Triangle-degrees in graphs and tetrahedron coverings in 3-graphs

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Abstract

We investigate a covering problem in 3-uniform hypergraphs (3-graphs): given a 3-graph F, what is $c_1(n, F)$, the least integer d such that if G is an n-vertex 3-graph with minimum vertex degree $\delta_1(G) > d$ then every vertex of G is contained in a copy of F in G ?

We asymptotically determine $c_1(n, F)$ when F is the generalised triangle $K_4^{(3)-}$, and we give close to optimal bounds in the case where F is the tetrahedron $K_4^{(3)}$ (the complete 3-graph on 4 vertices).

This latter problem turns out to be a special instance of the following problem for graphs: given an n-vertex graph G with $m > n^2/4$ edges, what is the largest t such that some vertex in G must be contained in t triangles? We give upper bound constructions for this problem that we conjecture are asymptotically tight. We prove our conjecture for tripartite graphs, and use flag algebra computations to give some evidence of its truth in the general case.

1 Introduction

Let F be a graph with at least one edge. What is the maximum number of edges $\mathrm{ex}(n, F)$ an n-vertex graph can have if it does not contain a copy of F as a subgraph? This is a classical question in extremal graph theory. If F is a complete graph, then the exact answer is given by Turán's theorem [63], one of the cornerstones of extremal graph theory. For other graphs F, the value of $\mathrm{ex}(n, F)$ is determined up to a $o(n^2)$ error term by the celebrated Erdős–Stone theorem [17].

Ever since Turán's foundational result, there has been significant interest in obtaining similar "Turán–type" results for r-uniform hypergraphs (r-graphs), with $r \geq 3$. The extremal theory of hypergraphs has however turned out to be much harder, and even the fundamental question of determining the maximum number of edges in a 3-graph with no copy of the tetrahedron $K_4^{(3)}$ (the complete 3-graph on 4 vertices) remains open — it is the subject of a 70-years old conjecture of Turán, and of an Erdős $ 1000 prize*. Most of the research efforts have focussed on the case of 3-graphs, where a small number of exact and asymptotic results are now known — see [3, 4, 12, 20, 24], as well as the surveys by Füredi [23], Sidorenko [61], and Keevash [34].

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*In fact, to earn this particular Erdős prize, it is sufficient to determine the limit $\lim_{n\to\infty} \mathrm{ex}(n, K_t^{(r)})/\binom{n}{r}$ for *any* integers $t > r \geq 3$.

It is well-known that the Turán problem for an r-graph F is essentially equivalent to identifying the minimum vertex-degree required to guarantee the existence of a copy of F. More recently [11, 44, 51], there has been interest in variants where one considers what minimum *i-degree* condition is required to guarantee the existence of a copy of F. Given an i-set $S \subseteq V(G)$ with $i \leq r$, its *neighbourhood* in G is the collection

$$\Gamma(S) = \Gamma_G(S) := \{T \subseteq V(G) \setminus S : \ S \cup T \in E(G)\}$$

of $(r-i)$-sets T whose union with S makes an edge of G. The neighbourhood of S defines an $(r-i)$-graph

$$G_S := (V(G) \setminus S, \Gamma_G(S)),$$

which is called the *link graph* of S. The *degree* of S in G is the size $\deg_G(S) = \deg(S) := |\Gamma(S)|$ of its neighbourhood. The *minimum i-degree* $\delta_i(G)$ of G is the minimum of $\deg(S)$ over all i-subsets $S \subseteq V(G)$. In particular, the case $i = r-1$ has received particular attention; $\delta_{r-1}(G)$ is known as the minimum *codegree* of G, and a minimum codegree condition is the strongest single degree condition one can impose on an r-graph. Determining what minimum codegree forces the existence of a copy of a fixed r-graph F is known as the codegree density problem [51]. A few results on the codegree density for various small 3-graphs are known, see [18, 19, 36, 49].

In a different direction, there has been significant recent research activity devoted to generalising another foundational result in extremal graph theory. Let F be a graph whose order divides n. What minimum degree condition is required to guarantee that a graph on n vertices contains an *F-tiling* — a collection of $n/v(F)$ vertex-disjoint copies of F? In the case of complete graphs, this was answered by the celebrated Hajnal–Szemerédi theorem [27], which (under the guise of equitable colourings) has applications to scheduling problems. For a general graph F, the Kühn–Osthus theorem [40] determines the minimum degree-threshold for F-tilings up to a constant additive error.

There has been a growing interest in determining analogous tiling thresholds in r-graphs for $r \geq 3$, see the surveys by Rödl and Ruciński [59], and Zhao [64] devoted to the subject. In an effort to generalise Dirac's theorem on Hamilton cycles to hypergraphs, Rödl, Ruciński and Szemerédi [60] determined the minimum codegree threshold for the existence of a perfect matching in r-graphs for $r \geq 3$. The paper also introduced the hugely influential *absorption method*, which has been used as a key ingredient in many of the results in the area obtained since. Beyond perfect matchings, codegree tiling thresholds have by now been determined for a number of small 3-graphs, including $K_4^{(3)}$ [35, 45], $K_4^{(3)-}$ [30, 43], and $K_4^{(3)--}$ ($K_4^{(3)}$ with two edges removed) [10, 39]. In addition, the codegree tiling thresholds for r-partite r-graphs have been studied recently [9, 25, 26, 29, 52]

Turning to minimum vertex-degree tiling thresholds, fewer results are known. The vertex-degree thresholds for perfect matchings were determined for 3-graphs by Han, Person, and Schacht [28] (asymptotically) and by Kühn, Osthus and Treglown [41] and Khan [38] (exactly). Han and Zhao [32] determined the vertex-degree tiling threshold for $K_4^{(3)--}$, while Han, Zang, andZhao [31] asymptotically determined the vertex-degree tiling threshold for all complete 3-partite 3-graphs.

As a key part of their argument, Han, Zang, and Zhao considered a certain 3-graph covering problem and showed it was distinct from the corresponding Turán-type existence problem. This stands in contrast with the situation for ordinary graphs, where existence and covering thresholds essentially coincide. Given an r-graph F, Falgas–Ravry and Zhao [21] introduced the notion of an F-covering, which is intermediate between that of the existence of a single copy of F and the existence of an F-tiling.

We say that an r-graph G has an *F-covering* if every vertex in G is contained in a copy of F in G. Equivalently an F-covering of G is a collection C of copies F whose union covers all of $V(G)$.

For every positive integer $i \leq r-1$, the *i-degree F-covering threshold* is the function

$$\mathrm{c}_i(n, F) := \max\left\{\delta_i(G) : \ v(G) = n, \ G \text{ has no } F-\text{covering}\right\}. \tag{1.1}$$

We further let the *i-degree F-covering density* to be the limit[†]

$$c_i(F) := \lim_{n\to\infty} c_i(n, F)/\binom{n-i}{r-i}. \tag{1.2}$$

Let $K_t^{(r)}$ denote the complete r-graph on t vertices and $K_t^{(r)-}$ denote the r-graph obtained by removing one edge from $K_t^{(r)}$. A tight r-uniform t-cycle $C_t^{(r)}$ is an r-graph with a cyclic ordering of its t vertices such that every r consecutive vertices under this ordering form an edge. Falgas–Ravry and Zhao [21] determined $c_2(F)$, where F is $K_4^{(3)}$, $K_4^{(3)-}$, $K_5^{(3)-}$, and $C_5^{(3)}$. Han, Lo, and Sanhueza-Matamala [29] determined $c_{r-1}(C_t^{(r)})$ for all $r \geq 3$ and $t > 2r^2$.

In this paper we investigate $c_1(n, F)$ and $c_1(F)$ for various 3-graphs F. We first consider $K_4^{(3)-}$. Let $f_n(d)$ be the function

$$f_n(d) := \binom{n-2}{2} + d - d(d-1) - \binom{d}{2} = \frac{1}{2}\left(n^2 - 5n + 6 - 3d^2 + 5d\right). \tag{1.3}$$

Observe that for fixed n, $f_n(d)$ is a decreasing function of d over the interval $[1, n-2]$. On the other hand $\frac{(n-1)}{2}d$ is an increasing function of d, so there exists a unique $d_\star = d_\star(n)$ such that $\frac{(n-1)}{2}d_\star = f(n, d_\star)$, namely

$$d_\star = \frac{1}{6}\left(\sqrt{13n^2 - 72n + 108} - n + 6\right) = \frac{\sqrt{13}-1}{6}n + O(1).$$

Theorem 1.1. *For all odd integer n, $\frac{n-1}{2}\lfloor d_\star \rfloor \leq c_1(n, K_4^{(3)-}) \leq \lfloor \frac{n-1}{2} d_\star \rfloor$. In particular, $c_1(K_4^{(3)-}) = \frac{\sqrt{13}-1}{6} = 0.4342\ldots$.*

The upper and lower bounds on $c_1(n, F)$ in Theorem 1.1 differ by less than $n/2$. However it seems much more work will be needed to determine $c_1(n, F)$ exactly. As a first step in this direction, we prove the following stability theorem characterising near-extremal configurations. Let $c_\star = \frac{\sqrt{13}-1}{6}$.

Theorem 1.2. *For every $\varepsilon > 0$, there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds: for every $n \geq n_0$, if H is a 3-graph on $n+1$ vertices with minimum vertex degree at least $(c_\star - \delta)\frac{n^2}{2}$ and $x \in V(H)$ is not covered by a copy of $K_4^{(3)-}$ in H, then the link graph H_x can be made bipartite by removing at most εn^2 edges.*

Next we consider $K_4^{(3)}$.

Theorem 1.3.

$$\frac{19}{27} = 0.7037\ldots \leq c_1(K_4^{(3)}) \leq \frac{19}{27} + 7.4 \times 10^{-9}.$$

The upper bound was derived from the flag algebra method. We believe that the lower bound is tight. As we show in Section 2.3, the problem of determining $c_1(K_4^{(3)})$ is equivalent to (a special case of) a problem about *triangle-degrees* in graphs.

[†]This limit can be shown to exist — see [21, Footnote 1].

