Codegree thresholds for covering 3-uniform hypergraphs

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Abstract

Given two 3-uniform hypergraphs F and $G = (V, E)$, we say that G has an *F-covering* if we can cover V with copies of F. The *minimum codegree* of G is the largest integer d such that every pair of vertices from V is contained in at least d triples from E. Define $c_2(n, F)$ to be the largest minimum codegree among all n-vertex 3-graphs G that contain no F-covering. Determining $c_2(n, F)$ is a natural problem intermediate (but distinct) from the well-studied Turán problems and tiling problems. In this paper, we determine $c_2(n, K_4)$ (for $n > 98$) and the associated extremal configurations (for $n > 998$), where K_4 denotes the complete 3-graph on 4 vertices. We also obtain bounds on $c_2(n, F)$ which are apart by at most 2 in the cases where F is K_4^- (K_4 with one edge removed), K_5^-, and the tight cycle C_5 on 5 vertices.

1 Introduction

1.1 Notation

Given a set A and a positive integer k, we write $A^{(k)}$ for the collection of k-element subsets of A. Let $[n] = \{1, 2, \dots, n\}$. We shall often consider pairs or triples of vertices; when there is no risk of confusion, we write ab and abc as a shorthand for $\{a, b\}$ and $\{a, b, c\}$ respectively. A *k-uniform hypergraph*, or *k-graph*, is a pair $G = (V, E)$ where V is a vertex set and $E \subseteq V^{(k)}$ is an edge set. Fix a k-graph $G = (V, E)$. A *subgraph* of G is a k-graph $H = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$. Given a vertex set $S \subseteq V$, the *induced subgraph* of G on S, denoted by $G[S]$, is $(S, S^{(k)} \cap E)$. The *degree* of a vertex x in G, which we denote by $d(x)$, is the number of edges of G containing x. The *minimum degree* $\delta_1(G)$ of G is the minimum of $d(x)$ over all vertices x in G.

In this paper, we will focus on 3-graphs $G = (V, E)$ and another degree-like quantity, and its minimum: the *codegree* of a pair $xy \in V^{(2)}$, denoted by $d(x, y)$, is the number of edges of G containing the pair xy. We write $\Gamma(x, y)$ for the *neighbourhood* of the pair xy, i.e. the set of $z \in V \setminus \{x, y\}$ such that $xyz \in E$. The *minimum codegree* of G is $\delta_2(G) = \min_{xy \in V^{(2)}} d(x, y)$. The *link graph* G_x of a vertex $x \in V$ is the (2-)graph $(V \setminus \{x\}, \Gamma(x))$, where $\Gamma(x)$ is the collection of all pairs uv such that $xuv \in E$. Finally, we define the *edit distance* between two 3-graphs $G = (V, E)$ and $G' = (V, E')$ on a common vertex set V to be $|E \triangle E'|$, that is, the minimum number of changes required to make G isomorphic to G', where a change consists in replacing an edge by a non-edge and vice-versa.

*Research partially supported by NSF Grant DMS-1400073.

Let us introduce the 3-graphs relevant to the present work. Let $K_t = ([t], [t]^{(3)})$ denote the complete 3-graph on t vertices, and let K_t^- denote the 3-graph obtained from K_t by removing one 3-edge. The *strong* or *tight* t-cycle is the 3-graph C_t on $[t]$ with 3-edges $\{123, 234, 345, \ldots, (t-2)(t-1)t, (t-1)t1, t12\}$. We denote by $F_{3,2}$ the 3-graph $([5], \{123, 124, 125, 345\})$. Finally a *Steiner Triple System* (STS) is a 3-graph in which every pair of vertices is contained in exactly one 3-edge; it is a 168 years old result of Kirkman [20] that a STS on t vertices exists if and only if $t \equiv 1, 3 \mod 6$. The *Fano plane* is the unique (up to isomorphism) STS on 7 vertices, which we will usually denote by Fano.

1.2 The problem

Let F be a fixed 3-graph on t vertices with at least one edge. A 3-graph G has an *F-covering* if all vertices of G are contained in copies of F. For $n \geq t$ and $i = 1, 2$, we define

$$c_i(n, F) = \max\{\delta_i(G) : |V(G)| = n \text{ and } G \text{ does not have an } F\text{-covering}\}.$$

and call $c_1(n, F)$ the *covering degree-threshold* and $c_2(n, F)$ the *covering codegree-threshold* of F.

The covering threshold $c_i(n, F)$ was introduced by Han, Zang and Zhao [15] when they studied the minimum degree in 3-graphs that guarantees a K-tiling for any fixed complete 3-partite 3-graph K. It was shown implicitly in [15] that $c_1(n, K) = (6 - 4\sqrt{2} + o(1))\binom{n}{2}$ if K has at least two vertices in each part (in contrast, it is easy to see that $c_1(n, K) = o(n^2)$ if some part of K has only one vertex).

The concluding remarks in [15] also noted that for graphs, the covering problem is essentially equivalent to the Turán problem, with $c_1(n, F) = (1 - 1/(\chi(F) - 1) + o(1))n$ for any *graph* F, where $\chi(F)$ is the chromatic number of F. In particular, it is straightforward to show that for the complete graph $K_t^{(2)}$ we have $c_1(n, K_t^{(2)}) = \lfloor (1 - 1/(t-1))n \rfloor$. Indeed, the lower bound comes from considering a complete balanced $(t-1)$-partite graph. For the upper bound we observe that in an n-vertex graph with minimum degree d, every i vertices have at least $i(d-i+1) - (i-1)(n-i) = id - (i-1)n$ neighbors in common. Therefore every vertex (in fact, every copy of $K_{t-1}^{(2)}$) can be greedily extended to a copy of $K_t^{(2)}$ if $d > (t-2)n/(t-1)$. The asymptotics of $c_1(n, F)$ for an arbitrary graph F can be deduced from this observation and standard tools in extremal graph theory.

Our objective in this paper is to study the behaviour of the function $c_2(n, F)$ for various 3-graphs F. When determining the exact value of $c_2(F, n)$ is difficult, we may ask instead for its asymptotic behaviour. It is easy to see that the limit

$$c_2(F) = \lim_{n \to \infty} \frac{c_2(n, F)}{n - 2}$$

exists.[1] We call $c_2(F)$ the *covering codegree-density* of F.

1.3 Motivation and related work in extremal hypergraph theory

Before we state our results, let us give some motivation and background for our problem. Let F be a fixed 3-graph on t vertices with at least one 3-edge. A 3-graph G is *F-free* if it does

[1] This is a direct corollary of the proof of Proposition 6 from [7] on the existence of conditional codegree density (with an uncovered vertex x used as the conditional subgraph H), or can be proved in the same way as the existence of the usual codegree density $\gamma(F)$ in [28].

not contain a copy of F as a subgraph. Further G has an *F-tiling*, or *F-factor*, if we can cover $V(G)$ with vertex-disjoint *F-subgraphs* (subgraphs that are isomorphic to F). There has been much research into the degree and/or codegree conditions needed to ensure the existence of an F-subgraph or of an F-factor in a 3-graph G. Determining the degree/codegree condition necessary to guarantee an F-covering is intermediate between these two well-studied problems. As we show in the next subsection, the existence, covering, and tiling problems give rise to different thresholds in their codegree versions, so that our work is novel. It is hoped that studying the properties of the covering codegree-threshold function $c_2(n, F)$ — such as *supersaturation*, discussed in Section 4, which could be useful for applying semi-random methods to tiling problems — will lead to insights about both the existence and tiling problems.

The *Turán number* $\mathrm{ex}(n, F)$ of F is the maximum number of 3-edges an F-free 3-graph on n vertices can have. It is well-known that $\lim_{n\to\infty} \mathrm{ex}(n, F)/\binom{n}{3}$ exists; the limit $\pi(F)$ is known as the *Turán density* of F. The extremal theory of 3-graphs and within it the study of Turán-type problems have received extensive attention from the combinatorics community since the 1950s , with strenuous efforts devoted in particular to the (in)famous and still-open conjecture of Turán [33] that $\pi(K_4) = 5/9$. See the surveys of Füredi [11] and Keevash [17] for an overview of techniques and results.

Mubayi and Zhao [28] introduced another extremal function, *codegree density*, for 3-graphs. The *codegree-threshold* $\mathrm{ex}_2(n, F)$ of F is the maximum of $\delta_2(G)$ over all F-free 3-graphs G on n vertices. It was shown in [28] that $\lim_{n\to\infty} \mathrm{ex}_2(n, F)/(n - 2)$ exists; the limit $\gamma(F)$, sometimes denoted by $\pi_2(F)$, became known as the *codegree density* of F. The first result on codegree density was due to Mubayi [26], who showed $\gamma(\mathrm{Fano}) = 1/2$. In general, the codegree problems seem not easier than the Turán problems; only a few results [19, 16, 5, 6] were obtained in the last ten years. Using flag algebra methods, Falgas-Ravry, Marchant, Pikhurko and Vaughan [7] determined $\mathrm{ex}_2(n, F_{3,2})$ exactly for sufficiently large n, and Falgas-Ravry, Pikhurko, Vaughan and Volec [8] showed $\gamma(K_4^-) = 1/4$, resolving a conjecture of Nagle [30]. The codegree density analogue [4] of Turán's conjecture remains open. Certainly $\gamma(F) \leq c_2(F)$ for any 3-graph F, and it may be hoped that giving good upper bounds for the latter may also help bounding the former.

