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Hamiltonian cycles with all small even chords

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ABSTRACT

Let G be a graph of order $n \geq 3$. An *even squared Hamiltonian cycle* (ESHC) of G is a Hamiltonian cycle $C = v_1v_2 \dots v_nv_1$ of G with chords v_iv_{i+3} for all $1 \leq i \leq n$ (where $v_{n+j} = v_j$ for $j \geq 1$). When n is even, an ESHC contains all bipartite 2-regular graphs of order n. We prove that there is a positive integer N such that for every graph G of *even* order $n \geq N$, if the minimum degree is $\delta(G) \geq \frac{n}{2} + 92$, then G contains an ESHC. We show that the condition of n being even cannot be dropped and the constant 92 cannot be replaced by 1. Our results can be easily extended to *even kth powered Hamiltonian cycles* for all $k \geq 2$.

1. Introduction

In this paper, we will only consider simple graphs — finite graphs without loops or multiple edges. The notations and definitions not defined here can be found in [7]. Let $G = (V, E)$ be a graph with vertex set V and edge set E. For a vertex $v \in V$ and a subset $S \subseteq V$, let $\Gamma(v, S)$ denote the set of neighbors of v in S, and $\deg(v, S) = |\Gamma(v, S)|$. Given another set $U \subseteq V$, define $\Gamma(U, S) = \cap_{u \in U} \Gamma(u, S)$ and $\deg(U, S) = |\Gamma(U, S)|$. When $U = \{v_1, \dots, v_k\}$, we simply write $\Gamma(U, S)$ and $\deg(U, S)$ as $\Gamma(v_1, \dots, v_k, S)$ and $\deg(v_1, \dots, v_k, S)$, respectively. When $S = V$, we only write $\Gamma(U)$ and $\deg(U)$.

A graph G is called *Hamiltonian* if it contains a spanning cycle. The Hamiltonian problem, determining whether a graph has a Hamiltonian cycle, has long been one of few fundamental problems in graph theory. In this paper, we fix G to be a graph of order $n \geq 3$. Dirac [8] proved that, if the minimum degree $\delta(G) \geq n/2$ then G is Hamiltonian. Ore [23] extended Dirac's result by replacing the minimum degree condition with that of $\deg(u) + \deg(v) \geq n$ for all nonadjacent vertices u and v. Many results have been obtained on generalizing these two classic results (see [13] for a recent survey in this area).

A 2-regular subgraph (2-factor) of G consists of disjoint cycles of G. Aigner and Brandt [2] proved that if the minimum degree $\delta(G) \geq \frac{2n-1}{3}$ then G contains all 2-factors as subgraphs (Alon and Fischer [3] proved this for sufficiently large n). If to-be-embedded 2-factors have at most k odd components, then by a conjecture of El-Zahar [9], the minimum degree condition can be reduced to $\delta(G) \geq (n + k)/2$ (Abbasi [1] announced a proof of El-Zahar's conjecture for large n). Another way to generalize Aigner and Brandt's result is to find one specific subgraph of G that contains all 2-factors of G. A *squared Hamiltonian cycle* of G is a Hamiltonian cycle $v_1v_2 \cdots v_nv_1$ together with edges v_iv_{i+2} for all $1 \leq i \leq n$. Note that we always assume that $v_{n+i} = v_i$ for $i \geq 1$. It is easy to see that a squared Hamiltonian cycle contains all 2-factors of G. Pósa (see [10]) conjectured that every graph G of order $n \geq 3$ with $\delta(G) \geq \frac{2}{3}n$ contains a squared Hamiltonian cycle. Fan and Kierstead [12] proved this conjecture approximately; Komlós et al. [15] proved the conjecture for sufficiently large n. More generally, the *k-th powered Hamiltonian cycle* is a Hamiltonian cycle $v_1v_2 \cdots v_nv_1$ with chords v_iv_{i+j} for all $1 \leq i \leq n$ and $1 \leq j \leq k$.

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Komlós et al. [17,18] proved a conjecture of Seymour for sufficiently large n: every n-vertex graph G with $\delta(G) \geq (k-1)n/k$ contains a kth powered Hamiltonian cycle.

Böttcher et al. [4] recently proved a conjecture of Bollobás and Komlós (see [14]), which asymptotically includes all the results mentioned above. Given an integer b, a graph H is said to have *bandwidth* at most b, if there exists a labeling of the vertices by $v_1, v_2, \dots, v_n$, such that $|j-i| \leq b$ whenever $v_iv_j \in E(H)$. It is shown in [4] that for any $\varepsilon > 0$ and integers r, Δ, there exists $\beta > 0$ with the following property. Let G and H be n-vertex graphs for sufficiently large n. If $\delta(G) \geq ((r-1)/r + \varepsilon)n$ and H is r-chromatic with maximum degree Δ and bandwidth at most βn, then G contains a copy of H. Note that the kth powered Hamiltonian cycle of order n has chromatic number $k+1$ or $k+2$ depending on the value of n. The authors of [4] make their result applicable even when H is $r+1$-chromatic but one of its color classes is fairly small, *e.g.*, the kth powered Hamiltonian cycle.

We are interested in the situation when the error term εn in the conjecture of Bollobás and Komlós can be reduced to a constant. According to the El-Zahar Conjecture, every n-vertex graph G with the minimum degree $\delta(G) \geq n/2$ contains all 2-factors with even components. Given a graph G, we define an *Even Squared Hamiltonian Cycle* (ESHC) as a Hamiltonian cycle $C = v_1v_2 \cdots v_nv_1$ of G with chords v_iv_{i+3} for all $1 \leq i \leq n$. When $n \geq 7$, an ESHC is 4-regular with chromatic number $\chi = 2$ for even n and $\chi = 3$ for odd n. It is not hard to check that an n-vertex ESHC contains all bipartite graphs of order n with maximum degree at most 2 (*e.g.*, by using the fact that every ESHC of even order contains a ladder graph defined below). Below is our main result.

Theorem 1.1. *There exists $N > 0$ such that for all even integers $n \geq N$, if G is a graph of order n with $\delta(G) \geq \frac{n}{2} + 92$, then G contains an ESHC.*

We show that the constant 92 in Theorem 1.1 cannot be replaced by 1.

Proposition 1.2. *Suppose that $n \geq 10$. Let G be the union of two copies of $K_{\frac{n}{2}+2}$ sharing 4 vertices. Then $\delta(G) = \frac{n}{2} + 1$ but G contains no ESHC.*

We also show that the condition of n being even is necessary for Theorem 1.1—even if we replace 92 by $\sqrt{n/8} - 1/2$.

Proposition 1.3. *There are infinitely many odd n and graphs G of order n such that $\delta(G) \geq \frac{n}{2} + \sqrt{n/8} - 1/2$ but G contains no ESHC.*

More generally, an *Even kth powered Hamiltonian Cycle* (EkHC) of a graph G is a Hamiltonian cycle $v_1v_2 \cdots v_nv_1$ with edges v_iv_{i+2j-1} for all $1 \leq i \leq n$ and $1 \leq j \leq k$. Then an E1HC is simply a Hamiltonian cycle while an E2HC is an ESHC. Using the same proof techniques for Theorem 1.1, we can derive the following result, whose proof is omitted.

Theorem 1.4. *For any positive integer k, there exist a constant $c = c(k)$ and a positive integer N such that if G is a graph of even order $n \geq N$ and $\delta(G) \geq \frac{n}{2} + c$ then G contains an* EkHC.

One may view an EkHC of order $n = 2N$ as the following bipartite graph. Let $B^k(N)$ be the bipartite graph $(X \cup Y, E)$ with $X = \{x_1, \dots, x_N\}$ and $Y = \{y_1, \dots, y_N\}$ such that $x_iy_j \in E$ if and only if

$$i - j (\text{mod } N) \in \{-k+1, \dots, -1, 0, 1, \dots, k\}.$$

In particular, $B^2(N)$, or ESHC, contains the ladder graph defined by Czygrinow and Kierstead [6], which has the same vertex sets X and Y but x_i is adjacent to y_j if and only if $i - j (\text{mod } N) \in \{-1, 0, 1\}$. Note that the ladder graph contains all 2-factors with bipartite components.

The structure of the paper is as follows. We prove two (easy) Propositions 1.2 and 1.3 in the next section. Following the approach of [15] on squared Hamiltonian cycles, we prove Theorem 1.1 by the regularity method. In Section 3 we state the Regularity Lemma and the Blow-up Lemma. In Section 4 we prove Theorem 1.1 by proving the non-extremal case and two extremal cases separately. It seems harder to handle the extremal cases here than in [15]; this is also the reason why we need a large constant 92 in Theorem 1.1. The last section gathers open problems with a remark.

2. Proofs of Propositions 1.2 and 1.3

Given a graph G, a pair (A, B) of vertex subsets is called a *separator* of G if $V(G) = A \cup B$, both $A - B$ and $B - A$ are non-empty and $E(A - B, B - A) = \emptyset$. It is easy to see that Proposition 1.2 follows from the following claim, which can be proved by a simple case analysis.

Claim 2.1. *Suppose that G is a graph with an ESHC. If (A, B) is a separator of G with $|A-B| \geq 3$ and $|B-A| \geq 3$, then $|A \cap B| \geq 6$.*

Proof of Claim 2.1. Let H be an ESHC of G with Hamiltonian cycle C. Assign C an orientation. A segment $P_1 = xP_1z$ of C is called an *AB-path* if $x \in A - B$ and $z \in B - A$. Now let $x_1P_1z_1$ be an AB-path such that $V(P_1) - \{x_1, z_1\} \subseteq A \cap B$ (this can be done by letting P_1 be minimal). Since $|A - B| \geq 3$, $|B - A| \geq 3$, there is an AB-path $x_2P_2z_2$ contained in $C - V(P_1)$ such that $V(P_2) - \{x_2, z_2\} \subseteq A \cap B$. If $e(P_i) \geq 4$ for $i = 1, 2$, then $|A \cap B| \geq e(P_1) + e(P_2) - 2 \geq 6$ and we are done. On the other hand, we know that $e(P_i) \notin \{1, 3\}$ for $i = 1, 2$ because $x_iz_i \notin E(G)$, which follows from $e(A - B, B - A) = \emptyset$. Without loss of generality, assume that $e(P_1) = 2$, or $P_1 = x_1y_1z_1$. By following the orientation of C, let x_1^- be the predecessor of x_1 and z_1^+ be the successor of z_1. Since $x_1^-z_1, x_1z_1^+ \in E(G)$, we have $x_1^-, z_1^+ \in A \cap B$ (see Fig. 1).

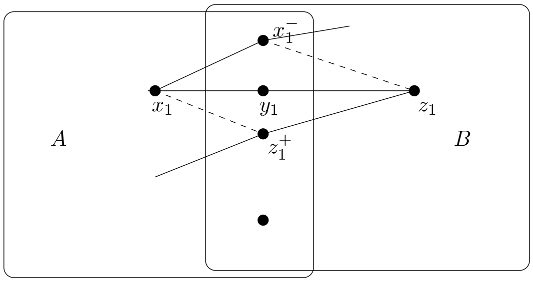

Fig. 1. A segment connecting A and B.

Since P_1 and P_2 are vertex disjoint AB-paths, we have $V(P_2) \cap \{x_1^-, y_1, z_1^+\} = \emptyset$. If P_2 contains at least three internal vertices, then $|A \cap B| \geq 6$ and we are done. So we may assume that $P_2 = x_2y_2z_2$, and consequently $\{x_2^-, y_2, z_2^+\} \subseteq A \cap B$.

