As noted by a referee, we could also use Farkas' lemma here (see [3]).

Proof. For each edge $e = v_1 \cdots v_k$ of K' with $v_i \in A_i$ for $i \in [k]$, we assign weight

$$h(v_1, \dots, v_k) = \left(\frac{1}{b^{k-1}}, \frac{1}{ab^{k-2}}, \dots, \frac{1}{ab^{k-2}} \right).$$

Then $h(e) = m/(ab^{k-1})$ and consequently, $w(h) = m$. □

In the rest of the proof, we will refer to the weight assignment in Fact 5.5 as the *standard weights*.

Let $\widehat{K'}$ be a k-graph obtained from K' by adding $k-1$ new vertices and a new edge that consists of the $k-1$ new vertices and a vertex from a large vertex class of K'.

Proposition 5.6. *There is a fractional* $\hom(K')$*-tiling of* $\widehat{K'}$ *such that* $w(h) \geq m + 1/(ab^{k-1})$ *and* $h_{\min} = b^{1-k}$.

Proof. Assume $V(K') = A_1 \cup \cdots \cup A_k$ with $|A_1| = a$ and $|A_i| = b$ for $2 \leq i \leq k$. Let $u_1, \dots, u_{k-1}$ be the vertices of $\widehat{K'}$ not in $V(K')$ and $e = \{v, u_1, \dots, u_{k-1}\}$ be the edge of $\widehat{K'}$ not in $E(K')$, where $v \in A_j$ for some $j > 1$. Fix any edge $e_1 = x_1 \cdots x_k$ of K' with $x_i \in A_i$ for $i \in [k]$ such that $x_j = v$. We assign the weight

$$h(v, u_1, \dots, u_{k-1}) = \left(\frac{1}{ab^{k-2}}, \frac{1}{a^2b^{k-3}}, \dots, \frac{1}{a^2b^{k-3}} \right).$$

to e and the standard weight to all the edges of K' except for e_1. Moreover, set e_1 as unweighted. Since $\sum_{e' \in E(K')} h(e') = m - m/(ab^{k-1})$ and $h(e) = m/(a^2b^{k-2})$, we have $h_{\min} = b^{1-k}$ and

$$w(h) = m - \frac{m}{ab^{k-1}} + \frac{m}{a^2b^{k-2}} = m + \frac{1}{ab^{k-1}} \frac{m(b-a)}{a} \geq m + \frac{1}{ab^{k-1}}.$$ □

The following proposition says that a fractional $\hom(K')$-tiling in the cluster hypergraph can be converted to an integer K'-tiling in the original hypergraph. It is almost the same as [15, Proposition 4.4], which covers the $k = 3$ case, so we omit its proof.

Proposition 5.7. *Suppose* $\epsilon, \phi > 0$, $d \geq 2\epsilon/\phi$ *and* $t > 0$ *is an integer, and* n *is sufficiently large. Let* H *be a* k*-graph on* n *vertices with an* (ϵ, t)*-regular partition* $\mathcal{Q}$ *and a cluster hypergraph* $\mathcal{R} = \mathcal{R}(\epsilon, d, \mathcal{Q})$. *Suppose there is a fractional hom*(K')*-tiling* h *of* $\mathcal{R}$ *with* $h_{\min} \geq \phi$. *Then there exists a* K'*-tiling of* H *that covers at least* $(1 - 2b\epsilon/\phi)\, w(h)n/t$ *vertices.* □

5.3. **Proof of the K'-tiling Lemma (Lemma 5.1).**

Proposition 5.8. *For all* $0 < \rho \leq 1/2$ *and all* $\xi, \beta, \delta, \epsilon > 0$ *the following holds. Suppose there exists an* n_0 *such that for every* k*-graph* H *on* $n > n_0$ *vertices satisfying* $\deg(S) \geq \rho n$ *for all but at most* ϵn^{k-1} $(k-1)$*-sets* S*, either* H *has a* β*-deficient* K'*-tiling, or* H *is* ξ*-extremal. Then every* k*-graph* H' *on* $n' > n_0$ *vertices with* $\deg(S) \geq (\rho - \delta)n'$ *for all but at most* $\epsilon(n')^{k-1}$ $(k-1)$*-sets* S *has a* $(\beta + 2\delta m)$*-deficient* K'*-tiling or is* ξ*-extremal.*

Proof. Given a k-graph H' on $n' > n_0$ vertices with $\deg(S) \geq (\rho - \delta)n'$ for all but at most $\epsilon(n')^{k-1}$ $(k-1)$-sets S. By adding a set A of $2\delta n'$ new vertices to H' and adding to $E(H)$ all k-subsets of $V(H') \cup A$ that intersect A, we obtain a new hypergraph H on $n = n' + 2\delta n'$ vertices. All $(k-1)$-subsets of $V(H)$ that intersect A have degree $n - k + 1$. All but at most $\epsilon(n')^{k-1}$ $(k-1)$-subsets S of $V(H')$ satisfy

$$\deg(S) \geq (\rho - \delta)n' + 2\delta n' = \rho n' + \delta n' = \rho n - 2\rho\delta n' + \delta n' \geq \rho n$$

because $\rho \leq 1/2$. By assumption, either H has a β-deficient K'-tiling, or H is ξ-extremal. If H has a β-deficient K'-tiling, then by removing all the K'-copies that intersect A, we obtain a $(\beta + 2\delta m)$-deficient K'-tiling of H'. Otherwise H is ξ-extremal, namely, there exists a set $B \subseteq V(H)$ of $\lfloor (1 - \sigma(K'))n \rfloor$ vertices such that $e(B) \leq \xi \binom{|B|}{k}$. We can assume that $A \cap B = \emptyset$ – otherwise we swap the vertices in

$A \cap B$ with the vertices in $V \setminus (A \cup B)$ and $e(B)$ will not increase. By averaging, there exists a subset $B' \subseteq B \subseteq V(H')$ of order exactly $\lfloor (1-\sigma(K'))n' \rfloor$ such that $e(B') \le \xi\binom{|B'|}{k}$. Thus, H' is ξ-extremal. $\square$

Now we are ready to prove Lemma 5.1.

Proof of Lemma 5.1. Fix positive integers $a < b$ and $k \ge 2$, and a real number $0 < \gamma \ll 1/m$. Let $m := a + (k-1)b$ and $\sigma := \sigma(K') = a/m$. Trivially the lemma works when $\alpha \ge 1$ or $\xi \ge 1$. Assume to the contrary, that there exist $\alpha \in (0,1), \xi \in [5bk^2\gamma, 1)$ such that for all $\epsilon_0 > 0$ and integers n_0, Lemma 5.1 fails, namely, there is a k-graph H on $n > n_0$ vertices which satisfies $\deg(S) \ge (\sigma - \gamma)n$ for all but at most $\epsilon_0 n^{k-1}$ $(k-1)$-sets S but which does not contain an α-deficient K'-tiling and is not ξ-extremal. Let Γ be the set of such pairs (α, ξ). Define $f(\alpha, \xi) = \alpha + \gamma\alpha^2\xi$, and let f_0 be the supremum of $f(\alpha, \xi)$ among all $(a, \xi) \in \Gamma$.

Let $\eta = \gamma^2 f_0^2/32$. By the definition of f_0, there exists a pair $(\alpha_0, \xi_0) \in \Gamma$ such that $f_0 - \eta \le f(\alpha_0, \xi_0) \le f_0$. Moreover, since $1 + \gamma\alpha_0\xi_0 \le 2$, we have that $\alpha_0 \ge (f_0 - \eta)/2 \ge f_0/4$. Let $d = (\gamma\alpha_0)^2/(4m)$. Since $f(\alpha_0, \xi_0) \ge f_0 - \eta$, we have

$$f(\alpha_0 + (\gamma\alpha_0)^2, \xi_0 - 5d) > \alpha_0 + (\gamma\alpha_0)^2 + \gamma\alpha_0^2(\xi_0 - 5d) \ge f_0 - \eta + \gamma\alpha_0^2(\gamma - 5d) \ge f_0$$

by $d \le \gamma^2 \le \gamma/10$ and $\eta = \gamma^2 f_0^2/32 \le \gamma^2\alpha_0^2/2$. This means that $(\alpha_0 + (\gamma\alpha_0)^2, \xi_0 - 5d) \notin \Gamma$, i.e., there exist $\epsilon_* > 0$ and $n_* \in \mathbb{N}$ such that for every k-graph H on $n > n_*$ vertices satisfying $\deg(S) \ge (\sigma-\gamma)n$ for all but at most $\epsilon_* n^{k-1}$ $(k-1)$-sets S, H has an $(\alpha_0 + (\gamma\alpha_0)^2)$-deficient K'-tiling or is $(\xi_0 - 5d)$-extremal. Since $\sigma < 1/k \le 1/2$, we can apply Proposition 5.8 with parameters $\delta = 2d$ and $\sqrt{\epsilon}$ and derive that

(†) for every k-graph H' on $n > n_*$ vertices satisfying $\deg(S) \ge (\sigma-\gamma-2d)n$ for all but at most $\epsilon_* n^{k-1}$ $(k-1)$-sets S, either H' has an $(\alpha_0 + (\gamma\alpha_0)^2 + 4dm)$-deficient K'-tiling, or it is $(\xi_0 - 5d)$-extremal.

Let $\epsilon > 0$ be such that $\epsilon \le \min\{\epsilon_*^2, d^k, (1/k-\sigma)/3, d/(2b^k)\}$ and $\epsilon \ll c, d, 1/k$ as required by Corollary 5.3. Let n_1 and T be the parameters returned by Corollary 5.3 with inputs $c = \sigma - \gamma$, ϵ, d, k and sufficiently large t_0 (in particular, $t_0 > n_*$).