Given a graph G, the *triangle-degree* of a vertex $x \in V(G)$ is the number of triangles that contains x. The well-studied *Rademacher–Turán* problem concerns the smallest *average* triangle-degree among all graphs with a given edge density (the edge density $\rho(G)$ is defined as $e(G)/\binom{v(G)}{2}$). This problem attracted significant attention (see [5, 14, 22, 46, 47]) until it was resolved asymptotically by Razborov [56] using the framework of his newly-developed theory of flag algebras. Different proofs expressed in the language of weighted graphs were later found by Nikiforov [53] and by Reiher [58] (who generalised Razborov's result to cliques of order 4 and of arbitrary order t, respectively).

Let $t_{\max}(G)$ denote the maximum triangle-degree in G. (This is related to but different from the well-studied *book number*, which is the maximum number of triangles containing a fixed edge of G, see the discussion in Section 4 for details.) For $\rho \in [0,1]$, we define

$$\tau(\rho) := \liminf_{n\to\infty} \min\{t_{\max}(G)/\binom{n-1}{2} : \ v(G) = n, \ \rho(G) \geq \rho\}, \tag{1.4}$$

which is the asymptotically smallest maximum scaled triangle-degree in a graph with edge density ρ. We derive the following upper bounds for $\tau(\rho)$ and conjecture that they are tight. If Conjecture 1.7 holds, then $c_1(K_4^{(3)}) = \frac{19}{27}$ (see Proposition 3.1).

Theorem 1.4. *Suppose $\rho \in [\frac{r-1}{r}, \frac{r}{r+1}]$, for some $r \in \mathbb{N}$. Then*

$$\tau(\rho) \leq \begin{cases} \frac{(r-1)(r-2)}{r^2} + \frac{3(r-1)}{r}\left(\rho - \frac{r-1}{r}\right) & \text{if } \frac{r-1}{r} \leq \rho \leq \frac{r}{r+1} - \frac{1}{3r(r+1)} \\ \frac{r(r-1)}{(r+1)^2} - \frac{3(r-1)}{r+1}\left(\frac{r}{r+1} - \rho\right) & \text{if } \frac{r}{r+1} - \frac{1}{3r(r+1)} \leq \rho \leq \frac{r}{r+1}. \end{cases}$$

The constructions underpinning Theorem 1.4 are very different from the extremal ones for the Rademacher–Turán problem, and are discussed in more detail in Section 3.

Construction 1.5 (Lower interval construction)**.** *Let $\rho \in [\frac{r-1}{r}, \frac{r}{r+1} - \frac{1}{3r(r+1)}]$ for some $r \in \mathbb{N}$. Suppose $n \in \mathbb{N}$ is divisible by $2r$. Consider a balanced complete r-partite graph on $[n]$ with parts $V_1, \dots V_r$. Add inside each V_i an arbitrary d-regular triangle-free graph H_i, where $d = \lfloor \left(\rho - \frac{r-1}{r}\right) n \rfloor$. Such triangle-free graphs exist since $d \leq \frac{2}{3(r+1)} \frac{n}{r}$ (by our upper bound on ρ), which is less than $n/(2r)$ (so one could take H_i to b a balanced bipartite graph, for example). The resulting graph is $\lfloor \rho n \rfloor$-regular. We denote by $\mathcal{G}^u_{\rho,n}$ the family of all graphs that can be constructed in this way.*

Construction 1.6 (Upper interval construction)**.** *Let $\rho \in [\frac{r}{r+1} - \frac{1}{3r(r+1)}, \frac{r}{r+1}]$ for some $r \in \mathbb{N}$. Suppose $n \in \mathbb{N}$ is divisible by $2(r+1)$. Consider a balanced complete $(r+1)$-partite graph on $[n]$ with parts $V_1, \dots V_{r+1}$. Equally divide each V_i into V_i' and V_i''. Let $\phi : [r+1] \to [r+1]$ be any bijection with the property that $\phi(i) \neq i$ for all $i \in [r+1]$ (any permutation of $[r+1]$ with no fixed point will do). Now for every $i \in [r+1]$, replace the complete bipartite graph between V_i' and $V''_{\phi(i)}$ by an arbitrary d-regular bipartite subgraph H_i, where $d = \left\lceil \left(\rho - \frac{r}{r+1} + \frac{1}{2(r+1)}\right) n \right\rceil$. The resulting graph is $\lceil \rho n \rceil$-regular. We denote by $\mathcal{G}^d_{\rho,n}$ the family of all graphs that can be constructed in this way.*

Conjecture 1.7. *The upper bounds on $\tau(\rho)$ given in Theorem 1.4 are tight for every $\rho \in [0,1]$.*

We use flag algebra computations to show the upper bounds from Conjecture 1.7 are not far from optimal when $\rho \in [\frac{1}{2}, \frac{2}{3}]$ (see Theorem 3.8).

Following on a beautiful result of Bondy, Shen, Thomassé and Thomassen [7] on a tripartite version of Mantel's theorem, Baber, Johnson and Talbot [2] gave a tripartite analogue of Razborov's triangle-density result. In a similar spirit, we prove Conjecture 1.7 holds for tripartite graphs. Note that a tripartite graph on n vertices can have between 0 and $\frac{n^2}{3}$ edges.

Theorem 1.8. *Let G be a tripartite graph on n vertices. Then*

$$t_{\max}(G) \geq \begin{cases} \frac{3}{2}\left(e(G) - \frac{n^2}{4}\right) & \text{if } \frac{e(G)}{n^2} < \frac{3}{10}, \\ e(G) - \frac{2}{9}n^2 & \text{if } \frac{3}{10} \leq \frac{e(G)}{n^2} \leq \frac{1}{3}. \end{cases}$$

Structure of the paper

In Section 2 we prove Theorems 1.1–1.3 along with bounds for $c_1(C_5^{(3)})$ and $c_1(K_t^{(3)})$ for $t \geq 5$. In Section 3 we prove Theorems 1.4 and 1.8, and give flag algebra bounds on $\tau(\rho)$. We end the paper in Section 4 with a discussion of book numbers in graphs and a comparison of known results and conjectures on minimal triangle density, triangle-degree and book-number as functions of edge density.

Notation

We use standard graph and hypergraph theory notation throughout the paper. In addition, we use $[n]$ to denote the set $\{1, 2, \dots n\}$ and $S^{(r)}$ to denote the collection of all r-subsets of a set S. Where there is no risk of confusion, we identify hypergraphs with their edge-sets.

2 Covering in 3-graphs

2.1 Proof of Theorem 1.1

Recall that $K_4^{(3)-}$ is the (unique up to isomorphism) 3-graph on 4-vertices spanning 3 edges, also known as the *generalised triangle*. In this subsection, we prove Theorem 1.1.

Proof of Theorem 1.1.
Lower bound: let n be odd, and let $d = \lfloor d_\star \rfloor \leq (n-1)/2$. We construct a 3-graph H on n vertices as follows. Set aside a vertex $v_\star$, and let $A \sqcup B$ be a bipartition of $V(H) \setminus \{v_\star\}$ into two sets of equal size. Let G be an arbitrary d-regular bipartite graph with partition $A \sqcup B$. Now let H be the 3-graph whose 3-edges are the union of the triples $\{v_\star xy : \ xy \in E(G)\}$ together with all the triples of vertices from $A \cup B$ inducing at most one edge in G.

Clearly, for every triple of vertices $S \subseteq A \cup B$, $S \cup \{v_\star\}$ induces at most two edges of H and $v_\star$ is not contained in any copy of $K_4^{(3)-}$. Thus $c_1(n, K_4^{(3)-}) \geq \delta_1(H)$. This latter quantity is easily calculated: the degree of $v_\star$ in H is $\frac{n-1}{2}d$. For any $a \in A$, there are exactly $d(d-1)$ pairs $(a', b) \in A \times B$ such that both $a'b$ and ab lie in G, and exactly $\binom{d}{2}$ pairs $(b, b') \in B^{(2)}$ such that both ab and ab' lie in G; such pairs are the only pairs from $((A \setminus \{a\}) \cup B)^{(2)}$ that do not form an edge of H with a. In addition, there are exactly d edges of H containing the pair $av_\star$. Thus the degree of a in H is

$$\deg(a) = \binom{n-2}{2} - d(d-1) - \binom{d}{2} + d = f_n(d).$$

By symmetry, the degree of any vertex in B is also $f_n(d)$. Thus $\delta_1(H) = \min(\frac{n-1}{2}d, f_n(d)) = \frac{n-1}{2}d$ because $d \leq d_\star$. Since H has no $K_4^{(3)-}$-covering, it follows that $c_1(n, K_4^{(3)-}) \geq \frac{n-1}{2}\lfloor d_\star \rfloor$.
Upper bound: suppose H is a 3-graph on n vertices with $\delta_1(H) = \frac{n-1}{2}d$ and no copy of $K_4^{(3)-}$ covering a vertex x (here n is not necessarily odd). We shall show that $\delta_1(H) \leq \frac{n-1}{2}d_\star$. Note that

the link graph H_x of x is triangle-free. Furthermore, $v_1v_2v_3 \notin E(H)$ for any triple $v_1v_2v_3$ spanning two edges in H_x. Let $F(v)$ denote the collection of pairs v_2v_3 such that vv_2v_3 induces two edges in H_x. We know that $vv_2v_3 \notin E(H)$ for every $v_2v_3 \in F(v)$. Observe that $F(v)$ consists of all pairs v_2v_3, where either $v_2, v_3 \in \Gamma(x, v)$ or $v_2v_3 \in H_x$ and exactly one of v_2, v_3 is in $\Gamma(x, v)$.