If $z_1^+ \neq x_2^-$ and $z_2^+ \neq x_1^-$, then all $x_1^-, y_1, z_1^+, x_2^-, y_2, z_2^+$ are distinct, and consequently $|A \cap B| \geq 6$. Otherwise, without loss of generality, assume that $z_1^+ = x_2^-$. Then $P_1' = P_1z_1^+P_2$ is an AB-path with two vertices in each of $A - B$ and $B - A$. Since $|A - B| \geq 3$ and $|B - A| \geq 3$, there is an AB-path $x_3P_3z_3$ which is vertex-disjoint from P_1' such that $V(P_3) - \{x_3, z_3\} \subseteq A \cap B$. If $e(P_3) \geq 4$, then $|A \cap B| \geq 6$ because P_1', P_3 are disjoint and P_1' contains three vertices y_1, z_1^+, y_2 from $A \cap B$. Otherwise $P_3 = x_3y_3z_3$ for some $y_3 \in A \cap B$, then $\{x_3^-, y_3, z_3^+\} \subseteq A \cap B$. Since six vertices $y_1, z_1^+, y_2, x_3^-, y_3, z_3^+$ are contained in $A \cap B$, we have $|A \cap B| \geq 6$. $\square$

Proof of Proposition 1.3. Let q be an odd prime power, by using projective planes, one can construct (e.g., [11]) C_4-free graphs H of order $h = q^2 + q + 1$ with $\delta(H) \geq q$. Let $G := H + \overline{K}_{h-q}$, i.e., a graph obtained from H by adding $h - q$ vertices such that each new vertex is adjacent to all vertices of H. Let $X := V(G) - V(H)$ and $n := |V(G)| = 2h - q$. Then n is odd and

$$\delta(G) \geq h = \frac{n+q}{2} \geq \frac{n}{2} + \sqrt{\frac{n}{8}} - \frac{1}{2}$$

because $n = 2q^2 + q + 2 < 2(q+1)^2$. To see that G does not have any ESHC, consider an arbitrary Hamiltonian cycle C of G (if it exists). Since X is an independent vertex set and n is odd, we conclude that $C - X$ is the union of vertex-disjoint paths such that $e(C - X)$ is odd. In particular, one path $P[x, y]$ of $C - X$ has odd length. If $|V(P)| \geq 4$, then $xx^{+++} \notin E(G)$, where $x^{+++} = ((x^+)^+)^+$, since H contains no C_4. Otherwise $|V(P)| = 2$, which in turn shows that $x^-, y^+ \in X$. Since X is independent, $x^-y^+ \notin E(G)$. In all cases G does not have an ESHC based on C. $\square$

3. The Regularity Lemma and Blow-up Lemma

As in [15,18], the Regularity Lemma of Szemerédi [24] and Blow-up Lemma of Komlós et al. [16] are main tools in our proof of Theorem 1.1. For any two disjoint vertex-sets A and B of a graph G, the *density* of A and B is the *ratio* $d(A, B) := e(A, B)/(|A| \cdot |B|)$, where $e(A, B)$ is the number of edges with one end vertex in A and the other in B. Let ε and δ be two positive real numbers. The pair (A, B) is called ε*-regular* if for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| > \varepsilon|A|$, $|Y| > \varepsilon|B|$, we have $|d(X, Y) - d(A, B)| < \varepsilon$. Moreover, the pair (A, B) is called (ε, δ)*-super-regular* if (A, B) is ε-regular and $\deg_B(a) > \delta|B|$ for all $a \in A$ and $\deg_A(b) > \delta|A|$ for all $b \in B$.

Lemma 3.1 (*Regularity Lemma—Degree Form*)**.** *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that, for any graph $G = (V, E)$ and any real number $d \in [0, 1]$, there is a partition of the vertex set V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- $\ell \leq M$,
- $|V_i| \leq \varepsilon|V|$ *for* $0 \leq i \leq \ell$, $|V_1| = |V_2| = \cdots = |V_\ell|$,
- $\deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)|V|$ *for all* $v \in V$,
- $G'[V_i] = \emptyset$ *(i.e. V_i is an independent set in G'), for all i,*
- *each pair* (V_i, V_j), $1 \leq i < j \leq \ell$, *is ε-regular with* $d(V_i, V_j) = 0$ *or* $d(V_i, V_j) \geq d$ *in* G'.

The Blow-up Lemma allows us to regard a super-regular pair as a complete bipartite graph when embedding a graph with bounded degree. We need a bipartite version of this lemma which also restricts the mappings of a small number of vertices.

Lemma 3.2 (*Blow-Up Lemma—Bipartite Version*)**.** *For every $\delta, \Delta, c > 0$, there exists an $\varepsilon = \varepsilon(\delta, \Delta, c) > 0$ and $\alpha = \alpha(\delta, \Delta, c) > 0$ such that the following holds. Let (X, Y) be an (ε, δ)-super-regular pair with $|X| = |Y| = N$. If a bipartite graph H with $\Delta(H) \leq \Delta$ can be embedded in $K_{N,N}$ by a function ϕ, then H can be embedded in (X, Y). Moreover, in each $\phi^{-1}(X)$ and $\phi^{-1}(Y)$, fix at most αN special vertices z, each of which is equipped with a subset S_z of X or Y of size at least cN. The embedding of H into (X, Y) exists even if we restrict the image of z to be S_z for all special vertices z.*

4. Proof of Theorem 1.1

Let V be the vertex set of a graph G of order n for some even n. A partition $V_1 \cup \cdots \cup V_k$ of V is called *balanced* if $||V_i| - |V_j|| \le 1$ for all $i \ne j$. In particular, a balanced *bipartition* $V_1 \cup V_2$ satisfies $|V_1| = |V_2| = n/2$. Given $0 \le \alpha \le 1$, we define two extremal cases with parameter α as follows.

Extremal Case 1: There exists a balanced partition of V into V_1 and V_2 such that the density $d(V_1, V_2) \ge 1 - \alpha$.
Extremal Case 2: There exists a balanced partition of V into V_1 and V_2 such that the density $d(V_1, V_2) \le \alpha$.

The following three theorems deal with the non-extremal case and two extremal cases separately.

Theorem 4.1. *For every $\alpha > 0$, there exist $\beta > 0$ and a positive integer n_0 such that the following holds for every even integer $n \ge n_0$. For every graph G of order n with $\delta(G) \ge (\frac{1}{2} - \beta)n$, either G contains an ESHC or G is in one of the extremal cases with parameter α.*

Theorem 4.2. *Suppose that $0 < \alpha \ll 1$ and n is a sufficiently large even integer. Let G be a graph on n vertices with $\delta(G) \ge \frac{n}{2}+3$. If G is in Extremal Case 1 with parameter α, then G contains an ESHC.*

Theorem 4.3. *Suppose that $0 < \alpha \ll 1$ and n is a sufficiently large even integer. Let G be a graph on n vertices with $\delta(G) \ge \frac{n}{2} + 92$. If G is in Extremal Case 2 with parameter α, then G contains an ESHC.*

It is easy to see that Theorem 1.1 follows from Theorems 4.1–4.3. For this purpose we can even use a weaker version of Theorem 4.1 with $\beta = 0$, but the current Theorem 4.1 may have other applications.

If G is in Extremal Case 2 with parameter α, then there exists $x \in V$ such that $\deg(x) < (1 + \alpha)n/2$ and in turn $\delta(G) < (1 + \alpha)n/2$. Theorems 4.1 and 4.2 together imply the following remark, which is a special case of the theorem of Böttcher et al. [4].

Remark 4.4. For any $\alpha > 0$, there exists a positive integer n_0 such that every graph G of even order $n \ge n_0$ with $\delta(G) \ge (\frac{1}{2} + \alpha)n$ contains an ESHC.

This remark will be used in the proof of Theorem 4.3.

4.1. Non-extremal case

In this section we prove Theorem 4.1.

Proof. We fix the following sequence of parameters

$$0 < \varepsilon \ll d \ll \beta \ll \alpha < 1 \tag{1}$$

and specify their dependence as the proof proceeds. Actually we let α be the minimum of the two parameters defined in the extremal cases. Then we choose $d \ll \beta$ such that they are much smaller than α. Finally we choose $\varepsilon = \frac{1}{2}\varepsilon(d, 4, \frac{d^2}{4})$ following the definition of ε in the Blow-up Lemma.

Choose n to be sufficiently large. In the proof we omit ceiling and floor functions if they are not crucial.

Let G be a graph of order n such that $\delta(G) \ge (\frac{1}{2} - \beta)n$ and G is *not* in either of the extremal cases. Applying the Regularity Lemma (Lemma 3.1) to G with parameters ε and d, we obtain a partition of $V(G)$ into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$ for some $\ell \le M = M(\varepsilon)$, and a subgraph G' of G with all described properties in Lemma 3.1. In particular, for all $v \in V$,

$$\deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)n \ge \left(\frac{1}{2} - \beta - \varepsilon - d\right) n \ge \left(\frac{1}{2} - 2\beta\right) n,$$

provided that $\varepsilon + d \le \beta$. On the other hand, $e(G') \ge e(G) - \frac{(d+\varepsilon)}{2}n^2 \ge e(G) - dn^2$ by using $\varepsilon < d$.

We further assume that $\ell = 2k$ is even; otherwise we eliminate the last cluster V_ℓ by removing all the vertices in this cluster to V_0. As a result, $|V_0| \le 2\varepsilon n$ and

$$(1 - 2\varepsilon)n \le \ell N = 2kN \le n. \tag{2}$$

For each pair i and j with $1 \le i \ne j \le \ell$, we write $V_i \sim V_j$ if $d(V_i, V_j) \ge d$. As in other applications of the Regularity Lemma, we consider the *reduced graph* G_r, whose vertex set is $\{1, \ldots, \ell\}$, and two vertices i and j are adjacent if and only if $V_i \sim V_j$. From $\delta(G') > (\frac{1}{2} - 2\beta)n$, a standard argument shows that $\delta(G_r) \ge (\frac{1}{2} - 2\beta)\ell$.

The rest of the proof consists of the following five steps.

Step 1: Show that G_r contains a Hamiltonian cycle $X_1Y_1 \cdots X_kY_k$.
Step 2: For each $1 \le i \le k$, initiate a connecting ESP (even squared path) P_i between Y_{i-1} and X_i (where $Y_0 = Y_k$) with two vertices from each Y_{i-1} and X_i.
Step 3: For each $1 \le i \le k$, move at most $2\varepsilon N$ vertices from $X_i \cup Y_i$ to V_0 such that the resulting graph $G'[X_i \cup Y_i]$ has the minimum degree at least $(d - 2\varepsilon)N$.

Step 4: Extend $P_1, \dots, P_k$ to include all the vertices in V_0 and some vertices in $V \setminus V_0$ such that $|X_i \cap (V(P_1) \cup \dots \cup V(P_k))| = |Y_i \cap (V(P_1) \cup \dots \cup V(P_k))| \le \frac{d^2}{2}N$ for all $1 \le i \le k$.

Step 5: Apply the Blow-up Lemma to each (X_i, Y_i) and obtain an ESP consisting of all the remaining vertices of $X_i \cup Y_i$. Concatenating these ESP's with $P_1, \dots, P_k$, we obtain the desired ESHC of G.