Let H be a k-graph on $n \ge n_1$ vertices which satisfies $\deg(S) \ge (\sigma - \gamma)n$ for all but at most ϵn^{k-1} $(k-1)$-sets S. Our goal is to show that either H contains an α_0-deficient K'-tiling or H is ξ_0-extremal. This implies that $(\alpha_0, \xi_0) \notin \Gamma$, contradicting the definition of (α_0, ξ_0). Let us apply Corollary 5.3 to H with the constants chosen above and obtain a cluster hypergraph $R = R(\epsilon, d, \mathcal{Q})$ on $t \ge t_0$ vertices such that the number of $(k-1)$-subsets $S \subseteq V(R)$ violating

$$\deg_R(S) \ge \left(\sigma - \gamma - d - 5\sqrt{\epsilon}\right) t \ge (\sigma - \gamma - 2d)\, t$$

is at most $\sqrt{\epsilon} t^{k-1} \le \epsilon_* t^{k-1}$. Let N be the number of vertices in each cluster except V_0 and thus $(1-\epsilon)\frac{n}{t} \le N \le \frac{n}{t}$.

Note that the reduced k-graph R satisfies the assumption of (†). If R is $(\xi_0 - 5d)$-extremal, then there exists a vertex set $B \subseteq V(R)$ of order $\lfloor (1-\sigma)t \rfloor$ such that $e(B) \le (\xi_0 - 5d)\binom{|B|}{k}$. Let $B' \subseteq V(H)$ be the union of the clusters in B. By regularity, we have

$$e(B') \le e(B) \cdot N^k + \binom{t}{k} \cdot d \cdot N^k + \epsilon\binom{t}{k} \cdot N^k + t\binom{N}{2}\binom{n-2}{k-2},$$

where the right-hand side bounds the number of edges from regular k-tuples with high density, edges from regular k-tuples with low density, edges from irregular k-tuples and edges that lie in at most $k-1$

clusters. Since $e(B) \le (\xi_0 - 5d)\binom{|B|}{k}$ and $1/t \le \epsilon$, we get

$$e(B') \le (\xi_0 - 5d)\binom{|B|}{k} N^k + (d+\epsilon)\binom{t}{k}\left(\frac{n}{t}\right)^k + t\binom{n/t}{2}\binom{n-2}{k-2}$$
$$\le (\xi_0 - 5d)\binom{|B'|}{k} + (d+2\epsilon)\binom{n}{k}.$$

Note that

$$|B'| = \lfloor (1-\sigma)t \rfloor N \ge (1-\sigma)(1-\epsilon)n - N \ge (1-\sigma-2\epsilon)n.$$

On the other hand, $|B'| \le (1-\sigma)n$ implies that $|B'| \le \lfloor (1-\sigma)n \rfloor$. By adding at most $2\epsilon n$ vertices of $V \setminus B'$ to B', we obtain a subset $B'' \subseteq V(H)$ of size exactly $\lfloor (1-\sigma)n \rfloor$, with $e(B'') \le e(B') + 2\epsilon n\binom{n-1}{k-1} \le e(B') + 2k\epsilon\binom{n}{k}$. Since $\sigma < 1/k$ and $\epsilon \le (1/k - \sigma)/3$, we have

$$\binom{|B'|}{k} \ge \binom{(1-\sigma-2\epsilon)n}{k} \ge (1-\sigma-2\epsilon)^k\binom{n}{k} - O(n^{k-1}) \ge \left(1 - \frac{1}{k}\right)^k \binom{n}{k}.$$

Since $(1-1/k)^k \ge 1/4$ for $k \ge 2$, it follows that $\binom{|B'|}{k} \ge \frac{1}{4}\binom{n}{k}$, and consequently,

$$e(B'') \le (\xi_0 - 5d)\binom{|B'|}{k} + (d + 2\epsilon + 2k\epsilon)\binom{n}{k} \le \Big(\xi_0 - 5d + 4(d + 2\epsilon + 2k\epsilon)\Big)\binom{|B'|}{k} \le \xi_0\binom{|B'|}{k}$$

by $\epsilon \le d^k$. Hence H is ξ_0-extremal, and we are done.

We thus assume that R is not $(\xi_0 - 5d)$-extremal. By (†), R has an $(\alpha_0 + (\gamma\alpha_0)^2 + 4dm)$-deficient K'-tiling. Let $\mathcal{K}$ be a largest K'-tiling of R, and let U be the set of vertices in R not covered by $\mathcal{K}$. Then $|U| \le (\alpha_0 + (\gamma\alpha_0)^2 + 4dm)t$. Let $q := |\mathcal{K}|$.

Now assume that H contains no α_0-deficient K'-tiling. The following proposition shows that there is no fractional hom(K')-tiling of R whose weight is substantially larger than $(1-\alpha_0)t$.

Claim 5.9. *If h is a fractional hom(K')-tiling of R with $h_{\min} \ge b^{1-k}$, then $w(h) \le (1 - \alpha_0 + 2b^k\epsilon)t \le qm + 3(\gamma\alpha_0)^2 t$.*

Proof. We first observe that H contains an α_0-deficient K'-tiling if there is a fractional hom(K')-tiling h of R with $h_{\min} \ge b^{1-k}$ and $w(h) \ge (1 - \alpha_0 + 2b^k\epsilon)t$. This indeed follows from Proposition 5.7 because $2b^k\epsilon \le d = (\gamma\alpha_0)^2/(4m) < \alpha_0$ and

$$\left(1 - \frac{2b\epsilon}{b^{1-k}}\right) w(h)\frac{n}{t} \ge \left(1 - 2b^k\epsilon\right)(1 - \alpha_0 + 2b^k\epsilon)t\frac{n}{t} > (1-\alpha_0)\,n.$$

It remains to show that $qm + 3(\gamma\alpha_0)^2 t \ge (1 - \alpha_0 + 2b^k\epsilon)t$. Since $|U| \le (\alpha_0 + (\gamma\alpha_0)^2 + 4dm)t$ and $|U| + qm = t$, we have

$$qm + 3(\gamma\alpha_0)^2 t \ge t - (\alpha_0 + (\gamma\alpha_0)^2 + 4dm)t + 3(\gamma\alpha_0)^2 t \ge (1 - \alpha_0 + 2b^k\epsilon)t$$

because $4dm = (\gamma\alpha_0)^2$ and $2b^k\epsilon < (\gamma\alpha_0)^2$. □

We claim the following for $|U|$:

$$|U| \ge \frac{\alpha_0}{2}t \quad \text{and} \quad e(U) \le \frac{\gamma}{k}\binom{|U|}{k}. \tag{5.1}$$

Indeed, if $|U| < \alpha_0 t/2$, then by applying Fact 5.5 to each member of $\mathcal{K}$, we obtain a fractional hom(K')-tiling h of R with $h_{\min} = b^{1-k}$ and weight $w(h) = (1 - \alpha_0/2)t > (1 - \alpha_0 + 2b^k\epsilon)t$. This contradicts Claim 5.9. If $e(U) \ge \gamma\binom{|U|}{k}/k$, then since $|U| \ge \alpha_0 t/2$ and t is sufficiently large, we can apply (1.4) to find a copy of K' in U, contradicting the maximality of $\mathcal{K}$.

Let D be the set of vertices $v \in V(R) \setminus U$ such that $|N(v) \cap \binom{U}{k-1}| \ge \gamma\binom{|U|}{k-1}$. Let $\mathcal{K}_1 \subseteq \mathcal{K}$ be the set of copies of K' that contain at least $a+1$ vertices from D. Let $\mathcal{K}_2 \subseteq \mathcal{K}$ be the set of copies of K' that contain exactly a vertices from D. Let $\mathcal{K}_3 \subseteq \mathcal{K}$ be the set of copies of K' that contain at most $a-1$ vertices from D.

Claim 5.10. $|\mathcal{K}_1| < \frac{\gamma}{m(k-1)^2}|U| < \frac{\gamma}{m(k-1)^2}t$.

Proof. Suppose $|\mathcal{K}_1| \geq \frac{\gamma}{m(k-1)^2}|U|$ instead. Let $K_1, \dots K_\ell$ be distinct members of $\mathcal{K}_1$, where $\ell = \lceil \frac{\gamma|U|}{m(k-1)^2} \rceil$. For each $j \leq \ell$, since $|V(K_j) \cap D| \geq a+1$, we can find a vertex $v_j \in D$ from a large vertex class of K_j. Since

$$\gamma \binom{|U|}{k-1} > \frac{\gamma|U|}{m(k-1)} \binom{|U|-1}{k-2} > (\ell-1)(k-1)\binom{|U|-1}{k-2},$$

we can greedily pick disjoint $(k-1)$-sets $S_1, \dots, S_\ell$ from U such that $S_j \in N(v_j)$. Thus, for each $j \leq \ell$, by Proposition 5.6, we get a fractional hom(K')-tiling h^j of $R[V(K_j) \cup S_j]$ such that $w(h^j) \geq m + 1/(ab^{k-1})$ and $h^j_{\min} \geq b^{1-k}$. We assign the standard weight to other members of $\mathcal{K}$ and thus obtain a fractional hom(K')-tiling h of R with $h_{\min} \geq b^{1-k}$ and weight

$$w(h) \geq qm + \frac{\gamma|U|}{m(k-1)^2}\frac{1}{ab^{k-1}} \geq qm + \frac{\gamma\alpha_0 t}{2m(k-1)^2 ab^{k-1}} > qm + 3(\gamma\alpha_0)^2 t,$$

where we used (5.1) and that γ is small. This contradicts Claim 5.9. □

We now find an upper bound for $|\mathcal{K}_3|$. First, by the definitions of $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ and $e(U) \leq \gamma\binom{|U|}{k}/k$, we have

$$\begin{aligned}\sum_{S\in\binom{U}{k-1}} \deg(S) &\leq (m|\mathcal{K}_1| + a|\mathcal{K}_2| + (a-1)|\mathcal{K}_3|)\binom{|U|}{k-1} + qm\gamma\binom{|U|}{k-1} + \gamma\binom{|U|}{k} \\ &\leq aq\binom{|U|}{k-1} + ((m-a)|\mathcal{K}_1| - |\mathcal{K}_3|)\binom{|U|}{k-1} + \gamma\binom{|U|}{k-1}t,\end{aligned}$$

where the second inequality follows from $|\mathcal{K}_1| + |\mathcal{K}_2| + |\mathcal{K}_3| = q$ and $qm + |U| = t$. Second, the degree condition of R implies that

$$\sum_{S\in\binom{U}{k-1}} \deg(S) \geq \left(\binom{|U|}{k-1} - \sqrt{\epsilon}t^{k-1}\right)(\sigma - \gamma - 2d)\,t.$$

