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BIPARTITE GRAPH TILING*

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Abstract. For each $s \geq 2$, there exists m_0 such that the following holds for all $m \geq m_0$: Let G be a bipartite graph with $n = ms$ vertices in each partition set. If m is odd and minimum degree $\delta(G) \geq \frac{n+3s}{2} - 2$, then G contains m vertex-disjoint copies of $K_{s,s}$. If m is even, the same holds under the weaker condition $\delta(G) \geq n/2 + s - 1$. This is sharp and much stronger than a conjecture of Wang [*Discrete Math.*, 187 (1998), pp. 221–231] (for large n).

Key words. graph packing, tiling, regularity lemma, blow-up lemma

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1. Introduction. Let H be a graph on h vertices and G be a graph on n vertices. The *tiling* (also called *packing*) problem in extremal graph theory is to find in G as many vertex-disjoint copies of H as possible. An H-factor (perfect tiling) of G is a subgraph of G which consists of $\lfloor n/h \rfloor$ copies of H. Dirac's theorem on Hamilton cycles [6] is one of the earliest tiling results. It implies that every n-vertex graph G with minimum degree $\delta(G) \geq n/2$ contains a perfect matching (usually called 1-factor, instead of K_2-factor). The problem of triangle-factors was solved by Corrádi and Hajnal [4], and the seminal result of Hajnal and Szemerédi [9] settled the problem of K_r-factors for all r. Using the celebrated regularity lemma of Szemerédi, Alon and Yuster [1, 2] found sufficient conditions for H-factors for arbitrary H. Their results were later improved by Komlós, Sárközy, and Szemerédi [14], Komlós [12], Shokoufandeh and the author [21], and Kühn and Osthus [16].

Tiling in a multipartite graph has a shorter history. The marriage theorem by König and Hall (see, e.g., [10]) implies that a bipartite graph G with two partition sets of size n contains a 1-factor if $\delta(G) \geq n/2$. In an r-partite graph G with $r \geq 2$, let $\bar{\delta}(G)$ be the minimum degree from a vertex in one partition set to another partition set (so $\bar{\delta}(G) = \delta(G)$ when $r = 2$). An r-partite graph is *balanced* if all partition sets have the same size.

Fischer [8] conjectured the following r-partite version of the Hajnal–Szemerédi theorem and proved it asymptotically for $r = 4, 5$: *if G is an r-partite graph with n vertices in each partition set and $\bar{\delta}(G) \geq \frac{r-1}{r}n$, then G contains a K_r-factor.* Magyar and Martin [17] showed that Fischer's conjecture is slightly wrong for $r = 3$ (off by only 1); Martin and Szemerédi [18] showed that the conjecture is true for $r = 4$. Csaba and Mydlarz [3] recently proved that the conclusion in Fischer's conjecture holds if $\bar{\delta}(G) \geq \frac{k_r}{k_r+1}n$, where $k_r = r + O(\log r)$. Another related result is given by Martin and the author [19] on $K_{s,s,s}$-factors in tripartite graphs. Note that, in general, a tiling result for multipartite graphs does not follow from a corresponding result for arbitrary graphs. On the other hand, given a graph G of order nr, we can easily

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obtain (by taking a random partition) an r-partite balanced spanning subgraph G' such that $\bar{\delta}(G') \geq \delta(G)/r - o(n)$. Therefore a tiling result for multipartite graphs immediately gives a slightly weaker tiling result for arbitrary graphs.

In this paper we consider $K_{s,s}$-tiling in large balanced bipartite graphs. Wang gave [25] the following conjecture and proved it for $n \leq 4s$ [26].

CONJECTURE 1.1 (Wang). *Fix $s \geq 2$. Let $G = (A, B; E)$ be a bipartite graph, and let $|A| = |B| = n$ be divisible by s. If $\delta(G) \geq \frac{s-1}{s}n + 1$, then G contains a $K_{s,s}$-factor.*

In this paper we prove a result much stronger than Conjecture 1.1, provided that n is large.

THEOREM 1.2. *For each $s \geq 2$, there exists m_0 such that the following holds for all $m \geq m_0$. Let $G = (A, B; E)$ be a bipartite graph with $|A| = |B| = n = ms$ such that*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + s - 1 & \text{if } m \text{ is even,} \\ \frac{n+3s}{2} - 2 & \text{if } m \text{ is odd.} \end{cases} \tag{1.1}$$

Then G contains m vertex-disjoint copies of $K_{s,s}$.

We also show that the minimal degree conditions (1.1) are best possible.

PROPOSITION 1.3. *Let $s \geq 2$, and $n = ms \geq 64s^2$. When m is even, there exists a bipartite graph $G = (A, B; E)$ with $|A| = |B| = n$ and $\delta(G) = n/2 + s - 2$ such that G contains no $K_{s,s}$-factor. When m is odd, the same holds when $\delta(G) = (n+3s)/2 - 3$.*

Note that $\frac{n+3s}{2} - 2 \geq \frac{n}{2} + s - 1$ if $s \geq 2$. When $s = 2$, (1.1) becomes one inequality $\delta(G) \geq n/2 + 1$. Wang [24] conjectured that every bipartite G with $2k$ vertices in each partition set and $\delta(G) \geq k + 1$ contains a $K_{2,2}$-factor. Define an n-ladder as a bipartite graph on $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ such that a_i is adjacent to b_j if and only if $|i - j| \leq 1$. Czygrinow and Kierstead [5] showed that for large n, every bipartite G with n vertices in each partition set and $\delta(G) \geq n/2 + 1$ contains an n-ladder, thus proving the conjecture of Wang. Our Theorem 1.2 provides another proof of this conjecture for large n.

The proof of Theorem 1.2 naturally falls into two stages as in other tiling results [14, 17, 21]. In the first stage we prove a result that resembles the stability theorem of Simonovits [22]; namely, any balanced bipartite graph with a slightly weaker degree condition either contains a $K_{s,s}$-factor or is close to the extremal graph. The main tools in this stage are the regularity lemma [23] and blow-up lemma [13]. In the second stage, we show that any graph close to the extremal graph contains a $K_{s,s}$-factor.

The structure of the paper is as follows. We first prove Proposition 1.3 in section 2. Next we state the regularity lemma and blow-up lemma in section 3. Then we prove the nonextremal case in section 4 and the extreme case in section 5. Some concluding remarks are given in section 6.