In addition to these Turán-type problems, there has been much research activity on the problem of determining thresholds for the existence of F-factors. The situation for ordinary (2-)graphs is now well-understood: the celebrated Hajnal–Szemerédi theorem [13] gives the exact minimum degree condition guaranteeing the existence of F-factors in an n-vertex graph when F is a clique, while Kühn and Osthus [22] determined the minimum degree condition for general graphs F up to an additive constant. On the other hand, until recently not much was known about tiling for k-graphs when $k \geq 3$. While there has been a spate of results in the last few years, see [2, 3, 12, 14, 18, 21, 23, 24, 29, 32], many more open problems remain. We refer to the surveys of Rödl and Ruciński [31] and Zhao [34] for a detailed discussion of the area, and briefly mention below four results relevant to the present work. For $i \in \{1, 2\}$ and $n \equiv 0 \mod |V(F)|$, let

$$t_i(n, F) = \max\{\delta_i(G) : \ |V(G)| = n \text{ and } G \text{ does not have an } F\text{-factor}\}.$$

Trivially $c_i(n, F) \leq t_i(n, F)$ for any 3-graph F with at least one edge. Lo and Markström [24, 23] determined $t_2(n, F)$ asymptotically when $F = K_4$ and $F = K_4^-$. Independently Keevash and Mycroft [18] determined $t_2(n, K_4)$ exactly, and recently Han, Lo, Treglown and Zhao [14] determined $t_2(n, K_4^-)$ exactly as well, in both cases for n sufficiently large. Finally in [15] Han, Zang and Zhao asymptotically determined $t_1(n, K)$ for all complete 3-partite 3-graphs K. In particular, they showed that $t_1(n,K) = c_1(n,K) = (6-4\sqrt{2}+o(1))\binom{n}{2}$ for *certain* K. This gives further motivation for the present paper: by determining $c_2(n,F)$ for 3-graphs F, we may hope likewise to shed light on $t_2(n,F)$ and facilitate its (asymptotic) computation.

1.4 Results

In this paper, we determine the covering codegree-threshold for K_4 for sufficiently large n.

Theorem 1.1. *For every $n \in \mathbb{N}$, $\lfloor \frac{2n-5}{3} \rfloor \le c_2(K_4, n) \le \lfloor \frac{2n-3}{3} \rfloor$. Furthermore, for every $n > 98$,*

$$c_2(n, K_4) = \left\lfloor \frac{2n-5}{3} \right\rfloor.$$

In addition, we determine $c_2(F)$ when F is K_4^-, the strong 5-cycle C_5, and K_5^- — in fact in each case we give upper and lower bounds on $c_2(n,F)$ differing by at most 2.

Theorem 1.2. *Suppose $n = 6m + r$ for some $r \in \{0,1,2,3,4,5\}$ and $m \in \mathbb{N}$, with $n \ge 7$. Then*

$$c_2(n, K_4^-) = \begin{cases} 2m-1 \text{ or } 2m & \text{if } r = 0 \\ 2m & \text{if } r \in \{1,2\} \\ 2m \text{ or } 2m+1 & \text{if } r \in \{3,4\} \\ 2m+1 & \text{if } r = 5. \end{cases}$$

In particular, $c_2(K_4^-) = \frac{1}{3}$.

Theorem 1.3. $\lfloor \frac{n-3}{2} \rfloor \le c_2(n, C_5) \le \lfloor \frac{n}{2} \rfloor$. *In particular, $c_2(C_5) = \frac{1}{2}$.*

Interestingly, there is no unique stable near-extremal configuration for Theorem 1.3: at least two configurations at edit distance $\Omega(n^3)$ of each other exist, see Remark 3.3.

Theorem 1.4. $\lfloor \frac{2n-5}{3} \rfloor \le c_2(n, K_5^-) \le \lfloor \frac{2n-2}{3} \rfloor$. *In particular, $c_2(K_5^-) = \frac{2}{3}$.*

Let us compare the (existence) codegree density γ, the Turán density π, the covering codegree density c_2, and the tiling codegree density t_2 of K_4, K_4^-, and C_5 in the following table (for a 3-graph F of order f, define $t_2(F) = \lim_{n=mf\to\infty} t_2(n,F)/(n-2)$ if this limit exists). In the table question marks indicate conjectures, except for $t_2(C_5)$, for which we are not aware of any conjecture.

	γ	π	c_2	t_2
K_4	1/2 ? [4]	5/9 ? [33]	2/3	3/4 [18, 24]
K_4^-	1/4 [8]	2/7 ? [10]	1/3	1/2 [23]
C_5	1/3 ? [25]	$2\sqrt{3}-3$? [27]	1/2	?

Finally we give bounds on $c_2(\text{Fano})$, $c_2(F_{3,2})$ and $c_2(K_t)$ for $t \ge 5$, and pose a number of questions.

Our paper is structured as follows. In Section 2, we determine the codegree covering threshold for K_4 and characterize the extremal configurations. In Section 3, we prove our bounds on $c_2(n,F)$ for the other 3-graphs F mentioned above. We end in Section 4 with some discussion and questions.

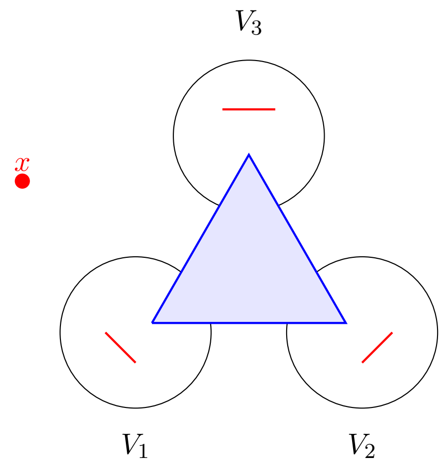

Figure 1: The complement of $F_1(n)$. The red pairs and the blue triples are absent from the link graph of x in F_1 and from $E(F_1)$ respectively.

2 The covering codegree-threshold for K_4

In this section we determine the covering codegree-threshold $c_2(n, K_4)$. We give a lower bound construction in Section 2.1 and prove the upper bound in Section 2.2. Finally, in Section 2.3 we provide other extremal constructions, and state a stability theorem that helps to show that these constructions are all possible extremal configurations; as the proofs of these latter results are similar to the proof of Theorem 1.1 we omit them here (we refer an interested reader to the appendix to the arxiv version of this paper [9] for the details).

2.1 Lower bound

Proof of the lower bound in Theorem 1.1. We construct a 3-graph $F_1(n)$ on $V = [n]$. Select a special vertex x. Split the remainder of the vertices into three parts $V_1 \sqcup V_2 \sqcup V_3 = V \setminus \{x\}$ with sizes as equal as possible,

$$|V_3| - 1 \leq |V_1| \leq |V_2| \leq |V_3|.$$

Put in as the link graph of x all pairs between distinct parts, i.e. add in all triples of the form xV_iV_j for $i \neq j$. Further, add in all triples not containing x and meeting at most two of the three parts $(V_i)_{i=1}^3$. Denote the resulting 3-graph by $F_1 = F_1(n)$. The complement of $F_1(n)$ is shown in Figure 1.

Observe that x is contained in no copy of K_4 in F_1: the only triangles in the link graph of x are tripartite, and thus are not covered by any triple of the 3-graph. Let us now compute the minimum codegree of F_1. For $u, u' \in V_i$ and $v \in V_j$ $(j \neq i)$, we have $d(u, x) = n - 1 - |V_i|$, $d(u, u') = n - 3$ and $d(u, v) = |V_i| + |V_j| - 1$. The minimum codegree $\delta_2(F_1)$ is thus $n - 2 - \lceil \frac{n-1}{3} \rceil$, attained by pairs $(v_1, v_2) \in V_1 \times V_2$ (and possibly by pairs from $V_2 \times V_3$ or $V_3 \times V_1$). Writing $n = 3m + r$ with $r \in \{0, 1, 2\}$ and $m \in \mathbb{N}$, we have shown that

$$c_2(3m + r, K_4) \geq \delta_2(F_1) = \begin{cases} 2m - 2 & \text{if } r = 0 \\ 2m - 1 & \text{if } r = 1 \text{ or } 2. \end{cases}$$

This lower bound can be expressed more compactly as $c_2(n, K_4) \geq \lfloor \frac{2n-5}{3} \rfloor$. $\square$

2.2 Upper bound

Let us give a general upper bound for $c_2(n, F)$ (in terms of the 'degeneracy' of F), which turns out to be surprisingly tight in some cases.