Since $|U| \geq \alpha_0 t/2$ and $\epsilon \leq d^k \leq (\gamma\alpha_0)^{2k}$, we have

$$\sqrt{\epsilon}t^{k-1} \leq (\gamma\alpha_0)^k \left(\frac{2|U|}{\alpha_0}\right)^{k-1} \leq \gamma\binom{|U|}{k-1}$$

as $\gamma \leq 1/(2k)$ and $|U|$ is sufficiently large. It follows that

$$\sum_{S\in\binom{U}{k-1}} \deg(S) \geq \left(\binom{|U|}{k-1} - \gamma\binom{|U|}{k-1}\right)\left(\frac{a}{m} - (\gamma+2d)\right)t \geq \binom{|U|}{k-1}aq - 2\gamma\binom{|U|}{k-1}t,$$

where the last inequality holds because $2d \leq \gamma(m-a)/m$ and $qm \leq t$. Comparing the upper and lower bounds for $\sum_{S\in\binom{U}{k-1}} \deg(S)$ gives

$$|\mathcal{K}_3| \leq (m-a)|\mathcal{K}_1| + 3\gamma t.$$

By Claim 5.10, it follows that

$$|\mathcal{K}_1| + |\mathcal{K}_3| \leq |\mathcal{K}_1| + (m-a)|\mathcal{K}_1| + 3\gamma t \leq 4\gamma t. \tag{5.2}$$

Thus we have $|\mathcal{K}_2| \geq q - 4\gamma t$.

Let A be the union of U and the vertices covered by $\mathcal{K}_2$ but not in D. Then

$$|A| \geq |U| + (m-a)(q-4\gamma t) \geq (1-\sigma)t - 4(m-a)\gamma t \geq \left(1 - \frac{1}{k}\right)t \tag{5.3}$$

because γ is small. We claim that we can find a set $\mathcal{M}$ of ℓ' disjoint edges in A, where $\ell' = \lceil \frac{\gamma|U|}{km^2} \rceil$. Indeed, by deleting some vertices or adding at most $4(m-a)\gamma t$ vertices from $V(R) \setminus A$ to A, we can obtain a set A' of size exactly $\lfloor (1-\sigma)t \rfloor$. Since R is not $(\xi_0 - 5d)$-extremal, we have that $e(A') \geq (\xi_0 - 5d)\binom{|A'|}{k}$. If $|A| > \lfloor (1-\sigma)t \rfloor$, then as $e(A') \geq (\xi_0 - 5d)\binom{|A'|}{k} > (\ell'-1)k\binom{|A'|-1}{k-1}$, we can find the desierd $\mathcal{M}$ in $A' \subseteq A$. Otherwise, for $|A| \leq \lfloor (1-\sigma)t \rfloor$, by (5.3), it follows that

$$
\begin{aligned}
e(A) &\geq e(A') - 4(m-a)\gamma t \cdot \binom{|A|-1}{k-1} \geq (\xi_0 - 5d)\binom{|A'|}{k} - 4(m-a)\gamma \frac{|A|}{1-1/k}\binom{|A|-1}{k-1} \\
&\geq (\xi_0 - 5d - 4bk^2\gamma)\binom{|A|}{k} > \gamma \binom{|A|}{k} > (\ell'-1)k\binom{|A|-1}{k-1}
\end{aligned}
$$

because $\xi_0 \geq 5bk^2\gamma$ and $5d < \gamma$. Thus we can greedily find the desired $\mathcal{M}$ in A.

Let $K_1, \dots, K_p$ denote the members of $\mathcal{K}_2$ that intersect the edges of $\mathcal{M}$, where $p \leq k\ell'$. For each $j \in [p]$, suppose $V(K_j) \cap D = \{v_{j,1}, \dots v_{j,a}\}$. For each $j \in [p]$ and $i \in [a]$, we claim that we can greedily find disjoint copies $K'_{j,i}$ of complete $(k-1)$-partite $(k-1)$-graphs $K^{(k-1)}(b, \dots, b)$ in $N(v_{j,i}) \cap \binom{U}{k-1}$. Indeed, during the process, at most $pa(k-1)b + k\ell'$ vertices of U are occupied and the number of $(k-1)$-subsets of U intersecting these vertices is at most

$$
(pa(k-1)b + k\ell')\binom{|U|-1}{k-2} \leq amk\ell'\binom{|U|-1}{k-2} \leq \frac{k-1}{k}\gamma\binom{|U|}{k-1}
$$

because $\ell' \leq \gamma|U|/(km^2) + 1$, $ak < m$, and $|U|$ is sufficiently large. Thus, since

$$
\left| N(v_{j,i}) \cap \binom{U}{k-1} \right| - \frac{k-1}{k}\gamma\binom{|U|}{k-1} \geq \gamma\binom{|U|}{k-1} - \frac{k-1}{k}\gamma\binom{|U|}{k-1} = \frac{\gamma}{k}\binom{|U|}{k-1},
$$

we can apply (1.4) to find the desired $K'_{j,i}$ for all $v_{j,i}$.

For each $j \in [p]$ and $i \in [a]$, $R[V(K'_{j,i}) \cup \{v_{j,i}\}]$ spans a copy of $K^{(k)}(1, b, \dots, b)$. We now assign the weight $(1/b^{k-1}, 1/(ab^{k-2}), \dots, 1/(ab^{k-2}))$ to each edge of this $K^{(k)}(1, b, \dots, b)$ such that the weight of $v_{j,i}$ is $1/b^{k-1}$. The total weight of $R[V(K'_{j,i}) \cup \{v_{j,i}\}]$ is thus $1 + b(k-1)/a = m/a$. Next we assign the standard weight to each member of $\mathcal{K} \setminus \{K_1, \dots, K_p\}$. Finally, we assign weight $(1, \dots, 1)$ to all the edges of $\mathcal{M}$. This gives a fractional hom(K')-tiling h with $h_{\min} = b^{1-k}$ and weight

$$
w(h) = pa\frac{m}{a} + (q-p)m + k\ell' \geq qm + \frac{\gamma|U|}{m^2} \geq qm + \frac{\gamma\alpha_0 t}{2m^2} > qm + 3(\gamma\alpha_0)^2 t
$$

where we used (5.1) and that the assumption γ is small. This contradicts Claim 5.9 and completes our proof. $\square$

6. The Extremal Case

In this section we prove Theorems 3.3, 3.4 and 3.5. We first give some notation. Given two disjoint vertex sets X and Y and two integers $i, j \geq 0$, a set $S \subset X \cup Y$ is called an X^iY^j-set if $|S \cap X| = i$ and $|S \cap Y| = j$. When X, Y are two disjoint subsets of $V(H)$ and $i + j = k$, we denote by $H(X^iY^j)$ the family of all edges of H that are X^iY^j-sets, and let $e_H(X^iY^j) = |H(X^iY^j)|$ (the subscript may be omitted if it is clear from the context). We use $\overline{e}_H(X^iY^{k-i})$ to denote the number of non-edges among X^iY^{k-i}-sets. Given a set $L \subseteq X \cup Y$ with $|L \cap X| = l_1 \leq i$ and $|L \cap Y| = l_2 \leq k-i$, we define $\deg(L, X^iY^{k-i})$ as the number of edges in $H(X^iY^{k-i})$ that contain L, and $\overline{\deg}(L, X^iY^{k-i}) = \binom{|X|-l_1}{i-l_1}\binom{|Y|-l_2}{k-i-l_2} - \deg(L, X^iY^{k-i})$. Our earlier notation $\deg(S, R)$ may be viewed as $\deg(S, S^{|S|}(R \setminus S)^{k-|S|})$.

6.1. **Tools and a general setup.** The following lemma deals with a special (ideal) case of Theorem 3.3. We postpone its proof to the end of this section.

Lemma 6.1. *Let $K := K^{(k)}(a_1, \dots, a_k)$ with $m := a_1 + \cdots + a_k$. Suppose $1/n \ll \rho \ll 1/m$ and $n \in m\mathbb{Z}$. Suppose H is a k-graph on n vertices with a partition of $V(H) = X \cup Y$ such that $a_1|Y| = (m - a_1)|X|$. Furthermore, assume that*

- *for every vertex $v \in X$, $\overline{\deg}(v, Y) \le \rho\binom{|Y|}{k-1}$,*
- *for every vertex $u \in Y$, $\overline{\deg}(u, XY^{k-1}) \le \rho\binom{|Y|}{k-1}$.*

Then H contains a K-factor.

We use the following simple fact in the proof.

Fact 6.2. *Let $t, m, n \in \mathbb{N}$ and F be an m-vertex k-partite k-graph. Let H be an n-vertex k-graph with maximum vertex degree Δ and $e(H) > (t-1)m\Delta + \mathrm{ex}(n', F)$, where $n' = n - (t-1)m$. Then H contains an F-tiling of size t.*

Proof. Assume to the contrary, that the largest F-tiling M in H has size at most $t-1$. Let V' be a set of $(t-1)m$ vertices containing $V(M)$. Thus, $V(H) \setminus V'$ spans no copy of F and thus spans at most $\mathrm{ex}(n', F)$ edges. So we have $e(H) \le (t-1)m\Delta + \mathrm{ex}(n', F)$, a contradiction. $\square$

Now we start with a general setup and prove some estimates. Assume that $k \ge 3$, $a_1 \le \cdots \le a_k$ and $m = a_1 + \cdots + a_k$. Suppose $1/n \ll \xi \ll 1/m$ such that $n \in m\mathbb{N}$. Let H be a k-graph on V of n vertices such that $\delta_{k-1}(H) \ge \frac{a_1}{m}n$. Furthermore, assume that there is a set $B \subseteq V(H)$, such that $|B| = \frac{m-a_1}{m}n$ and $e(B) \le \xi\binom{|B|}{k}$. Let $A = V \setminus B$. Then $|A| = \frac{a_1}{m}n$. Note that we only require $\delta_{k-1}(H) \ge \frac{a_1}{m}n$, so that we can use the estimates in all proofs.