We gather our notation as follows. For $x > 0$, let $[x] = \{1, 2, \ldots, \lfloor x \rfloor\}$. We write $A+B$ instead of $A \cup B$ when two sets A and B are disjoint. Given a graph $G = (V, E)$, let $v(G) = |V|$ and $e(G) = |E|$. For $v \in V$ and $X \subseteq V$, the neighborhood $N(v, X)$ consists of all vertices $x \in X$ that are adjacent to v. Let $\deg(v, X) = |N(v, X)|$. In particular, $N(v) = N(v, V)$ and $\deg(x) = \deg(x, V)$. The maximum degree and minimum degree in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For two (not necessary disjoint) subsets A and B of V, we define

$$\delta(A, B) = \min_{a \in A} \deg(a, B), \quad \Delta(A, B) = \max_{a \in A} \deg(a, B), \quad e(A, B) = \sum_{a \in A} \deg(a, B).$$

The *density* between A and B is defined as $d(A,B) = e(A,B)/(|A||B|)$. For disjoint vertex-sets A and B, $G[A,B]$ is the bipartite subgraph on A and B with all the edges of G between A and B. As usual, $G[A]$ denotes the induced subgraph of G on A.

2. Proof of Proposition 1.3. In this section we prove Proposition 1.3 by using two constructions. Let $1 \le r < n$. The key ingredient in our constructions is an r-regular bipartite graph $P(n,r)$ with n vertices in each partition set containing no $K_{2,2}$. One way to achieve this is by using a Sidon set (also called B_2 set), a set S of integers such that sums $a+b$ are distinct for distinct pairs $a, b \in S$. In other words, for $a,b,c,d \in S$, if $a+b = c+d$, then $\{a,b\} = \{c,d\}$. It is well known (e.g., [7]) that $[n]$ contains a Sidon set of size about $\sqrt{n}$ for large n. A construction of Ruzsa [20] shows that one can find a Sidon set of size r in $[n]$ if $n \ge 2r(2r-1)$ for any r. When $n \ge 8r^2$, we thus find an r-element Sidon subset S of $[\frac{n}{2}-1]$. Let $P = P(n,r)$ be a bipartite graph on $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ such that $x_i y_j \in E(P)$ if and only if $j - i \in S$ or $n + j - i \in S$. It is easy to see that P is r-regular. We claim that P contains no $K_{2,2}$. Suppose, instead, that two vertices in X have two common neighbors. Since the vertices in X are identical, without loss of generality, we assume that the two vertices are x_1 and x_i such that $1 < i \le n/2+1$. Since $S \subset [\frac{n}{2}-1]$, the neighbors of x_1 and x_i are y_{1+a} and y_{i+a}, $a \in S$, respectively. If y_{j_1} and y_{j_2} are two common neighbors of x_1 and x_i, then

$$j_1 = 1 + a = i + d, \quad j_2 = 1 + c = i + b$$

for some $a,b,c,d \in S$. This implies that $a+b = c+d$. Since $a \notin \{c,d\}$, this contradicts the definition of S. We thus know that if $K_{a,b} \subseteq P(n,r)$ for some $a > b > 0$, then $a + b \le r + 1$, where equality holds only if $a = r$ and $b = 1$. For convenience, we also define $P(n,r)$ for $r \le 0$ to be the empty graph on $2n$ vertices.

CONSTRUCTION 2.1. *Suppose that $s \ge 2$, $n \ge 16s^2$, and n is divisible by $2s$. Let $G = (A,B;E)$ be a bipartite graph with the following properties:*

- $|A| = |B| = n$.
- $A = A_1 + A_2$, $B = B_1 + B_2$, $|A_1| = |B_2| = n/2+1$, *and* $|A_2| = |B_1| = n/2-1$.
- $G[A_i, B_i]$ *is a complete bipartite graph for* $i = 1,2$.
- $G[A_1, B_2] \cong P(n/2+1,\, s-1)$ *and* $G[A_2, B_1] \cong P(n/2-1,\, s-3)$.

Note that $P(n/2+1, s-1)$ and $P(n/2-1, s-3)$ are well defined when $n \ge 16s^2$. It is easy to see that $\delta(G) = n/2 + s - 2$ (by checking the cases of $s \ge 3$ and $s = 2$ separately). We claim that G does not contain a $K_{s,s}$-factor. Since s divides $\frac{n}{2}$, s does not divide $\frac{n}{2}+1$. Therefore a $K_{s,s}$-factor of G must contain some $F \cong K_{s,s}$ such that $V(F)$ is not a subset of $A_1 \cup B_1$ or $A_2 \cup B_2$. Let $X_i = V(F) \cap A_i$, $Y_i = V(F) \cap B_i$ for $i = 1,2$. If $X_2 \ne \emptyset$ and $Y_1 \ne \emptyset$, then $|X_2| + |Y_1| \le s-2$ because $G[A_2, B_1] \cong P(n/2-1,\, s-3)$. Otherwise, at least one of X_2 and Y_1 is empty, and clearly $|X_2| + |Y_1| \le s$. Putting the cases together, we have $|X_2| + |Y_1| \le s$, where equality holds only if one of X_2 and Y_1 is empty. Since $G[A_1, B_2] \cong P(n/2+1, s-1)$, we have $|X_1| + |Y_2| \le s$. In order to have $\sum(|X_i| + |Y_i|) = 2s$, it must be the case that $|X_2| = s$ and $|Y_1| = 0$ or $|X_2| = 0$ and $|Y_1| = s$. But this implies that either $|X_1| = |Y_1| = 0$ or $|X_2| = |Y_2| = 0$, a contradiction to our assumption on F.

CONSTRUCTION 2.2. *Suppose that $s \ge 2$ and $n = ms \ge 64s^2$ with odd m. Let $G = (A,B;E)$ be a bipartite graph with the following properties:*

- $|A| = |B| = n$.
- $A = A_1 + A_2$, $B = B_1 + B_2$, $|A_1| = |B_2| = \frac{n+s}{2} - 1$, *and* $|A_2| = |B_1| = \frac{n-s}{2} + 1$.
- $G[A_i, B_i]$ *is a complete bipartite graph for* $i = 1,2$.
- $G[A_1, B_2] \cong P(\frac{n+s}{2} - 1,\, 2s-4)$ *and* $G[A_2, B_1] \cong P(\frac{n-s}{2} + 1,\, s-2)$.