Counting non-edges of H over all $v \in V \setminus \{x\}$, we thus have

$$
\begin{aligned}
\sum_{v\in V\setminus\{x\}} \left(\binom{n-1}{2} - \deg(v)\right) &\geq \sum_{v\in V\setminus\{x\}} n-2-\deg(x,v)+|F(v)| \\
&\geq \sum_{v\in V\setminus\{x\}} \left(n-2-\deg(x,v)+\binom{\deg(x,v)}{2}+\sum_{v_2\in\Gamma(x,v)}(\deg(x,v_2)-1)\right) \\
&= (n-1)(n-2)+\sum_{v\in V\setminus\{x\}} \frac{1}{2}\left(3(\deg(x,v))^2-5\deg(x,v)\right) \\
&\geq (n-1)(n-2)+\frac{n-1}{2}\left(3d^2-5d\right) = (n-1)\left(n-2+\frac{3d^2-5d}{2}\right).
\end{aligned}
$$

where in the last line we used Jensen's inequality and our minimum degree assumption $\deg(x) \geq \frac{n-1}{2}d$. By averaging, there exists a vertex $v \in v \in V \setminus \{x\}$ with

$$
\deg(v) \leq \binom{n-1}{2} - n + 2 - \frac{3d^2-5d}{2} = f_n(d).
$$

Applying our minimum degree assumption $\deg(v) \geq \frac{n-1}{2}d$ yields $\frac{n-1}{2}d \leq f_n(d)$ and hence $d \leq d_\star$. Thus $\delta_1(H) \leq \frac{n-1}{2}d_\star$ as claimed.

□

2.2 Proof of Theorem 1.2

Our proof shall make use of a consequence of Karamata's inequality. Let $a_n \geq a_{n-1} \geq \ldots \geq a_1$ and $b_n \geq b_{n-1} \geq \ldots \geq b_1$ be real numbers. We say that $\mathbf{a} = (a_n, \ldots, a_1)$ *majorises* $\mathbf{b} = (b_n, \ldots, b_1)$ if $\sum_{i\geq k} a_i \geq \sum_{i\geq k} b_i$ for all $1 \leq k \leq n$, with equality attained in the case $k = 1$. Karamata's inequality states that if $\mathbf{a}$ majorises $\mathbf{b}$ and f is a convex function then $\sum_i f(a_i) \geq \sum_i f(b_i)$.

Lemma 2.1. *Suppose $f: \mathbb{R} \to \mathbb{R}$ is a convex function. Let $a_1 \leq a_2 \leq \ldots \leq a_n$ be real numbers such that $\sum_i a_i = \bar{a}n$, and let $\eta > 0$. Set $\mathcal{B} := \{i : a_i \leq (1-\eta)\bar{a}\}$. Then*

$$
\sum_i f(a_i) \geq |\mathcal{B}| \cdot f((1-\eta)\bar{a}) + (n-|\mathcal{B}|) \cdot f\left(\left(1+\frac{\eta|\mathcal{B}|}{n-|\mathcal{B}|}\right)\bar{a}\right). \tag{2.1}
$$

Proof. Since $\eta > 0$, our assumption on $\sum_i a_i$ tells us that $[n] \setminus \mathcal{B} \neq \emptyset$. If $\mathcal{B} = \emptyset$, then the claimed inequality is just Jensen's inequality. So assume $\mathcal{B}$ is nonempty and set $|\mathcal{B}| = \beta n$ for some $\beta > 0$.

Let $a'_1, a'_2, \ldots, a'_n$ be given by

$$
a'_i = \begin{cases} (1-\eta)\bar{a} & \text{if } i \in [\beta n] \\ \left(1+\frac{\eta\beta}{1-\beta}\right)\bar{a} & \text{if } i \in [n] \setminus [\beta n]. \end{cases}
$$

Observe that $\sum_i a'_i = \sum_i a_i = \bar{a}n$. Setting

$$x = \frac{1}{|\mathcal{B}|}\sum_{i\in\mathcal{B}} a_i \quad \text{and} \quad y = \frac{1}{n-|\mathcal{B}|}\sum_{i\in[n]\setminus\mathcal{B}} a_i,$$

we have

$$x \le (1-\eta)\bar{a} < \left(1+\frac{\eta\beta}{1-\beta}\right)\bar{a} \le \frac{\bar{a}-\beta x}{1-\beta} = y.$$

It follows readily from this that the n-tuple $(a_n, a_{n-1}, a_{n-2}, \ldots, a_1)$ majorises $(a'_n, a'_{n-1}, a'_{n-2} \ldots, a'_1)$. Applying Karamata's inequality to the convex function f we obtain

$$\sum_i f(a_i) \ge \sum_i f(a'_i) = \beta n \cdot f\left((1-\eta)\bar{a}\right) + (1-\beta)\, n \cdot f\left(\left(1+\frac{\eta\beta}{1-\beta}\right)\bar{a}\right). \qquad \square$$

Another ingredient in the proof of Theorem 1.2 is a classical result of Andrásfai, Erdős and Sós.

Theorem 2.2 (Andrásfai, Erdős, Sós [1])**.** *Let G be a triangle-free graph on n vertices with minimum degree $\delta(G) > \frac{2n}{5}$. Then G is bipartite.*

With these two preparatory results in hand, the proof of Theorem 1.2 is straightforward: we first use Lemma 2.1 to show that the overwhelming majority of vertices in the link graph H_x have degree much larger than $\frac{2}{5}n$, whereupon we deduce from the Andrásfai–Erdős–Sós theorem that H_x is almost bipartite.

Proof of Theorem 1.2. Recall $c_\star = \frac{\sqrt{13}-1}{6} = 0.43\ldots > \frac{2}{5}$. Fix $\varepsilon > 0$. Without loss of generality, assume that $\varepsilon < \frac{1}{3}\left(c_\star - \frac{2}{5}\right)$. Pick $0 < \eta < \frac{1}{3c_\star}\left(c_\star - \frac{2}{5}\right)$ and $\delta > 0$ such that

$$\delta < \frac{1}{3}\left(c_\star - \frac{2}{5}\right) < \frac{c_\star}{6} \qquad \text{and} \qquad \left(\frac{1+6c_\star}{2c_\star^2\eta^2}\right)\delta < \frac{\varepsilon}{2} \tag{2.2}$$

both hold.

Let H be a 3-graph with $v(H) = n+1$, $\delta_1(H) \ge (c_\star - \delta)\frac{n^2}{2}$. Suppose x is a vertex in H not covered by any copy of $K_4^{(3)-}$. Without loss of generality, assume $V(H) = [n] \cup \{x\}$. By the vertex-degree assumption, $e(H_x) = cn^2$, for some $c \ge c_\star - \delta$. Let $\mathcal{B} = \{y \in V(H) : \deg(xy) \le c(1-\eta)n\}$ be the collection of vertices in H whose codegree with x is smaller than average by a multiplicative factor of $(1-\eta)$. Set $|\mathcal{B}| = \beta n$.

Since x is not covered by a copy of $K_4^{(3)-}$ in H, the following hold:

(i) H_x is triangle-free;

(ii) for every triple of vertices $\{y_1, y_2, y_3\}$ inducing two edges in H_x, the 3-edge $y_1y_2y_3$ is missing from $E(H)$.

Property (i) implies that for every $y \in [n]$, the neighbourhood $\Gamma(xy)$ is an independent set in H_x, while property (ii) implies that for every $z, z' \in H_{xy}$ and every $w \in H_{xz}$, the 3-edges $zz'y$ and zwy are both missing from $E(H)$. In particular for every $y \in [n]$, we have

$$(1-c_\star+\delta)\frac{n^2}{2} > \binom{n}{2} - e(H_y) \ge \binom{|H_{xy}|}{2} + \sum_{z\in H_{xy}} \left(|H_{xz}| - 1\right).$$

Summing this inequality over all $y \in [n]$ and using the fact $\sum_{y\in[n]} \sum_{z\in H_{xy}} (|H_{xz}|-1) = 2\sum_{y\in[n]} \binom{|H_{xy}|}{2}$, we get

$$(1 - c_\star + \delta)\frac{n^3}{2} > \sum_{y\in[n]} 3\binom{|H_{xy}|}{2}. \tag{2.3}$$

Since the function $f(t) = \binom{t}{2}$ is convex and $\sum_{y\in[n]} |H_{xy}| = 2|H_x| = cn^2$, we can apply Lemma 2.1 to bound from below the right-hand side of (2.3) by

$$3\left(\beta n\binom{c(1-\eta)n}{2} + (1-\beta)n\binom{\frac{cn^2-\beta n\cdot c(1-\eta)n}{(1-\beta)n}}{2}\right) = \frac{3c^2}{2}\left(\beta(1-\eta)^2 + \frac{(1-\beta(1-\eta))^2}{1-\beta}\right)n^3 + O(n^2).$$

Inserting this inequality back into (2.3), dividing through by n^3 and using $c \ge c_\star - \delta$ yields

$$\begin{aligned}
1 - c_\star + \delta &\ge 3(c_\star-\delta)^2\left(\beta(1-\eta)^2 + \frac{(1-\beta(1-\eta))^2}{1-\beta}\right) + O(n^{-1}) \\
&= 3(c_\star-\delta)^2\left(1 + \eta^2\beta + \frac{\eta^2\beta^2}{1-\beta}\right) + O(n^{-1}) && (2.4)\\
&> (3c_\star^2 - 6\delta c_\star)\left(1+\eta^2\beta\right) + O(n^{-1}) \\
&\ge 3c_\star^2 - 6\delta c_\star + 2c_\star^2\eta^2\beta + O(n^{-1}), && (2.5)
\end{aligned}$$

where the last inequality holds because our choice of δ in (2.2) ensures $\delta < c_\star/6$. Note that $c_\star$ satisfies $1 - c_\star = 3c_\star^2$. Rearranging terms in inequality (2.5) gives

$$(1 + 6c_\star)\delta > 2c_\star^2\eta^2\beta + O(n^{-1}).$$

By the second part of (2.2) and the assumption that n is sufficiently large, we have

$$\beta < \left(\frac{1+6c_\star}{2c_\star^2\eta^2}\right)\delta + O(n^{-1}) < \frac{\varepsilon}{2} + O(n^{-1}) < \varepsilon$$

and $|\mathcal{B}| = \beta n < \varepsilon n$. Remove from H_x all vertices from $\mathcal{B}$. By the definitions of $\delta, \eta, \varepsilon$, the resulting triangle-free graph G has at most n vertices and minimum degree at least

$$c(1-\eta)n - \varepsilon n \ge (c_\star - \delta)(1-\eta)n - \varepsilon n > (c_\star - \eta c_\star - \delta - \varepsilon)\, n > \frac{2}{5}n.$$

By Theorem 2.2, G is bipartite. Since we removed only at most εn vertices from H_x to obtain G, it follows that H_x can be made bipartite by removing at most εn^2 edges, as claimed. This concludes the proof of Theorem 1.2.