We now give details of each step.

The assumption that G is not in either of the extremal cases leads to the following claim, which will be used in Step 1 and Step 4.

Claim 4.5. (a) *G_r contains no independent set U_1 of size at least $(\frac{1}{2} - 8\beta)\ell$.*
(b) *G_r contains no two disjoint subsets U_1, U_2 of size at least $(\frac{1}{2} - 6\beta)\ell$ such that $e_{G_r}(U_1, U_2) = 0$.*

Proof. (a) Suppose instead, that G_r contains an independent set U_1 of size $(\frac{1}{2} - 8\beta)\ell$. We will show that G is in the Extremal Case 1 with parameter α. Let $A = \bigcup_{i \in U_1} V_i$ and $B = V(G) - A$. By (2),

$$\left(\frac{1}{2} - 9\beta\right) n \le \left(\frac{1}{2} - 8\beta\right) N\ell = |U_1|N = |A| < \left(\frac{1}{2} - 2\beta\right) n.$$

For each $x \in A$, since $\deg_G(x, A) \le \deg_{G'}(x, A) + (d + \varepsilon)n < \beta n$, we have $\deg_G(x, B) > (\frac{1}{2} - \beta)n - \beta n \ge (\frac{1}{2} - 2\beta)n$. Hence $e_G(A, B) \ge (\frac{1}{2} - 9\beta)n(\frac{1}{2} - 2\beta)n > (\frac{1}{4} - \frac{11}{2}\beta)n^2$. Now move at most $9\beta n$ vertices from B to A such that A and B are of size $n/2$. We still have

$$e_G(A, B) > \left(\frac{1}{4} - \frac{11}{2}\beta\right) n^2 - 9\beta n \frac{n}{2} = \left(\frac{1}{4} - 10\beta\right) n^2 = (1 - 40\beta)\left(\frac{n}{2}\right)^2.$$

By specializing $40\beta \le \alpha$ in (1), we see that G is in the Extremal Case 1 with parameter α.

(b) Suppose instead, that G_r contains two disjoint subsets U_1, U_2 of size $(\frac{1}{2} - 6\beta)\ell$ such that $e_{G_r}(U_1, U_2) = 0$. We will show that G is in the Extremal Case 2 with parameter α. Let $A = \bigcup_{i \in U_1} V_i$ and $B = \bigcup_{i \in U_2} V_i$. Since $e_{G_r}(U_1, U_2) = 0$, we have $e_{G'}(A, B) = 0$. Since $e(G) \le e(G') + dn^2$, we have $e_G(A, B) \le e_{G'}(A, B) + dn^2 = dn^2$. Note that $|A| = |U_1|N = (\frac{1}{2} - 6\beta)\ell N > (\frac{1}{2} - 7\beta)n$. Similarly, $|B| > (\frac{1}{2} - 7\beta)n$. By adding at most $7\beta n$ vertices to each of A and B, we obtain two subsets of size $n/2$ and still name them as A and B, respectively. Then, $e(A, B) \le dn^2 + 2 \cdot (7\beta n)(n/2) = 8\beta n^2$, which in turn shows the density $d(A, B) = e(A, B)/(\frac{n}{2})^2 \le 32\beta$. Since $\alpha > 32\beta$, we obtain that G is in the Extremal Case 2 with parameter α. □

Step 1. To show that G_r is Hamiltonian, we need the following theorem of Nash-Williams.

Theorem 4.6 (*Nash-Williams [22]*). *Let G be a 2-connected graph of order n. If minimum degree $\delta(G) \ge \max\{(n+2)/3, \alpha(G)\}$, then G contains a Hamiltonian cycle.*

We first show that G_r is $\beta\ell$-connected. Suppose, to the contrary, let S be a cut of G_r such that $|S| < \beta\ell$ and let U_1 and U_2 be two components of $G_r - S$. Since $\delta(G_r) \ge (\frac{1}{2} - 2\beta)\ell$, we have $|U_i| \ge (\frac{1}{2} - 3\beta)\ell$ for $i = 1, 2$. Since $e(U_1, U_2) = 0$, we obtain a contradiction to Claim 4.5(b). Since $n = N\ell + |V_0| \le (\ell + 2)\varepsilon n$, we have $\ell \ge 1/\varepsilon - 2 \ge 3/\beta$, provided that $\beta \ge 5\varepsilon$. Then $\beta\ell \ge 3$, and G_r is 3-connected.

By Claim 4.5(a), we have $\alpha(G) \le (\frac{1}{2} - 8\beta)\ell < \delta(G_r)$. By Theorem 4.6, G_r is Hamiltonian.

Following the order of a Hamiltonian cycle of G_r, we denote all the clusters of G except for V_0 by $X_1, Y_1, \dots, X_k, Y_k$ (recall that $\ell = 2k$ is even). We call X_i, Y_i *partners* of each other and write $P(X_i) = Y_i$ and $P(Y_i) = X_i$.

Step 2. For each $i = 1, \dots, k$, we initiate an ESP P_i of G connecting X_i and Y_{i-1} (with $Y_0 = Y_k$) as follows.

Given an ε-regular pair (X, Y) of clusters and a subset $Y' \subseteq Y$, we call a vertex $x \in X$ *typical* to Y' if $\deg(x, Y') \ge (d - \varepsilon)|Y'|$. By the regularity of (X, Y), at most εN vertices of X are not typical to Y' whenever $|Y'| > \varepsilon N$. Fix $1 \le i \le k$. First let $a_i \in X_i$ be a vertex typical to both Y_{i-1} and Y_i and let $b_i \in X_i$ be a vertex typical to both $\Gamma(a_i, Y_{i-1})$ and $\Gamma(a_i, Y_i)$. Since both pairs (Y_{i-1}, X_i) and (X_i, Y_i) are ε-regularity with density at least d, all but $2\varepsilon N$ vertices of X can be chosen as a_i. Since $|\Gamma(a_i, Y_{i-1})|, |\Gamma(a_i, Y_i)| \ge (d - \varepsilon)N > \varepsilon N$, all but at most $2\varepsilon N + 1$ vertices of X can be chosen as b_i (note that $b_i \ne a_i$). Recall that $\Gamma(a_i b_i, Y_{i-1}) = \Gamma(a_i, Y_{i-1}) \cap \Gamma(b_i, Y_{i-1})$. The way we select a_i and b_i guarantees that

$$|\Gamma(a_i b_i, Y_{i-1})| \ge (d - \varepsilon)^2 N \ge 2\varepsilon N + 2.$$

Now let $c_{i-1}, d_{i-1} \in \Gamma(a_i b_i, Y_{i-1})$ be two (distinct) vertices of Y_{i-1} such that c_{i-1} is typical to both X_{i-1} and X_i, and d_{i-1} is typical to both $\Gamma(c_{i-1}, X_{i-1})$ and $\Gamma(c_{i-1}, X_i)$. All but at most $2\varepsilon N$ vertices of $\Gamma(a_i b_i, Y_{i-1})$ can be chosen as c_{i-1} and d_{i-1}.

In summary $P_i = c_{i-1} a_i d_{i-1} b_i$ is an ESP with $c_{i-1}, d_{i-1} \in Y_{i-1}$, $a_i, b_i \in X_i$ such that

$$\begin{aligned} &\deg(c_{i-1} d_{i-1}, X_{i-1}) \ge (d - \varepsilon)^2 N, \qquad \deg(a_i, Y_{i-1}) \ge (d - \varepsilon)N, \\ &\deg(a_i b_i, Y_i) \ge (d - \varepsilon)^2 N, \qquad \deg(d_{i-1}, X_i) \ge (d - \varepsilon)N. \end{aligned} \tag{3}$$

Step 3. For each $i \geq 1$, let

$$X_i' := \{x \in X_i, \deg(x, Y_i) \geq (d-\varepsilon)N\} \quad \text{and}$$
$$Y_i' := \{y \in Y_i, \deg(y, X_i) \geq (d-\varepsilon)N\}.$$

Since (X_i, Y_i) is ε-regular, we have $|X_i'|, |Y_i'| \geq (1-\varepsilon)N$. If $|X_i'| \neq |Y_i'|$, say $|X_i'| > |Y_i'|$, then we pick an arbitrary subset of X_i' of size $|Y_i'|$ and still name it X_i'. As a result, we have $|X_i'| = |Y_i'|$. Let $V_0' := V_0 \cup \bigcup_{i=1}^{k}(X_i - X_i') \cup (Y_i - Y_i')$. From $|X_i - X_i'| + |Y_i - Y_i'| \leq 2\varepsilon N$, we derive that $|V_0| \leq 2\varepsilon n + (2\varepsilon N)k = 3\varepsilon n$ by using $2Nk \leq n$ from (2). In addition, the minimum degree $\delta(G[X_i', Y_i']) \geq (d-\varepsilon)N - \varepsilon N$. It is easy to see that (X_i', Y_i') is 2ε-regular [19, Slicing Lemma].

Step 4. Consider a vertex $x \in V(G)$ and an original cluster A (X_i or Y_i for some i), we say that x is *adjacent to* A, denoted by $x \sim A$, if $\deg(x, A) \geq (d-\varepsilon)N$. Given two vertices u, w, we define a u, w-chain of length $2t$ as distinct clusters $A_1, B_1, \ldots, A_t, B_t$ such that $u \sim A_1 \sim B_1 \sim \cdots A_t \sim B_t \sim w$ and each A_j and B_j are partners, in other words, $\{A_j, B_j\} = \{X_{i_j}, Y_{i_j}\}$ for some i_j.

Claim 4.7. *Let L be a list of at most $2\varepsilon n$ pairs of vertices of G. For each $\{u, w\} \in L$, we can find u, w-chains of length at most four such that every cluster is used in at most $d^2N/20$ chains.*

Proof. Suppose that we have found chains of length at most four for the first $m < 2\varepsilon n$ pairs such that no cluster is contained in more than $d^2N/20$ chains. Let Ω be the set of all clusters that are used in exactly $d^2N/20$ chains. Since each chain uses at most four clusters, we have

$$\frac{d^2}{20}N|\Omega| \leq 4m \leq 8\varepsilon n \leq 8\varepsilon\frac{2kN}{1-2\varepsilon},$$

where the last inequality follows from (2). Therefore $|\Omega| \leq \frac{320\varepsilon}{(1-2\varepsilon)d^2}k \leq \frac{320\varepsilon}{d^2}\ell \leq \beta\ell$ provided that $1-2\varepsilon \geq \frac{1}{2}$ and $320\varepsilon \leq d^2\beta$.

Now consider a pair $\{u, w\} \in L$. Our goal is to find a u, w-chain of length at most four by using clusters not in Ω. Let $\mathcal{U}$ be the set of all clusters adjacent to u but not in Ω, and $\mathcal{W}$ be the set of all clusters adjacent to w but not in Ω. Let $P(\mathcal{U})$ and $P(\mathcal{W})$ be the set of the partners of clusters in $\mathcal{U}$ and $\mathcal{W}$, respectively. The definition of chains implies that a cluster $A \in \Omega$ if and only if its partner $P(A)$ is in Ω. Therefore $(P(\mathcal{U}) \cup P(\mathcal{W})) \cap \Omega = \emptyset$.