Note that $P(\frac{n+s}{2}-1, 2s-4)$ and $P(\frac{n-s}{2}+1, s-2)$ are well defined when $n \geq 64s^2$. It is easy to see that $\delta(G) = \frac{n+3s}{2} - 3$. Suppose that G contains a $K_{s,s}$-factor $\mathcal{K}$. For $F \in \mathcal{K}$ (then $F \cong K_{s,s}$), we let $X_i = V(F) \cap A_i$ and $Y_i = V(F) \cap B_i$ for $i = 1, 2$, and let $\vec{v}(F) = (|X_1|, |X_2|, |Y_1|, |Y_2|)$. We claim that $(1, s-1, 0, s)$, $(s, 0, s-1, 1)$, $(s, 0, s, 0)$, and $(0, s, 0, s)$ are all possible values for $\vec{v}(F)$. In fact, since $G[A_2, B_1] \cong P(\frac{n-s}{2}+1,\ s-2)$, we have either $|X_2| + |Y_1| \leq s-1$ or one of X_2, Y_1 is empty and the other one is of size s. In the latter case $\vec{v}(F)$ is either $(s, 0, s, 0)$ or $(0, s, 0, s)$. Now assume that $|X_2|+|Y_1| \leq s-1$. Since $G[A_1, B_2] \cong P(\frac{n+s}{2}-1,\ 2s-4)$ contains no $K_{2,2}$, we have $|X_1|+|Y_2| \leq s+1$, where equality holds only if $|X_1| = s$ or $|Y_2| = s$. Since $\sum_{i=1,2}(|X_i|+|Y_i|) = 2s$, it must be the case that $|X_1| = s$ or $|Y_2| = s$. Consequently, $\vec{v}(F)$ is either $(1, s-1, 0, s)$ or $(s, 0, s-1, 1)$. Assume that among all $\vec{v}(F)$ with $F \in \mathcal{K}$, i_1 of them are $(1, s-1, 0, s)$, i_2 of them are $(s, 0, s-1, 1)$, i_3 of them are $(s, 0, s, 0)$, and i_4 of them are $(0, s, 0, s)$. This gives

$$|A_1| = \frac{n+s}{2} - 1 = i_1 + i_2 s + i_3 s, \quad |B_1| = \frac{n-s}{2} + 1 = i_2(s-1) + i_3 s,$$

which implies that $s - 2 = i_1 + i_2$ and $n = i_1 - i_2 + 2(i_2 + i_3)s$. If $i_1 = i_2$, then $n = 2(i_2 + i_3)s$, a contradiction to the assumption that n is an odd multiple of s. Otherwise, $|i_1 - i_2| \geq s$, contradicting $s - 2 = i_1 + i_2$.

3. Main tools.

The regularity lemma and the blow-up lemma are main tools in the proof of the nonextremal case. Let us first define ε-regularity and (ε, δ)-super-regularity.

DEFINITION 3.1. *Let $\varepsilon > 0$. Suppose that a graph G contains disjoint vertex-sets A and B.*

1. *The pair (A, B) is* ε-regular *(otherwise* ε-irregular*) if for every $X \subseteq A$ and $Y \subseteq B$, satisfying $|X| > \varepsilon|A|$, $|Y| > \varepsilon|B|$, we have $|d(X, Y) - d(A, B)| < \varepsilon$.*
2. *The pair (A, B) is* (ε, δ)-super-regular *if (A, B) is ε-regular and $\deg(a, B) > \delta|B|$ for all $a \in A$ and $\deg(b, A) > \delta|A|$ for all $b \in B$.*

The celebrated regularity lemma of Szemerédi [23] has a multipartite version (see the survey paper [15]), which guarantees that when applying the lemma to a multipartite graph, every resulting cluster is from some original partition set.

LEMMA 3.2 (regularity lemma—bipartite version). *For every positive ε there is an $M = M(\varepsilon)$ such that if $G = (A, B; E)$ is any bipartite graph with $|A| = |B| = n$, and $d \in [0, 1]$ is any real number, then there are a partition of A into clusters $A_0, A_1, \ldots, A_k$, a partition of B into $B_0, B_1, \ldots, B_k$, and a subgraph $G' = (A, B; E')$ with the following properties:*

- *$k \leq M$.*
- *$|A_0| \leq \varepsilon n$, $|B_0| \leq \varepsilon n$.*
- *$|A_i| = |B_i| = N \leq \varepsilon n$ for all $i \geq 1$.*
- *$\deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)n$ for all $v \notin A_0 \cup B_0$.*
- *All pairs (A_i, B_j), $1 \leq i, j \leq k$, are ε-regular under G', each with density either 0 or exceeding d.*

We will also need the blow-up lemma of Komlós, Sárközy, and Szemerédi [13].

LEMMA 3.3 (blow-up lemma). *Given a graph R of order r and positive parameters δ, Δ, there exists an $\varepsilon > 0$ such that the following holds: Let N be an arbitrary positive integer, and let us replace the vertices of R with pairwise disjoint N-sets $V_1, V_2, \ldots, V_r$. We construct two graphs on the same vertex-set $V = \cup V_i$. The graph $R(N)$ is obtained by replacing all edges of R with copies of the complete bipartite graph $K_{N,N}$, and a sparser graph G is constructed by replacing the edges of R with some*

(ε, δ)-super-regular pairs. If a graph H with maximum degree $\Delta(H) \le \Delta$ can be embedded into $R(N)$, then it can be embedded into G.

4. Nonextremal case. In this section we prove the following theorem.

THEOREM 4.1. *For every $\alpha > 0$ and every positive integer s, there exist $\beta > 0$ and positive integer m_1 such that the following holds for all $n = ms$ with $m \ge m_1$. Given a bipartite graph $G = (A, B; E)$ with $|A| = |B| = n$, if $\delta(G) \ge (\frac{1}{2} - \beta)n$, then either G contains a $K_{s,s}$-factor, or there exist*

$$A_1 \subset A, \quad B_1 \subset B \quad \text{such that} \quad |A_1| = |B_1| = \lfloor n/2 \rfloor, \quad d(A_1, B_1) \le \alpha. \tag{4.1}$$

We say that the graphs G satisfying (4.1) are in the *extremal case*.