Let $\epsilon_1 = \xi^{1/7}$, $\epsilon_2 = \xi^{1/3}$ and $\epsilon_3 = 2\xi^{2/3}$. We now define

$$A' := \left\{v \in V : \overline{\deg}(v, B) \le \epsilon_2\binom{|B|}{k-1}\right\},$$

$$B' := \left\{v \in V : \deg(v, B) \le \epsilon_1\binom{|B|}{k-1}\right\}, \; V_0 := V \setminus (A' \cup B').$$

Claim 6.3. $\{|A \setminus A'|, |B \setminus B'|, |A' \setminus A|, |B' \setminus B|\} \le \epsilon_3|B|$ *and* $|V_0| \le 2\epsilon_3|B|$.

Proof. First assume that $|B \setminus B'| > \epsilon_3|B|$. By the definition of B', we get that

$$e(B) > \frac{1}{k}\epsilon_1\binom{|B|}{k-1} \cdot \epsilon_3|B| > 2\xi\binom{|B|}{k},$$

which contradicts $e(B) \le \xi\binom{|B|}{k}$.

Second, assume that $|A \setminus A'| > \epsilon_3|B|$. Then by the definition of A', for any vertex $v \notin A'$, we have that $\overline{\deg}(v, B) > \epsilon_2\binom{|B|}{k-1}$. So we get

$$\overline{e}(AB^{k-1}) > \epsilon_3|B| \cdot \epsilon_2\binom{|B|}{k-1} = 2\xi|B|\binom{|B|}{k-1}.$$

Together with $e(B) \le \xi\binom{|B|}{k}$, this implies that

$$\begin{aligned}\sum_{S \in \binom{B}{k-1}} \overline{\deg}(S) &= k\overline{e}(B) + \overline{e}(AB^{k-1}) \\ &> k(1-\xi)\binom{|B|}{k} + 2\xi|B|\binom{|B|}{k-1} \\ &= ((1-\xi)(|B| - k + 1) + 2\xi|B|)\binom{|B|}{k-1} > |B|\binom{|B|}{k-1}.\end{aligned}$$

where the last inequality holds because n is large enough. By the pigeonhole principle, there exists a set $S \in \binom{B}{k-1}$, such that $\overline{\deg}(S) > |B| = \frac{m-a_1}{m}n$, contradicting $\delta_{k-1}(H) \geq \frac{a_1}{m}n$.

Consequently,

$$\begin{aligned} |A' \setminus A| &= |A' \cap B| \leq |B \setminus B'| \leq \epsilon_3 |B|, \\ |B' \setminus B| &= |A \cap B'| \leq |A \setminus A'| \leq \epsilon_3 |B|, \\ |V_0| &\leq |A \setminus A'| + |B \setminus B'| \leq \epsilon_3|B| + \epsilon_3|B| = 2\epsilon_3|B|. \end{aligned} \qquad \square$$

Note that by $|B \setminus B'| \leq \epsilon_3|B|$, we have

$$\deg(w, B') \geq \deg(w, B) - |B \setminus B'|\binom{|B|}{k-2} \geq \frac{\epsilon_1}{2}\binom{|B'|}{k-1} \text{ for any vertex } w \in V_0, \tag{6.1}$$

and by $|B' \setminus B| \leq \epsilon_3|B|$, we have

$$\overline{\deg}(v, B') \leq \overline{\deg}(v, B) + |B' \setminus B|\binom{|B'|}{k-2} \leq 2\epsilon_2\binom{|B'|}{k-1} \text{ for any vertex } v \in A', \tag{6.2}$$

and

$$\deg(v, B') \leq \deg(v, B) + |B' \setminus B|\binom{|B'|}{k-2} \leq 2\epsilon_1\binom{|B'|}{k-1} \text{ for any vertex } v \in B'. \tag{6.3}$$

Moreover, for any $(k-1)$-set $S \subseteq B'$, by $\deg(S, A') + \deg(S, B') + \deg(S, V_0) \geq \delta_{k-1}(H)$ and $\overline{\deg}(S, A') = |A'| - \deg(S, A')$, we have

$$\overline{\deg}(S, A') \leq |A'| - \delta_{k-1}(H) + \deg(S, B') + \deg(S, V_0) \leq \deg(S, B') + 3\epsilon_3|B|,$$

where we used $\deg(S, V_0) \leq |V_0| \leq 2\epsilon_3|B|$, $|A'| \leq \frac{a_1}{m}n + \epsilon_3|B|$ and $\delta_{k-1}(H) \geq \frac{a_1}{m}n$. Furthermore, for any $v \in B'$, by (6.3), we have

$$\sum_{S: v \in S \in \binom{B'}{k-1}} \deg(S, B') = (k-1)\deg(v, B') \leq 2(k-1)\epsilon_1\binom{|B'|}{k-1}.$$

Putting these together gives that for any $v \in B'$,

$$\overline{\deg}(v, A'B'^{k-1}) = \sum_S \overline{\deg}(S, A') \leq \sum_S \deg(S, B') + 3\epsilon_3|B|\binom{|B'|-1}{k-2} \leq 2k\epsilon_1\binom{|B'|}{k-1}, \tag{6.4}$$

where the sums are on S such that $v \in S \in \binom{B'}{k-1}$. Let $\mathcal{B}$ be the set of $(k-1)$-sets $S \subseteq B'$ such that $\overline{\deg}_H(S, A') > \sqrt{2\epsilon_2}|A'|$. By (6.2), we have that $\overline{e}_H(A'B'^{k-1}) \leq 2\epsilon_2|A'|\binom{|B'|}{k-1}$ and thus

$$|\mathcal{B}| \leq \sqrt{2\epsilon_2}\binom{|B'|}{k-1}. \tag{6.5}$$

In all three proofs we will define $\epsilon_1, \epsilon_2, \epsilon_3$ and A', B', V_0 in the same way and thus Claim 6.3 and (6.1) – (6.5) hold.

6.2. **Proof of Theorem 3.3.** Assume that $k \geq 3$, $a_1 \leq \cdots \leq a_k$ and $m = a_1 + \cdots + a_k$. Let $K := K^{(k)}(a_1, \ldots, a_k)$ such that $\gcd(K) = 1$. Suppose $1/n \ll \xi \ll 1/m$ such that $n \in m\mathbb{N}$. Assume H is a ξ-extremal k-graph on n vertices that satisfies (1.3). Let B be a set of $\frac{m-a_1}{m}n$ vertices such that $e(B) \leq \xi\binom{|B|}{k}$. Let $A = V \setminus B$. Define $\epsilon_1, \epsilon_2, \epsilon_3, A', B', V_0$ as in Section 1.1 and thus Claim 6.3 and (6.1) – (6.5) hold.

In the following proof we will build four vertex-disjoint K-tilings $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$, whose union is a K-factor of H. The ideal case is when $(m-a_1)|A'| = a_1|B'|$ and $V_0 = \emptyset$ – in this case we apply Lemma 6.1 to obtain a K-factor of H such that each copy of K has a_1 vertices in A' and $m - a_1$ vertices in B'. So the purpose of the K-tilings $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ is to cover the vertices of V_0 and adjust the sizes of A' and B'

so that we can apply Lemma 6.1 (and obtain $\mathcal{K}_4$) after $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ are removed. More precisely, we cover the vertices of V_0 by $\mathcal{K}_2$ and let

$$q := |B'| - |B| = \frac{a_1}{m} n - |A'| - |V_0|$$

denote the *discrepancy* between the current and ideal sizes of B'. If $q > 0$, then we apply the minimum codegree condition to find copies of K from B'. Since removing a copy of K from B' reduces the discrepancy by a_1, we can not reduce the discrepancy to zero unless a_1 divides q. Therefore we remove enough copies of K from B' (denoted by $\mathcal{K}_1$) such that the discrepancy is less than or equal to $-C$. This allows us to apply the definition of Frobenius numbers and remove more copies of K (denoted by $\mathcal{K}_3$) to "increase" the discrepancy to zero.

The K-tilings $\mathcal{K}_1, \mathcal{K}_2$. Our goal is to find K-tilings $\mathcal{K}_1, \mathcal{K}_2$ such that $V_0 \subseteq V(\mathcal{K}_2)$,

$$|\mathcal{K}_1| + |\mathcal{K}_2| \le 4\epsilon_3 |B| \text{ and} \tag{6.6}$$

$$-2a_1\epsilon_3|B| \le q_1 \le -C, \tag{6.7}$$

where $q_1 := \frac{a_1}{m}|V \setminus V(\mathcal{K}_1 \cup \mathcal{K}_2)| - |A' \setminus V(\mathcal{K}_1 \cup K_2)|$.

When $q \le -C$, let $\mathcal{K}_1 = \emptyset$. When $q \ge 1 - C$, $\mathcal{K}_1$ consists of $q + C$ copies of K obtained from $H[B']$ as follows.[4] Note that $\delta_{k-1}(H[B']) \ge \delta_{k-1}(H) - |A'| - |V_0| \ge q + C + f(n)$ by (1.3) and the definition of q. We claim that

$$\frac{f(n)}{k}\binom{|B'|}{k-1} \ge \mathrm{ex}\Big(|B'| - (q + C - 1)m, K\Big). \tag{6.8}$$

Indeed, when $q \ge 1$, we have $|B'| \ge |B| + 1$ and $\binom{|B'|}{k-1} \ge \binom{|B|+1}{k-1}$. Since $f(n) \ge k\,\mathrm{ex}(|B| + 1, K)/\binom{B|+1}{k-1}$, it follows that $\frac{f(n)}{k}\binom{|B'|}{k-1} \ge \mathrm{ex}(|B| + 1, K)$. Since $|B| + 1 = |B'| - q + 1 \ge |B'| - (q + C - 1)m$, (6.8) follows from the monotonicity of the Turán number. When $q \le 0$, we have $|B| - C < |B'| \le |B|$. The definition of $f(n)$ implies that $\frac{f(n)}{k}\binom{|B'|}{k-1} \ge \mathrm{ex}(|B'|, K)$. Again, (6.8) follows from the monotonicity of the Turán number.