We claim that $|P(\mathcal{U})| = |\mathcal{U}| \geq (\frac{1}{2} - 3\beta)\ell$. To see it, we first observe that any vertex $v \in V$ is adjacent to at least $(\frac{1}{2} - 2\beta)\ell$ clusters. For instead,

$$\left(\frac{1}{2}-\beta\right)n \leq \deg_G(v) \leq \left(\frac{1}{2}-2\beta\right)\ell N + dN\ell + 3\varepsilon n < \left(\frac{1}{2}-\frac{3}{2}\beta\right)n,$$

a contradiction, provided that $\frac{\beta}{2} \geq d + 3\varepsilon$. Since $|\Omega| \leq \beta\ell$, we thus have $|\mathcal{U}| \geq (\frac{1}{2} - 3\beta)\ell$. Similarly $|P(\mathcal{W})| = |\mathcal{W}| \geq (\frac{1}{2} - 3\beta)\ell$.

If $E_{G_r}(P(\mathcal{U}), P(\mathcal{W})) \neq \emptyset$, then there exist two adjacent clusters $B_1 \in P(\mathcal{U}), A_2 \in P(\mathcal{W})$. If B_1, A_2 are partners of each other, then $u \sim A_2 \sim B_1 \sim w$ gives a u, w-chain of length two. Otherwise assume that $A_1 = P(B_1)$ and $B_2 = P(A_2)$. Then $u \sim A_1 \sim B_1 \sim A_2 \sim B_2 \sim w$ gives a u, w-chain of length four. Note that all $A_i, B_i \notin \Omega$. We may thus assume that

$$E_{G_r}(P(\mathcal{U}), P(\mathcal{W})) = \emptyset. \tag{4}$$

If $P(\mathcal{U}) \cap P(\mathcal{W}) = \emptyset$, then (4) contradicts with Claim 4.5(b) because $|P(\mathcal{U})| \geq (\frac{1}{2} - 3\beta)\ell$ and $|P(\mathcal{W})| \geq (\frac{1}{2} - 3\beta)\ell$. Otherwise assume that $A \in P(\mathcal{U}) \cap P(\mathcal{W})$. Then by (4), A is not adjacent to any cluster in $P(\mathcal{U}) \cup P(\mathcal{W})$. Since $\deg_{G_r}(A) \geq (\frac{1}{2} - 2\beta)\ell$, we derive that $|P(\mathcal{U}) \cup P(\mathcal{W})| \leq (\frac{1}{2} + 2\beta)\ell$. Since $|P(\mathcal{U})| \geq (\frac{1}{2} - 3\beta)\ell$ and $|P(\mathcal{W})| \geq (\frac{1}{2} - 3\beta)\ell$, then $|P(\mathcal{U}) \cap P(\mathcal{W})| \geq (\frac{1}{2} - 8\beta)\ell$. By (4), $P(\mathcal{U}) \cap P(\mathcal{W})$ is an independent set in G_r, which contradicts with Claim 4.5(a). □

We arbitrarily partition V_0 into at most $2\varepsilon n$ pairs (note that $|V_0|$ is even because $|X_i'| = |Y_i'|$ for all i). Applying Claim 4.7, we construct chains of length at most four for each pair such that every cluster is used in at most $d^2N/20$ chains. For each i let m_i denote the number of chains containing X_i and Y_i.

Claim 4.8. *We can extend connecting ESP's to include all the vertices in V_0 such that the following holds for all i. The resulting ESP's $P_i = u_1v_1 \cdots u_tv_t$ satisfies $u_1, u_2 \in Y_{i-1}$, $v_{t-1}, v_t \in X_i$ and*

$$\begin{aligned} &\deg(u_1u_2, X_{i-1}) \geq (d-\varepsilon)^2N, \qquad \deg(v_1, Y_{i-1}) \geq (d-\varepsilon)N, \\ &\deg(v_tv_{t-1}, Y_i) \geq (d-\varepsilon)^2N, \qquad \deg(u_t, X_i) \geq (d-\varepsilon)N. \end{aligned} \tag{5}$$

The sets $X_i^* = X_i' - \cup_j V(P_j)$ and $Y_i^* = Y_i' - \cup_j V(P_j)$ satisfy

$$|X_i^*| = |Y_i^*| \geq (1-\varepsilon)N - 2 - 7m_i. \tag{6}$$

Proof. We prove by induction on $m := |V_0|/2$. When $m = 0$, by (3), the initial $P_i = c_{i-1}a_id_{i-1}b_i$ satisfies (5). The initial $X_i^* = X_i' - \{a_i, b_i\}$ and $Y_i^* := Y_i' - \{c_i, d_i\}$) satisfy (6).

Suppose that $m \geq 1$ and we have extended the connecting ESP's to include $m-1$ pairs from V_0 such that (5) and (6) hold for all i. Let $\{x, y\}$ be the last pair from V_0. We first consider the case when the x, y-chain has length two.

Without loss of generality, assume that $x \sim Y_i \sim X_i \sim y$ for some i. Let $P_i = u_1v_1 \cdots u_tv_t$ be the current connecting ESP between Y_{i-1} and X_i and $Y_i^* := Y_i' - \cup_j V(P_j)$ and $X_i^* := X_i' - \cup_j V(P_j)$. To include x, we extend P_i to $P_i' = P_iy_1x_1y_2x_2y_3x_3y_4x$ with four vertices $y_1, y_2, y_3, y_4 \in Y_i^*$ and three vertices $x_1, x_2, x_3 \in X_i^*$ such that in addition

$$y_1 \in \Gamma(v_{t-1}v_t), \qquad y_2 \in \Gamma(v_t), \qquad y_3, y_4 \in \Gamma(x), \qquad x_1 \in \Gamma(u_t),$$
$$y_4 \text{ is typical to } X_i, \quad \text{and} \quad x_3 \text{ is typical to } \Gamma(x, Y_i). \tag{7}$$

This is possible by using Lemma 3.2 (actually we only need the regularity between X_i and Y_i; but applying the Blow-up Lemma makes our proof shorter). To see it, first note that (6) implies that $|X_i^*|, |Y_i^*| > (1 - d^2/2)N$ because $m_i \leq d^2N/20$ by Claim 4.7. Then, by (5),

$$|\Gamma(v_{t-1}v_t, Y_i^*)| \geq \deg(v_{t-1}v_t, Y_i) - \frac{d^2}{2}N \geq (d-\varepsilon)^2N - \frac{d^2}{2}N > \frac{d^2}{4}N.$$

Similarly we can show that $|\Gamma(v_t, Y_i^*)|, |\Gamma(u_t, X_i^*)| \geq (d - d^2)N$. The definition of $x \sim Y_i$ also guarantees that $|\Gamma(x, Y_i^*)| \geq (d - d^2)N$. Finally (7) only forbids additional εN vertices when choosing y_4 and x_3. Therefore we can apply Lemma 3.2 to find such an P_i'. By (7), we have $\deg(y_4, X_i) \geq (d-\varepsilon)N$ and $\deg(x_3x, Y_i) \geq (d-\varepsilon)^2N$. Consequently P_i' satisfies (5).

Since x behaves like a vertex of X_i in P_i', we call such a procedure *inserting x into* X_i (by extending P_i). Since $y \sim X_i$, we can similarly insert y to Y_i by extending P_{i+1} to $P_{i+1}' = yx_4y_5x_5y_6x_6y_7x_7P_{i+1}$ with $x_j \in X_i^*$ and $y_j \in Y_i^*$. Since each X_i^* and Y_i^* lose seven vertices totally, (6) holds.

Now consider the case when the x, y-chain has length four. Assume that $x \sim A_1 \sim B_1 \sim A_2 \sim B_2 \sim y$. We first insert x to B_1, then pick any (available) vertex in B_1 that is typical to A_2 and insert it to B_2, and finally insert y to A_2. As a result, A_1, B_1 each loses four vertices to some connecting paths while A_2, B_2 each loses seven vertices to some connecting paths. Thus (6) holds. □

Step 5. Fix $1 \leq i \leq k$. Suppose that at present $P_i = u_1v_1 \cdots u_tv_t$ and $P_{i+1} = w_1z_1 \cdots w_sz_s$ for some integers $s, t \geq 2$, and $X_i^* = X_i' - \cup_j V(P_j)$ and $Y_i^* = Y_i' - \cup_j V(P_j)$. By Claim 4.8, P_i and P_{i+1} satisfy (5) and $|X_i^*|, |Y_i^*| \geq (1 - d^2/2)N$. Since (X_i, Y_i) is ε-regular, the Slicing Lemma of [19] says that (X_i^*, Y_i^*) is $(2\varepsilon, d/2)$-super-regular (note that $(d-\varepsilon)N - d^2N/2 > dN/2$).

We now apply the Blow-up Lemma to each $G'[X_i^*, Y_i^*]$ to obtain an spanning ESP $y_1x_1y_2 \cdots x_{N_i-1}, y_{N_i}, x_{N_i}$ (see Fig. 2), where $N_i = |X_i^*| = |Y_i^*|$ such that

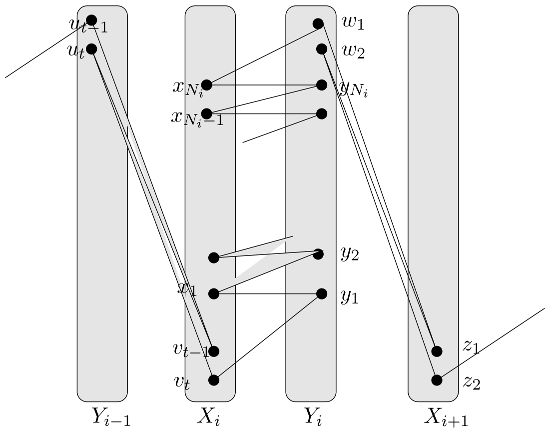

Fig. 2. An ESP covering X_i^* and Y_i^*.

$$y_1 \in \Gamma(v_tv_{t-1}, Y_i^*), \qquad x_1 \in \Gamma(u_t, X_i^*), \qquad y_2 \in \Gamma(v_t, Y_i^*),$$
$$x_{N_i-1} \in \Gamma(w_1, X_i^*), \qquad y_{N_i} \in \Gamma(z_1, Y_i^*), \qquad x_{N_i} \in \Gamma(w_1w_2, X_i^*).$$

The restrictive mapping of $y_1, x_1, y_2, x_{N_i-1}, y_{N_i}, x_{N_i}$ is possible because by (5), all the targeting sets are of size at least $(d-\varepsilon)^2N - d^2N/2 > d^2N/4$.

We now complete the proof of the non-extremal case. □

4.2. Extremal Case 1

In this subsection we prove Theorem 4.2.

We start with a lemma which gives a balanced spanning bipartite subgraph that we will use throughout the section.