□

2.3 Proof of Theorem 1.3

Given an r-graph G, write $t_G(x)$ for the number of copies of $K_{r+1}^{(r)}$ in G that cover x.

Proposition 2.3. *There exists an r-graph H on $n+1$ vertices with minimum vertex-degree $\delta_1(H) \ge \alpha\binom{n-1}{r-1}$ and no $K_{r+1}^{(r)}$-covering if and only if there exists an $(r-1)$-graph G on n vertices with at least $\alpha\binom{n-1}{r-1}$ edges such that for every vertex $x \in V(G)$, $t_G(x) - \deg_G(x) \le (1-\alpha)\binom{n-1}{r-1}$.*

Proof. In one direction, let H be an r-graph on $n+1$ vertices with minimum degree $\alpha\binom{n-1}{r-1}$. Suppose $v_\star$ is not covered by any $K_{r+1}^{(r)}$ in H. By the minimum degree condition on $v_\star$, the $(r-1)$-uniform link graph $G = H_{v_\star}$ contains at least $\alpha\binom{n-1}{r-1}$ edges. Also, every copy of $K_r^{(r-1)}$ in the $(r-1)$-graph G must be a non-edge in the r-graph H, else together with $v_\star$ it would make a copy of $K_{r+1}^{(r)}$ in H covering $v_\star$. The minimum degree condition in H then implies that for every vertex x in the n-vertex $(r-1)$-graph G,

$$\alpha\binom{n-1}{r-1} \leq \deg_H(x) \leq \binom{n-1}{r-1} + \deg_G(x) - t_G(x),$$

implying $t_G(x) - \deg_G(x) \leq (1-\alpha)\binom{n-1}{2}$ as desired.

In the other direction, let G be an $(r-1)$-graph on n vertices with at least $\alpha\binom{n-1}{r-1}$ edges such that $t_G(x) - \deg_G(x) \leq (1-\alpha)\binom{n-1}{r-1}$ for all $x \in V(G)$. We add a new vertex $v_\star$ to G and define an r-graph H on $V(G) \sqcup \{v_\star\}$ by setting the link graph of $v_\star$ be equal to G, and adding in as edges all r-sets from $V(G)^{(r)}$ which do not induce a copy of $K_r^{(r-1)}$ in G. This yields an r-graph on $n+1$ vertices in which $v_\star$ is not covered by a copy of $K_{r+1}^{(r)}$, $\deg_H(v_\star) = e(G) \geq \alpha\binom{n-1}{r-1}$, and for every $x \in V(H) \setminus \{v_\star\}$,

$$\deg_H(x) = \binom{n-1}{r-1} - t_G(x) + \deg_G(x) \geq \alpha\binom{n-1}{r-1},$$

so $\delta_1(H) \geq \alpha\binom{n-1}{r-1}$ as desired. □

Corollary 2.4. *For any $r \in \mathbb{N}$, the 1-degree covering density $c_1(K_{r+1}^{(r)})$ is the least $\alpha > 0$ such that if G is an $(r-1)$-graph on n vertices with at least $\alpha\binom{n}{r-1}$ edges, then there is a vertex $x \in G$ contained in $t_G(x) \geq (1-\alpha+o(1))\binom{n-1}{r-1}$ copies of $K_r^{(r-1)}$ in G.*

Proof of Theorem 1.3.
Lower bound: suppose $3|n$ and partition $[n]$ into three sets V_1, V_2, V_3 of size $n/3$. Further partition each V_i into two sets $V_{i,1}$ and $V_{i,2}$ of size $n/6$. Now let G be the 2-graph on $[n]$ obtained by putting in all edges of the form V_iV_j, with $1 \leq i < j \leq 3$ and adding for each $i \in [3]$ an arbitrary $n/27$-regular bipartite graph with partition $V_{i,1} \sqcup V_{i,2}$. An easy calculation shows G is both regular and triangle-degree regular, with every vertex x satisfying $\deg(x) = 19n/27$ and $t(x) = 4n^2/27$. We have thus $t(x) - \deg(x) = \frac{8}{27}\binom{n-1}{2} + O(n)$. It follows from Proposition 2.3 that there exists a 3-graph H on $n+1$ vertices with minimum degree $\left(\frac{19}{27} + O(\frac{1}{n})\right)\binom{n-1}{2}$ and no K_4-covering, establishing the desired lower bound on $c_1(K_4^{(3)})$.
Upper bound: set $\alpha = \frac{19}{27} + 7.4 \times 10^{-9}$. By Proposition 2.3, it is enough to show that if G is an n-vertex graph with $t_{\max} \leq (1-\alpha+o(1))\binom{n-1}{2}$, then $e(G) \leq (\alpha+o(1))\binom{n-1}{2}$. This is done in Proposition 3.10 in the next section via a simple flag algebra calculation. □

2.4 $C_5^{(3)}$

Theorem 2.5. $0.55\ldots = \frac{5}{9} \leq c_1(C_5^{(3)}) \leq 2 - \sqrt{2} = 0.58\ldots$

Proof. **Lower bound:** we construct a 3-graph on $[3n+1]$ as follows. Set aside $v_\star = 3n+1$, and partition the remaining vertices into an n-set A and a $2n$-set B. Let H be the 3-graph on $[3n+1]$ obtained by setting the link graph of $v_\star$ to be the union of a clique on A and a clique on B, and adding all triples of the form AAB or ABB. Every path of length 3 in the link graph of $v_\star$ gives rise to an independent set in H, hence there is no copy of the strong 5-cycle C_5 covering $v_\star$ in H. The degree of $v_\star$ in H is $\binom{n}{2}+\binom{2n}{2}=\frac{5}{9}\binom{3n}{2}-\frac{2n}{3}$, and the degree in the rest of the graph are all at least

$$\min\left((|A|-1)\,|B|+\binom{|B|}{2},(|B|-1)\,|A|+\binom{|A|}{2}\right)=n(2n-1)+\binom{n}{2}=\frac{5}{9}\binom{3n}{2}-\frac{2n}{3}.$$

Thus $c_1(3n+1,C_5)\geq\frac{5}{9}\binom{3n}{2}-\frac{2n}{3}$, as desired.

Upper bound: Mubayi and Rödl [50, Theorem 1.9] proved that $\pi(C_5^{(3)})\leq 2-\sqrt{2}$. An easy modification of their proof shows that $\alpha=2-\sqrt{2}$ is in fact an upper bound for the covering threshold. Indeed, let H be a 3 graph on n vertices with $\delta_1(H)\geq\alpha\binom{n-1}{2}+c(2n-1)$, for some $c\geq 10$. Let x be an arbitrary vertex in $V(H)$. By averaging, there exists $y\in V(H)$ such that $\deg(xy)\geq\alpha n$. Form the multigraph $G=H_x\cup H_y$ as in [50, Proof of Theorem 1.9, p 151]. Then [50, Claim, p 151] shows that if there is no copy of $C_5^{(3)}$ covering the pair xy, then G satisfies the conditions of [50, Lemma 6.2, p 149], and one can conclude as Mubayi and Rödl do that one of x and y has degree at most $\alpha\binom{n-1}{2}+c(2n-1)-n$ in H , contradicting our minimum degree assumption.

□

2.5 $K_t^{(3)}$, $t\geq 5$

Proposition 2.6. *For all $t\geq 4$, $c_1(K_{t+1}^{(3)})\leq\frac{-1+\sqrt{3-2c_1(K_t^{(3)})}}{1-c_1(K_t^{(3)})}$.*

Proof. Let $\varepsilon>0$ and n be sufficiently large. Suppose that H is a 3-graph on n vertices with $\delta_1(H)\geq\alpha\frac{n^2}{2}$ for some $\alpha>0$ satisfying $1+\frac{2}{\alpha}-\frac{2}{\alpha^2}=c_1(K_t^{(3)})+\varepsilon$. Let $v_\star$ be an arbitrary vertex. By averaging, there exists a vertex $x\in V\setminus\{v_\star\}$ and an αn-set V' such that $V'\subseteq\Gamma(x,v_\star)$. Observe that

$$e(H_x[V'])\geq e(H_x)-|V'|(n-|V'|)-\binom{n-|V'|}{2}$$

and an analogous bound holds for $e(H_{v_\star}[V'])$. Thus

$$\begin{aligned}e(H_x[V']\cap H_{v_\star}[V'])&\geq e(H_x)+e(H_{v_\star})-2|V'|(n-|V'|)-2\binom{n-|V'|}{2}-\binom{|V'|}{2}\\&\geq\left(\alpha^2+2\alpha-2\right)\frac{n^2}{2}+O(n).\end{aligned}\tag{2.6}$$

On the other hand, for any $y\in V'$, we have

$$\begin{aligned}|\Gamma(y)\cap(V'\cup\{x,v_\star\})^{(2)}|&\geq\deg(y)-(|V'|+1)(n-|V'|-2)-\binom{n-|V'|-2}{2}\\&\geq\left(\alpha^2+\alpha-1\right)\frac{n^2}{2}+O(n).\end{aligned}\tag{2.7}$$

Note that

$$\min\left(\alpha^2+2\alpha-2,\alpha^2+\alpha-1\right)=\alpha^2+2\alpha-2=(c_1(K_t^{(3)})+\varepsilon)\alpha^2.\tag{2.8}$$

Let H' be the 3-graph obtained by taking $H[V']$ and adding a new vertex z whose link graph consists precisely of those pairs $yy'\in E(H_x[V']\cap H_{v_\star}[V'])$. By (2.6), (2.7) and (2.8), $\delta_1(H')\geq$

$\left(c_1(K_t) + \frac{\varepsilon}{2}\right)\binom{v(H')}{2}$. Thus provided $\alpha n = v(H')$ is sufficiently large, there must be a set $S \subseteq V'$ such that $S \cup \{z\}$ induces a copy of $K_t^{(3)}$ in H' covering z. But then by construction of H', this implies that $S \cup \{x, v_\star\}$ induces a copy of $K_{t+1}^{(3)}$ covering $v_\star$ in H. It follows that $\alpha \geq c_1(K_{t+1}^{(3)})$, and hence (since $\varepsilon > 0$ was arbitrary) that $c_1(K_{t+1}^{(3)}) \leq \frac{-1+\sqrt{3-2c_1(K_t^{(3)})}}{1-c_1(K_t^{(3)})}$. □

Proposition 2.7. *Suppose there exists a 3-graph H on $[N]$ such that*

(i) every vertex of H has degree at most d;

(ii) every t-set of vertices from $V(H)$ spans at least one edge.