Proof of Theorem 4.1. We assume that n is large and use the following sequence of parameters (without specifying the actual dependence of them):

$$\varepsilon \ll d \ll \beta \ll \alpha.$$

For simplicity, we will omit round-offs if they are not crucial.

Let $G = (A, B; E)$ be a bipartite graph with $|A| = |B| = n$ and $\delta(G) \ge (\frac{1}{2} - \beta)n$. We apply the regularity lemma (Lemma 3.2) to G with parameters ε and d. The lemma partitions sets A, B into $A_0, A_1, \ldots, A_k$ and $B_0, B_1, \ldots, B_k$, respectively, such that $|A_i| = |B_i| = N \le \varepsilon n$ for all $i \ge 1$ and $|A_0| = |B_0| \le \varepsilon n$. Under the subgraph G' mentioned in the lemma, for any $i > 1$, $j > 1$, (A_i, B_j) is ε-regular with density either 0 or exceeding d. We write $A_i \sim B_j$ if (A_i, B_j) is ε-regular with density exceeding d. The regularity lemma also gives that $\deg_{G'}(x) \ge \left(\frac{1}{2} - \beta\right) n - (d + \varepsilon)n$ for all vertices x and $e(G') \ge e(G) - (d + \varepsilon)n^2$. From now on, we conveniently treat G' as our underlying graph and specify G or G' in subscripts only if necessary.

As in many applications of the regularity lemma, it is convenient to consider the *reduced graph* G_r. In our case, G_r is bipartite with two vertex-sets $U_1 = \{a_1, \ldots, a_k\}$ and $U_2 = \{b_1, \ldots, b_k\}$ such that a_i is adjacent to b_j if and only if $A_i \sim B_j$. It is easy to see that the degree condition in G forces that $\delta(G_r) \ge (\frac{1}{2} - 2\beta)k$.

CLAIM 4.2. *If G_r contains two subsets $X \subseteq U_1$, $Y \subseteq U_2$ such that $|X| \ge (\frac{1}{2} - 3\beta)k$, $|Y| \ge (\frac{1}{2} - 3\beta)k$, and no edge exists between X and Y, then G is in the extremal case.*

Proof. Without loss of generality, assume that $|X| = (\frac{1}{2} - 3\beta)k$ and $|Y| = (\frac{1}{2} - 3\beta)k$. Let $A' = \bigcup_{a_i \in X} A_i$ and $B' = \bigcup_{b_i \in Y} B_i$. We have

$$\left(\frac{1}{2} - 4\beta\right) n < \left(\frac{1}{2} - 3\beta\right) kN = |X|N = |A'| \le \left(\frac{1}{2} - 3\beta\right) n.$$

The same holds for $|B'|$. Since there is no edge between X and Y, then $e_{G'}(A', B') = 0$. Consequently $e_G(A', B') \le e_{G'}(A', B') + (d + \varepsilon)n|A| < dn^2$. By adding at most $4\beta n$ vertices to each of A' and B', we obtain two subsets of size $\lfloor n/2 \rfloor$ with at most $dn^2 + 4\beta n^2 + 4\beta n^2$ edges. Since $d \ll \beta \ll \alpha$, G is in the extremal case. ∎

From now on, we assume that G is *not* in the extremal case.

We first claim that G_r contains a perfect matching. Indeed, let M be a matching of G_r with the maximum size. After relabeling indices if necessary, we may assume that $M = \{a_i b_i : i = 1, \ldots, \ell\}$. Suppose that $x \in U_1$ and $y \in U_2$ are not in the vertex-set $V(M)$ of M. Then the neighborhood $N(x)$ is a subset of $V(M)$; otherwise we can enlarge M by adding an edge xz for any $z \in N(x) - V(M)$. We have $N(y) \subseteq V(M)$ for the same reason. Now let $I = \{i : b_i \in N(x)\}$ and $J = \{j : a_j \in N(y)\}$. If

$I \cap J \neq \emptyset$, that is, there exists i such that xb_i and ya_i are both edges, then we can obtain a larger matching by replacing a_ib_i in M by xb_i and ya_i. Otherwise, assume that $I \cap J = \emptyset$. Since $|I|, |J| \geq \delta(G_r) \geq (\frac{1}{2} - 2\beta)k$ and G is not in the extremal case, by the contrapositive of Claim 4.2, there exists an edge between $\{a_i : i \in I\}$ and $\{b_j : j \in J\}$. This implies that there exists $i \neq j$ such that xb_i, a_ib_j, and ya_j are edges. Replacing a_ib_i, a_jb_j in M by xb_i, a_ib_j, and ya_j, we obtain a larger matching, a contradiction.

We therefore assume that $A_i \sim B_i$ for all $i \geq 1$.

If each ε-pair (A_i, B_i) is also super-regular and s divides N, then the blow-up lemma (Lemma 3.3) guarantees that $G'(A_i, B_i)$ contains a $K_{s,s}$-factor (since $K_{N,N}$ contains a $K_{s,s}$-factor). If we also know that $A_0 = B_0 = \emptyset$, then we obtain a $K_{s,s}$-factor in G' (consequently, in G). Otherwise we do the following steps (details of these steps are given next). *Step* 1: For each $i \geq 1$, we move vertices from A_i to A_0 and from B_i to B_0 such that each remaining vertex in (A_i, B_i) has at least $(d-2\varepsilon)N$ neighbors. *Step* 2: We eliminate A_0 and B_0 by removing copies of $K_{s,s}$, each of which contains at most one vertex of $A_0 \cup B_0$. *Step* 3: We make sure that for each $i \geq 1$, $|A_i| = |B_i| > (1-d)N$ and $|A_i|$ is divisible by s. Finally we apply the blow-up lemma to each (A_i, B_i) (which is still super-regular) to finish the proof. Note that we always refer to the clusters as A_i, B_i, $i \geq 0$, even though they may gain or lose vertices during the process.

Step 1. For each $i \geq 1$, we remove all $v \in A_i$ such that $\deg(v, B_i) < (d-\varepsilon)N$ and all $v \in B_i$ such that $\deg(v, A_i) < (d-\varepsilon)N$. The definition of regularity guarantees that the number of removed vertices is at most εN. We then remove more vertices from either A_i or B_i to make sure A_i and B_i still have the same number of vertices. All removed vertices are added to A_0 and B_0. As a result, we have $|A_0| = |B_0| \leq 2\varepsilon n$.