By (6.8), we have

$$\begin{aligned} e_H(B') &\ge \frac{\delta_{k-1}(H[B'])}{k}\binom{|B'|}{k-1} \ge \frac{q+C}{k}\binom{|B'|}{k-1} + \mathrm{ex}(|B'| - (q + C - 1)m, K) \\ &> (q + C - 1)m \cdot 2\epsilon_1\binom{|B'|}{k-1} + \mathrm{ex}(|B'| - (q + C - 1)m, K). \end{aligned}$$

By (6.3), we can apply Fact 6.2 to obtain $q + C$ vertex-disjoint copies of K in $H[B']$, denoted by $\mathcal{K}_1$.

Next we choose a K-tiling $\mathcal{K}_2$ such that each copy of K contains $a_1 - 1$ vertices in A', one vertex in V_0 and $m - a_1$ vertices in B'. By (6.1) and (6.5) we derive that for any vertex $w \in V_0$,

$$|N(w, B') \setminus \mathcal{B}| \ge \frac{\epsilon_1}{2}\binom{|B'|}{k-1} - \sqrt{2\epsilon_2}\binom{|B'|}{k-1} \ge \frac{\epsilon_1}{3}\binom{|B'|}{k-1}$$

by the choice of ϵ_1 and ϵ_2. Let $V_0 = \{w_1, \dots, w_{|V_0|}\}$. For each w_i, by (1.4) we can find a copy T_i of $K^{(k-1)}(a_2, \dots, a_k)$ in (the $(k-1)$-graph) $N(w, B') \setminus \mathcal{B}$ such that these copies are vertex disjoint, and are also vertex disjoint from $V(\mathcal{K}_1)$. This is possible because the number of vertices in B' that we need to avoid is at most $|V(\mathcal{K}_1)| + (m - a_1)|V_0| \le (\epsilon_3|B| + C) \cdot m + (m - a_1)\epsilon_3|B| \le 2m\epsilon_3|B|$, and so we have

$$|N(w, B') \setminus \mathcal{B}| - 2m\epsilon_3|B|\binom{|B'|}{k-2} \ge \frac{\epsilon_1}{3}\binom{|B'|}{k-1} - 2m\epsilon_3|B|\binom{|B'|}{k-2} \ge \frac{\epsilon_1}{4}\binom{|B'|}{k-1}.$$

Thus, (1.4) implies the existence of the desired T_i. Note that each $\{w_i\} \cup T_i$ spans a copy of $K^{(k)}(1, a_2, \dots, a_k)$ in H. To obtain copies of K, we extend each of them (one by one) by adding $a_1 - 1$ vertices from A'. Note

[4]It suffices to find $\lceil (q + C)/a_1 \rceil$ copies of K but since Fact 6.2 provides (at least) $q + C$ copies, we choose to use all of them to simplify later calculations.

that each such vertex from A' needs to be the common neighbor of $a' := \prod_{2\le i\le k} a_i$ $(k-1)$-sets, which is possible since by our choice, these $(k-1)$-sets are not in $\mathcal{B}$, and thus they have at least $(1-a'\sqrt{2\epsilon_2})|A'|$ common neighbors in A'. Since $(1-a'\sqrt{2\epsilon_2})|A'| > (a_1-1)|V_0|$, we can greedily extend each $\{w_i\}\cup T_i$ into a copy of K. Denote the resulting K-tiling by $\mathcal{K}_2$.

By definitions, we have $|\mathcal{K}_1| \le |q| + C$ and $|\mathcal{K}_2| = |V_0|$. The result in [7] (see also [33]) mentioned in Section 1 implies that $C \le (a_k - a_1)^2$. By Claim 6.3, $|\mathcal{K}_1|+|\mathcal{K}_2| \le \epsilon_3|B| + C + 2\epsilon_3|B| \le 4\epsilon_3|B|$, i.e., (6.6) holds. Let A_1 and B_1 be the sets of vertices in A' and B' not covered by $\mathcal{K}_1\cup\mathcal{K}_2$, respectively, and $V_1 := A_1\cup B_1$. So $q_1 = \frac{a_1}{m}|V_1| - |A_1|$. Note that $|A_1| = |A'| - (a_1-1)|V_0|$ and $|V_1| = n - m|\mathcal{K}_1| - m|V_0|$. So we have

$$q_1 = \frac{a_1}{m}n - a_1|\mathcal{K}_1| - |V_0| - |A'| = q - a_1|\mathcal{K}_1| \le q - |\mathcal{K}_1|.$$

Recall that $|\mathcal{K}_1| = q+C$ if $q \ge 1-C$ and $|\mathcal{K}_1| = 0$ if $q \le -C$. So in both cases we get $q_1 \le q - |\mathcal{K}_1| \le -C$. Moreover, by $-\epsilon_3|B| \le q \le \epsilon_3|B|$ and that n is sufficiently large, we have $q_1 = q - a_1|\mathcal{K}_1| \ge q - a_1|q+C| \ge -2a_1\epsilon_3|B|$. So (6.7) holds.

The K-tiling $\mathcal{K}_3$. Next we build our K-tiling $\mathcal{K}_3$. Since $-q_1 \ge C > g(a_2-a_1, a_3-a_1, \dots, a_k-a_1)$ and $\gcd(a_2-a_1, a_3-a_1,\dots,a_k-a_1) = 1$, there exists nonnegative integers $\ell_1,\dots,\ell_{k-1}$ such that

$$\ell_1(a_2-a_1) + \ell_2(a_3-a_1) + \cdots + \ell_{k-1}(a_k-a_1) = -q_1$$

(here we let $\ell_i = 0$ if $a_i - a_1 = 0$). For each $i\in[k-1]$, we pick ℓ_i vertex disjoint copies of K each with a_{i+1} vertices in A_1 and $m - a_{i+1}$ vertices in B_1 such that all copies of K are vertex disjoint. Denote by $\mathcal{K}_3$ as the set of copies of K. Note that we can choose these desired copies of K by Proposition 4.3 and the fact that $H(A_1B_1^{k-1})$ is dense (see (6.9)). By the definition of $\mathcal{K}_3$, we have $|\mathcal{K}_3| = \sum_{i\in[k-1]}\ell_i$ and

$$\begin{aligned}\frac{a_1}{m}|V_1\setminus V(\mathcal{K}_3)| - |A_1\setminus V(\mathcal{K}_3)| &= \frac{a_1}{m}(|V_1| - m|\mathcal{K}_3|) - (|A_1| - |V(\mathcal{K}_3)\cap A_1|)\\ &= q_1 + (|V(\mathcal{K}_3)\cap A_1| - a_1|\mathcal{K}_3|)\\ &= q_1 + \sum_{i\in[k-1]}\ell_i(a_{i+1}-a_1) = 0.\end{aligned}$$

Let $A_2 = A_1\setminus V(\mathcal{K}_3)$ and $B_2 = B_1\setminus V(\mathcal{K}_3)$. So we have $\frac{a_1}{m}(|A_2|+|B_2|) = |A_2|$, i.e., $(m-a_1)|A_2| = a_1|B_2|$.

Note that $|\mathcal{K}_3| = \sum_{i\in[k-1]}\ell_i \le -q_1 \le 2a_1\epsilon_3|B|$ by (6.7). Together with (6.6) and Claim 6.3, we have

$$|B_2| \ge |B'| - |V(\mathcal{K}_1\cup\mathcal{K}_2)| - m|\mathcal{K}_3| \ge |B'| - 4m\epsilon_3|B| - 2a_1m\epsilon_3|B| > (1-\epsilon_1)|B'|.$$

Hence, for every vertex $v\in A_2$, by (6.2),

$$\overline{\deg}(v, B_2) \le \overline{\deg}(v, B') \le 2\epsilon_2\binom{|B'|}{k-1} \le 2\epsilon_2\binom{\frac{1}{1-\epsilon_1}|B_2|}{k-1} < \epsilon_1\binom{|B_2|}{k-1}.$$

By (6.4) and $|B_2| \ge (1-\epsilon_1)|B'|$, for every $v\in B_2$ we have

$$\overline{\deg}(v, A_2B_2^{k-1}) \le \overline{\deg}(v, A'B'^{k-1}) \le 2k\epsilon_1\binom{|B'|}{k-1} \le 3k\epsilon_1\binom{|B_2|}{k-1}. \tag{6.9}$$

The K-tiling $\mathcal{K}_4$. At last, we apply Lemma 6.1 with $X = A_2$, $Y = B_2$ and $\rho = 3k\epsilon_1$ and get a K-factor $\mathcal{K}_4$ on $A_2\cup B_2$.

So $\mathcal{K}_1\cup\mathcal{K}_2\cup\mathcal{K}_3\cup\mathcal{K}_4$ is a K-factor of H. This concludes the proof of Theorem 3.3.

6.3. **Proof of Theorem 3.4.** As mentioned in Section 1, (1.3) reduces to $\delta_{k-1}(H) \ge \frac{n}{k+1} + 1$ when $K = K^{(k)}(1,\dots,1,2)$. Thus Theorem 3.4 follows from Theorem 3.3 if Condition (i) holds. Now assume Condition (ii), that is, $\delta_{k-1}(H) \ge \frac{n}{k+1}$ and $k - i \nmid \binom{n'-i}{k-1-i}$ for some $0 \le i \le k-2$ and $n' = \frac{kn}{k+1} + 1$. The proof follows the proof of Theorem 3.3 (with $C = 0$) closely and the only difference is the existence of $\mathcal{K}_1$ when $q = |B'| - |B| \ge 1$. Note that $\delta_{k-1}(H[B']) \ge q$. Since $\mathrm{ex}(n, K^{(k)}(1,\dots,1,2)) \le \binom{n}{k-1}/k$, when $q \ge 2$ we can find q copies of $K^{(k)}(1,\dots,1,2)$ in B' by Fact 6.2. Otherwise $q = 1$, i.e., $|B'| = n'$. Assume to the contrary that $H[B']$ is $K^{(k)}(1,\dots,1,2)$-free, i.e., every $k-1$ vertices in B' has degree at most 1 in B'. Then by $\delta_{k-1}(H[B']) \ge 1$ we derive that every $k-1$ vertices in B' has degree exactly 1 in B'. This means that a Steiner system $S(k-1,k,n')$ exists, contradicting the divisibility conditions.