Lemma 2.1. *Suppose that F is a 3-graph of order f with at least one edge. Let r be* degeneracy *of F, namely, the maximum of $\delta_1(F')$ among all subgraphs F' of F. Then $c_2(n, F) \leq \lfloor (1-1/r)n + (f-2r-1)/r \rfloor$.*

Proof. We order the vertices of F as $x_1, \ldots, x_f$ such that x_i is a vertex of minimum degree in the subgraph induced by $\{x_1, \ldots, x_i\}$. As $r = \max \delta_1(F')$ among all subgraphs F' of F, we know that x_i has at most r *neighbours* (namely, the pairs of vertices that form edges with x_{i+1} in F) among $x_1, \ldots, x_i$.

Let $G = (V, E)$ be a 3-graph on n vertices with minimum codegree

$$\delta_2 := \delta_2(G) > \frac{r-1}{r} n + \frac{f-1}{r} - 2.$$

Fix two arbitrary vertices v_1, v_2 of G. We will find a copy of F in G by first mapping x_1 to v_1, x_2 to v_2, and x_3 to any $v_3 \in \Gamma_G(v_1, v_2)$. Suppose that $x_1, \ldots, x_i$ have been embedded to $v_1, \ldots, v_i$. In order to embed x_{i+1}, we consider the neighbours of x_{i+1} among $x_1, \ldots, x_i$. There are $t \leq r$ such neighbours. Assume that they have been mapped to pairs $p_1, \ldots p_t$ of $v_1, \ldots, v_i$. Since each pair p_j has degree at least δ_2 in G, its neighbourhood $\Gamma(p_i)$ misses at most $n - 2 - \delta_2$ vertices in $V \setminus \{v_1, \ldots, v_i\}$. By the definition of δ_2 and $i \leq f-1$, we have $r(n-2-\delta_2) < n - i$. Hence there exists a vertex $v_{i+1} \in V \setminus \{v_1, \ldots, v_i\}$ such that v_{i+1} is a common neighbour of $p_1, \ldots p_t$. Mapping x_{i+1} to v_{i+1} and continuing the embedding process, we obtain a copy of F as desired. $\square$

Remark 2.2. *The proof of Lemma 2.1 actually shows that if $\delta_2(G) > (1-1/r)n + (f-2r-1)/r$ then every* edge *of G is covered by a copy of F.*

Applying Lemma 2.1 with $F = K_4$ and $r = 3$, we obtain that $c_2(n, K_4) \leq \lfloor \frac{2n-3}{3} \rfloor$. When $n \equiv 1 \mod 3$, this implies that $c_2(n, K_4) \leq \lfloor \frac{2n-5}{3} \rfloor$. Together with the lower bound $c_2(n, K_4) \geq \lfloor \frac{2n-5}{3} \rfloor$, we obtain $c_2(n, K_4) = \lfloor \frac{2n-5}{3} \rfloor$ immediately.

When $n \equiv 0$ or $2 \mod 3$, more work is required to reduce the upper bound to $\lfloor \frac{2n-5}{3} \rfloor$. In both cases, we shall make use of the following simple observation.

Lemma 2.3. *Let G be a 3-graph on $n \geq 4$ vertices. Suppose that $x \in V(G)$ is not covered by any copy of K_4 and there exist $a, b, c \in V(G)$ such that $abx, bcx, acx \in E(G)$ (thus $abc \notin E(G)$). Let $S = \{a, b, c, x\}$ and for each vertex $y \in V(G) \setminus S$, let S_y consist of all the pairs of S that make a 3-edge with y in G. Then S_y must be a subset of one of the following sets:*

$$S^{1,c} = \{ax, bx, ac, bc\}, \quad S^{1,b} = \{ax, cx, ab, bc\}, \quad S^{1,a} = \{bx, cx, ab, ac\},$$

$$S^{2,a} = \{ab, ac, bc, ax\},\ S^{2,b} = \{ab, ac, bc, bx\},\ S^{2,c} = \{ab, ac, bc, cx\},\ S^3 = \{ax, bx, cx\}.$$

In particular, $|S_y| \leq 4$. $\square$

Proof of $c_2(n, K_4) \le \lfloor (2n-5)/3 \rfloor$ when 3 divides n. Since 3 divides n, we have $\lfloor (2n-5)/3 \rfloor = 2n/3 - 2$. Let $G = (V, E)$ be a 3-graph on n vertices with $\delta_2(G) \ge 2n/3 - 1$. We claim that all vertices of G are covered by copies of K_4. Suppose instead, that some vertex $x \in V$ is not contained in a copy of K_4. Since the minimum degree in the link graph G_x of x is at least $2n/3 - 1 > (n-1)/2$, there exists a triangle $\{ab, bc, ac\}$ in G_x. This implies that $abc \notin E$. Set $S = \{a, b, c, x\}$. For each vertex $y \in V \setminus S$, by Lemma 2.3, at most four pairs of S form edges of G with y. Thus, by the codegree assumption,

$$6\left(\frac{2n}{3} - 1\right) \le d(a,x) + d(b,x) + d(c,x) + d(a,b) + d(b,c) + d(c,a) \le 4(n-4) + 9,$$

a contradiction. □

When $n \equiv 2 \mod 3$, we start the proof in the same way. However, since we only have $\delta_2(G) \ge \lfloor (2n-5)/3 \rfloor + 1 = (2n-4)/3$, we will not obtain a contradiction until we prove that G has a similar structure as the 3-graph $F_1(n)$ given in Section 2.1. Part of the difficulty is that in this case there is some 'slack' in the codegree condition, so that more extremal constructions are possible (see Section 2.3).

Proof of $c_2(n, K_4) \le \lfloor \frac{2n-5}{3} \rfloor$ when $n \equiv 2 \mod 3$. Suppose $n = 3m + 2 > 98$. Consider a 3-graph $G = (V, E)$ on n vertices satisfying $\delta_2(G) \ge (2n-4)/3$.

Suppose that a vertex x of G is not contained in any copy of K_4. As $(2n-4)/3 > (n-1)/2$, the link graph G_x contains a triangle $\{ab, bc, ac\}$. Set $S = \{a, b, c, x\}$ and for each $y \in V \setminus S$, define S_y as in Lemma 2.3. By Lemma 2.3, S_y is a subset of $S^{1,c}, S^{1,b}$, $S^{1,a}$, $S^{2,a}$, $S^{2,b}$, $S^{2,c}$ or S^3. For $i \in \{1, 2\}$ and $j \in \{a, b, c\}$, write $s_{i,j}$ for the number of vertices $y \in V \setminus S$ for which $S_y = S^{i,j}$, and write s_i for the sum $s_{i,a} + s_{i,b} + s_{i,c}$. Finally let s_0 be the number of vertices $y \in V \setminus S$ such that $S_y \ne S^{i,j}$ for any $i \in \{1, 2\}$ and $j \in \{a, b, c\}$. Note that $|S_y| \le 3$ for such y. We know that $s_1 + s_2 + s_0 = n - 4$. Furthermore, by the codegree assumption,

$$3\left(\frac{2n-4}{3}\right) \le d(a,x) + d(b,x) + d(c,x) \le 2s_1 + s_2 + 3s_0 + 6, \tag{1}$$

$$6\left(\frac{2n-4}{3}\right) \le d(a,x) + d(b,x) + d(c,x) + d(a,b) + d(b,c) + d(c,a) \le 4s_1 + 4s_2 + 3s_0 + 9. \tag{2}$$

Substituting $s_0 = n - 4 - s_1 - s_2$ into (1) and (2) yields that $s_1 + 2s_2 \le n - 2$ and $s_1 + s_2 \ge n - 5$, respectively. Combining the two inequalities we have just obtained, we get

$$s_2 \le 3 \quad \text{and} \quad s_1 \ge n - 8.$$

We now show that the weight of s_1 splits almost equally between $s_{1,a}$, $s_{1,b}$, $s_{1,c}$. Note that

$$\frac{2n-4}{3} \le d(b,c) \le n - 3 - s_{1,a},$$

from which it follows that $s_{1,a} \le \frac{n-5}{3}$. Similarly we derive that $s_{1,b}, s_{1,c} \le (n-5)/3$. Consequently

$$s_{1,a} = s_1 - s_{1,b} - s_{1,c} \ge n - 8 - 2\left(\frac{n-5}{3}\right) = \frac{n-14}{3}.$$

Similarly $s_{1,b}$ and $s_{1,c}$ satisfy the same lower bound. Let $A = \{y \in V \setminus S : \; S_y = S^{1,a}\} \cup \{a\}$, $B = \{y \in V \setminus S : \; S_y = S^{1,b}\} \cup \{b\}$ and $C = \{y \in V \setminus S : \; S_y = S^{1,c}\} \cup \{c\}$. Set $V' = A \cup B \cup C \cup \{x\}$. Then we have just shown the following lemma.