Lemma 4.9. *Suppose that $0 \le \alpha \le (1/9)^3$. Let $G = (V, E)$ be a graph on n vertices with $\delta(G) \ge \frac{n}{2} + 3$ and a balanced partition $V_1 \cup V_2$ such that $d(V_1, V_2) \ge 1 - \alpha$. Then G contains a balanced spanning bipartite subgraph G' with parts U_1, U_2 such that*

- *There is a set W of at most $\alpha^{2/3}n$ vertices such that we can find vertex disjoint 4-stars (stars with four edges) in G' with the vertices of W as their centers.*
- $\deg_{G'}(x) \ge (1 - \alpha^{1/3} - 2\alpha^{2/3})n/2$ *for all $x \notin W$.*

Proof. For simplicity, let $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. For each $i = 1, 2$, we define

$$V_i' = \left\{x \in V : \deg(x, V_{3-i}) \ge (1 - \alpha_1)\frac{n}{2}\right\}.$$

Since $d(V_1, V_2) \ge 1 - \alpha$, we have $|V_i - V_i'| \le \alpha_2 n/2$ and consequently $|V_i'| \ge (1 - \alpha_2)n/2$ for $i = 1, 2$. For any $x \in V_i'$,

$$\deg(x, V_{3-i}') > (1 - \alpha_1)\frac{n}{2} - \alpha_2\frac{n}{2}. \tag{8}$$

Let $V_0 = V - V_1' - V_2'$. Then $|V_0| \le \alpha_2 n$. For each $v \in V_0$ and $i = 1, 2$, we have $\deg(v, V_i) \le (1 - \alpha_1)\frac{n}{2}$, which implies that $\deg(v, V_i) \ge \alpha_1\frac{n}{2}$ and

$$\deg(v, V_i') \ge (\alpha_1 - \alpha_2)\,\frac{n}{2}. \tag{9}$$

We now separate cases based on $|V_1'|$ and $|V_2'|$.

Case 1: $|V_1'|, |V_2'| \le n/2$. In this case we partition V_0 into $W_1 \cup W_2$ such that $|W_i| = n/2 - |V_i'|$ for $i = 1, 2$. For each vertex $w \in W_i$, we greedily find four neighbors from V_{3-i}' such that the neighbors for all the vertices of W_i are distinct. This is possible for $i = 1, 2$ because of (9) and

$$(\alpha_1 - \alpha_2)\frac{n}{2} \ge 4\alpha_2 n \ge 4|V_0|$$

provided that $\alpha_1 \ge 9\alpha_2$ or $\alpha_1 \le 1/9$. Define $U_i = V_i' \cup W_i$ for $i = 1, 2$. Then $|U_1| = |U_2| = n/2$. With $W = V_0$, the second assertion of Lemma 4.9 follows from (8).

Case 2: *one of* $|V_1'|, |V_2'|$, *say*, $|V_1'|$ is greater than $n/2$. Let V_1^0 be the set of vertices $v \in V_1'$ such that $\deg(v, V_1') \ge \alpha_1 n/2$. First assume that $|V_1^0| \ge |V_1'| - n/2$. Then we form a set W with $|V_1'| - n/2$ vertices of V_1^0 and all the vertices of V_0. Let $U_1 = V_1' - W$ and $U_2 = V_2' \cup W$. Then $|U_1| = |U_2| = n/2$. Since $(1 - \alpha_2)n/2 \le |V_2'| \le n/2$, we have $|W| \le \alpha_2 n/2$. Then $|V_1' - U_1| \le \alpha_2 n/2$, by (9) and the definition of V_1^0, we have

$$\deg(v, U_1) \ge \deg(v, V_1') - \alpha_2 n/2 \ge (\alpha_1 - \alpha_2)n/2 - \alpha_2 n/2 \tag{10}$$

for all $v \in W$. With $\alpha_1 \ge 6\alpha_2$, we have $(\alpha_1 - 2\alpha_2)n/2 \ge 4\alpha_2 n/2 \ge 4|W|$. Therefore we can greedily find four neighbors for each vertex $v \in W$ such that the neighbors for all the vertices of W are distinct. The second assertion of Lemma 4.9 follows from (8) and $|V_1' - U_1| \le \alpha_2 n/2$.

Otherwise assume that $|V_1^0| < |V_1'| - n/2$. In this case let $U_1 = V_1' - V_1^0$ and $U_2 = V_2' \cup V_1^0 \cup V_0$. Then $|U_1| = \frac{n}{2} + t_1$ for some positive integer $t_1 \le \alpha_2 n/2$. Since $\deg(v, V_1') < \alpha_1 n/2$ for every $v \in U_1$, the induced graph $G[U_1]$ has the maximum degree $\Delta \le \alpha_1 n/2$. By the minimum degree condition $\delta(G) \ge n/2 + 3$, $G[U_1]$ has the minimum degree at least

$$\delta(G) - |U_2| \ge \left(\frac{n}{2} + 3\right) - \left(\frac{n}{2} - t_1\right) \ge t_1 + 3.$$

We now need the following simple fact.

Fact 4.10. *In a graph G_1 of order n_1 with the maximum degree $\Delta(G_1) \le \Delta$ and the minimum degree $\delta(G_1) \ge t$, the number of disjoint 4-stars is at least $\frac{(t-3)n_1}{5(\Delta+t-3)}$.*

To see it, suppose G_1 has a largest family of disjoint 4-stars on some vertex set M of size m. Then $(t - 3)(n_1 - 5m) \le e(M, V(G) - M) \le 5m\Delta$ and the fact follows.

Applying Fact 4.10, there are at least

$$\frac{(t_1 + 3 - 3)|U_1|}{5(\Delta + t_1 + 3 - 3)} \ge \frac{t_1|U_1|}{5\left(\alpha_1\frac{n}{2} + t_1\right)} \ge \frac{t_1\frac{n}{2}}{5(\alpha_1 + \alpha_2)\frac{n}{2}} \ge t_1$$

vertex disjoint 4-starts in $G[U_1]$. Pick t_1 such 4-stars and move their centers to U_2. As a result, $|U_1| = |U_2| = n/2$. Let $W_0 = V_1^0 \cup V_0$ and W be the union of W_0 with the new vertices of U_2.

Since $|V_1' - U_1| = |V_1'| - n/2 \le \alpha_2 n/2$, we have (10) for all $v \in W_0$. Since $|W| = n/2 - |V_2'| \le \alpha_2 n/2$, we can find disjoint 4-stars in $G[U_1, W_0]$ with all the vertices of W_0 as centers such that these 4-stars are also disjoint from the existing 4-stars. In addition, the second assertion of Lemma 4.9 holds as before. □

Proposition 4.11 shows that if G is a graph or a bipartite graph with a large minimum degree and it contains not many vertex disjoint 4-stars, then we can find an ESC or ESP containing all the vertices in these stars. We need its part (2) for this subsection, and part (1) for Extremal Case 2.

Proposition 4.11. *Fix* $0 < \varepsilon_1 \le 1/5$.

(a) *Let G be a graph of order N with a subset W of size $t \le \varepsilon_1 N/8$. Suppose that G contains t vertex-disjoint 4-stars with the vertices of W as centers, and $\deg(x) \ge (1-\varepsilon_1)N$ for every vertex $x \notin W$. Then G contains an ESC C of length $8t$ which contains all the vertices of W such that any two nearest vertices of W are separated by exactly seven vertices not in W.*

(b) *Let G be a bipartite graph on two parts U_1, U_2 of size N. Let W be a vertex subset of size $t \le \varepsilon_1 N/5$. Suppose that G contains t vertex-disjoint 4-stars with the vertices of W as centers, and $\deg(x) \ge (1-\varepsilon_1)N$ for every vertex $x \notin W$. Then G contains an ESP of length $8t+4$ which contains all the vertices of W and whose first and last three vertices are not from W.*

Proof. (a) Suppose that $W = \{w_1, \ldots, w_t\}$, and denote the four leaves under w_i by a_i, b_i, c_i, d_i. For each i, we greedily choose three new vertices u_i, v_i, x_i that are not contained in any existing star such that

$$u_i \in \Gamma(c_{i-1}d_{i-1}a_ib_i), \qquad v_i \in \Gamma(d_{i-1}a_ib_ic_i), \qquad x_i \in \Gamma(b_ic_id_ia_{i+1}),$$

in which the indices are modulo t. This is possible because each a_i, b_i, c_i, d_i, $1 \le i \le t$, has at least $(1-\varepsilon_1)N$ neighbors and any four of them have at least $(1-4\varepsilon_1)N \ge \varepsilon_1 N \ge 8t$ common neighbors (where $8t$ is the total number of vertices used at the end of this greedy algorithm). We thus obtain an ESC $u_ia_iv_ib_iw_ic_ix_id_i : i = 1, \ldots, t$, in which the vertices of W are distributed evenly.

(b) Partition $W = W_1 \cup W_2$ with $W_1 = U_1 \cap W$ and $W_2 = U_2 \cap W$. For each W_i, we follow the procedure in (1) to find two vertex disjoint ESC's C_1 and C_2 of length $8|W_1|$ and $8|W_2|$ in which the vertices of W are distributed evenly. The calculation is similar except that any four vertices in $U_1 - W$ (or $U_2 - W$) have at least $(1-4\varepsilon_1)N \ge \varepsilon_1 N > 4t$ common neighbors in U_2 (or U_1), where $4t$ is the total number of the vertices used in one partition set.

We next break C_1 into $P_1 = x_1x_2x_3\cdots u_3u_2u_1$ and break C_2 into $P_1 = v_1v_2v_3\cdots y_3y_2y_1$ such that $x_i, u_i, v_i, y_i \notin W$ for $i = 1, 2, 3$ and $u_1, u_3, v_2 \in U_1$, $u_2, v_1, v_3 \in U_2$. Assume that $t \ge 2$ otherwise we are done. Choose four new vertices not in W (in this order) $z_1 \in \Gamma(u_1u_3)$, $z_3 \in \Gamma(u_1v_2)$ and $z_2 \in \Gamma(u_2z_1z_3v_1)$, $z_4 \in \Gamma(z_1z_3v_1v_3)$. This is possible because the number of common neighbors of any four vertices not in W is at least $(1-4\varepsilon_1)N \ge 5t \ge 4t+2$, where $4t+2$ is the total number of the vertices used in one partition set. As a result, $P_1z_1z_2z_3z_4P_2$ is an ESP which contains all the vertices of W and whose first and last three vertices are not from W. □

We finally observe that a bipartite graph with very large minimum degree is super-regular.

Proposition 4.12. *Given $0 < \rho < 1$, let G be a bipartite graph on $X \cup Y$ such that*

$$\delta(X, Y) \ge (1-\rho)|Y|, \qquad \delta(Y, X) \ge (1-\rho)|X|. \tag{11}$$

Then G is $(\sqrt{\rho}, 1-\rho)$-super-regular.

Proof. It suffices to show that G is $\sqrt{\rho}$-regular. Consider subsets $A \subseteq X$, $B \subseteq Y$ with $|A| = \varepsilon_1|X|$ and $|B| = \varepsilon_2|Y|$ for some $\varepsilon_1, \varepsilon_2 > \sqrt{\rho}$. By (11), we have $\delta(A, Y) \ge |Y| - \rho|Y|$ and consequently $\delta(A, B) \ge |B| - \rho|Y| = (\varepsilon_2 - \rho)|Y|$. The density between A and B satisfies

$$d(A, B) \ge \frac{\delta(A, B)|A|}{|A||B|} \ge \frac{(\varepsilon_2 - \rho)|Y|}{|B|} = \frac{\varepsilon_2 - \rho}{\varepsilon_2} > 1 - \frac{\rho}{\sqrt{\rho}} = 1 - \sqrt{\rho}.$$

Since $1 - \sqrt{\rho} < d(A, B) \le 1$ and in particular, $1 - \sqrt{\rho} < d(X, Y) \le 1$, we have $|d(A, B) - d(X, Y)| < \sqrt{\rho}$. □

We are ready to prove Theorem 4.2 now.