6.4. **Proof of Theorem 3.5.** Recall that a loose cycle C_s^k has a vertex set $[s(k-1)]$ and s edges $\{\{j(k-1)+1,\dots,j(k-1)+k\} \text{ for } 0 \le j < s\}$, where we treat $s(k-1)+1$ as 1. When $s = 2,3$, C_s^k has a unique k-partite realization: $C_2^k \subset K^{(k)}(1,1,2,\dots,2)$ and $C_3^k \subset K^{(k)}(2,2,2,3,\dots,3)$. When $s \ge 4$, we 3-color the vertices $j(k-1)+1$, $0 \le j < s$ (these are the vertices of degree two) with $\lfloor s/2 \rfloor$ red vertices, $\lfloor s/2 \rfloor - 1$ blue vertices and the remaining one or two vertices green. We complete the k-coloring of C_s^k by coloring the $k-2$ uncolored vertices of each edge of C_s^k with the $k-2$ colors not used to color the two vertices of degree two. In this coloring, there are $\lceil s/2 \rceil$ red vertices, $\lceil s/2 \rceil + 1$ blue vertices, $s-1$ or $s-2$ green vertices, and s vertices in other color classes. Furthermore, since each vertex in C_s^k has degree at most 2, in any k-coloring of C_s^k, each color class has size at least $\lceil s/2 \rceil$. Thus, $\sigma(C_s^k) = \frac{\lceil s/2 \rceil}{s(k-1)}$.

We summarize above arguments into a proposition.

Proposition 6.4. *For any $k \ge 4$ and $s \ge 2$ we have $\sigma(C_s^k) = \frac{\lceil s/2 \rceil}{s(k-1)}$. Moreover, there exists a k-partite realization of C_s^k, in which the smallest part is of size $\lceil s/2 \rceil$ and there is a part of size $\lceil s/2 \rceil + 1$. In particular,* $\gcd(C_s^k) = 1$. □

In order to prove Theorem 3.5, we use upper bounds for $\mathrm{ex}(n, P_2^k)$ and $\mathrm{ex}(n, C_2^k)$ from [8]. Note that the results in [8] are in the language of extremal set theory, but it is easy to formalize the results for our purpose: a k-graph is C_2^k-free if and only if the size of the intersection of any two edges is not 2; a k-graph is P_2^k-free if and only if the size of the intersection of any two edges is not 1.

Theorem 6.5. [8] *For $k \ge 4$, there exists a constant d_k such that* $\mathrm{ex}(n, C_2^k) \le d_k n^{\max\{2,k-3\}}$ *and* $\mathrm{ex}(n, P_2^k) \le \binom{n-2}{k-2}$.

Proof of Theorem 3.5. Let $K^{(k)}(a_1,\dots,a_k)$ be a k-partite realization of C_s^k such that $a_1 = \lceil s/2 \rceil$ and $a_{k'} = \lceil s/2 \rceil + 1$ for some $k' \in [k]$. Then we have that $\gcd(K^{(k)}(a_1,\dots,a_k)) = 1$ and $C = g(a_2 - a_1, \dots, a_k - a_1) + 1 = 0$. Let $m = s(k-1)$. Suppose $1/n \ll \xi \ll 1/m$. Assume H is an n-vertex k-graph which is ξ-extremal and $\delta_{k-1}(H) \ge \frac{a_1}{m} n$. Define $\epsilon_1, \epsilon_2, \epsilon_3, A', B', V_0$ as in Section 1.1 and thus Claim 6.3 and (6.1) – (6.5) hold.

The proof follows the one of Theorem 3.3 by constructing C_s^k-tilings $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$, whose union forms a perfect C_s^k-tiling of H. We will only show the first step, the existence of $\mathcal{K}_1, \mathcal{K}_2$, because it is the only part different from that in the proof of Theorem 3.3.

We here need a stronger control on the 'good' $(k-1)$-sets in B', i.e., every vertex in B' is in many such good $(k-1)$-sets (note that this is stronger than $\mathcal{B}$, which we only have control on the total number of 'bad' sets). Let G be the $(k-1)$-graph on B' whose edges are all $(k-1)$-sets $S \subseteq B'$ such that $\overline{\deg}_H(S, A') < \sqrt{2k\epsilon_1}|A'|$. We claim that

$$\delta_1(G) \ge (1 - m\sqrt{2k\epsilon_1})\binom{|B'|-1}{k-2}, \text{ and thus, } \overline{e}(G) \le m\sqrt{2k\epsilon_1}\binom{|B'|}{k-1}. \tag{6.10}$$

Suppose instead, some vertex $v \in B'$ satisfies $\overline{\deg}_G(v) > m\sqrt{2k\epsilon_1}\binom{|B'|-1}{k-2}$. Since every non-neighbor S' of v in G satisfies $\overline{\deg}_H(S' \cup \{v\}, A') \geq \sqrt{2k\epsilon_1}|A'|$, we have

$$\overline{\deg}_H(v, A'B'^{k-1}) > m\sqrt{2k\epsilon_1}\binom{|B'|-1}{k-2}\sqrt{2k\epsilon_1}|A'| > 2k\epsilon_1\binom{|B'|}{k-1},$$

where we used $m|A'| > |B'|$. This contradicts (6.4).

The K-tilings $\mathcal{K}_1, \mathcal{K}_2$. Assume that $q = |B'| - \frac{m-a_1}{m}n$. Similar as in the proof of Theorem 3.3, our goal is to find C_s^k-tilings $\mathcal{K}_1, \mathcal{K}_2$ such that $V_0 \subseteq V(\mathcal{K}_2)$, and (6.6) and (6.7) hold (with $C = 0$).

We first construct a C_s^k-tiling $\mathcal{K}_1$ such that $|\mathcal{K}_1| = \max\{q, 0\}$ and each copy of C_s^k in $\mathcal{K}_1$ contains exactly $m - a_1 + 1$ vertices in B'. Let $\mathcal{K}_1 = \emptyset$ if $q \leq 0$. Then assume $q \geq 1$ and note that $\delta_{k-1}(H[B']) \geq q$. Thus

$$e_H(B') \geq \frac{1}{k}\delta_{k-1}(H[B'])\binom{|B'|}{k-1} \geq \frac{q}{k}\binom{|B'|}{k-1} > (q-1)m \cdot 2\epsilon_1\binom{|B'|}{k-1} + \frac{q}{2k}\binom{|B'|}{k-1}.$$

Since $k \geq 4$, by Theorem 6.5, we know that $\mathrm{ex}(|B'|, C_2^k) \leq \frac{q}{2k}\binom{|B'|}{k-1}$ and $\mathrm{ex}(|B'|, P_2^k) \leq \frac{q}{2k}\binom{|B'|}{k-1}$. First assume $s = 2$. Note that if $s = 2$, then $a_1 = 1$, that is, $m - a_1 + 1 = m$. By (6.3) and Fact 6.2, $H[B']$ contains a set of q vertex disjoint copies of C_2^k. Denote it by $\mathcal{K}_1$ and we are done. Second assume $s \geq 3$, then by (6.3) and Fact 6.2, $H[B']$ contains a collection of q vertex disjoint copies of P_2^k denoted by $Q_1, \dots, Q_q$.

For each $i \in [q]$, we extend Q_i (or only one edge of Q_i) to a copy of C_s^k such that all copies of C_s^k are vertex disjoint and each copy contains exactly $m - a_1 + 1$ vertices in B'. Indeed, for $i \in [q]$, let E_1, E_s be the two edges in Q_i. If s is even, let $Q_i' = Q_i$ and pick $u \in E_1 \setminus E_s$, $u' \in E_s \setminus E_1$; if s is odd, let $Q_i' = E_1$ and let u, u' be two distinct vertices from Q_i'. Let $s' = 2\lceil s/2 \rceil$. We pick vertex sets $S_2, \dots, S_{s'-1}$ of size $k-2$, and vertices $u_3, u_5, \dots, u_{s'-3}$ from the unused vertices in B' such that the following $(k-1)$-sets

$$\begin{aligned}
&F_2 := S_2 \cup \{u\},\ F_{s'-1} := S_{s'-1} \cup \{u'\},\\
&F_{2j-1} := S_{2j-1} \cup \{u_{2j-1}\},\ F_{2j} := S_{2j} \cup \{u_{2j-1}\} \text{ for } 2 \leq j \leq (s'-2)/2
\end{aligned}$$

are in $E(G)$. This is possible by (6.10) (we pick an edge that contains u, an edge that contains u' and then some copies of P_2^{k-1} such that all these are vertex disjoint and vertex disjoint from other existing vertices). Then for each $2 \leq j \leq s'/2$, pick $u_{2j-2} \in N_H(F_{2j-2}) \cap N_H(F_{2j-1}) \cap A'$, which is possible since $F_i \in E(G)$ and thus $\overline{\deg}_H(F_i, A') < \sqrt{2k\epsilon_1}|A'|$. Note that $Q_i' \cup \bigcup_{2 \leq j \leq s'-1} S_j \cup \{u_2, \dots, u_{s'-2}\}$ spans a loose cycle of length $s'-1$ if s is odd and s' if s is even, i.e., it spans a copy of C_s^k. Moreover, each such copy contains exactly $s'/2 - 1 = a_1 - 1$ vertices in A', and thus exactly $m - a_1 + 1$ vertices in B'.

Next we choose a K-tiling $\mathcal{K}_2$ such that each copy of K contains $a_1 - 1$ vertices in A', one vertex in V_0 and $m - a_1$ vertices in B'. This can be done by the same argument as in the proof of Theorem 3.3.