Then we have

$$c_1\left(K_{t+1}^{(3)}\right) \geq \min\left(1 - \frac{1}{N}, 1 - \frac{2d}{N^2}\right)$$

Proof. We construct a 3-graph G on $[Nn+1]$ as follows. Set aside $v_\star = Nn+1$, and partition the remaining vertices into n-sets $V_1, V_2, \ldots V_N$. Now let the link graph of $v_\star$ in G be the complete N-partite graph on $[Nn]$ with partition $\sqcup_{i=1}^N V_i$. To make up the remainder of the edges of G, add in all triples $v_1v_2v_3$ from $[Nn]^{(3)}$ with $v_j \in V_{i_j}$ for $j = 1, 2, 3$ and $i_1i_2i_3 \notin E(H)$.

Clearly $\deg_G(v_\star) = \binom{nN}{2} - N\binom{n}{2} = \left(1 - \frac{1}{N}\right)\binom{nN}{2} + O(n)$, and every other vertex $x \in [nN]$ with $x \in V_i$ has degree

$$\deg_G(x) = n(N-1) + \binom{nN-1}{2} - \deg_H(i)n^2 \geq \left(1 - \frac{2d}{N^2}\right)\binom{nN}{2} + O(n).$$

Thus $\delta_1(G) \geq \min\left(1 - \frac{1}{N}, 1 - \frac{2d}{N^2}\right)\binom{nN}{2} + O(n)$. Furthermore, every complete graph $G_{v_\star}[T]$ on $|T| = t$ vertices in the link graph of $v_\star$ in G meets t different parts $V_{i_1}, \ldots, V_{i_t}$ from our partition of $[nN]$. By assumption, $i_1i_2, \ldots i_t$ spans at least one edge of H, whence we have that at least one of the triples from $T^{(3)}$ is missing from $E(G)$. In particular $\{v_\star\} \cup T$ does not span a copy of $K_{t+1}^{(3)}$ in G, and G fails to have a $K_{t+1}^{(3)}$-cover. The proposition follows. □

A natural family of 3-graphs for applications of Proposition 2.7 are *Steiner triple systems* (STS), where each pair of vertices is contained in a unique edge. Let α_t denote the minimum of the independence number over all STS of order t. The unique (up to isomorphism) STS of orders 3 and 7 are the 3-edge $K_3^{(3)}$ and the Fano plane S_7 respectively, which give $\alpha_3 = 2$, $\alpha_7 = 4$. The affine plane of order 9, S_9, is the unique up to isomorphism STS of order 9 and has $\alpha(S_9) = \alpha_9 = 4$. It is further known that $\alpha_{13} = 6$, $\alpha_{15} = 6$ [48], and $\alpha_{19} = 7$ [8] (see also the monograph of Kaski and Östergård [33]).

Proposition 2.8.

$$\begin{aligned} 0.8888\ldots = \frac{8}{9} &\leq c_1\left(K_6^{(3)}\right) \leq 0.947962\ldots \\ 0.9333\ldots = \frac{14}{15} &\leq c_1\left(K_8^{(3)}\right) \leq 0.98793\ldots \\ 0.9473\ldots = \frac{18}{19} &\leq c_1\left(K_9^{(3)}\right) \leq 0.99404\ldots\ . \end{aligned}$$

Proof.
Lower bound: apply Proposition 2.7 to STS of orders 9, 15 and 19 with minimum independence numbers, and observe that an STS of order t is a $d = \frac{t-1}{2}$-regular 3-graph, so that $\min(1-\frac{1}{t}, 1-\frac{2d}{t^2}) = 1 - \frac{1}{t}$.
Upper bound: repeatedly apply Proposition 2.6 with our upper bound $c_1(K_4) \leq \frac{19}{27} + 7.4 \times 10^{-9}$ from Theorem 1.3. □

Remark 2.9. *The lower bounds on the covering densities in Proposition 2.8 above are strictly stronger than the bounds one gets from the conjectured values of the corresponding Turán densities.*

In each case, they are about 5×10^{-2} below our upper bounds. Note that if one applies Proposition 2.7 to the unique STS on 3-vertices, one gets a lower bound of $2/3$ for $c_1(K_4^{(3)})$. We obtained an improvement of this bound in Theorem 1.3 by almost 5×10^{-2} by adding a few edges in the link graph of $v_\star$ and deleting a few triples meetings the corresponding pairs. It seems natural to believe a similar (albeit significantly more intricate) process would similarly improve the lower bounds in Proposition 2.8. If we had to guess, we would thus say that the true value of $c_1(K_t^{(3)})$ for $t = 6, 8, 9$ probably lies closer to the upper bounds we give.

For completeness, we give (very weak) bounds on $c_1\left(K_5^{(3)}\right)$, which show $c_1\left(K_4^{(3)}\right) < c_1\left(K_5^{(3)}\right) < c_1\left(K_6^{(3)}\right)$.

Proposition 2.10. $\frac{3}{4} \leq c_1\left(K_5^{(3)}\right) \leq 0.8842\ldots$

Proof.
Lower bound: consider a partition of $[2n]$ into n-sets, $[2n] = V_1 \sqcup V_2$. Let G be the 3-graph on $[2n]$ whose edge-set consists of all triples meeting both V_1 and V_2. It is easily checked that G is $K_5^{(3)}$-free and has minimum degree $\binom{2n-1}{2} - \binom{n-1}{2} = \frac{3}{4}\binom{2n-1}{2} + O(n)$, giving us the required lower bound.
Upper bound: apply Proposition 2.6 with our upper bound $c_1(K_4^{(3)}) \leq \frac{19}{27} + 7.4 \times 10^{-9}$ from Theorem 1.3. □

3 Triangle-degree in graphs

In this section, we investigate the problem of minimising the maximum triangle-degree $\tau(\rho)n^2/2$ in a 2-graph with a given edge density ρ. We give upper bound constructions for $\tau(\rho)$, which we conjecture are best possible. We show our conjecture holds for tripartite graphs and use flag algebra computations to bound below $\tau(\rho)$ for general graphs with $1/2 < \rho \leq 2/3$.

3.1 Proof of Theorem 1.4

Proposition 3.1. *Conjecture 1.7 implies $c_1(K_4^{(3)}) = \frac{19}{27}$.*

Proof. Suppose $\rho = c_1(K_4^{(3)})$. By Proposition 2.3, there exist a sequence $(G_n)_{n\in\mathbb{N}}$ of 2-graphs with $v(G_n) \to \infty$, $\rho(G_n) \geq \rho + o(1)$ and $t_{\max}(G_n) \leq (1-\rho+o(1))\binom{v(G_n)-1}{2}$. In particular, this implies that $\tau(\rho) \leq 1-\rho$. If Conjecture 1.7 is true, then since $\frac{19}{27} \in (\frac{2}{3}, \frac{3}{4})$, we have $\tau(\frac{19}{27}) = \frac{8}{27}$ and $\tau(\frac{19}{27}+\varepsilon) > \frac{8}{27}$ for sufficiently small $\varepsilon > 0$. Hence $\rho \leq \frac{19}{27}$. Together with the lower bound from Theorem 1.3, we conclude that $c_1(K_4^{(3)}) = \frac{19}{27}$. □

We now give constructions for two families of graphs used in the proof of Theorem 1.4.

Construction 3.2 (Lower interval construction)**.** *Let $\rho \in [\frac{r-1}{r}, \frac{r}{r+1} - \frac{1}{3r(r+1)}]$ for some $r \in \mathbb{N}$. Suppose $n \in \mathbb{N}$ is divisible by $2r$. Consider a balanced complete r-partite graph on $[n]$ with parts $V_1, \dots V_r$. Add inside each V_i an arbitrary d-regular triangle-free graph H_i, where $d = \lfloor (\rho - \frac{r-1}{r}) n \rfloor$. Such triangle-free graphs exist since $d \leq \frac{2}{3(r+1)} \frac{n}{r}$ (by our upper bound on ρ), which is less than $n/(2r)$ (so one could take H_i to b a balanced bipartite graph, for example). The resulting graph is $\lfloor \rho n \rfloor$-regular. We denote by $\mathcal{G}^u_{\rho,n}$ the family of all graphs that can be constructed in this way.*

Construction 3.3 (Upper interval construction)**.** *Let $\rho \in [\frac{r}{r+1} - \frac{1}{3r(r+1)}, \frac{r}{r+1}]$ for some $r \in \mathbb{N}$. Suppose $n \in \mathbb{N}$ is divisible by $2(r+1)$. Consider a balanced complete $(r+1)$-partite graph on $[n]$ with parts $V_1, \dots V_{r+1}$. Equally divide each V_i into V_i' and V_i''. Let $\phi : [r+1] \to [r+1]$ be any bijection with the property that $\phi(i) \neq i$ for all $i \in [r+1]$ (any permutation of $[r+1]$ with no fixed point will do). Now for every $i \in [r+1]$, replace the complete bipartite graph between V_i' and $V''_{\phi(i)}$ by an arbitrary d-regular bipartite subgraph H_i, where $d = \left\lceil \left(\rho - \frac{r}{r+1} + \frac{1}{2(r+1)}\right) n \right\rceil$. The resulting graph is $\lceil \rho n \rceil$-regular. We denote by $\mathcal{G}^d_{\rho,n}$ the family of all graphs that can be constructed in this way.*

Remark 3.4. *The choices of the graphs H_i in both Construction 3.2 and 3.3 give rise to very different graphs (lying at edit distance $\Omega(n^2)$ from each other). In particular if Conjecture 1.7 is correct, then the problem of minimising the maximum triangle-degree is not stable. This stands in some contrast with the Rademacher–Turán problem for triangles, for which Pikhurko and Razborov [54] obtained a stability result, establishing that there is an asymptotically unique way of minimising the number of triangles for a given edge-density. This instability is observed even at the level of subgraph frequencies, as e.g. in the first construction we could take as H_i a subgraph of a blow-up of the five-cycle instead of a bipartite graph, provided $\rho \leq \frac{r-1}{r} + \frac{2}{5r}$.*

In particular, this suggests Conjecture 1.7 may be harder to resolve than the Rademacher–Turán problem for triangles, and might not amenable to standard flag algebraic approaches due to the instability of the extremal examples.