Lemma 2.4.

$$|V'| = 1 + |A| + |B| + |C| \geq n - 4, \quad \text{and} \quad \frac{n-11}{3} \leq |A|, |B|, |C| \leq \frac{n-2}{3}. \quad \square$$

Let $\mathcal{B}$ be the collection of 3-edges of G of the form xAA, xBB, xCC (the 'bad' triples). Let $\mathcal{M}$ be the collection of non-edges of G of the form xAB, xAC, xBC (the 'missing' triples). Viewing $\mathcal{B}$ and $\mathcal{M}$ as 3-graphs on V', for two distinct vertices $v_1, v_2 \in V'$, we let $d_{\mathcal{B}}(v_1, v_2)$ denote their codegree in $\mathcal{B}$ and $d_{\mathcal{M}}(v_1, v_2)$ their codegree in $\mathcal{M}$.

Claim 2.5. *For every $v \in V' \setminus \{x\}$, $d_{\mathcal{B}}(v, x) \leq 4$.*

Proof. Suppose without loss of generality that $v \in A$. If $v = a$, then $d_{\mathcal{B}}(v, x) = 0$ because G contains no 3-edges of the form xaA. We thus assume that $v \neq a$. The bad triples for the pair (v, x) are triples of the form $a'vx$ for $a' \in A \setminus \{a, v\}$. Suppose $a'vx \in \mathcal{B}$. Then since there is no K_4 in G containing x, and since, by the definition of A, $a'bx$, vbx, $a'cx$ and vcx are all in G, it must be the case that both of $a'vb$ and $a'vc$ are missing from G. Further if $c' \in C \cap \Gamma(v, x)$ then all of $c'vx, bvx, c'bx$ are in G, whence $bc'v$ is absent from G. Similarly for any $b' \in B$, at most one of $b'cv$, $b'xv$ is in G. Finally since $bcv \notin E(G)$, b and c are contained in exactly one of $\Gamma(b, v)$, $\Gamma(c, v)$, and $\Gamma(x, v)$. To summarize, a vertex y in V' can lie in at most two of $\Gamma(b, v)$, $\Gamma(c, v)$ and $\Gamma(x, v)$ unless y is in $\Gamma_{\mathcal{B}}(x, v)$ (and lies in exactly one of those joint neighbourhoods) or is in $\{b, c, v\}$ (and lies in at most one of those joint neighbourhoods). Together with our codegree assumption, this gives us

$$\begin{aligned} 3\left(\frac{2n-4}{3}\right) \leq d(b, v) + d(c, v) + d(x, v) &\leq 2|V'| - d_{\mathcal{B}}(v, x) - 4 + 3(n - |V'|) \\ &= 3n - |V'| - 4 - d_{\mathcal{B}}(v, x) \leq 2n - d_{\mathcal{B}}(v, x), \end{aligned}$$

where we apply $|V'| \geq n - 4$ from Lemma 2.4 in the last inequality. It follows that $d_{\mathcal{B}}(v, x) \leq 4$, as claimed. $\square$

Claim 2.6. *For every $v \in V' \setminus \{x\}$, $d_{\mathcal{M}}(v, x) \leq 8$.*

Proof. Suppose without loss of generality that $v \in A$. Then by the codegree assumption, Claim 2.5 and the bound on $|A|$ from Lemma 2.4 we have

$$\frac{2n-4}{3} \leq d(v, x) \leq n - 1 - |A| + d_{\mathcal{B}}(v, x) - d_{\mathcal{M}}(v, x) \leq n - 1 - \frac{n-11}{3} + 4 - d_{\mathcal{M}}(v, x),$$

which gives that $d_{\mathcal{M}}(v, x) \leq 8$ as claimed. $\square$

Claim 2.7. *For every $y \in V(G) \setminus \{x\}$, $\Gamma(x, y)$ has a non-empty intersection with exactly two of the parts A, B and C.*

Proof. Let $y \in V(G) \setminus \{x\}$. Set $A_y = A \cap \Gamma(x,y)$, $B_y = B \cap \Gamma(x,y)$, and $C_y = C \cap \Gamma(x,y)$. We first observe that at most one of A_y, B_y, C_y is empty. Indeed, suppose that two of them, say A_y and B_y, are empty. Then by the codegree condition and Lemma 2.4,

$$\frac{2n-4}{3} \le d(x,y) \le |C_y| + \big(n - |V'|\big) \le \frac{n-2}{3} + 4,$$

which implies that $n \le 14$, a contradiction.

Now suppose none of A_y, B_y, C_y is empty. Fix $a' \in A_y$. For $b' \in B_y$, if $b' \in \Gamma(a', x)$, then $a'b'y \notin E(G)$ – otherwise $\{a', b', x, y\}$ spans a copy of K_4. Similarly, for $c' \in C_y \cap \Gamma(a', x)$, we have $a'c'y \notin E(G)$. Hence,

$$\frac{2n-4}{3} \le d(a', y) \le n - 2 - |B_y \cap \Gamma(a', x)| - |C_y \cap \Gamma(a', x)|.$$

Claim 2.6 gives that $d_{\mathcal{M}}(a', x) \le 8$. Consequently,

$$|B_y \cap \Gamma(a', x)| + |C_y \cap \Gamma(a', x)| = |B_y| + |C_y| - d_{\mathcal{M}}(a', x) \ge |B_y| + |C_y| - 8.$$

This implies that

$$\frac{2n-4}{3} \le n - 2 - |B_y| - |C_y| + 8,$$

which yields $|B_y| + |C_y| \le (n+22)/3$. Similarly by considering any vertex $b' \in B_y$ and any vertex $c' \in C_y$ we obtain that

$$|A_y| + |C_y| \le \frac{n+22}{3} \quad \text{and} \quad |A_y| + |B_y| \le \frac{n+22}{3}.$$

Summing these three inequalities and dividing by 2, we obtain that

$$|A_y| + |B_y| + |C_y| \le \frac{n+22}{2}.$$

Furthermore, by the codegree condition,

$$\frac{2n-4}{3} \le d(x,y) \le |A_y| + |B_y| + |C_y| + \big(n - |V'|\big) \le \frac{n+22}{2} + 4,$$

where we apply $|V'| \ge n - 4$ from Lemma 2.4. Rearranging terms yields $\frac{n}{6} \le \frac{49}{3}$, which contradicts our assumption $n > 98$. $\square$

Set $V_1 = \{y \in V \setminus \{x\} : \ \Gamma(x,y) \cap A = \emptyset\}$, $V_2 = \{y \in V \setminus \{x\} : \ \Gamma(x,y) \cap B = \emptyset\}$ and $V_3 = \{y \in V \setminus \{x\} : \ \Gamma(x,y) \cap C = \emptyset\}$. Without loss of generality, assume that

$$|V_1| \le |V_2| \le |V_3|. \tag{3}$$

Claim 2.7 shows that $V_1 \sqcup V_2 \sqcup V_3$ is a partition of $V(G) \setminus \{x\}$. We now show that $A \subseteq V_1$, $B \subseteq V_2$, and $C \subseteq V_3$. Consider $a' \in A$. By Claim 2.6, $a'xv \in E(G)$ for all but at most 8 vertices $v \in B \cup C$. By Lemma 2.4, we have $|B|, |C| \ge (n-11)/3 > 8$ (as $n > 35$). Thus $\Gamma(a', x)$ has non-empty intersections with B and C. We then deduce from Claim 2.7 that $\Gamma(a', x) \cap A = \emptyset$ and that $A \subseteq V_1$. Similarly we have $B \subseteq V_2$ and $C \subseteq V_3$.

Let $c' \in C$. By the definition of V_3, we have $\Gamma(c', x) \subseteq V_1 \cup V_2$. By the codegree assumption, it follows that

$$\frac{2n-4}{3} \le d(c', x) \le |V_1| + |V_2| = n - 1 - |V_3|, \tag{4}$$

from which we get that $|V_3| \le (n+1)/3$. Since $n = 3m+2$ and $|V_3| \ge \lceil (n-1)/3 \rceil = (n+1)/3$ by our assumption (3), we deduce that $|V_3| = (n+1)/3 = m+1$ and $|V_1| \le |V_2| \le (n+1)/3$.

Claim 2.8. *Let $y \in V_i$. Then $\Gamma(x, y)$ contains all but at most 6 vertices from $\bigcup_{j \ne i} V_j$ and no vertex from V_i.*

Proof. Suppose without loss of generality that $y \in V_1$. Then by Claim 2.7, $A \cap \Gamma(x, y) = \emptyset$. Thus

$$\frac{2n-4}{3} \le d(x, y) \le |\Gamma(x, y) \cap (V_2 \cup V_3)\,| + |\Gamma(x, y) \cap (V_1 \setminus A)\,| \le |\Gamma(x, y) \cap (V_2 \cup V_3)\,| + 4$$

since $|V_1 \setminus A| \le n - |V'| \le 4$ by Lemma 2.4. Hence $|\Gamma(x, y) \cap (V_2 \cup V_3)\,| \ge (2n-16)/3$. Since $|V_i| \le (n+1)/3$ for all i,

$$|\,(V_2 \cup V_3) \setminus \Gamma(x, y)| \le 2\,\frac{n+1}{3} - \frac{2n-16}{3} = 6.$$

This establishes the first part of our claim.