Proof of Theorem 4.2. Let $0 \le \alpha \ll 1$, in particular $\alpha \le (1/9)^3$. Write $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. Let $G = (V, E)$ be a graph on n vertices with $\delta(G) \ge \frac{n}{2} + 3$. Suppose G is in Extremal Case 1 with parameter α. We first apply Lemma 4.9 to G and obtain a bipartite subgraph G' with two partition sets U_1, U_2 of size $n/2$ which contains at most $\alpha^{2/3}n$ vertex disjoint 4-stars. Denote by W the set of the centers of the 4-stars. We also have

$$\deg_{G'}(x) \ge (1 - \alpha_1 - 2\alpha_2)n/2 \quad \text{for all } x \notin W. \tag{12}$$

Since $\alpha_1 + 2\alpha_2 \le 1/5$ and $\alpha_2 n \le \frac{\alpha_1+\alpha_2}{5}\frac{n}{2}$, we may apply Proposition 4.11(b) to G' with $\varepsilon_1 = \alpha_1 + 2\alpha_2$ and $N = n/2$. We thus obtain an ESP $P_0 = x_1x_2x_3\cdots y_3y_2y_1$ of length $8|W|+4$ which contains all the vertices of W such that $x_i, y_i \in V_1' \cup V_2'$ for $1 \le i \le 3$. In order to find an ESHC of G, it suffices to find an ESP $P = u_1u_2u_3\cdots v_3v_2v_1$ on $V(G) - V(P_0)$ such that

$$x_3x_2x_1Py_1y_2y_3 = x_3x_2x_1u_1u_2u_3\cdots v_3v_2v_1y_1y_2y_3$$

is also an ESP.

Let $U_i' = U_i - V(P_0)$ for $i = 1, 2$, and $n' = |U_1'| = |U_2'|$. Then $n' = n/2 - (4|V_0| + 2) \geq n/2 - (4\alpha_2 n + 2)$. By (12), the bipartite subgraph $G[U_1', U_2']$ has its minimum degree at least

$$(1 - \alpha_1 - 2\alpha_2)\frac{n}{2} - (4\alpha_2 n + 2) \geq (1 - 3\alpha_1)\frac{n}{2} \geq (1 - 3\alpha_1)n'$$

by using $\alpha_1 \geq 9\alpha_2$. Similarly the degree from x_i or y_i, $i = 1, 2, 3$, to U_1' or U_2' is at least $(1 - 3\alpha_1)n'$. By Proposition 4.12, (U_1', U_2') is $(\sqrt{3\alpha_1}, \frac{2}{3})$-super-regular (using $\alpha_1 \leq 1/9$ again). Since $\sqrt{3\alpha_1} \ll 1$, we can apply the Blow-up Lemma to obtain an ESHP $u_1u_2u_3 \cdots v_3v_2v_1$ of $G[U_1', U_2']$ such that

$$u_1 \in \Gamma(x_1, x_3), \quad u_2 \in \Gamma(x_2), \quad u_3 \in \Gamma(x_1),$$
$$v_1 \in \Gamma(y_1, y_3), \quad v_2 \in \Gamma(y_2), \quad v_3 \in \Gamma(y_1).$$

Since $|\Gamma(x_i)|, |\Gamma(y_i)| \geq (1 - 3\alpha_1)n'$ for $i = 1, 2, 3$ and $|\Gamma(x_1, x_3)|, |\Gamma(y_1, y_3)| \geq (1 - 6\alpha_1)n'$, the restrictive mapping of u_1, u_2, u_3 and v_1, v_2, v_3 is possible. □

4.3. Extremal Case 2

In this subsection we prove Theorem 4.3.

In Extremal Case 1, we used the Blow-up Lemma to find an ESHP with certain properties in a bipartite graph with very large minimum degree. In this subsection we first prove such a lemma for arbitrary graphs.

Lemma 4.13. *Let $k \geq 3$ and n_1 be sufficiently large. Let G be a graph of order $k + n_1 + 6$. Suppose that G contains an ESP $P_0 = u_1 \cdots u_k$. Let $X = V(G) - V(P_0)$. Suppose that $x_1x_2x_3$ and $y_1y_2y_3$ are two paths in X and let X' be the set of the remaining vertices of X (then $|X'| = n_1$). If $\deg(x, X') \geq \frac{7}{8}n_1 + 1$ for all vertices $x \in X$ and $\deg(u_j, X) \geq \frac{7}{8}|X| + 1$ for $j = 1, 2, 3, k-2, k-1, k$, then G contains an ESHP that starts with $x_3x_2x_1$, finishes with $y_1y_2y_3$ and contains P_0 as an internal path.*

Proof. Our proof consists of three steps.

Step 1: we find an ESC on X'. Let $G_1 = G[X']$. Since $\delta(G_1) \geq \frac{7}{8}n_1$ and n_1 is sufficiently large, by Remark 4.4, G_1 contains an ESHC.

Step 2: we find an ESP on X such that it starts with $x_3x_2x_1$ and finishes with $y_1y_2y_3$. Let $v_1, \ldots, v_{n_1}$ be the ESC given by Step 1. We will form an ESP

$$x_3x_2x_1v_i \cdots v_1v_{n_1}v_{n_1-1} \cdots v_{i+1}y_1y_2y_3$$

for some $1 \leq i \leq n_1$. It suffices to have the following adjacencies.

$$\begin{array}{llll} x_3 \sim v_i, & x_2 \sim v_{i-1}, & x_1 \sim v_i, & x_1 \sim v_{i-2}, \\ y_1 \sim v_{i+1}, & y_1 \sim v_{i+3}, & y_2 \sim v_{i+2}, & y_3 \sim v_{i+1}, \end{array} \tag{13}$$

in which we assume that $v_j = v_{j+n_1}$ for all integers j. Since $\deg(x, X') \geq 7n_1/8 + 1$ for any vertex $x \in \{x_1, x_2, x_3, y_1, y_2, y_3\}$, the number of $1 \leq i \leq n_1$ satisfying (13) is at least $n_1 - 8(n_1/8 - 1) = 8$. Thus (13) holds for some $1 \leq i \leq n_1$.

Step 3: we find an ESHP of G that starts with $x_3x_2x_1$, finishes with $y_1y_2y_3$ and contains P_0 as an internal path. Let $n_2 = |X| = n_1 + 6$. Denote the ESP found in Step 2 by $v_1, \ldots, v_{n_2}$, where

$$v_1 = x_3, \quad v_2 = x_2, \quad v_3 = x_1, \quad v_{n_2-2} = y_1, \quad v_{n_2-1} = y_2, \quad v_{n_2} = y_3.$$

Our goal is to find an index $3 \leq i \leq n_2 - 3$ such that $v_1 \cdots v_iP_0v_{i+1} \cdots v_{n_2}$ is an ESP. Since $P_0 = u_1 \cdots u_k$, it suffices to have the following adjacencies.

$$\begin{array}{llll} u_1 \sim v_i, & u_2 \sim v_{i-1}, & u_3 \sim v_i, & u_1 \sim v_{i-2}, \\ u_k \sim v_{i+1}, & u_k \sim v_{i+3}, & u_{k-1} \sim v_{i+2}, & u_{k-2} \sim v_{i+1}, \end{array} \tag{14}$$

for some $3 \leq i \leq n_2 - 3$. Since $\deg(u_j, X) \geq 7n_2/8 + 1$ for $j = 1, 2, 3, k-2, k-1, k$, the number of $1 \leq i \leq n_2$ satisfying (13) is at least $n_2 - 8(n_2/8 - 1) = 8$. Thus (13) holds for some $3 \leq i \leq n_2 - 3$. □

Proof of Theorem 4.3. We start with defining two new sets, which are variants of V_1 and V_2. Let $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. We define

$$V_i' = \left\{x \in V : \deg(x, V_{3-i}) < \alpha_1 \frac{n}{2}\right\}$$

for $i = 1, 2$. Since $\delta(G) > n/2$, we have $\deg(x, V_i) > (1 - \alpha_1)n/2$ for every $x \in V_i'$. Since $d(V_1, V_2) \leq \alpha$, we have $|V_i - V_i'| \leq \alpha_2 n/2$ and $|V_i'| \geq (1 - \alpha_2)n/2$ for $i = 1, 2$. Consequently,

$$\deg(x, V_i') > \deg(x, V_i) - \alpha_1 \frac{n}{2} \geq (1 - \alpha_1 - \alpha_2)\frac{n}{2} \quad \text{for all } x \in V_i'. \tag{15}$$

Let $V_0 = V - V_1' - V_2'$. Then $|V_0| \leq \alpha_2 n$ and $\deg(x, V_i') \geq (\alpha_1 - \alpha_2)n/2$ for all $x \in V_0$.

Our proof consists of the following two steps which together provide an ESHC of G.

Step 1. Find two disjoint ESP's $x_1 \cdots x_p$ and $y_1 \cdots y_p$ of length $6 \le p \le 14$ such that $x_1, x_2, x_3, y_1, y_2, y_3 \in V_1'$ and $x_{p-2}, x_{p-1}, x_p, y_{p-2}, y_{p-1}, y_p \in V_2'$.

Step 2. Find two ESP's P_1 and P_2 consisting of all the remaining vertices in V_1' and $V_2' \cup V_0$, respectively, such that $x_3x_2x_1P_1y_1y_2y_3$, and $x_{p-2}x_{p-1}x_pP_2y_py_{p-1}y_{p-2}$ are also ESP's.

While Step 2 follows from Proposition 4.11 and Lemma 4.13 easily, Step 1 is much harder (at least from our point of view)—it is where we need the large constant 92 in the min-degree condition. Below we present Step 2 first.

4.3.1. *Step 2: Find two ESP's covering the remaining vertices*

Let $P^1 = x_1 \cdots x_p$ and $P^2 = y_1 \cdots y_p$ be the two ESP's of length $6 \le p \le 14$ provided by Step 1. Let $S = V(P^1) \cup V(P^2)$.

Let $U_1' = V_1' - S$ and $U_1 = U_1' \cup \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Fix $x \in U_1$, by (15), we have

$$\deg(x, U_1') \ge (1 - \alpha_1 - \alpha_2)n/2 - 2p \ge (1 - \alpha_1 - 2\alpha_2)n/2 + 1, \tag{16}$$

where the second inequality follows from $\alpha_2 n/2 \ge 29 \ge 2p + 1$. Using $|U_1'| \le (1 + \alpha_2)n/2$ and $\alpha_1 \ge 3\alpha_2$, we derive that $\deg(x, U_1') \ge (1 - 2\alpha_1)|U_1'| + 1$. Using $\alpha_1 \le \frac{1}{16}$, we have $\deg(x, U_1') \ge \frac{7}{8}|U_1'| + 1$. We then apply Lemma 4.13 to $G[U_1]$ with $P_0 = \emptyset$ and obtain an ESP $x_3x_2x_1 \cdots y_1y_2y_3$ on U_1.