Step 2. This step implies that a vertex in A_0, B_0 can be viewed as a vertex in A_i or B_i for some $i \geq 1$. For a vertex $x \in V$ and a cluster C, we say x is adjacent to C, or $x \sim C$ if and only if $\deg_G(x, C) \geq dN$. We claim that, at present, each vertex is adjacent to at least $(\frac{1}{2} - 2\beta)k$ clusters. If this is not true for some $x \in A$, then we obtain a contradiction

$$\left(\frac{1}{2} - \beta\right) n \leq \deg_G(x) \leq \left(\frac{1}{2} - 2\beta\right) kN + dNk + 2\varepsilon n < \left(\frac{1}{2} - \frac{3}{2}\beta\right) n.$$

Assign an arbitrary order to the vertices in A_0. For each $v \in A_0$, we pick some B_i adjacent to v. The selection of B_i is arbitrary, but no B_i is selected more than $\frac{dN}{6s}$ times. Such B_i exists even for the last vertex of A_0 because $|A_0| \leq 2\varepsilon n < (\frac{1}{2} - 2\beta)k\frac{dN}{6s}$. For each $v \in A_0$ and its corresponding B_i, we remove a copy of $K_{s,s}$ containing s vertices in B_i, and $s-1$ vertices in A_i and v. Such a copy of $K_{s,s}$ can always be found even if v is the last vertex in A_0 because (A_i, B_i) is ε-regular and $\deg_G(v, B_i) \geq dN > \varepsilon N + \frac{dN}{6s}s$. As a result, A_i now has one more vertex than B_i, so one may view this process as moving v to A_i. We repeat this process for all $v \in B_0$ as well. By the end of this step, we have $A_0 = B_0 = \emptyset$, and each A_i, B_i, $i \geq 1$, contains at least $N - \varepsilon N - dN/3$ vertices (for example, A_i may have lost $\frac{dN(s-1)}{6s}$ vertices because of A_0 and $dN/6$ vertices because of B_0). Note that the sizes of A_i and B_i may be different.

Step 3. We show that for any $i \neq j$, there is a path $A_iB_{i_1}A_{i_1} \cdots B_{i_t}A_{i_t}B_jA_j$ for some $0 \leq t \leq 2$. If such a path exists, then for each i_ℓ, $1 \leq \ell \leq t+1$ (assume that $i = i_0$ and $j = i_{t+1}$), we remove a copy of $K_{s,s}$ containing one vertex from $A_{i_{\ell-1}}$, s vertices from B_{i_ℓ}, and $s-1$ vertices from A_{i_ℓ}. This removal reduces the size of A_i by one, increases the size of A_j by one, but does not change the sizes of other clusters (modulo s). We may therefore adjust the sizes of A_i and B_i (for $i \geq 1$) such that they are equal and divisible by s. Now we show how to find this path from A_1 to A_2. First, if $A_1 \sim B_2$, then $A_1 B_2 A_2$ is a path. Let $I = \{i : A_1 \sim B_i\}$ and $J = \{i : A_i \sim B_2\}$. If there exists $i \in I \cap J$, then we find a path $A_1 B_i A_i B_2 A_2$. Otherwise $I \cap J = \emptyset$. Since both $|I|$ and $|J|$ are greater than $(1-2\beta)k$, Claim 4.2 guarantees that there exist $i \in I$ and $j \in J$ such that $a_i b_j$ is an edge of G_r, or $A_i \sim B_j$. We thus have a path $A_1 B_i A_i B_j A_j B_2 A_2$. Note that in this step we require that a cluster is contained in at most $\frac{dN}{3s}$ paths. This restriction has little impact on the arguments above: we have $|I|, |J| > (1-3\beta)k$ instead, still satisfying the conditions of Claim 4.2.

Now $A_0 = B_0 = \emptyset$, and for all $i \geq 1$, $|A_i| = |B_i|$ is divisible by s. Furthermore, A_i and B_i each contain at least $N - \varepsilon N - 2dN/3$ vertices, and each pair (A_i, B_i) is $(\frac{\varepsilon}{2}, \frac{d}{4})$-super-regular. Applying the blow-up lemma to each (A_i, B_i), we find the desired $K_{s,s}$-factor. ∎

5. Extremal case.

THEOREM 5.1. *Theorem* 1.2 *holds if* $G = (A, B; E)$ *satisfies* (4.1) *with sufficiently small* α.

Proof. We define

$$A_1' = \left\{ x \in A : \deg(x, B_1) < \alpha^{\frac{1}{3}} \frac{n}{2} \right\} \quad \text{and} \quad B_1' = \left\{ x \in B : \deg(x, A_1) < \alpha^{\frac{1}{3}} \frac{n}{2} \right\}.$$

We claim that

$$|A_1'|, |B_1'| \geq (1 - \alpha^{\frac{2}{3}}) \frac{n}{2}. \tag{5.1}$$

In fact, this follows from

$$\alpha^{\frac{1}{3}} \frac{n}{2} \ |A_1 - A_1'| \leq e(A_1 - A_1', B_1) \leq e(A_1, B_1) \leq \alpha \frac{n^2}{4}, \tag{5.2}$$

which implies that $|A_1 - A_1'| \leq \alpha^{\frac{2}{3}} \frac{n}{2}$, or $|A_1'| \geq (1-\alpha^{\frac{2}{3}}) \frac{n}{2}$ (the same holds for B_1'). More precisely we have $|A_1'| \geq \lfloor n/2 \rfloor - \alpha^{\frac{2}{3}} n/2$. As in section 4, we omit floor functions if they are not crucial to our calculation.

Let $A_2 = A - A_1$ and $B_2 = B - B_1$. We further define

$$A_2' = \left\{ x \in A : \deg(x, B_1) > (1 - \alpha^{\frac{1}{3}}) \frac{n}{2} \right\}, B_2' = \left\{ x \in B : \deg(x, A_1) > (1 - \alpha^{\frac{1}{3}}) \frac{n}{2} \right\}.$$

Apparently A_1' and A_2', B_1' and B_2' are disjoint. We claim that

$$|A_2'|, |B_2'| \geq (1 - \alpha^{\frac{2}{3}}) \frac{n}{2}. \tag{5.3}$$

In fact, the degree condition $\delta(G) \geq \frac{n}{2}$ (the extra constants are not needed) together with $e(B_1, A_1) \leq \alpha \frac{n^2}{4}$ implies that $e(B_1, A_2) \geq (1-\alpha) \frac{n^2}{4}$. From similar inequalities on $\bar{e}(B_1, A_2)$, the number of nonedges between B_1 and A_2, as in (5.2), we derive that $|A_2 - A_2'| \leq \alpha^{\frac{2}{3}} \frac{n}{2}$ or $|A_2'| \geq (1 - \alpha^{\frac{2}{3}}) \frac{n}{2}$.