For the second part of our claim (namely, $\Gamma(y, x) \cap V_1 = \emptyset$), suppose that $yy'x \in E(G)$ for some $y' \in V_1$. Then $\Gamma(y, y') \cap \Gamma(y, x) \cap \Gamma(y', x) = \emptyset$. Consequently,

$$\frac{2n-4}{3} \le d(y, y') \le 1 + |V_1| - 2 + \left|(V_2 \cup V_3) \setminus \big(\Gamma(y, x) \cap \Gamma(y', x)\big)\right| \le 1 + \frac{n+1}{3} - 2 + 2 \cdot 6$$

where in the last inequality we apply $|V_1| \le (n+1)/3$ and the first part of the claim. This implies that $n \le 38$, a contradiction. □

Claim 2.8 implies that $\Gamma(v_3, x) \subseteq V_1 \cup V_2$ for all $v_3 \in V_3$, and in particular $d(v_3, x)$ satisfies (4) with two inequalities replaced by equalities. Consequently all triples of the form xvv_3 with $v_3 \in V_3$ and $v \in V_1 \cup V_2$ are in $E(G)$. Claim 2.8 also implies that most $v_1 \in V_1$ and $v_2 \in V_2$ satisfy $xv_1v_2 \in E(G)$. Fix such v_1 and v_2. Then $v_1v_2v_3 \notin E(G)$ for any $v_3 \in V_3$ otherwise $xv_1v_2v_3$ induces a copy of K_4. We thus have

$$2m \le d(v_1, v_2) \le |V_1| + |V_2| - 2 = 2m - 2,$$

a contradiction. This completes the proof of Theorem 1.1 in the case $n = 3m+2$. □

2.3 Other extremal constructions and stability

Recall the construction of $F_1(n)$ described in Section 2.1. There are other codegree-extremal families of 3-graphs for K_4-covering that are not isomorphic to subgraphs of $F_1(n)$. These families can be obtained by perturbing $F_1(n)$ slightly. Explicitly, let $V = [n]$. Select a special vertex x, and split the remainder of the vertices into three parts V_1, V_2, V_3 with part sizes m_1, m_2, m_3 summing to $n-1$. Put in as the link graph of x all pairs between distinct parts, and then add in all triples not containing x and meeting at most two of the three parts $(V_i)_{i=1}^3$. Let $F_1(m_1, m_2, m_3)$ denote the resulting 3-graph. Now, given a collection of $\mathcal{E}$ of pairs of vertices from the link graph of x, let $F_1(m_1, m_2, m_3, \mathcal{E})$ denote the 3-graph obtained from $F_1(m_1, m_2, m_3)$ by deleting xuv and adding all tripartite triples that contain u and v for all $uv \in \mathcal{E}$ (a triple is *tripartite* if it meets all three parts $(V_i)_{i=1}^3$). It is easy to see that $F_1(m_1, m_2, m_3, \mathcal{E})$ contains no K_4 covering x. A suitable choice of $\mathcal{E}$ will ensure we also maintain a high minimum codegree.

Case 1: $n = 3m$**.** Take $m_1 = m - 1$ and $m_2 = m_3 = m$. Call a collection $\mathcal{E}$ of pairs of vertices from the link graph of x *admissible* if (i) every vertex $v_1 \in V_1$ is contained in at most two pairs from $\mathcal{E}$, and (ii) every vertex $v \in V_2 \sqcup V_3$ is contained in at most one pair from $\mathcal{E}$. Then for any admissible $\mathcal{E}$, we have $\delta_2(F_1(m-1, m, m, \mathcal{E})) = \delta_2(F_1(m-1, m, m)) = 2m - 2$.

Case 2: $n = 3m + 1$**.** Take $m_1 = m_2 = m_3 = m$. Call a collection $\mathcal{E}$ of pairs of vertices from the link graph of x *admissible* if every vertex is contained in at most one pair from $\mathcal{E}$. Then for any admissible $\mathcal{E}$, we have $\delta_2(F_1(m, m, m, \mathcal{E})) = \delta_2(F_1(m, m, m)) = 2m - 1$.

Case 3a: $n = 3m + 2$**.** Take $m_1 = m_2 = m$, and $m_3 = m + 1$. Call a collection $\mathcal{E}$ of pairs of vertices from the link graph of x *admissible* if (i) every vertex $v \in V_1 \sqcup V_2$ is contained in at most 2 pairs from $\mathcal{E}$ and (ii) every vertex $v_3 \in V_3$ is contained in at most 1 pair from $\mathcal{E}$. Then for any admissible $\mathcal{E}$, we have $\delta_2(F_1(m, m, m + 1, \mathcal{E})) = \delta_2(F_1(m, m, m + 1)) = 2m - 1$.

Case 3b: $n = 3m + 2$**.** Take $m_1 = m - 1$ and $m_2 = m_3 = m + 1$. Call a collection $\mathcal{E}$ of pairs of vertices from the link graph of x *admissible* if (i) every vertex $v_1 \in V_1$ is contained in at most 3 pairs from $\mathcal{E}$ and (ii) every vertex $v \in V_2 \sqcup V_3$ is contained in at most 1 pair from $\mathcal{E}$. Then for any admissible $\mathcal{E}$, we have $\delta_2(F_1(m-1, m+1, m+1, \mathcal{E})) = \delta_2(F_1(m-1, m+1, m+1)) = 2m - 1$.

We can show that the above constructions are *all* extremal configurations for n sufficiently large ($n > 998$). This can be done by first proving the following stability theorem.

Theorem 2.9. *Suppose $n \geq 4$ and $0 < \delta \leq \frac{1}{429}$. Suppose that G is a 3-graph on n vertices with minimum codegree $\delta_2(G) \geq \left(\frac{2}{3} - \delta\right) n$ and that there is a vertex $x \in V(G)$ not contained in any copy of K_4 in G. Then there exists a tripartition $V_1 \sqcup V_2 \sqcup V_3$ of $V(G) \setminus \{x\}$ such that the following holds for all $i \in [3]$ and $j \neq i$:*

(i) there is no triple in G of the form xV_iV_i;

(ii) all but at most $9\delta n^2$ triples of the form xV_iV_j are in G;

(iii) there are at most $4\delta n^3$ triples in G of the form $V_1V_2V_3$;

(iv) all but at most $6\delta n^3$ triples of the form $V_iV_iV_j$ are in G;

(v) $\left||V_i| - \frac{n-1}{3}\right| \leq 2\delta n$.

Note that Theorem 2.9 implies not only that G is 'close' to $F_1(n)$, but also that the link graphs of the uncovered vertices in G and $F_1(n)$ are 'close' to each other. We elaborate on the implications of this fact in Section 4.

Theorem 2.10. *1. For $n = 3m > 855$, the extremal configurations for $c_2(n, K_4)$ are isomorphic to a subgraph of $F_1(m-1, m, m, \mathcal{E})$ for some admissible $\mathcal{E}$.*

2. For $n = 3m+1 > 712$, the extremal configurations for $c_2(n, K_4)$ are isomorphic to a subgraph of $F_1(m, m, m, \mathcal{E})$ for some admissible $\mathcal{E}$.

3. *For $n = 3m+2 > 998$, the extremal configurations for $c_2(n, K_4)$ are isomorphic to a subgraph of $F_1(m, m, m+1, \mathcal{E})$ or to a subgraph of $F_1(m-1, m+1, m+1, \mathcal{E})$ for some admissible $\mathcal{E}$.*

The proof of Theorem 2.9 is very similar to that of the case $n = 3m + 2$ of Theorem 1.1, while the proof of Theorem 2.10 is a straightforward application of parts (i) and (ii) of Theorem 2.9. We thus omit proofs here, and refer an interested reader to the appendix of [9] for details.

3 Covering thresholds for other 3-graphs

3.1 K_4^-

Proof of the lower bound in Theorem 1.2. We construct a 3-graph $F_2(n)$ on $V = [n]$. Select a special vertex x. Split the remainder of the vertices into six parts $\sqcup_{i=1}^{6} V_i = V \setminus \{x\}$ with sizes as equal as possible, as follows: $|V_1| - 1 \leq |V_6| \leq |V_5| \leq |V_4| \leq |V_3| \leq |V_2| \leq |V_1|$.

Put as the link of x the blow-up of a 6-cycle through the six parts, i.e. add all triples of the form xV_iV_{i+1} for $i \in [6]$, winding round modulo 6 as necessary (identifying V_7 with V_1, and so on). Finally add those triples not involving x which are not of type $V_iV_iV_{i+1}$, $V_iV_{i+1}V_{i+1}$ or $V_iV_{i+1}V_{i+2}$ for $i \in [6]$ (winding round modulo 6) to form the 3-graph $F_2(n)$.