Proof of Theorem 1.4. Assume that $\rho \in [\frac{r-1}{r}, \frac{r}{r+1}]$ for some $r \in \mathbb{N}$. When $r = 1$, a ρn-regular bipartite graph on n vertices (we may use $\mathcal{G}^u_{\rho,n}$ and $\mathcal{G}^d_{\rho,n}$ as well) shows that $\tau(\rho) = 0$. So we may assume that $r \geq 2$.

First assume that $\rho \in [\frac{r-1}{r}, \frac{r}{r+1} - \frac{1}{3r(r+1)}]$. Consider an arbitrary graph G of $\mathcal{G}^u_{\rho,n}$, for some n divisible by $2r$. Pick a vertex $x \in V_i$. Let us compute the triangle-degree of x. There are at most $(n - |V_i|)d$ pairs (y, x') with $x' \in V_i \setminus \{x\}$, $y \in [n] \setminus V_i$ and $xx'y$ forming a triangle in G. Further, there are at most $\frac{1}{2}\sum_{j \neq i} |V_j| d$ pairs (y, y') with $y, y' \in V_j \neq V_i$ and xyy' forming a triangle in G. Finally, there are at most $\frac{1}{2}\sum_{j:\ j\neq i}\sum_{k:\ k\neq i,j} |V_j||V_k|$ pairs (y, z) with $y \in V_j$, $z \in V_k$, V_i, V_j, V_k all distinct and xyz forming a triangle in G. Since each part V_i is triangle-free by construction, there are no other triangles in G containing x, and the triangle-degree of x is thus at most

$$\begin{aligned} t_G(x) &= \frac{r-1}{r} n \left\lfloor \left(\rho - \frac{r-1}{r}\right) n \right\rfloor + \frac{r-1}{2r} n \left\lfloor \left(\rho - \frac{r-1}{r}\right) n \right\rfloor + \frac{(r-1)(r-2)}{2r^2} n^2 \\ &= \left(\frac{(r-1)(r-2)}{r^2} + \frac{3(r-1)}{r} \left(\rho - \frac{r-1}{r}\right)\right) \frac{n^2}{2} + O(n). \end{aligned}$$

This gives the claimed upper bound on $\tau(\rho)$ for $\rho \in [\frac{r-1}{r}, \frac{r}{r+1} - \frac{1}{3r(r+1)}]$.

Next, assume that $\rho \in [\frac{r}{r+1} - \frac{1}{3r(r+1)}, \frac{r}{r+1}]$. Consider an arbitrary graph G of $\mathcal{G}^d_{\rho,n}$, for some n divisible by $2(r+1)$. Pick a vertex $x \in V_i'$ (the case when $x \in V_i''$ is analogous) . When computing the triangle-degree of x, it is more convenient to count the number of triangles containing x in the balanced complete $(r+1)$-partite graph from which an edge was deleted when constructing G. Observe that every triangle has lost at most one edge.

First of all, we have lost $(|V''_{\phi(i)}| - d)\left(\frac{r-1}{r+1}\right) n$ triangles of the form xyz with $y \in V''_{\phi(i)}$. Secondly, for every $y \in [n] \setminus \left(V_i \cup V''_{\phi(i)} \cup V'_{\phi^{-1}(i)}\right)$, there are $\frac{n}{2(r+1)} - d$ vertices $z \in [n] \setminus \left(V_i \cup V''_{\phi(i)} \cup V'_{\phi^{-1}(i)}\right)$ such that the edge yz was lost. This results in $\frac{r-1}{r+1}\frac{n}{2}(\frac{n}{2(r+1)} - d)$ lost triangles xyz. In total there are

$$\left(\frac{n}{2(r+1)} - d\right)\left(\frac{r-1}{r+1}\right) n + \frac{n}{2}\left(\frac{r-1}{r+1}\right)\left(\frac{n}{2(r+1)} - d\right) = 3\left(\frac{r-1}{r+1}\right)\left(\frac{r}{r+1} - \rho\right)\frac{n^2}{2} + O(n)$$

lost triangles for x. Subtracting this quantity from the triangle-degree of x in the original complete balanced $(r+1)$-partite graph, we get

$$t_G(x) = \left(\frac{r(r-1)}{(r+1)^2} - 3\left(\frac{r-1}{r+1}\right)\left(\frac{r}{r+1} - \rho\right)\right)\frac{n^2}{2} + O(n).$$

This gives the claimed upper bound on $\tau(\rho)$ for $\rho \in [\frac{r}{r+1} - \frac{1}{3r(r+1)}, \frac{r}{r+1}]$. □

3.2 Proof of Theorem 1.8

For this range of $e(G)$, Conjecture 1.7 states that for any n-vertex graph G,

$$t_{\max}(G) \geq \begin{cases} 0 & \text{if } e(G) \leq \frac{n^2}{4}, \\ \frac{3}{2}\left(e(G) - \frac{n^2}{4}\right) + O(n) & \text{if } \frac{n^2}{4} \leq e(G) \leq \frac{11}{36}n^2, \\ e(G) - \frac{2}{9}n^2 + O(n) & \text{if } \frac{11}{36}n^2 \leq e(G) \leq \frac{1}{3}n^2. \end{cases}$$

Remark 3.5. *Since $\frac{3}{10} < \frac{11}{36}$ and since for $e(G) < \frac{11}{36}n^2$ we have*

$$e(G) - \frac{2}{9}n^2 > \frac{3}{2}\left(e(G) - \frac{n^2}{4}\right),$$

Theorem 1.8 implies that Conjecture 1.7 holds true for all tripartite graphs.

Proof of Theorem 1.8. Let G be an n-vertex tripartite graph with partition $A \sqcup B \sqcup C$. Since $t_{\max}(G)$ is nonnegative, we only need to consider the case when $e(G) > \frac{n^2}{4}$. Assume without loss of generality that

$$|A| \geq |B| \geq |C|.$$

Suppose $|A| = xn$ and $|B| = yn$ (and so $|C| = (1-x-y)n$). Then $x \geq y \geq \frac{1-x}{2} \geq 0$, and in particular $x \geq \frac{1}{3}$. Since $|B||C| \leq (\frac{1-x}{2})^2$, we have

$$e(G) \leq |A|(n - |A|) + |B||C| \leq \left(x(1-x) + \left(\frac{1-x}{2}\right)^2\right) n^2.$$

The function of x on the right-hand side has derivative $\frac{3}{2}\left(\frac{1}{3} - x\right) n^2 \leq 0$ for $x \geq \frac{1}{3}$, and attains the value $\frac{n^2}{4}$ at $x = \frac{2}{3}$. Since $e(G) > n^2/4$, we must have $x < \frac{2}{3}$.

Write α for the edge density of G between parts B and C, β for the edge density between parts A and C, and γ for the edge density between parts A and B. So we have

$$\frac{e(G)}{n^2} = \gamma xy + \beta x(1-x-y) + \alpha y(1-x-y).$$

Since $x \geq y \geq 1-x-y$, if $\alpha+\beta+\gamma = S \leq 2$ then $e(G)/n^2$ is maximised by letting $\gamma = \min(S,1)$, $\beta = S-\gamma$, and $\alpha = 0$, i.e. by making G bipartite. But a bipartite graph contains at most $\frac{n^2}{4}$ edges, contradicting our lower bound on $e(G)$. Thus we assume $\alpha+\beta+\gamma = 2+s$ for some s with $0 < s \leq 1$. Further, if x, s are fixed with $x \geq y \geq 1-x-y$, then $e(G)/n^2$ is maximised by letting $\gamma = 1$, $\beta = 1$, $\alpha = s$ and $y = \frac{1-x}{2}$. In other words, we have

$$\frac{e(G)}{n^2} \leq f_1(x,s) := x - x^2 + \frac{s}{4}(1-x)^2. \tag{3.1}$$

Since

$$\frac{\partial}{\partial x} f_1(x,s) = 1 - 2x - \frac{s}{2}(1-x) = \left(\frac{2-s}{2}\right) - \left(\frac{4-s}{2}\right)x,$$

when s is fixed, $f_1(x,s)$ attains a maximum at $x_\star = \frac{2-s}{4-s} \in [\frac{1}{3}, \frac{1}{2}]$ (as $0 \leq s \leq 1$). Consequently,

$$\frac{e(G)}{n^2} \leq f_1(x,s) \leq f_1(x_\star, s) = \frac{1}{4-s}. \tag{3.2}$$

On the other hand, we can give a lower bound on $t_{\max}(G)/n^2$ as follows. Select vertices $a \in A$, $b \in B$ and $c \in C$ uniformly at random. By the union bound,

$$\mathbb{P}(abc \text{ induces a triangle}) \geq \mathbb{P}(ab \in E(G)) - \mathbb{P}(bc \notin E(G)) - \mathbb{P}(ac \notin E(G)) = \alpha+\beta+\gamma-2 = s.$$

In particular, G must contain at least $sxy(1-x-y)n^3$ triangles. By averaging over all vertices $c \in C$ we have

$$\frac{t_{\max}(G)}{n^2} \geq \frac{sxy(1-x-y)n^3}{|C|n^2} = sxy.$$

Since $x \geq y \geq 1-x-y$, for fixed s and x, sxy is minimised by setting $y = \frac{1-x}{2}$. Thus

$$\frac{t_{\max}(G)}{n^2} \geq f_2(x,s) := \frac{sx(1-x)}{2}. \tag{3.3}$$

Having done these preparatory work, we can now prove the theorem by using the following claim.