We call the vertices in A_i' and B_i', $i = 1, 2$, *typical* vertices. For $j \neq i$, A_i' and B_j' are called *diagonal* sets to each other. We claim that for $j \neq i$,

$$\delta(A_i', B_j'), \delta(B_j', A_i') > \frac{n}{2} - \alpha^{\frac{1}{3}} n. \tag{5.4}$$

In other words, every typical vertex in a set is adjacent to almost all vertices in its diagonal set. In fact, the definition of A_1' and the degree condition $\delta(G) \geq \frac{n}{2}$ force that $\delta(A_1', B_2) > (1-\alpha^{\frac{1}{3}})\frac{n}{2}$. Consequently, $\delta(A_1', B_2') > (1-2\alpha^{\frac{1}{3}})\frac{n}{2}$ because $|B_2'| \geq |B_2| - \alpha^{\frac{2}{3}}\frac{n}{2}$. Furthermore, the definition of A_2' and the fact that $|B_1'| \geq |B_1| - \alpha^{\frac{2}{3}}\frac{n}{2}$ together imply that $\delta(A_2', B_1') > (1-2\alpha^{\frac{1}{3}})\frac{n}{2}$. Similarly we have $\delta(B_1', A_2'), \delta(B_2', A_1') > (1-2\alpha^{\frac{1}{3}})\frac{n}{2}$.

Let $A_0 = A - A_1' - A_2'$ and $B_0 = B - B_1' - B_2'$. We call the vertices in A_0 and B_0 *special* vertices. We claim that

$$(5.5) \quad |A_0|, |B_0| \leq \alpha^{\frac{2}{3}} n, \quad \text{and} \quad \delta(A_0, B_i'), \delta(B_0, A_i') \geq \left(\alpha^{\frac{1}{3}} - \alpha^{\frac{2}{3}}\right)\frac{n}{2} \quad \text{for } i = 1, 2.$$

In fact, $|A_0|, |B_0| \leq \alpha^{\frac{2}{3}} n$ because of (5.1) and (5.3). Every vertex $x \in A_0$ satisfies $\alpha^{\frac{1}{3}}\frac{n}{2} \leq \deg(x, B_1) \leq (1-\alpha^{\frac{1}{3}})\frac{n}{2}$, and, consequently, $\deg(x, B_1') \geq (\alpha^{\frac{1}{3}} - \alpha^{\frac{2}{3}})\frac{n}{2}$ and $\deg(x, B_2') \geq (\alpha^{\frac{1}{3}} - \alpha^{\frac{2}{3}})\frac{n}{2}$. The vertices in B_0 have a similar property.

The main idea of finding a $K_{s,s}$-factor in G can be seen from the following *ideal* case, in which $|A_i'| = |B_i'| = n/2 = sk$ for $i = 1, 2$ (thus $A_0 = B_0 = \emptyset$). We find a $K_{s,s}$-factor in $G[A_1', B_2']$ and a $K_{s,s}$-factor in $G[B_1', A_2']$ in the same way. In order to tile $G[A_1', B_2']$, we arbitrarily partition each of A_1' and B_2' into sets of size s, denoted by $X_1, \dots, X_k$ and $Y_1, \dots, Y_k$, respectively. Create an auxiliary bipartite graph H with vertices $X_1, \dots, X_k$, $Y_1, \dots, Y_k$ such that X_i is adjacent to Y_j whenever $G[X_i, Y_j]$ is a complete bipartite graph $K_{s,s}$. We have $\delta(H) \geq k - s\alpha^{\frac{1}{3}} n \geq k/2$ because of (5.4). Then the marriage theorem guarantees that H contains 1-factor; in turn, $G[A_1', B_1']$ contains a $K_{s,s}$-factor.

Case I. n/s is even.

Let $\mathcal{V} = \{A_1', B_1', A_2', B_2'\}$. First assume that no member of $\mathcal{V}$ contains more than $n/2$ vertices. If $A_0 \neq \emptyset$, then we move the vertices of A_0 to A_1' or A_2' such that at the end both A_1' and A_2' have $n/2$ vertices. We do the same for B_0. For each special vertex x, we find a copy of $K_{s,s}$ containing x and $2s-1$ typical vertices. Suppose that at present $x \in A_1'$. We first pick s unoccupied vertices $y_1, \dots, y_s$ from $N(x, B_2')$—this is possible because $\deg(x, B_2') \geq (\alpha^{\frac{1}{3}} - \alpha^{\frac{2}{3}})\frac{n}{2} > s(|A_0| + |B_0|)$ by (5.5). Then pick $s-1$ unoccupied vertices from the common neighborhood of $y_1, \dots, y_s$ in A_1'—this is possible because the size of their common neighborhood is at least $\frac{n}{2} - (s-1)\alpha^{\frac{1}{3}} n > s(|A_0| + |B_0|)$ by (5.4) and (5.5). After removing these copies of $K_{s,s}$, we can find a $K_{s,s}$-factor in the remaining graph as in the ideal case.

Now assume that some member of $\mathcal{V}$ contains more than $n/2$ vertices. We need to reduce its size to $n/2$ before handling special vertices. If $|A_2'| > \frac{n}{2}$, then we move a vertex $x \in A_2'$ to A_0 if $\deg(x, B_2') \geq \alpha^{\frac{1}{3}}\frac{n}{2}$. Repeat this process until either $|A_2'| = \frac{n}{2}$ or the maximum degree from A_2' to B_2', $\Delta(A_2', B_2') < \alpha^{\frac{1}{3}}\frac{n}{2}$. We do the same for B_2'. The claim (5.5) is still valid because (5.1) and (5.3) are not changed, and new vertices in A_0 and B_0 also have large degrees in B_1', B_2' and A_1', A_2', respectively. We therefore handle these new vertices in the same way as the old vertices in A_0 and B_0.