Observe that the link graph of x in $F_2(n)$ is triangle-free (being the blow-up of a 6-cycle). Thus a putative K_4^- containing x would have to be induced by a 4-set $\{a, b, c, x\}$, with abc, abx and acx all being triples of $F_2(n)$. Since ab is in the link graph of x, we must have that a, b come from different but adjacent parts V_i, V_{i+1}; by symmetry of $F_2(n)$, we may assume without loss of generality that $a \in V_1$ and $b \in V_2$. Since $acx \in E(F_2(n))$, it follows that $c \in V_2$ or $c \in V_6$. But by construction of $F_2(n)$, there are no triples of type $V_6V_1V_2$ or $V_1V_2V_2$, so that we cannot have in $abc \in E(F_2(n))$. Thus there is no copy of K_4^- in $F_2(n)$ covering x.

Let us now compute the minimum codegree of $F_2(n)$. Consider vertices $a_i, a_i' \in V_i$, $a_{i+1} \in V_{i+1}$, $a_{i+2} \in V_{i+2}$ and $a_{i+3} \in V_{i+3}$. We have that $d(a_i, a_i') = n - 3 - |V_{i-1}| - |V_{i+1}|$, $d(a_i, a_{i+2}) = n - 3 - |V_{i+1}|$, $d(a_i, a_{i+3}) = n - 3$, and, lastly,

$$d(a_i, x) = |V_{i-1}| + |V_{i+1}| \qquad \text{and} \qquad d(a_i, a_{i+1}) = 1 + |V_{i+3}| + |V_{i+4}|.$$

Up to the choice of i, this covers all possible pairs in $F_2(n)$. The first three quantities are at least $n - 3 - 2\lceil \frac{n-1}{6} \rceil \geq \frac{2n}{3} - \frac{13}{3}$, which for $n \geq 12$ is greater than $\lfloor \frac{n-1}{3} \rfloor$. The last two quantities are both of order $\frac{n}{3} + O(1)$, however, and we analyse them more closely. Set $n = 6m + r$ for some $r \in \{0, 1, 2, 3, 4, 5\}$. Then

$$d(a_i, x) \geq \min_i \bigl(|V_{i-1}| + |V_{i+1}|\bigr) = |V_6| + |V_4| = \begin{cases} 2m - 1 & \text{if } r = 0 \\ 2m & \text{if } 0 < r < 5 \\ 2m + 1 & \text{if } r = 5, \end{cases}$$

and

$$d(a_i, a_{i+1}) \geq \min_i \bigl(1 + |V_{i+3}| + |V_{i+4}|\bigr) = 1 + |V_5| + |V_6| = \begin{cases} 2m & \text{if } r = 0 \\ 2m + 1 & \text{if } 0 < r \leq 5. \end{cases}$$

Thus

$$c_2(n, K_4^-) \geq \delta_2(F_2(n)) = \begin{cases} 2m - 1 & \text{if } r = 0 \\ 2m & \text{if } 0 < r < 5 \\ 2m + 1 & \text{if } r = 5. \quad \square \end{cases}$$

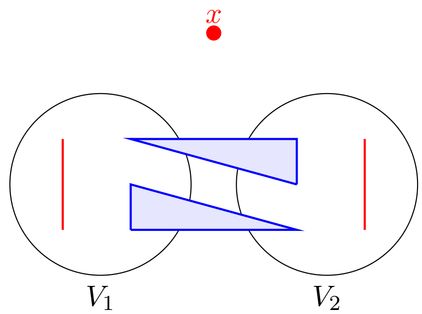

Figure 2: The 3-graph $F_3(n)$. The red pairs and the blue triples make up the link graph of x and the remainder of $E(F_3)$ respectively.

Proof of the upper bound in Theorem 1.2. Let G be a 3-graph on $n \geq 4$ vertices. Suppose $\delta_2(G) > \frac{n}{3}$. Pick an arbitrary vertex $x \in V(G)$. Let abx be any 3-edge containing x. We have $d(a,b) + d(a,x) + d(b,x) - 3 > n - 3$. So by the pigeonhole principle, there exists $c \in V(G) \setminus \{a,b,x\}$ which makes a 3-edge of G with at least two of ab, ax, bx. The 4-set $abcx$ then contains a copy of K_4^- in G covering x, as required. This shows that $c_2(n, K_4^-) \leq \lfloor \frac{n}{3} \rfloor$. □

Matching the upper and lower bounds obtained above, we obtain the set of possible values for $c_2(n, K_4^-)$ claimed in Theorem 1.2.

Remark 3.1. 1. *Again we actually proved something stronger here: our argument establishes that for $\delta_2(G)$ above $\lfloor \frac{n}{3} \rfloor$, every edge of G can be extended to a copy of K_4^-.*

2. *We believe the gap between the upper and lower bounds for $c_2(n, K_4^-)$ could be closed using similar (but more involved) stability arguments to those we used on to determine $c_2(n, K_4)$. However since such arguments would be non-trivial (the conjectured extremal configurations in this case are 6-partite) and would greatly increase the length of this paper, we do not pursue them here and leave open the determination of $c_2(n, K_4^-)$ in the case where $n \equiv 0, 3, 4 \bmod 6$.*

3.2 C_5

Proof of the lower bound in Theorem 1.3. We construct a 3-graph $F_3(n)$ on $V = [n]$. Select a special vertex x. Split the remainder of the vertices into two parts $V \setminus \{x\} = V_1 \sqcup V_2$ with sizes as equal as possible, $|V_2| - 1 \leq |V_1| \leq |V_2|$. Form the link graph of x by adding in all pairs internal to one of the parts, i.e. all pairs of the form xV_1V_1 or xV_2V_2. Next, add in all triples not containing x and meeting both of the parts, i.e. all triples of the form $V_1V_1V_2$ or $V_1V_2V_2$. This yields a 3-graph $F_3(n)$ with minimum codegree $\delta_2(F_3(n)) = |V_1| - 1 = \lfloor \frac{n-3}{2} \rfloor$, attained by x and any vertex $a \in V_1$; see Figure 2.

Now there is no copy of C_5 covering $x \in F_3(n)$. Indeed, let $S = \{a_1, a_2, b_1, b_2\}$ be a set of four distinct vertices in $V \setminus \{x\}$ such that all of a_1a_2x, a_1b_1x and b_1b_2x are triples of $F_3(n)$. Then by construction these four vertices must all lie within the same part of $F_3(n)$. But by construction again we have that S spans no triple of $F_3(n)$, whence $S \cup \{x\}$ does not contain a copy of C_5. □

Proof of the upper bound in Theorem 1.3. Let $G = (V, E)$ be a 3-graph on n vertices with minimum codegree $\delta_2(G) > \frac{n}{2}$. Let x be any vertex. Fix an edge ab in the link graph G_x. Since $\delta_1(G_x) > n/2$, a and b each have at least $\frac{n}{2} - 1$ neighbours in $V \setminus \{x, a, b\}$. Hence a and b have a common neighbour c in G_x. We shall use the triangle $\{a, b, c\}$ to find a copy of C_5 covering x.

Claim 3.2. *There is either a copy of C_5 or a copy of K_4 covering x in G.*

Proof. If $abc \in E$ then the claim is immediate since $S = \{a, b, c, x\}$ induces a complete 3-graph. Assume therefore that $abc \notin E$. By our codegree assumption,

$$d(a, b) + d(a, c) + d(b, c) + d(a, x) + d(b, x) + d(c, x) - 9 \geq 6\delta_2(G) - 9 > 3(n - 4).$$

Thus there exists $y \in V \setminus S$ which makes a 3-edge with at least four of the pairs ab, ac, bc, ax, bx, cx. We claim that $S \cup \{y\}$ contains either a K_4 or a C_5 covering x. Indeed, without loss of generality, assume that G_y misses exactly two edges on S.

- If the two missing edges form a matching, then $G_y[S]$ contains a cycle of length four, in particular, a path of length three starting from x, say, $xabc$. Then $xaybc$ gives a copy of C_5 in G.