By definitions, we have $|\mathcal{K}_1| \leq |q|$ and $|\mathcal{K}_2| = |V_0|$. By Claim 6.3, $|\mathcal{K}_1| + |\mathcal{K}_2| \leq |q| + |V_0| \leq 4\epsilon_3|B|$, i.e., (6.6) holds. Let A_1 and B_1 be the sets of vertices in A' and B' not covered by $\mathcal{K}_1 \cup \mathcal{K}_2$, respectively. Let $V_1 := A_1 \cup B_1$. Let $q_1 = \frac{a_1}{m}|V_1| - |A_1|$. Recall that if $s = 2$, then $a_1 = 1$. Therefore for any $s \geq 2$, by the definitions of A_1, B_1, we have $|A_1| = |A'| - (a_1 - 1)|\mathcal{K}_1| - (a_1 - 1)|V_0|$ and $|V_1| = n - m|\mathcal{K}_1| - m|V_0|$. Together with $q = \frac{a_1}{m}n - (|A'| + |V_0|)$, we get

$$q_1 = \frac{a_1}{m}n - |\mathcal{K}_1| - |V_0| - |A'| = q - |\mathcal{K}_1| = q - \max\{0, q\} \leq 0.$$

Since $-\epsilon_3|B| \leq q \leq \epsilon_3|B|$, we get $q_1 = q - \max\{0, q\} \geq -|q| \geq -2a_1\epsilon_3|B|$. Thus (6.7) holds. The rest of the proof is similar to the previous and is omitted. $\square$

6.5. **Proof of Lemma 6.1.** In this subsection we prove Lemma 6.1 by following the proof of [16, Lemma 4.4], which proves the case when $K = K^{(3)}(1,1,2)$.

We need the following result of Lu and Székely [27, Theorem 3].

Theorem 6.6. [27] *Let F be a k-graph in which each edge intersects at most d other edges. If H is an n-vertex k-graph such that $|V(F)|$ divides n and*

$$\delta_1(H) \geq \left(1 - \frac{1}{e(d+1+xk^2)}\right)\binom{n-1}{k-1},$$

where $e = 2.718...$ and $x = |E(F)|/|V(F)|$, then H contains an F-factor.

Proof of Lemma 6.1. Let $t = |X|/a_1$. Let $\mathcal{G}$ be the $(k-1)$-graph on Y whose edges are all $(k-1)$-sets $S \subseteq Y$ such that $\overline{\deg}_H(S, X) < \sqrt{\rho}t$. First we claim that

$$\delta_1(\mathcal{G}) \geq (1 - m\sqrt{\rho})\binom{|Y|-1}{k-2}, \tag{6.11}$$

and consequently,

$$\overline{e}(\mathcal{G}) \leq m\sqrt{\rho}\binom{|Y|}{k-1}. \tag{6.12}$$

Suppose instead, some vertex $v \in Y$ satisfies $\overline{\deg}_{\mathcal{G}}(v) > m\sqrt{\rho}\binom{|Y|-1}{k-2}$. Since every non-neighbor S' of v in $\mathcal{G}$ satisfies $\overline{\deg}_H(S' \cup \{v\}, X) \geq \sqrt{\rho}t$, we have $\overline{\deg}_H(v, XY^{k-1}) > m\sqrt{\rho}\binom{|Y|-1}{k-2}\sqrt{\rho}t$. Since $|Y| = (m-a_1)t$, we have

$$\overline{\deg}_H(v, XY^{k-1}) > m\rho\frac{|Y|}{m-a_1}\binom{|Y|-1}{k-2} > \rho\binom{|Y|}{k-1},$$

contradicting our assumption.

Let Q be an $(m-a_1)$-subset of Y. We call Q *good* (otherwise *bad*) if every $(k-1)$-subset of Q is an edge of $\mathcal{G}$, i.e., Q spans a *clique* of size $m-a_1$ in $\mathcal{G}$. Furthermore, we say Q is *suitable* for a vertex $x \in X$ if $x \cup T \in E(H)$ for every $(k-1)$-set $T \subset Q$. Note that if an $(m-a_1)$-set is good, by the definition of $\mathcal{G}$, it is suitable for at least $(1 - \binom{m-a_1}{k-1}\sqrt{\rho})t$ vertices of X.

Claim 6.7. *For any $x \in X$, at least $(1-\rho^{1/4})\binom{|Y|}{m-a_1}$ $(m-a_1)$-subsets of Y are good and suitable for x.*

Proof. First by the degree condition of H, namely, for any $x \in X$, the number of $(m-a_1)$-sets in Y that are not suitable for x is at most $\rho\binom{|Y|}{k-1}\binom{|Y|-k+1}{m-a_1-k+1} \leq \sqrt{\rho}\binom{|Y|}{m-a_1}$. Second, by (6.12), at most

$$\overline{e}(\mathcal{G})\binom{|Y|-k+1}{m-a_1-k+1} \leq m\sqrt{\rho}\binom{|Y|}{k-1}\binom{|Y|-k+1}{m-a_1-k+1} \leq \frac{1}{2}\rho^{1/4}\binom{|Y|}{m-a_1}$$

$(m-a_1)$-subsets of Y contain a non-edge of $\mathcal{G}$. Since $\rho^{1/2} + \frac{1}{2}\rho^{1/4} \leq \rho^{1/4}$, the claim follows. □

Let $\mathcal{F}_0$ be the set of good $(m-a_1)$-sets in Y. We will pick a family of disjoint good $(m-a_1)$-sets in Y such that for any $x \in X$, many members of this family are suitable for x. To achieve this, we pick a family $\mathcal{F}$ by selecting each member of $\mathcal{F}_0$ randomly and independently with probability $p = 4\binom{m-a_1}{k-1}\sqrt{\rho}|Y|/\binom{|Y|}{m-a_1}$. Then $|\mathcal{F}|$ follows the binomial distribution $B(|\mathcal{F}_0|, p)$ with expectation $\mathbb{E}(|\mathcal{F}|) = p|\mathcal{F}_0| \leq p\binom{|Y|}{m-a_1}$. Furthermore, for every $x \in X$, let $f(x)$ denote the number of members of $\mathcal{F}$ that are suitable for x. Then $f(x)$ follows the binomial distribution $B(N, p)$ with $N \geq (1-\rho^{1/4})\binom{|Y|}{m-a_1}$ by Claim 6.7. Hence $\mathbb{E}(f(x)) \geq p(1-\rho^{1/4})\binom{|Y|}{m-a_1}$. Since there are at most $\binom{|Y|}{m-a_1} \cdot (m-a_1) \cdot \binom{|Y|-1}{m-a_1-1}$ pairs of intersecting $(m-a_1)$-sets in Y, the expected number of intersecting pairs of $(m-a_1)$-sets in $\mathcal{F}$ is at most

$$p^2\binom{|Y|}{m-a_1} \cdot (m-a_1) \cdot \binom{|Y|-1}{m-a_1-1} = 16\binom{m-a_1}{k-1}^2(m-a_1)^2\rho|Y|.$$

By Chernoff's bound (the first two properties) and Markov's bound (the last one), we can find a family $\mathcal{F}$ of good $(m-a_1)$-subsets of Y that satisfies

- $|\mathcal{F}| \le 2p\binom{|Y|}{m-a_1} \le 8\binom{m-a_1}{k-1}\sqrt{\rho}|Y|$,
- for any vertex $x \in X$, at least $\frac{p}{2}(1-\rho^{1/4})\binom{|Y|}{m-a_1} = 2\binom{m-a_1}{k-1}(1-\rho^{1/4})\sqrt{\rho}|Y|$ members of $\mathcal{F}$ are suitable for x.
- the number of intersecting pairs of $(m-a_1)$-sets in $\mathcal{F}$ is at most $32\binom{m-a_1}{k-1}^2(m-a_1)^2\rho|Y|$.

After deleting one $(m-a_1)$-set from each of the intersecting pairs from $\mathcal{F}$, we obtain a family $\mathcal{F}' \subseteq \mathcal{F}$ consisting of at most $8\binom{m-a_1}{k-1}\sqrt{\rho}|Y|$ disjoint good $(m-a_1)$-subsets of Y and for each $x \in X$, at least

$$2\binom{m-a_1}{k-1}(1-\rho^{1/4})\sqrt{\rho}|Y| - 32\binom{m-a_1}{k-1}^2(m-a_1)^2\rho|Y| \ge \binom{m-a_1}{k-1}\sqrt{\rho}|Y| \tag{6.13}$$

members of $\mathcal{F}'$ are suitable for x.

Denote $\mathcal{F}'$ by $\{Q_1, Q_2, \dots, Q_q\}$ for some $q \le 8\binom{m-a_1}{k-1}\sqrt{\rho}|Y|$. Let $Y_1 = Y \setminus V(\mathcal{F}')$ and $\mathcal{G}' = \mathcal{G}[Y_1]$. Then $|Y_1| = |Y| - (m-a_1)q$. Since $\overline{\deg}_{\mathcal{G}'}(v) \le \overline{\deg}_{\mathcal{G}}(v)$ for every $v \in Y_1$, we have, by (6.11),

$$\delta_1(\mathcal{G}') \ge \binom{|Y_1|-1}{k-2} - m\sqrt{\rho}\binom{|Y|-1}{k-2} \ge (1-2m\sqrt{\rho})\binom{|Y_1|-1}{k-2}.$$

By the choice of ρ and Theorem 6.6, $\mathcal{G}'$ contains a perfect tiling $\{Q_{q+1}, \dots, Q_t\}$ such that each Q_i is a clique on $m-a_1$ vertices for $q+1 \le i \le t$.