Claim 3.6.

$$t_{\max}(G) \geq \begin{cases} e(G) - \frac{2}{9}n^2 & \textit{if } s \geq \frac{2}{3}, \\ \frac{3}{2}\left(e(G) - \frac{n^2}{4}\right) & \textit{if } s < \frac{2}{3}. \end{cases}$$

To see why Claim 3.6 implies Theorem 1.8, first assume $e(G) \geq \frac{3}{10}n^2$. By (3.2), we have $s \geq 2/3$. Then Claim 3.6 gives that $t_{\max}(G) \geq e(G) - \frac{2}{9}n^2$. Now assume $e(G) < \frac{3}{10}n^2$. If we still have $s \geq 2/3$, then by Claim 3.6,

$$t_{\max}(G) \geq e(G) - \frac{2}{9}n^2 > \frac{3}{2}\left(e(G) - \frac{n^2}{4}\right)$$

because $e(G) < \frac{11}{36}n^2$. Otherwise $s < 2/3$ and Claim 3.6 implies that $t_{\max}(G) \geq \frac{3}{2}\left(e(G) - \frac{n^2}{4}\right)$, as desired. □

Proof of Claim 3.6.
Case 1: $s \geq \frac{2}{3}$. By inequalities (3.1) and (3.3), we have

$$\frac{e(G)}{n^2} - \frac{t_{\max}(G)}{n^2} \leq f_1(x,s) - f_2(x,s) = x - x^2 + \frac{s}{4}(1-x)^2 - \frac{s}{2}x(1-x)$$

It is an easy exercise in calculus to show that as a function of $x \in (0,1)$, the right-hand side is maximized at $x_\star = \frac{2-2s}{4-3s} \leq \frac{1}{3}$ (as $s \geq \frac{2}{3}$), and is decreasing in $[x_\star, +\infty)$. Under our assumption $x \geq 1/3$, we thus have

$$f_1(x,s) - f_2(x,s) \leq f_1\left(\tfrac{1}{3}, s\right) - f_2\left(\tfrac{1}{3}, s\right) = \frac{2}{9}.$$

This implies that $t_{\max}(G) \geq e(G) - \frac{2}{9}n^2$.
Case 2: $0 < s < \frac{2}{3}$. By inequalities (3.1) and (3.3) we have

$$\frac{3}{2}\frac{e(G)}{n^2} - \frac{3}{8} - \frac{t_{\max}(G)}{n^2} \leq \frac{3}{2}f_1(x,s) - \frac{3}{8} - f_2(x,s) = \frac{3}{2}\left(x(1-x) - \frac{1}{4}\right) + \frac{s}{8}(1-x)(3-7x). \quad (3.4)$$

If $x \in [\frac{3}{7}, 1]$, then both terms on the right-hand side are non-positive. Assume now that $x \in [\frac{1}{3}, \frac{3}{7})$. Then for such values of x, the right-hand side is an increasing function of s. Applying our assumption on s, its value is at most

$$\frac{3}{2}f_1(x, \tfrac{2}{3}) - \frac{3}{8} - f_2(x, \tfrac{2}{3}) = -\frac{1}{8} + \tfrac{2}{3}x - \frac{11}{12}x^2.$$

The discriminant of this quadratic is $\frac{4}{9} - 4 \cdot \frac{1}{8} \cdot \frac{11}{12} = -\frac{1}{72} < 0$, so the expression above is (strictly) non-positive. We deduce that the right-hand side of (3.4) is non-positive for every value of $x \in [0,1]$. This yields $t_{\max}(G) \geq \frac{3}{2}\left(e(G) - \frac{1}{4}n^2\right)$. □

3.3 Flag algebra bounds

In this section we will employ Razborov's [55] flag algebra framework, and more specifically his semidefinite method, to obtain bounds for some of the problems we study. The semi-definite method has become a fairly standard tool in extremal combinatorics — see e.g. [57] for a survey of some of the early applications. As the method is well established and we have only obtained non-sharp bounds using it, we give only minimal details here, without expounding on the underlying theoretical machinery.

We have used *Flagmatic* to perform our flag algebra computations; this is an open source program written by Emil Vaughan, and later developed further by Jakub Sliacan [62], who currently maintains a Flagmatic page on GitHub [62]. We have used Vaughan's Flagmatic 2.0 in this paper. We refer the reader to [20] and to the Flagmatic 2.0 section on the webpage [62] for a description of the inner workings of *Flagmatic* and download links for the program. Our calculations involve the use of flag inequalities given as 'axioms'. The use of such 'axioms' first appeared in [18], where an edge-maximisation problem was solved subject to a codegree constraint. We refer a reader interested in the details to either Section 3 in that paper or to the Flagmatic 2.0 webpage [62].

Let T_1 denote the $([1], \emptyset)$-flag consisting of a triangle with one vertex labelled 1. Let ρ denote the $(\emptyset, \emptyset)$-flag consisting of a single 2-edge (this flag corresponds to the edge density). Let $f(\rho)$ be denote the upper bound on $\tau(\rho)$ given in Theorem 1.4.

The function $f(\rho)$ is piecewise linear, continuous and strictly increasing in the interval $(\frac{1}{2}, 1]$. In particular, it has a piecewise linear inverse. Over any subinterval $I \subseteq [\frac{1}{2}, 1]$ on which is f is linear, we can use semidefinite method to obtain an upper bound on how much $\tau(\rho)$ can deviate from $f(\rho)$ on I by giving an upper bound for the following problem.

Problem 3.7. *Maximise $\rho - f^{-1}(y)$ over $y \in f(I)$ subject to the constraint $T_1 \leq y$.*

Note the constraint we have given corresponds to requiring that all but $o(1)$ proportion of the vertices have triangle-degree at most $y\frac{n^2}{2} + o(n)$ (which is slightly weaker than what we require for τ). A standard flag algebra computation will give us an upper bound $\varepsilon_I > 0$ on the solution to Problem 3.7. If $f(x) = ax + b$ over the interval I, then this tells us that $f(x - \varepsilon_I) = a(x - \varepsilon_I) + b$ is a lower bound for $\tau(x)$ on the interval $I - \varepsilon_I := \{x \in I : \ x \leq \sup I - \varepsilon_I\}$, i.e that $f(\rho)$ is a most $a\varepsilon_I$ away from the true value of $\tau(\rho)$ on $I - \varepsilon$. Using this technique, we obtain the following:

Theorem 3.8.

$$\tau(\rho) \geq \begin{cases} f(\rho) - 0.0010705 & \text{if } \rho \in [\frac{1}{2}, \frac{29}{54}] \\ f(\rho) - 0.0044863 & \text{if } \rho \in [\frac{29}{54}, \frac{31}{54}] \\ f(\rho) - 0.0106917 & \text{if } \rho \in [\frac{31}{54}, \frac{11}{18}] \\ f(\rho) - 0.0106917 & \text{if } \rho \in [\frac{11}{18}, \frac{17}{27}] \\ f(\rho) - 0.0058198 & \text{if } \rho \in [\frac{17}{27}, \frac{35}{54}] \\ f(\rho) - 0.0002057 & \text{if } \rho \in [\frac{35}{54}, \frac{2}{3}] \\ f(\rho) - 0.00123143 & \text{if } \rho \in [\frac{2}{3}, \frac{25}{36}] \\ f(\rho) - 0.00534603 & \text{if } \rho \in [\frac{25}{36}, \frac{13}{18}] \\ f(\rho) - 0.00534583 & \text{if } \rho \in [\frac{13}{18}, \frac{53}{72}] \\ f(\rho) - 0.00189005 & \text{if } \rho \in [\frac{53}{72}, \frac{3}{4}] \end{cases}$$

Proof. The theorem follows from standard algebra computations using the method outline above. Running the script `theorem38.sage` which is found in the auxiliary files of this ArXiv submission on Flagmatic 2.0 yields the bounds claimed above. (The resulting computation certificates are somewhat large, but the computation itself can easily be run on a modern laptop computer.) □

We also 'zoom in' on the value $\rho = \rho_\star$ at which $\tau(\rho)$ becomes greater than $1 - \rho$, and which we conjecture is equal to $\frac{19}{27}$. This is done by giving an upper bound for the following variant of Problem 3.7:

Problem 3.9. *Maximise $1 - x - \rho$ subject to the constraint $T_1 \leq x$.*

Suppose for some fixed x we perform a flag algebra calculation and get a non-positive upper bound for the solution to Problem 3.9 . This implies that any n-vertex graph with at least $(1 - x)\frac{n^2}{2} + o(n^2)$ edges must have a positive proportion of its vertices having triangle-degree greater than $x\frac{n^2}{2} + o(n^2)$. In particular we must have $\rho_\star \leq x$. Using this technique, we obtain the following bounds on $\rho_\star$

Proposition 3.10.

$$\rho_\star \leq \frac{19}{27} + 7.4 \times 10^{-9}.$$

Proof. The theorem follows from standard algebra computations using the method outline above. Running the script `theorem310.sage` which is found in the auxiliary files of this ArXiv submission on Flagmatic 2.0 yields the bounds claimed above. (This is a much smaller computation than the one required for Theorem 3.8.) □

4 Concluding remarks

In earlier sections we showed that

$$c_1(K_4^{(3-)}) = \frac{\sqrt{13}-1}{6}, \quad \frac{19}{27} \leq c_1(K_4^{(3)}) \leq \frac{19}{27} + 7.4 \times 10^{-9}, \quad \text{and} \quad \frac{5}{9} \leq c_1(C_5^{(3)}) \leq 2 - \sqrt{2}.$$

We conjecture that $c_1(K_4^{(3)}) = \frac{19}{27}$ and $c_1(C_5^{(3)}) = \frac{5}{9}$.