Let $U_2' = V_2' \cup V_0 - S$ and $U_2 = U_2' \cup \{x_{p-2}, x_{p-1}, x_p, y_{p-2}, y_{p-1}, y_p\}$. Partition U_2' into $V_2'' = V_2' - S$ and $V_0' = V_0 - S$. We have $|V_0'| \le \alpha_2 n$ and for each $x \in V_0'$,

$$\deg(x, V_2'') \ge (\alpha_1 - \alpha_2)\frac{n}{2} - 2p \ge \frac{\alpha_1}{2}\frac{n}{2} \ge 4|V_0'|,$$

by using $\alpha_2 n/2 \ge 2p$ and $\alpha_1 \ge 16\alpha_2$. We then greedily find $|V_0'|$ disjoint 4-stars with the vertices of V_0' as centers and vertices in V_2'' as leaves. Let $N = (1 + \alpha_2)n/2$. Then $|U_2'| \le |V_2' \cup V_0| = n - |V_1'| \le N$. For any vertex $x \in V_2'$, the arguments above for $G[U_1]$ give that

$$\deg(x, V_2'') \ge (1 - 2\alpha_1)N + 1. \tag{17}$$

We then apply Proposition 4.11(a) to $G[U_2']$ and obtain an ESC C_0 of length $k = 8|V_0'| \le 8\alpha_2 n$ such that it contains all the vertices of V_0' in a way that every two nearest vertices of V_0' are separated by exactly seven vertices of V_2''. We then break C_0 into an ESP $P_0 = u_1 \cdots u_k$ such that $u_1, u_2, u_3, u_{k-2}, u_{k-1}, u_k \in V_2''$.

Let $X' = U_2' - V(P_0)$ and $X = X' \cup \{x_{p-2}, x_{p-1}, x_p, y_{p-2}, y_{p-1}, y_p\}$. For any vertex $x \in V_2'$, by (17) and using $8\alpha_2 n \le \alpha_1 \frac{n}{2} \le \alpha_1 N$,

$$\deg(x, X') \ge (1 - 2\alpha_1)N + 1 - 8\alpha_2 n \ge (1 - 3\alpha_1)N + 1.$$

Since $X \subset V_2'$, $u_1, u_2, u_3, u_{k-2}, u_{k-1}, u_k \in V_2'$ and $|X'| < |X| \le N$, by letting $\alpha_1 \le \frac{1}{24}$, the degree conditions in Lemma 4.13 hold. We then apply Lemma 4.13 to $G[U_2]$ and obtain the desired ESHP

$$x_{p-2}x_{p-1}x_p \cdots P_0 \cdots y_py_{p-1}y_{p-2}.$$

4.3.2. *Step 1: Connect V_1' and V_2'*

Given two disjoint sets A and B, an ESP on $A \cup B$ is called an (A, B)-connector if its first three vertices are from A, and the last three vertices are from B. Our goal is to find two disjoint (V_1', V_2')-connectors of length at most 14. The simplest connector is an ESP $x_1 \cdots x_6$ with $x_1, x_2, x_3 \in V_1'$ and $x_4, x_5, x_6 \in V_2'$. Unfortunately such a simple connector may not exist if $e(V_1', V_2')$ is very small but $e(V_i', V_0)$ is relatively large.

Let us sketch our proof. We first separate the vertices of V_0 with large degree to both V_1' and V_2':

$$V_0' = \{x \in V_0 : \deg(x, V_1'), \deg(x, V_2') \ge n/6 - 2\alpha_2 n\}.$$

The reason why we choose $n/6$ can be seen from (20), in which we use $n/2 = 3(n/6)$. If $|V_0'| > 165$, then we can find two disjoint copies of $T_{2,3,2}$ from the 3-partite subgraph $G[V_1', V_0', V_2']$, where $T_{2,3,2}$ is the union of two copies $K_{2,3}$ sharing the three vertices in one partition set. Each copy of $T_{2,3,2}$ can be easily extended to an (V_1', V_2')-connector. If $|V_0'| \le 165$, then V_0' will not be used any more. We add the vertices of $V_0 - V_0'$ into V_1' or V_2' forming two new (disjoint) sets U_1, U_2 such that any three vertices in U_i, $i = 1, 2$, have many common neighbors. What remains is to find two disjoint (U_1, U_2)-connectors by using the minimum degree condition $\delta(G) \ge (n + |V_0'|)/2 + 9$. One way to construct such a connector is to find two adjacent vertices $x \in U_1, y \in U_2$ such that there is 4-vertex path between $\Gamma(x, U_1)$ and $\Gamma(y, U_2)$. If this cannot be done, then we find six vertices $x_1, x_2, x_3 \in U_1$ and $x_4, x_5, x_6 \in U_2$ such that x_1x_4 and x_3x_6 are edges and x_2, x_3, x_4, x_5 form a copy of $K_{2,2}$ in $G[U_1, U_2]$. After finding one connector, we remove all or some of its vertices and repeat the procedure above. Note that ignoring V_0' of size at most 165 is the major reason for the large constant 92 in $\delta(G)$; for example, when $V_0' = \emptyset$, then $\delta(G) \ge n/2 + 9$ suffices.

We need the following propositions on the existence of $T_{2,3,2}$ and $K_{2,2}$, whose proofs are standard counting arguments. Given two disjoint vertex sets A, B in a graph H, we write $K_{s,t} \subseteq (A, B)$ if $H[A, B]$ contains a copy of $K_{s,t}$ with s vertices from A and t vertices from B. Similarly $T_{2,3,2} \subseteq (A, B, C)$ means that there are subsets $X \subseteq A$, $Y \subseteq B$ and $Z \subseteq C$ such that $H[X, Y] \cong H[Z, Y] \cong K_{2,3}$.

We denote by $\delta(A, B)$ the minimum degree $\deg(a, B)$ over all $a \in A$.

Proposition 4.14. (a) *Given any integer $t > 0$, there exist $\varepsilon_0 > 0$ and n_0 such that the following holds for any $\varepsilon \le \varepsilon_0$ and $n \ge n_0$. Let A, B be two disjoint vertex sets in a graph such that $\delta(B, A) \ge n/6 - 3\varepsilon n$, $|A| \le (1+\varepsilon)n/2$ and $|B| > 9(t-1)$. Then $K_{2,t} \subseteq (A, B)$.*

(b) *There exist $\varepsilon_0 > 0$ and n_0 such that the following holds for any $\varepsilon \le \varepsilon_0$ and $n \ge n_0$. Let A, B, C be three disjoint vertex set in a graph such that $\delta(B, A), \delta(B, C) \ge n/6 - 3\varepsilon n$, $|A|, |C| \le (1+\varepsilon)n/2$ and $|B| > 162$. Then $T_{2,3,2} \subseteq (A, B, C)$.*

Proof. (a) If the graph contains no $K_{2,t}$ with 2 vertices in A and t vertices in B, then

$$\sum_{x\in B} \binom{\deg(x, A)}{2} \le (t-1)\binom{|A|}{2}. \tag{18}$$

Since $\delta(B, A) \ge n/6 - 3\varepsilon n$ and $|A| \le (1+\varepsilon)n/2$,

$$|B|\binom{n/6 - 3\varepsilon n}{2} \le (t-1)\binom{(1+\varepsilon)n/2}{2}.$$

As $n \to \infty$ and $\varepsilon \to 0$, we obtain $|B| \le 9(t-1)$, contradiction.

(b) Since $|B| > 9(19-1)$, we can apply (1) with $t = 19$ to (A, B) and obtain a copy of $K_{2,19}$ on $X \subset A$ of size 2 and $Y' \subset B$ of size 19. Then since $|Y'| = 19 > 9(3-1)$, we can apply (1) again with $t = 3$ to (C, Y') and obtain a copy of $K_{2,3}$ on $Z \subset C$ of size 2 and $Y \subset Y'$ of size 3. □

The next proposition easily follows from a classical result of Kővári et al. [20].

Proposition 4.15. *Let $H = (A \cup B, E)$ be a bipartite graph such that $|A| = n$, $|B| = m$. Then H contains a copy of $K_{2,2}$ if either of the following holds.*

(a) $\deg(x) \ge \sqrt{m}$ *for all $x \in A$ and $n > m + \sqrt{m}$,*
(b) $e := |E| \ge \max\{3n, m^2/2\}$.

Now we start our proof. First assume that $|V_0'| > 165$. Since $\delta(V_0', V_i') \ge n/6 - 2\alpha_2 n$ and $|V_i'| \le (1+\alpha_2)n/2$, we can apply Proposition 4.14(b) to the 3-partite subgraph on $V_1' \cup V_0 \cup V_2'$ and find a copy of $T_{2,3,2}$ on $X \subset V_1'$, $Y \subset V_0'$ and $Z \subset V_2'$ such that $|X| = |Z| = 2$, $|Y| = 3$. Let $V_2'' = V_1' - X$, $V_0'' = V_0' - Y$, and $V_2'' = V_2' - Z$. Then $|V_0''| > 162$ and $\delta(V_0'', V_i'') \ge n/6 - 2\alpha_2 n - 2$. We apply Proposition 4.14(b) again to 3-partite subgraph on $V_1'' \cup V_0'' \cup V_2''$ and find another copy of $T_{2,3,2}$. We next extend each copy of $T_{2,3,2}$ to a (V_1', V_2')-connector of length 11 as follows. Assume $X = \{x_3, x_5\}$, $Y = \{x_4, x_6, x_8\}$, and $Z = \{x_7, x_9\}$. Then $x_3, x_4, \ldots, x_8, x_9$ is an ESP but it is not a (V_1', V_2')-connector because $x_4 \notin V_1'$ and $x_8 \notin V_2'$. We extend this ESP by adding two vertices from V_1' in the beginning and two vertices from V_2' at the end. Since $x_3, x_5 \in V_1'$, by (15), we can find a vertex $x_2 \in \Gamma(x_3x_5, V_1')$. Since $\deg(x_2, V_1') > |V_1'| - \alpha_1 n/2$ and $\deg(x_4, V_1') > n/6 - 2\alpha_2 n$, we can find a vertex $x_1 \in \Gamma(x_2x_4, V_1')$, which is different from x_3, x_5. Therefore $x_1x_2\cdots x_9$ is an ESP. Similarly we find $x_{10}, x_{11} \in V_2'$ such that $x_1x_2, \ldots, x_{10}x_{11}$ is an ESP, which is a (V_1', V_2')-connector.

Now assume that $c_0 := |V_0'| \le 165$. We will not use the vertices of V_0' any more. Since $|V_0| \le \alpha_2 n$, all vertices $x \in V_0$ satisfy $\deg(x, V_1' \cup V_2') > n/2 - \alpha_2 n$. If $x \in V_0 - V_0'$, then exactly one of $\deg(x, V_1)$ and $\deg(x, V_2)$ is less than $n/6 - 2\alpha_2 n$. We thus partition $V_0 - V_0'$ into W_1 and W_2 such that $W_i = \{x \in V_0 - V_0' : \deg(x, V_{3-i}') < n/6 - 2\alpha_2 n\}$. For $i = 1, 2$, we have

$$\delta(W_i, V_i') \ge \delta(W_i, V_1' \cup V_2') - \frac{n}{6} + 2\alpha_2 n \ge \frac{n}{2} - \alpha_2 n - \frac{n}{6} + 2\alpha_2 n = \frac{n}{3} + \alpha_2 n.$$

Let $U_i = V_i' \cup W_i$ for $i = 1, 2$. The above bound for $\delta(W_i, V_i')$ and (15) together imply that

$$\delta(U_i, V_i') \ge \frac{n}{3} + \alpha_2 n. \tag{19}$$

Since $|V_i'| \le (1+\alpha_2)n/2$, for any three vertices $x_1, x_2, x_3 \in U_i$, we have

$$\deg(x_1x_2x_3, V_i') \ge 3\left(\frac{n}{3} + \alpha_2 n\right) - 2|V_i'| \ge 2\alpha_2 n. \tag{20}$$

Without loss of generality, assume that $|U_1| \le |U_2|$. Since $|U_1| \ge (1-\alpha_2)n/2$, we have $|U_2| \le (1+\alpha_2)n/2$. It suffices to find two disjoint (U_1, U_2)-connectors $u_1 \cdots u_q$ and $v_1 \cdots v_q$ for some $q \le 8$. In fact, if any of u_1, u_2, u_3 is not from V_1' (note that $u_1, u_2, u_3 \in U_1$ by the definition of (U_1, U_2)-connectors), then we find at most three new vertices $x_1, x_2, x_3 \in V_1'$ such that $x_1x_2x_3u_1u_2u_3$ is an ESP. For example, assume that $u_1 \notin V_1'$. Then we first find $x_3 \in \Gamma(u_1u_3, V_1')$, then $x_2 \in \Gamma(x_3u_2, V_1')$, and finally $x_1 \in \Gamma(x_2u_1, V_1')$ by applying (20) three times. Similar we add at most three new vertices $x_{q+1}, x_{q+2}, x_{q+3}$ from V_2' such that $u_{q-2}\, u_{q-1}\, u_q\, x_{q+1}\, x_{q+2}\, x_{q+3}$ is an ESP. The resulting ESP $x_1x_2x_3u_1\cdots u_qx_{q+1}x_{q+2}x_{q+3}$ is a (V_1', V_2')-connector of length at most 14. We obtain the other connector similarly.