Consider the bipartite graph Γ between X and $\mathcal{Q} := \{Q_1, Q_2, \dots, Q_t\}$ such that $x \in X$ and $Q_i \in \mathcal{Q}$ are adjacent if and only if $\mathcal{Q}_i$ is suitable for x. For every $i \in [t]$, since each Q_i is a clique in $\mathcal{G}$, we have $\deg_\Gamma(Q_i) \ge |X| - \binom{m-a_1}{k-1}\sqrt{\rho}t$ by the definition of $\mathcal{G}$. Let $\mathcal{Q}' = \{Q_{q+1}, \dots, Q_t\}$ and X_0 be the set of $x \in X$ such that $\deg_\Gamma(x, \mathcal{Q}') \le |\mathcal{Q}'|/2$. Then

$$|X_0|\frac{|\mathcal{Q}'|}{2} \le \sum_{x\in X} \overline{\deg}_\Gamma(x, \mathcal{Q}') \le \binom{m-a_1}{k-1}\sqrt{\rho}t \cdot |\mathcal{Q}'|,$$

which implies that $|X_0| \le 2\binom{m-a_1}{k-1}\sqrt{\rho}t = 2\binom{m-a_1}{k-1}\sqrt{\rho}\frac{|Y|}{m-a_1} \le \binom{m-a_1}{k-1}\sqrt{\rho}|Y|$ (since $m-a_1 \ge 2$).

We now find a perfect tiling of $K^{(2)}(1, a_1)$ in Γ such that the center of each $K^{(2)}(1, a_1)$ is in $\mathcal{Q}$.

Step 1: Each $x \in X_0$ is matched to some Q_i, $i \in [q]$ that is suitable for x – this is possible because of (6.13) and $|X_0| \le \binom{m-a_1}{k-1}\sqrt{\rho}|Y|$.

Step 2: Each Q_i, $i \in [q]$ is matched with a_1-1 or a_1 more vertices in $X \setminus X_0$ – this is possible because $\deg_\Gamma(Q_i) \ge |X| - \binom{m-a_1}{k-1}\sqrt{\rho}t \ge |X_0| + a_1 q$. Thus, all $\mathcal{Q}_i$, $i \in [q]$ are covered by vertex-disjoint copies of $K^{(2)}(1, a_1)$.

Step 3: Let X' be the set of uncovered vertices in X and note that $|X'| = a_1 t - a_1 q = a_1|\mathcal{Q}'|$. Partition X' arbitrarily into $X_1, \dots, X_{a_1}$ each of size $|\mathcal{Q}'|$. Note that for each $i \in [a_1]$, we have for all $x \in X_i$, $\deg_\Gamma(x, \mathcal{Q}') > |\mathcal{Q}'|/2$ and for all $Q_j \in \mathcal{Q}'$, $\deg_\Gamma(Q_j, X_i) \ge |X_i| - \binom{m-a_1}{k-1}\sqrt{\rho}t \ge |X_i|/2$. So the Marriage Theorem provides a perfect matching in each $\Gamma[X_i, \mathcal{Q}']$, $i \in [a_1]$ and thus we get a perfect tiling of $K^{(2)}(1, a_1)$ on $X' \cup \mathcal{Q}'$ in Γ.

The perfect tiling of $K^{(2)}(1, a_1)$ in Γ gives rise to the desired K-factor in H. □

7. Concluding Remarks

In this paper we study the minimum codegree threshold $\delta(n, K)$ for tiling complete k-partite k-graphs K perfectly when $\gcd(K) = 1$. By Proposition 1.2, $\delta(n, K) \ge n/m + \mathrm{coex}(\frac{m-1}{m}n + 1, K)$ when $a_1 = 1$. In view of this and Theorem 1.3, it is interesting to know if one can replace the second term in (1.3) by a term similar to $\mathrm{coex}(\frac{m-a_1}{m}n+1, K)$. Moreover, it is interesting to know if $\mathrm{ex}(n, K)/\binom{n}{k-1}$ (or $\mathrm{coex}(n, K)$) is monotone on n so that the maximization in (1.3) could be avoided.

Following the notion in [29], we call k-partite k-graphs satisfying the three lines of (1.1) type 0, type 1 and type d, respectively. For complete k-partite k-graphs K of type d for an even $d \ge 0$, a simple application of the absorbing method together with Lemma 3.2 implies that $\delta(n, K) = n/2 + o(n)$, which gives a reproof of the result of Mycroft [29] without using the Hypergraph Blow-up Lemma. We think that a further sharpening is possible by a careful analysis on the extremal case, to which we shall return in the near future.

Suppose $K := K^{(k)}(a_1, \dots, a_k)$. When K is type 1 (namely, $\gcd(K) = 1$), Proposition 1.2 and Theorem 1.5 settle Conjecture 1.1 (either negatively or positively). Now we give a construction showing that Conjecture 1.1 is false for many other complete k-partite k-graphs, for which we need to recall a construction by Mycroft [29]. Fix a prime number p. Let $\mathbf{u}_i \in \mathbb{Z}_p^p$ be the unit vector whose ith coordinate is one. For $1 \le i < p$, let $\mathbf{v}_i = \mathbf{u}_i + (i-1)\mathbf{u}_p$. Let L be the (proper) sublattice of $\mathbb{Z}_p^p$ generated by $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$. The following property was proved in [29, Section 2].

(†) For any vector $\mathbf{v} \in \mathbb{Z}_p^p$, there exists precisely one $i \in [p]$ such that $\mathbf{v} + \mathbf{u}_i \in L$.

Let $\mathcal{P} = \{V_1, V_2, \dots, V_p\}$ be a partition of V such that $|V_1| + |V_2| + \cdots + |V_p| = n$, $|V_i| = n/p \pm 1$ for $i \in [p]$ and $\mathbf{i}_{\mathcal{P}}(V) \notin L \pmod p$ (recall that $\mathbf{i}_{\mathcal{P}}(S)$ is the vector of $\mathbb{Z}^p$ whose ith coordinate is $|S \cap V_i|$). Let H_p be the k-graph on V whose edges are k-sets e such that $\mathbf{i}_{\mathcal{P}}(e) \in L \pmod p$. Observe that (†) implies that $\delta_{k-1}(H_p) \ge n/p - k$.

Proposition 7.1. [29] *Suppose that K is a complete k-partite k-graph of type d for some $d \neq 1$. Let p be the smallest prime factor of d (thus $p = 2$ when $d = 0$). Then H_p contains no K-factor.*

Using H_p and the construction behind Proposition 2.3, we disprove Conjecture 1.1 for many complete k-partite k-graphs of type $d \neq 1$.

Proposition 7.2. *Let $K := K^{(k)}(a_1, \dots, a_k)$ be type $d \neq 1$ and let p be the smallest prime factor of d. Then Conjecture 1.1 is false for K if $a_{k-2} \ge p + 1$.*

Proof. Let $n \in (a_1 + \cdots + a_k)\mathbb{Z}$ and let G be an n-vertex $K^{(k)}(1, \dots, 1, 2, 2)$-free k-graph with $\delta_{k-1}(G) = (1 - o(1))\sqrt{n}$ provided by Proposition 2.3. We first take a copy of an n-vertex k-graph H_p under a vertex partition $V_1, V_2, \dots, V_p$. Then we take a random permutation of $V(G)$ and then add $E(G)$ on top of H_p. Denote the resulting graph by H. By standard concentration results, we have $\delta_{k-1}(H) \ge n/p + (1 - o(1))\sqrt{n}/p$. We claim that

(‡) every copy of K in H has each of its vertex classes completely in V_i for some $i \in [p]$.

Suppose instead, there exist distinct $i_1, i_2 \in [p]$ and a copy of K in H with vertex classes $U_1, \dots, U_k$ such that $U_l \cap V_{i_1} \neq \emptyset$ and $U_l \cap V_{i_2} \neq \emptyset$ for some $l \in [k]$. For $i \in [k] \setminus \{l\}$, let $C_i = U_i \cap V_j$ such that $|U_i \cap V_j| \ge |U_i \cap V_{j'}|$ for all $j' \in [p]$ (if more than one j satisfies this, choose any of them). Since $a_k \ge a_{k-1} \ge a_{k-2} \ge p+1$, by the pigeonhole principle, we have $|C_i| \ge 2$ for any $i \in \{k-2, k-1, k\} \setminus \{l\}$. Thus both $(U_l \cap V_{i_1}) \cup \bigcup_{j \neq l} C_j$ and $(U_l \cap V_{i_2}) \cup \bigcup_{j \neq l} C_j$ contain $K^{(k)}(1, \dots, 1, 2, 2)$ as a subgraph. Since G is $K^{(k)}(1, \dots, 1, 2, 2)$-free, both copies of $K^{(k)}(1, \dots, 1, 2, 2)$ contain an edge in H_p. The index vectors of these two edges can be written as $\mathbf{v} + \mathbf{u}_{i_1}$ and $\mathbf{v} + \mathbf{u}_{i_2}$ for some $\mathbf{v}$, contradicting (†).

Since G is K-free, each copy K_1 of K in H must contain some edge e in H_p. Since K is a blow-up of e, (‡) implies that $\mathbf{i}_{\mathcal{P}}(e') = \mathbf{i}_{\mathcal{P}}(e) \in L \pmod p$ for all $e' \in E(K_1)$. Thus, all copies of K in H are actually in H_p. Therefore, H has no K-factor by Proposition 7.1. $\square$

When $K = K^{(k)}(a_1, \dots, a_k)$ is type $d \neq 1$, Proposition 7.2 leaves out the following unsettled cases for Conjecture 1.1: $K^{(k)}(2, \dots, 2, 2s, 2t)$ for $t \ge s \ge 1$ (type 0), and $K^{(k)}(a_1, \dots, a_1, a_{k-1}, a_k)$ (type $d \ge 2$). To see how to derive the second case, we note that if K is type $d \ge 2$ and p is the smallest prime factor of d, then $a_1 \ge 1$ and $a_{k-2} \le p$ force that $a_1 = a_{k-2}$.

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Department of Mathematics and Statistics, Auburn University, Auburn, AL 36830
Email address, Wei Gao: `wzg0021@auburn.edu`

Department of Mathematics, University of Rhode Island, 5 Lippitt Road, Kingston, RI, USA, 02881
Email address, Jie Han: `jie_han@uri.edu`

Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303
Email address, Yi Zhao: `yzhao6@gsu.edu`