In order to further reduce the size of A_i' or B_i', we will find some vertex-disjoint s-stars ($K_{1,s}$'s) and relocate the centers of these stars. This is possible because of the following technical lemma, which roughly says that in every almost balanced bipartite graph on V_1 and V_2, if the minimum degree from V_1 to V_2 is not small and the maximum degree from V_2 to V_1 is not large, then this graph contains many vertex-disjoint stars, some of which are from A and B (i.e., centered at A), and some from B to A. We postpone its proof to the end of this section.

LEMMA 5.2. *Let $1 \le h \le \delta \le M$ and $0 < c < \frac{1}{6h+7}$. Suppose that $F = (V_1, V_2; E)$ is a bipartite graph such that $||V_i| - M| \le cM$ for $i = 1, 2$. If $\delta = \delta(V_1, V_2) \le cM$ and $\Delta = \Delta(V_2, V_1) \le cM$, then we can find a family of vertex-disjoint h-stars, $2(\delta - h + 1)$ of which have centers in V_1 and $2(\delta - h + 1)$ of which have centers in V_2.*

We first assume that only one member of $\mathcal{V}$ contains more than $n/2$ vertices. Suppose $|A'_1| = \frac{n}{2} + t$ with $t > 0$ (the case when $|B'_1| > n/2$ is similar). The min-degree condition $\delta(G) \ge n/2 + s - 1$ forces $\delta(B'_1, A'_1) \ge t + s - 1$. We also know that $\Delta(A'_1, B'_1) \le (\alpha^{\frac{1}{3}} + \alpha^{\frac{2}{3}})\frac{n}{2}$ from the definition of A'_1 and (5.1). Applying Lemma 5.2 with $V_1 = B'_1$ and $V_2 = A'_1$, we obtain t vertex-disjoint s-stars whose centers are in A'_1. Then we move the centers of these stars to A'_2 and immediately remove t disjoint copies of $K_{s,s}$ from $G[A'_2, B'_1]$, each of which contains one of the stars—this can be done because the leaves of the stars are typical vertices. Now suppose that $|A'_2| = \frac{n}{2} + t$ with $t > 0$ (the case when $|B'_2| > n/2$ is similar). Since $\Delta(A'_2, B'_2) < \alpha^{\frac{1}{3}}\frac{n}{2}$ (recall that the vertices of A'_2 with larger degree in B'_2 have been moved to A_0), we can follow the same procedure.

Now assume that two members of $\mathcal{V}$ each contain more than $n/2$ vertices. If they are diagonal sets, such as A'_2 and B'_1, then we apply Lemma 5.2 to $G[A'_1, B'_1]$ and $G[A'_2, B'_2]$ separately. Otherwise, say, $|A'_2| = n/2 + t_1$ and $|B'_2| = n/2 + t_2$ with $t_1, t_2 > 0$. Lemma 5.2 guarantees that we can find $t_1 + t_2$ vertex-disjoint s-stars in $G[A'_2, B'_2]$, t_1 of which have centers in A'_2 and t_2 of which have centers in B'_2. After moving these centers to A'_1 and B'_1, we proceed as in the previous case.

Case II. n/s is odd.

Let $n/s = 2k + 1$ for some integer k. We have $\delta(G) \ge \frac{n+3s}{2} - 2 = (k+2)s - 2$. The main procedure of finding a $K_{s,s}$-factor is the same as in the case when m is even: we first adjust the sizes of A'_i or B'_i, then add special vertices to A'_i or B'_i, and finally complete the $K_{s,s}$-tiling by the marriage theorem. The only difference here is in how to adjust the sizes of A'_i or B'_i.

Let $G_1 = G[A'_1, B'_1]$ and $G_2 = G[A'_2, B'_2]$. After moving vertices from A'_2 to A_0, from B'_2 to B_0 if necessary, we have $|A'_2| > ks$ only if $\Delta(A'_2, B'_2) < \alpha^{\frac{1}{3}}\frac{n}{2}$, and $|B'_2| > ks$ only if $\Delta(B'_2, A'_2) < \alpha^{\frac{1}{3}}\frac{n}{2}$. This makes it possible to apply Lemma 5.2 to G_2 whenever $|A'_2| > ks$ or $|B'_2| > ks$. Note that all inequalities in (5.1), (5.3), (5.4), (5.5) still hold.

Let $a_i = |A'_i| - ks$, $b_i = |B'_i| - ks$ for $i = 1, 2$. We have $a_1 + a_2 \le s$ and $b_1 + b_2 \le s$. Note that a_i or b_i could be negative or greater than s. Without loss of generality, we may assume that

$$\max(a_1, b_1) \ge \max(a_2, b_2).$$

Our goal is that after adjusting the sizes of A'_i and B'_i and adding the vertices of A_0 and B_0 appropriately, we either have $|A'_1| = |B'_2| = ks$ and $|A'_2| = |B'_1| = (k+1)s$, or $|A'_1| = |B'_2| = (k+1)s$ and $|A'_2| = |B'_1| = ks$. As seen in the case when n/s is even, we can adjust the sizes of A'_i and B'_i by relocating the centers of s-stars. Here the difficulty is that, when $a_i, b_i \le 1$, we may not be able to find any s-star in G_i. In this case we will find two $(s-1)$-stars, one in G_1 the other in G_2 such that together they form a copy of $K_{s,s}$, and then relocate the centers of these two $(s-1)$-stars.

Let us consider all possible values of a_i and b_i.

Case 1. $\max(a_1, b_1) \le 0$ (and therefore $\max(a_2, b_2) \le 0$).

We simply add the vertices of A_0 and B_0 to A'_i and B'_i such that $|A'_1| = |B'_2| = ks$ and $|A'_2| = |B'_1| = (k+1)s$.

Case 2. $\max(a_1, b_1) \ge 2$.

Without loss of generality, assume that $a_1 \geq b_1$. We have $\delta(B'_1, A'_1) = (k+2)s - 2 - (n - ks - a_1) = a_1 + (s-2)$. By Lemma 5.2, G_1 contains a family of vertex-disjoint s-stars with $2(a_1 + (s-2) - s + 1) = 2(a_1 - 1) \geq a_1$ of them from A'_1 to B'_1, and a_1 of them from B'_1 to A'_1. We now consider the following subcases.