- If the two missing edges are adjacent, then $G_y[S]$ contains either a path of length three starting from x, or a triangle containing x. In the former case, $S \cup \{y\}$ spans a copy of C_5 in G; in the latter case, $S \cup \{y\}$ contains a copy of K_4 covering x in G. □

With a view towards proving Theorem 1.3, we may thus assume that G contains a copy of K_4 covering x. Let $S = \{a, b, c, x\}$ be a 4-set of vertices inducing such a K_4. By the codegree assumption,

$$d(a, b) + d(a, c) + d(b, c) + d(a, x) + d(b, x) + d(c, x) - 12 \geq 6\delta_2(G) - 12 > 3(n - 4).$$

Thus there exists $y \in V \setminus S$ which makes a 3-edge with at least four of the pairs ab, ac, bc, ax, bx, cx. It is now easy to check that $S \cup \{y\}$ contains a copy of C_5 covering x because $G_y[S]$ always contains a path of length three (it is no longer necessary that the path starts with x because x, a, b, c spans a copy of K_4). The claimed upper bound on $c_2(n, C_5)$ follows. □

Remark 3.3. *Interestingly, as pointed out to us by Jie Han and Allan Lo, another very different construction attains the lower bound in Theorem 1.3. Take a balanced bipartition of $[n]$ into two sets V_1 and V_2, with $|V_1| \leq |V_2|$. Now take all triples meeting V_1 in an even number of vertices to form a 3-graph $F_4(n)$. Note that $\delta_2(F_4(n)) = \min(|V_1| - 1, |V_2| - 2)$ (attained by pairs from $V_1 \times V_2$ and $V_2^{(2)}$ respectively), which is exactly equal to $\lfloor \frac{n-3}{2} \rfloor$. Now, it is an easy exercise to check that every vertex $x \in V_1$ fails to be covered by a C_5, giving us a second proof that $c_2(n, C_5) \geq \lfloor \frac{n-3}{2} \rfloor$. In particular, we do not have stability for this problem: we have two near-extremal constructions which are easily seen to lie at edit distance $\Omega(n^3)$ from each other. Also we have that just below the codegree-threshold for covering by C_5, we could have as many as $\lfloor \frac{n}{2} \rfloor$ uncovered vertices. This stands in sharp contrast with the situation for K_4 (see the discussion in Section 4).*

3.3 K_5^-

Proof of Theorem 1.4. For the lower bound, note that

$$c_2(n, K_5^-) \geq c_2(n, K_4) \geq \delta_2(F_1(n)) = \left\lfloor \frac{2n-5}{3} \right\rfloor.$$

For the upper bound, let $G = (V, E)$ be a 3-graph on n vertices with $\delta_2(G) > \frac{2n-2}{3}$. By Theorem 1.1, for any vertex $x \in V$ there is a triple a_1, a_2, a_3 such that $S = \{x, a_1, a_2, a_3\}$ induces a copy of K_4 in G. Now

$$d(x, a_1) + d(x, a_2) + d(x, a_3) + d(a_1, a_2) + d(a_1, a_3) + d(a_2, a_3) - 12 > 4(n - 4),$$

whence there exists $a_4 \in V \setminus S$ which makes a 3-edge with at least 5 of the pairs from $S^{(2)}$. Thus $S \cup \{a_4\}$ contains a copy of K_5^- covering x. This shows that $c_2(n, K_5^-) \leq \lfloor \frac{2n-2}{3} \rfloor$. □

3.4 The Fano plane

Proposition 3.4. $\lfloor \frac{n}{2} \rfloor \leq c_2(n, \text{Fano}) \leq \lfloor \frac{2n}{3} \rfloor$.

Proof. The lower bound is from the codegree-threshold of the Fano plane: consider a bipartition of $[n]$ into two sets $V_1 \sqcup V_2$ with $|V_1| = \lfloor \frac{n}{2} \rfloor$ and $|V_2| = \lceil \frac{n}{2} \rceil$, and adding all triples meeting both parts. The resulting 3-graph is easily seen to be Fano-free (it is 2-colourable, whereas the Fano plane is not) and has codegree $\lfloor \frac{n}{2} \rfloor$. For the upper bound, apply Lemma 2.1 with $F = \text{Fano}$ and $r = 3$. □

3.5 $F_{3,2}$

Recall that $F_{3,2}$ denotes the 3-graph $([5], \{123, 124, 125, 345\})$.

Theorem 3.5. *$1/3 \leq c_2(F_{3,2}) \leq 3/7$.*

Proof. The lower bound is from the codegree density of $F_{3,2}$. An $F_{3,2}$-free construction on n vertices with codegree $\lfloor \frac{n}{3} \rfloor - 1$ is obtained by considering a tripartition of $[n]$ into three parts with sizes as equal as possible, $|V_3| - 1 \leq |V_1| \leq |V_2| \leq |V_3|$ and adding all triples of the form $V_i V_i V_{i+1}$.[2]

For the upper bound, let $G = (V, E)$ be a 3-graph on n vertices with $\delta_2(G) = cn$. Without loss of generality, assume that $c \geq 2/5$. Suppose there exists $x \in V$ such that there is no copy of $F_{3,2}$ in G covering x. This means that for every vertex $v \in V \setminus \{x\}$, $\Gamma(x, v)$ is an independent set in G, and moreover that for every 4-set $\{a, b, y, z\} \subseteq V$, at least one of the triples $\{xab, xyz, aby, abz\}$ is not in E. For convenience, we shall write $ab|xyz$ as a short-hand for the statement that all $\{abx, aby, abz, xyz\}$ are edges of G.

We use the following technical lemma to deduce $c \leq 3/7 + o(1)$.

Claim 3.6. *If there exists two disjoint sets $A, B \subseteq V \setminus \{x\}$ of size $|A| = |B| = cn$ such that*

1. *$A \subseteq \Gamma(x, y)$ for some $y \in V \setminus \{x\}$, and*

2. *B is independent in both G and the link graph G_x,*

[2] This is not actually best possible — see [7] for a determination of the precise codegree-threshold and the extremal constructions attaining it.

then $c \leq 3/7 + o(1)$.

Proof of Claim 3.6. Note that $c \leq 1/2$ because A and B are disjoint. Let $C = V \setminus (A \cup B)$. We have $|C| = n(1-2c)$. By our assumption, A is independent in G. By the codegree assumption, at least $\binom{|A|}{2}(cn - |C|)$ triples of G have two vertices in A and one vertex in B. Consequently, at most $\binom{|A|}{2}|C| = \binom{cn}{2}(1-2c)n$ triples of the form AAB are missing from G.

On the other hand, let $b, b' \in B$. Since B is independent in G and G_x, we have $\Gamma(b, b') \subseteq A \cup C$ and $\Gamma(b, x) \subseteq A \cup C$. Consequently $|\Gamma(b', x) \cap A| \geq cn - |C| \geq (3c-1)n$ and

$$|\Gamma(b, b') \cap \Gamma(b, x) \cap A| \geq 2(cn - |C|) - |A| \geq (5c-2)n.$$

For any $a \in \Gamma(b, b') \cap \Gamma(b, x) \cap A$ and any $a' \in \Gamma(b', x) \cap A$, the triple $aa'b$ must be absent from G – otherwise $ab|a'bx$. There are at least $\binom{(3c-1)n}{2} - \binom{(1-2c)n}{2}$ such pairs $\{a, a'\}$ because, in general, there are at least $\binom{|A_1|}{2} - \binom{|A_1|-|A_2|}{2}$ unordered pairs $\{a_1, a_2\}$ with $a_1 \in A_1$ and $a_2 \in A_2$ for arbitrary sets A_1, A_2 satisfying $|A_1| \geq |A_2|$. There are thus at least $\binom{(3c-1)n}{2} - \binom{(1-2c)n}{2}$ distinct pairs $\{a, a'\}$ for which $aa'b \notin E$. Summing over all $b \in B$, this gives us a total of at least $\left(\binom{(3c-1)n}{2} - \binom{(1-2c)n}{2}\right) cn$ AAB triples missing from G. Combining this with our upper bound on the number of missing AAB triples yields the inequality

$$\left(\binom{(3c-1)n}{2} - \binom{(1-2c)n}{2}\right) cn \leq \binom{cn}{2}(1-2c)n,$$

which implies that

$$\left((3c-1)^2 - (1-2c)^2\right) \frac{n^2}{2} cn \leq \frac{c^2 n^2}{2} (1-2c)\, n + O(n^2).$$

This inequality in turn gives $c \leq 3/7 + O(n^{-1})$. $\square$

We now show that we can find $A, B \subseteq V$ satisfying the properties in Claim 3.6.

Suppose first of all that G_x is not triangle-free. Let ya_1a_2 be vertices spanning a triangle in G_x. Let A be a subset of $\Gamma(x, y)$ of size cn. Then A must be an independent set in G. Let B be a subset of $\Gamma(a_1, a_2)$ in $V \setminus \{x\}$ of size cn. Then B is disjoint from A — otherwise $xy|a_1a_2a_3$ for any $a_3 \in A \cap B$. In addition, B is an independent set in G_x — indeed if $b_1b_2 \in G_x$ for some $b_1, b_2 \in B$ then $a_1a_2|xb_1b_2$. We now show that B is an independent set in G. Indeed, for every $b \in B$, $\Gamma(b, x)$ is a subset of $V \setminus B$ of size at least cn. Consider an arbitrary triple $\{b_1, b_2, b_3\}$ of distinct vertices from B. Since

$$d(b_1, x) + d(b_2, x) + d(b_3, x) - 2\left(n - |B|\right) \geq \left(3c - 2(1-c)\right) n = (5c-2)n > 0,$$

by the pigeon-hole principle there exists $a \in V \setminus (B \cup \{x\})$ with xab_1, xab_2, xab_3 all in E. In particular $b_1b_2b_3 \notin E$, as otherwise we would have $ax|b_1b_2b_3$. It follows that B must be an independent set in G. The sets A, B then satisfy the two properties required to apply Claim 3.6, and thus $c \leq 3/7 + o(1)$.