4.1 Book numbers of graphs

In Section 3, we investigated the following question: let G be a graph on n vertices with $m > \mathrm{ex}(n, K_3^{(2)})$ edges. What is the largest t such that G must have some *vertex* contained in at least t triangles? A different but equally natural question is to ask: what is the largest b such that G must have some *edge* contained in at least b triangles? This is in fact a well-studied problem in graph theory.

Definition 4.1. *Let G be a 2-graph, and $xy \in E(G)$. The* book size *of xy in G is $\mathrm{bk}(xy) = \mathrm{bk}_G(xy) := |\Gamma(x) \cap \Gamma(y)|$, the number of triangles in G containing the edge xy. The* book number *of G is*

$$\mathrm{bk}(G) := \max\{\mathrm{bk}(xy) : \ xy \in E(G)\}.$$

The study of book numbers in graphs was initiated by Erdős in 1962 [14], and has attracted considerable attention in extremal graph theory and Ramsey theory. Set

$$\beta(n, m) := \min\{\mathrm{bk}(G) : \ v(G) = n, \ e(G) = m\},$$

and

$$\beta(x) := \inf_n\{\mathrm{bk}(G)/n : v(G) = n, \ e(G) \geq x\binom{n}{2}\},$$

Erdős conjectured that $\beta(n, \mathrm{ex}(n, K_3^{(2)}) + 1) > \frac{n}{6}$. This was proved by Edwards [13] and independently by Khadžiivanov–Nikiforov [37]. Bollobás and Nikiforov [6] determined $\beta(n, m)$ exactly for infinitely many value of m with $\frac{n^2}{4} < m < \frac{n^2}{3}$.

A construction giving the best known lower bound on $\beta(n, m)$ was given by Erdős, Faudree and Györi [15], generalising an earlier construction due to Erdős, Faudree and Rousseau [16].

Construction 4.2 (Erdős, Faudree and Györi [15])**.** *Suppose $n = r_1 \cdot r_2 \cdot r_3 \cdots r_{k-1} \cdot r_k t$, where $r_1, r_2, \ldots, r_k, t$ are strictly positive integers satisfying $(r_{i-1} - 1)^2 < r_i$ for every $i \in [k]$. Set*

$$V = \{(i_1, i_2, \ldots, i_k, i_{k+1}) : \ i_j \in [r_j] \text{ for all } j \in [k], \ i_{k+1} \in [t]\}.$$

Define a graph G on V by joining pairs of vectors from V by an edge if and only if they differ in each of the first k coordinates.

This construction gives rise to a d-regular graph with book number b, where $d = \prod_{i=1}^k \left(\frac{r_i - 1}{r_i}\right) n$ and $b = \prod_{i=1}^k \left(\frac{r_i - 2}{r_i}\right) n$. Erdős, Faudree and Györi conjectured this gives the correct behaviour for the minimum value of the book number in graphs subject to a minimum degree condition.

Conjecture 4.3 (Erdős, Faudree and Györi [15]). *Let $x \in \mathbb{Q}$ with $\frac{1}{2} < x < 1$. Let*

$$x = \prod_{i=1}^{k} \frac{r_i - 1}{r_i}$$

with $3 \leq r_1$ and $(r_{i-1} - 1)^2 < r_i$ for $2 \leq i \leq k$ be the (unique) "greedy representation" of x. Set

$$b(x) = \prod_{i=1}^{k} \frac{r_i - 2}{r_i}.$$

Then every graph on n vertices with minimum degree $d \geq xn$ has book number at least $b(x)n$.

We believe that the minimum degree condition in Conjecture 4.3 can be replaced by a size condition, and this belief seemed to be borne out by flag algebra computations we ran for this problem.

Conjecture 4.4. *Let $x \in \mathbb{Q} \cap (\frac{1}{2}, 1)$ and $b(x)$ be as above. Then $\beta(x) = b(x)$, i.e. any graph on n vertices with at least $x\frac{n^2}{2}$ edges has book number at least $b(x)n$.*

4.2 Maximal triangle-degree, book number and triangle density

In Sections 3 and 4.1, we discussed the maximum triangle-degree of a vertex and the book number (i.e. maximum triangle-degree of an edge) in graphs, giving conjectures on their minimum value for a given edge-density or minimum degree condition. Here we compare the conjectured behaviour of these two triangle-related extremal quantities with each other and with the minimal triangle density in graphs G with $\rho\binom{n}{2}$ edges for $\frac{1}{2} \leq \rho \leq \frac{2}{3}$.

Razborov [56] showed that such a graph G contain at least $\kappa(\rho)\binom{n}{3} + o(n^3)$ triangles, where

$$\kappa(\rho) = \frac{1}{18}\left(1 - \sqrt{2(2-3\rho)}\right)\left(2 + \sqrt{2(2-3\rho)}\right)^2.$$

In addition, Lo [42] showed that if the minimum degree of G is at least ρn, then it contains at least $\lambda(\rho)\binom{n}{3} + o(n^3)$ triangles, where

$$\lambda(\rho) = 3\rho(1-\rho)(2\rho - 1).$$

Conjecture 1.7 implies that every n-vertex graph with edge density ρ contains a vertex with triangle-degree at least $\tau'(\rho)\binom{n}{2} + o(n^2)$, where

$$\tau'(\rho) = \begin{cases} \frac{3}{2}\left(\rho - \frac{1}{2}\right) & \text{if } \frac{1}{2} \leq \rho \leq \frac{11}{18}, \\ \rho - \frac{4}{9} & \text{if } \frac{11}{18} \leq \rho \leq \frac{2}{3}. \end{cases}$$

Finally, let $\beta'(x)$ denote the function obtained by extending the function $b(x)$ from Conjecture 4.3 from the rationals in $(\frac{1}{2}, \frac{2}{3}]$ to a monotonically increasing left-continuous function on the whole interval. This last function unfortunately does not have a nice closed form, but we can plot an approximation of it (or rather: $\rho\beta'(\rho)$) along the other three in Figure 1, allowing for a visual comparison of the four functions κ, λ, τ' and $\rho\beta'$ in the interval $\rho \in (\frac{1}{2}, \frac{2}{3}]$.

Clearly by averaging we have that $\kappa(\rho)$ is the smallest of the functions in Figure 1. Assume now that Conjecture 1.7 is true. Then Constructions 3.2 and 3.3 provide ρn-regular graphs G of order n with $t_{\max}(G) = \tau(\rho)n^2/2 + o(n)$. Averaging the triangle-degree over all vertices, this would give that $\lambda(\rho) \leq \tau(\rho)$.

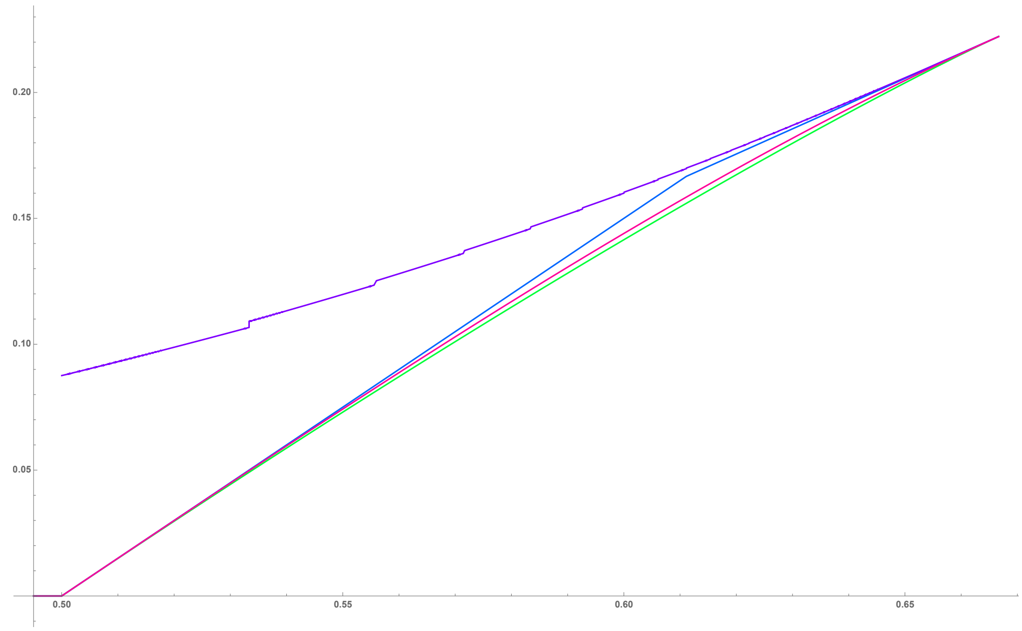

Figure 1: The functions $\rho \cdot \beta'(\rho)$, $\tau'(\rho)$, $\lambda(\rho)$, $\kappa(\rho)$ (from top to bottom)

Further assuming Conjecture 4.3 is true, Construction 4.2 gives a ρn-regular graph G of order n with book number βn that is triangle-degree regular with $t_{\max}(G) = \rho\beta(\rho)\binom{n}{2} + o(n^2)$. This would imply that $\tau(\rho) \leq \rho\beta(\rho)$, and all together,

$$\kappa(\rho) \leq \lambda(\rho) \leq \tau(\rho) \leq \rho\beta(\rho). \tag{4.1}$$

In Figure 1 we plotted the four functions $\kappa(\rho)$, $\lambda(\rho)$, $\tau'(\rho)$ and $\rho\beta'(\rho)$ in the interval $[\frac{1}{2}, \frac{2}{3}]$. As the plot shows, the inequalities in (4.1) with τ' and β' taking the place of τ and β all hold in $[\frac{1}{2}, \frac{2}{3}]$ with equality if and only if $\rho = \frac{1}{2}$ (for the first two inequalities) or $\frac{2}{3}$ (for all three).

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