The following technical lemma is the main step in our proof, we postpone its proof to the end.

Lemma 4.16. *Let $\varepsilon \ll 1$ and n be sufficiently large. Suppose that G is a graph of order n with a vertex partition $U_0 \cup U_1 \cup U_2$ such that*

- $|U_0| = c_0 \ll n$; $|U_1| \le |U_2| \le (1+\varepsilon)n/2$;
- U_1 *contains a subset* V_1' *such that* $\delta(V_1', V_1') \ge (1-\varepsilon)n/2$;
- $\delta(U_2, U_2) \ge n/3$.

If $\delta(G) \ge \frac{n+c_0}{2} + 6$, *then* $G[U_1, U_2]$ *contain either of the following* 6*-vertex subgraphs.*

H_1: *Two vertices $x_2 \in U_1, x_5 \in U_2$ are adjacent; two vertices $x_1, x_3 \in \Gamma(x_2, V_1')$ and two vertices $x_4, x_6 \in \Gamma(x_5, U_2)$ form a path $x_1x_4x_3x_6$.*

H_2: *Four vertices $x_2, x_3 \in U_1$ and $x_4, x_5 \in U_2$ form a copy of $K_{2,2}$; a vertex $x_1 \in \Gamma(x_4, U_1)$, and a vertex $x_6 \in \Gamma(x_3, U_2)$.*

We observe that subgraphs H_1 and H_2 given by Lemma 4.16 can be easily converted to (U_1, U_2)-connectors. In the case of H_1, $x_1x_2x_3x_4x_5x_6$ is an ESP and thus a (U_1, U_2)-connector. In the case of H_2, by (20), x_1, x_2, x_3 have a common neighbor $x_7 \in V_1'$, and x_4, x_5, x_6 have a common neighbor $x_8 \in V_2'$. The $x_1x_7x_2x_4x_3x_5x_8x_6$ is an ESP and thus a (U_1, U_2)-connector (see Fig. 3).

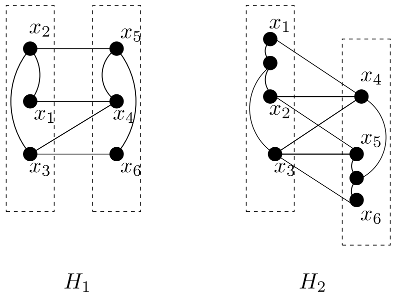

Fig. 3. Convert H_1 and H_2 to ESP's.

Recall that $\delta(G) \ge n/2 + 92 \ge (n + |V_0'|)/2 + 9$. With $\varepsilon = \alpha_1 + \alpha_2$, the partition $V_0' \cup U_1 \cup U_2$ satisfies the condition of Lemma 4.16 because of (15) and (19). Applying Lemma 4.16, we obtain either H_1 or H_2 with vertex set $\{x_1, \dots, x_6\}$. Now let $V_0'' = V_0' \cup \{x_1, \dots, x_6\}$, $U_1' = U_1 - \{x_1, x_2, x_3\}$, and $U_2' = U_2 - \{x_4, x_5, x_6\}$. Then $\delta(G) \ge (n + |V_0''|)/2 + 6$ and the new partition $V_0'' \cup U_1' \cup U_2'$ still satisfies the condition of Lemma 4.16. By Lemma 4.16, we can find a copy of H_1 or H_2 from $G[U_1', U_2']$. We finally convert the two copies of H_1 or H_2 to two disjoint (U_1, U_2)-connectors.

This complete the proof of Theorem 4.3 and the main theorem. □

Proof of Lemma 4.16. Define $k := c_0/2 + 6$. By assumption $\delta(G) \ge n/2 + k$. Let $|U_1| = n/2 - b$. Since $|U_1| \le (n - c_0)/2$, we have $b \ge c_0/2$.

$$\delta(U_1, U_2) \ge \frac{n}{2} + k - \left(\frac{n}{2} - b\right) - c_0 = k + b - c_0 \ge k - \frac{c_0}{2} = 6. \tag{21}$$

However, we do not have a nontrivial lower bound for $\delta(U_2, U_1)$ because it may be the case that $k \le b$. Define $U_2^* = \{u \in U_2 : u \sim x \text{ for some } x \in V_1'\}$. Note that $U_2^* \ne \emptyset$ because of (21). Select $x_5 \in U_2^*$ such that $\deg(x_5, U_2) = \max_{u \in U_2^*} \deg(u, U_2)$. Pick an arbitrary vertex $x_2 \in \Gamma(x_5, V_1')$. Let $B_1 = \Gamma(x_2, V_1')$ and $B_2 = \Gamma(x_5, U_2)$. By assumptions, $|B_1| \ge (1-\varepsilon)n/2$ and $|B_2| \ge n/3$. If the bipartite graph $G[B_1, B_2]$ contains a 4-vertex path $x_1x_4x_3x_6$, then we immediately obtain the desired graph H_1.

We may therefore assume that $G[B_1, B_2]$ contains no 4-vertex path. This implies $G[B_1, B_2]$ consists of disjoint stars, in particular, $e(B_1, B_2) < |B_1| + |B_2|$. Let $B_1' = \{x \in B_1 : \deg(x, B_2) \le 1\}$. The vertices in $B_1 - B_1'$ thus have disjoint neighborhoods in B_2 of size at least 2. Consequently $|B_1 - B_1'| \le |B_2|/2 \le |U_2|/2 \le (1+\varepsilon)n/4$. Therefore $|B_1'| \ge n/4 - \varepsilon n$ (in particular $B_1' \ne \emptyset$).

Let $A_2 = U_2 - B_2$ and set $m = |A_2|$. Since $|B_2| \ge n/3$, then $m \le (1+\varepsilon)n/2 - n/3 \le n/6 + \varepsilon n/2$. By (21) and the definition of B_1', we have

$$\delta(B_1', A_2) \ge k + b - c_0 - 1. \tag{22}$$

In particular, $m \ge k + b - c_0 - 1$. On the other hand, the definition of x_5 says that $\deg(u, U_2) \le \deg(x_5, U_2) = n/2 + b - c_0 - m$ for every $u \in U_2^*$. By using $m \ge k + b - c_0 - 1$, we obtain

$$\delta(U_2^*, U_1) \ge \frac{n}{2} + k - \left(\frac{n}{2} + b - c_0 - m\right) - c_0 = k - b + m \tag{23}$$

$$\ge 2k - c_0 - 1 = 11. \tag{24}$$

We observe that it suffices to find a copy of $K_{2,2}$ from $G[U_1, U_2^*]$. In fact, assume that $x_2, x_3 \in U_1$ and $x_4, x_5 \in U_2^*$ are the four vertices of $K_{2,2}$. By (21), x_3 has a neighbor x_6 in $U_2 - \{x_4, x_5\}$; by (24), x_4 has a neighbor $x_1 \in U_1 - \{x_2, x_3\}$. This gives the desired graph H_2.

We now separate cases by whether $b \geq \sqrt{m} + \frac{c_0}{2}$.

Case 1: $b \geq \sqrt{m} + \frac{c_0}{2}$.

By (22), $\delta(B_1', A_2) \geq k + \sqrt{m} + \frac{c_0}{2} - c_0 - 1 > \sqrt{m}$ as $k > c_0/2 + 1$. Since $|B_1'| \geq n/4 - \varepsilon n$ and $m \leq n/6 - \varepsilon n/2$, we have $|B_1'| > |A_2| + \sqrt{|A_2|}$. By Proposition 4.15, $G[B_1', A_2]$ contains a copy of $K_{2,2}$. Note that the two vertices of this $K_{2,2}$ in A_2 belong to U_2^* because they have neighbors in $B_1' \subseteq V_1'$.

Case 2: $b < \sqrt{m} + \frac{c_0}{2}$. Let $A_2^* = U_2^* \cap A_2$. By (23), $\delta(A_2^*, U_1) > k - (\sqrt{m} + \frac{c_0}{2}) + m \geq 1 + m - \sqrt{m} \geq m/2$, where the last inequality holds for any $m \geq 0$. This implies $e(A_2^*, U_1) \geq |A_2^*|^2/2$. On the other hand, (21) implies that $e(B_1, U_2) \geq 6|B_1|$ because $k - \frac{c_0}{2} \geq 6$. Recall that $e(B_1, B_2) < |B_1| + |B_2|$. By the definition of A_2^*,

$$\begin{aligned} e(B_1, A_2^*) &= e(B_1, A_2) > 6|B_1| - (|B_1| + |B_2|) \\ &> 5(1-\varepsilon)\frac{n}{2} - (1+\varepsilon)\frac{n}{2} > \frac{3n}{2}. \end{aligned}$$

Consequently $e(U_1, A_2^*) \geq e(B_1, A_2^*) > 3n/2 \geq 3|U_1|$. We can apply Proposition 4.15 and obtain a copy of $K_{2,2}$ in $G[U_1, A_2^*]$. □

5. Open problems

- What is the smallest integer C such that every graph of even order n with $\delta(G) \geq n/2 + C$ contains an ESHC? Theorem 1.1 and Proposition 1.2 together show that $2 \leq C \leq 92$. We think the lower bound is closer to the truth. To justify it, one needs to improve the constants in Theorems 4.2 and 4.3.
- What is the minimum degree threshold $\delta(n)$ for ESHC's of odd order? The (general) theorem of Böttcher, Schacht and Taraz (see the footnote on page 2) implies that $\delta(n) \leq n/2 + \varepsilon n$ for any $\varepsilon > 0$; our Proposition 1.3 shows that $\delta(n) \geq (n + \sqrt{n/2} - 1)/2$. Probably $\delta(n) = n/2 + c\sqrt{n}$ for some constant c. It seems that Theorem 4.1 for the non-extremal case remains valid when n is odd; the difficulty again is on the extremal cases.
- Can one find a proof of Theorem 1.1 (in fact, only Theorem 4.1) without using the Regularity Lemma? Recently Pósa's conjecture has been (re)proved [5,21] without the Regularity Lemma (thus it holds for all $n \geq n_0$ with some modest n_0). However, it is not clear if similar approaches work on Theorem 1.1 or Theorem 1.4.

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