On the other hand, suppose G_x was triangle-free. Let $y \in V \setminus \{x\}$, and let A be a subset of $\Gamma(x, y)$ of size cn. Since G_x is triangle-free and x is not covered by an $F_{3,2}$-subgraph, A forms an independent set in both G_x and G. Let $a \in A$ be arbitrary, and let B be a subset of $\Gamma(a, x)$ of size cn. Then B is disjoint from A and independent in G_x (since G_x is triangle-free). Thus A, B satisfy the two properties in Claim 3.6, and $c \leq 3/7 + o(1)$. $\square$

3.6 K_t, $t \geq 5$

Proposition 3.7. *For all $t \geq 5$, $c_2(K_t) \leq 1 - 1/\binom{t-1}{2}$.*

Proof. Applying Lemma 2.1 with $F = K_t$ and $r = \binom{t-1}{2}$, we get

$$c_2(n, K_t) \leq \left\lfloor \left(1 - \frac{1}{\binom{t-1}{2}}\right) n - \frac{2t-6}{t-2} \right\rfloor. \qquad \square$$

We now derive a lower bound for the covering codegree density of K_t by using (small) lower-bound constructions for the codegree-threshold of K_{t-1}

Proposition 3.8. *Suppose there exists a K_{t-1}-free 3-graph $H = (V(H), E(H)$ on $[m]$ with minimum codegree δ. Then $c_2(K_t) \geq (\delta + 2)/m$.*

Proof. We build a 3-graph G on $n = Nm+1$ vertices as follows. Set $V = [n]$ and set aside a special vertex x. Partition $V \setminus \{x\}$ into m sets $V_1, \dots, V_m$, each of size N. Set as the link graph of x all pairs of vertices from distinct parts. For every triple $ijk \in E(H)$, add to G all 3-edges of the form $V_iV_jV_k$. Finally, add all triples of $V \setminus \{x\}$ that contain at least two vertices from one part. The minimum codegree of G is

$$\delta_2(G) = (\delta + 2)N - 1 \geq \frac{\delta + 2}{m} n - 2.$$

Now consider a $(t-1)$-set $S \subseteq V \setminus \{x\}$ that induce a $(t-1)$-clique in the link graph of x. By construction, these vertices must come from $t-1$ different parts of $V \setminus \{x\}$. Since H is K_{t-1}-free, by our construction, some triple of S is absent from G. Thus $S \cup \{x\}$ does not induce a copy of K_t in G. Taking the limit as $n \to \infty$, the result follows. $\square$

Corollary 3.9. *1. $c_2(K_5) \geq \frac{3}{4}$, and*

2. $c_2(K_t) \geq \frac{2t-6}{2t-5}$ if $t \equiv 0, 1 \mod 3$ and $t \geq 4$.

Proof. For part 1, we apply Proposition 3.8 with $H = K_4^-$ (thus $m = 4$ and $\delta = 1$) and obtain $c_2(K_5) \geq \frac{3}{4}$.

For part 2, since $t \equiv 0, 1 \mod 3$, we have $2t - 5 \equiv 1, 3 \mod 6$, whence there exists a Steiner triple system $\mathcal{S}$ on the vertex set $[2t-5]$. It is easy to see that every set $T \subset [2t-5]$ of $t-1$ vertices spans at least one triple from $\mathcal{S}$. Indeed, suppose not and fix a vertex $a \in T$. All pairs of vertices from T containing a must have distinct neighbours under $\mathcal{S}$ in $[2t-5] \setminus T$. Since $t-2 > (2t-5)-(t-1)$, this is impossible. Therefore the complement 3-graph $\overline{\mathcal{S}}$ is K_{t-1}-free. Applying Proposition 3.8 with $H = \overline{\mathcal{S}}$ (thus $m = 2t-5$ and $\delta = 2t-8$), we obtain that $c_2(K_t) \geq \frac{2t-6}{2t-5}$. $\square$

Remark 3.10. *1. Combining Proposition 3.7 and Corollary 3.9 gives that $\frac{3}{4} \leq c_2(K_5) \leq \frac{5}{6}$ and $\frac{6}{7} \leq c_2(K_6) \leq \frac{9}{10}$.*

2. For $t \equiv 2 \mod 3$, Corollary 3.9 part 2 gives the lower bound $c_2(K_t) \geq c_2(K_{t-1}) \geq (2t-8)/(2t-7)$. When $t \geq 7$, this bound is better than $c_2(K_t) \geq \frac{t-2}{t-1}$, which is obtained by directly applying Proposition 3.8 with $H = K_{t-1}^-$.

3. Theorem 1.1 shows that the lower bound in Corollary 3.9 part 2 is tight in the case $t = 4$. The bound is also tight in the trivial case $t = 3$, since $c_2(n, K_3) = 1 = o(n)$. If this bound is tight in general, then we do not have stability for the covering codegree-threshold problem: while the

3-edge K_3 and the Fano plane are the unique (up to isomorphism) Steiner triple systems on 3 *and* 7 *vertices respectively, there are for example* $11,084,874,829$ *non-isomorphic Steiner triple systems on* 19 *vertices (see [1, Section 4.5]).*

4 Concluding remarks

There are many questions arising from our work. To begin with, we may ask which of the following properties of Turán density and codegree density does the covering codegree density c_2 share.

1. Do we have *supersaturation*? That is, if $\delta_2(G) \geq c_2(n, F) + \varepsilon n$ for some fixed $\varepsilon > 0$, is it the case that every vertex in G is contained in $\Omega(n^{|V(F)|-1})$ copies of F?
2. Do we have *blow-up invariance*? Given a 3-graph F, we define the blow-up $F(t)$ to be the 3-graph on $V(F) \times [t]$ with 3-edges $\{(u,i)(v,j)(w,k) : \ uvw \in E(F), \ i,j,k \in [t]\}$. Is it the case that for every F and every fixed t we have $c_2(F) = c_2(F(t))$?
3. Is the set of covering codegree densities $\{c_2(F) : \ F \text{ a 3-graph}\}$ dense in $[0,1]$, or does it have jumps?

The first two of these questions are addressed in a forthcoming work of the authors. In addition there are some natural variants of the covering codegree-threshold $c_2(n, F)$ which may be interesting. What if instead of covering every vertex by a copy of F we wanted to cover every pair? What if we wanted instead to be able to extend every 3-edge to a copy of F? It is not immediately clear whether the corresponding codegree-extremal functions behave similarly to $c_2(n, F)$ or not.

In a different direction, what if we asked for the threshold for covering all but at most k vertices, for some $k \geq 1$? On the one hand, in the case of C_5 we observed in Remark 3.3 that this does not affect the value of the covering threshold very much. On the other hand, Theorem 2.9 implies that the threshold for covering all but at most 1 vertex with a copy of K_4 is at most $(2/3 - c)\,n$ for some $c > 0$ (in particular the threshold is genuinely different from $c_2(n, K_4)$). Let us sketch a proof. Let $G = (V, E)$ be a 3-graph on n vertices with $\delta_2(G) \geq (2/3 - c)\,n$ for some $c > 0$ sufficiently small. Suppose that $x \in V$ is not covered by any copy of K_4. By Theorem 2.9, there is a partition $V_1 \sqcup V_2 \sqcup V_3$ of $V \setminus \{x\}$ satisfying (i)–(v). If another vertex y is not covered by any copy of K_4, then there is a partition $V_1' \sqcup V_2' \sqcup V_3'$ of $V \setminus \{y\}$ satisfying (i)–(v) as well. Because of (iii) and (iv), these two partitions essentially coincide. Now consider $\Gamma(x, y)$, which has size at least $(2/3 - c)n$. There are about $(2/3 - c)(1/3 - 3c)n^2/2$ pairs $u, v \in \Gamma(x, y)$ coming from different parts of $V_1 \cap V_1', V_2 \cap V_2', V_3 \cap V_3'$. Since $(2/3 - c)(1/3 - 3c)n^2/2 > 2 \cdot 10cn^2$ (for c sufficiently small), by (ii), there exists a pair $u, v \in \Gamma(x, y)$ such that both uvx and uvy are edges of G. This implies that $\{u, v, x, y\}$ spans a copy of K_4, a contradiction. The authors note that the bound on c given by this argument can be significantly improved; this is the subject of future work.

Finally, it would be interesting to determine the value of $c_2(F)$ when F is the Fano plane or $F_{3,2}$, and to have if not a tight result then at least a reasonable guess as to the value of $c_2(K_t)$ for $t \geq 5$. An investigation of $c_1(n, F)$ when $F = K_4^-$ and $F = K_4$ would also be desirable. We remark that for small 3-graphs F the problem of proving upper bounds for c_1 or c_2 should be amenable to flag algebra computations by following the approach of [7] to encode the minimum degree/codegree constraint. Note that one would need to do computation with non-uniform hypergraphs, containing a mixture of 2-edges (from the link graph of an uncovered vertex x) and 3-edges.

5 Acknowledgements

The authors thank two anonymous referees for their careful work on the paper, which helped improve its clarity and presentation, and Jie Han and Allan Lo for pointing out the alternative lower bound construction for Theorem 1.3. The first author is also grateful for the hospitality of Georgia State University during a research visit which led to the present paper.

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