- $b_2 \leq 0$. We move a_1 centers of the stars (later simply called *centers*) from A'_1 to A'_2 and $\max(b_1 - s, 0)$ centers from B'_1 to B'_2. As a result, $|A'_1| = ks$, $|B'_1| \leq ks + s$, and $|B'_2| \leq ks$. Next add the vertices of A_0 and B_0 such that $|A'_1| = |B'_2| = ks$ and $|A'_2| = |B'_1| = (k+1)s$.
- $b_2 > 0$ and $\max\{a_2, b_2\} \geq 2$. By Lemma 5.2, G_2 contains a family of vertex-disjoint s-stars with b_2 of them from A'_2 to B'_2, and b_2 of them from B'_2 to A'_2. We move a_1 centers from A'_1 to A'_2, and b_2 centers from B'_2 to B'_1. The sizes of A'_1 and B'_2 thus become ks. We then add the vertices of A_0 and B_0 such that $|A'_2| = |B'_1| = (k+1)s$.
- $b_2 = 1$ and $a_2 \leq 0$. We move $\max(a_1 - s, 0)$ centers from A'_1 to A'_2 and $\max(b_1, 0)$ centers from B'_1 to B'_2. As a result, $|A'_1| \leq (k+1)s$, $|B'_1| \leq ks$, and $|B'_2| \leq (k+1)s$ (since $b_2 = 1 \leq s$). Next add the vertices of A_0 and B_0 such that $|A'_2| = |B'_1| = ks$ and $|A'_1| = |B'_2| = (k+1)s$.
- $b_2 = a_2 = 1$. If there exists an s-star from A'_2 to B'_2, then we move the center of this star to A'_1 and move $\max(b_1, 0)$ centers from B'_1 to B'_2. We apply a similar procedure when there exists an s-star from B'_2 to A'_2. Now assume that there exists no s-star in G_2, or $\Delta(G_2) \leq s - 1$. Since $\delta(G) \geq (k+2)s - 2$ and $|A'_2| = |B'_2| = ks + 1$, it forces that every vertex in $A'_2 \cup B'_2$ is adjacent to all the vertices in $A - A'_2$, or all the vertices in $B - B'_2$; i.e., $G[A'_2, B - B'_2]$ and $G[A - A'_2, B'_2]$ become two complete bipartite graphs. Fix an $(s-1)$-star S_2 from B'_2 to A'_2 (any vertex $x \in B'_2$ can be the center because $\deg(x, A'_2) \geq s - 1$). Recall that there are a_1 vertex-disjoint s-stars from A'_1 to B'_1. We pick one of them and obtain an $(s-1)$-star S_1 as its subgraph. Note that S_1 and S_2 together form a copy of $K_{s,s}$. We relocate the centers of S_1 and S_2, i.e., the center of S_1 to A'_2 and the center of S_2 to B'_1. Next we move the centers of the remaining $a_1 - 1$ s-stars to A'_2 such that $|A'_1| = |B'_2| = ks$.

Case 3. $\max(a_1, b_1) = 1$ (and therefore $\max(a_2, b_2) \leq 1$).

In this case G_1 or G_2 need not contain any s-star. Lemma 5.2 provides only four vertex-disjoint $(s-1)$-stars G_1, two of them from A'_1 to B'_1 and the other two from B'_1 to A'_1. Without loss of generality, assume that $a_1 = 1$. We consider the following subcases.

- $b_1 \leq 0$ and $a_2 \leq 0$. We simply add the vertices of A_0 and B_0 such that $|A'_1| = |B'_2| = (k+1)s$ and $|A'_2| = |B'_1| = ks$.
- $b_1 \leq 0$ and $a_2 = 1$. If there exists an s-star from A'_2 to B'_2, then we move its center to A'_1 and we are in the previous subcase. We then assume that there exists no s-star from A'_2 to B'_2. Fix an $(s-1)$-star with center $x \in A'_1$ and a set L of leaves in B'_1. Let $B''_2 = N(x, B'_2)$ and $A''_2 = \bigcap_{y \in L} N(y, A'_2)$. By (5.4), we have $|B''_2| \geq \frac{n}{2} - \alpha^{\frac{1}{3}} n$ and $|A''_2| \geq \frac{n}{2} - (s-1)\alpha^{\frac{1}{3}} n$. We claim that there is an $(s-1)$-star from B''_2 to A''_2. Otherwise, since $\delta(B'_2, A'_2) \geq s - 1$ (using $a_2 = 1$), each vertex in B''_2 must have at least one neighbor in $A'_2 - A''_2$. By averaging, there exists $u \in A'_2 - A''_2$ with $\deg(u, B''_2) \geq |B''_2| / |A'_2 - A''_2| > s$, contradicting the assumption that no s-star exists from A'_2 to B'_2. One $(s-1)$-star from B''_2 to A''_2 together with x and L form a copy of $K_{s,s}$. We relocate the centers of these two $(s-1)$-stars. After adding some vertices of A_0 and B_0, we obtain that $|A'_1| = |B'_2| = ks$.

- $b_1 = 1$ and $\min(a_2, b_2) \le 0$. Without loss of generality, assume that $b_2 \le 0$. We now separate the cases when $a_2 = 1$ and when $a_2 \le 0$. If $a_2 = 1$, then after switching G_1 and G_2, we are in the previous case when $b_1 \le 0$ and $a_2 = 1$. If $a_2 \le 0$, then we follow the arguments for $a_2 = b_2 = 1$ in Case 2. In other words, if G_1 contains an s-star, then we move its center to G_2 and are done. Otherwise, the minimum degree condition $\delta(G) \ge (k+2)s - 2$ forces that $G[A'_1, B - B'_1]$ and $G[B'_1, A - A'_1]$ become complete bipartite graphs, and $\deg(x, B'_1) = s - 1$ for all $x \in A'_1$. We form a copy of $K_{s,s}$ with an arbitrary vertex $x \in A'_1$, $s-1$ vertices from $N(x, B'_1)$, an arbitrary vertex $y \in B_0$, and $s-1$ vertices from $N(y, A'_2)$. We move x to A'_2 and y to B'_1. After adding some vertices of A_0 and B_0, we obtain that $|A'_1| = |B'_2| = ks$.
- $a_1 = b_1 = a_2 = b_2 = 1$. If there exist an s-star from A'_1 to B'_1 and an s-star from B'_2 to A'_2, then we relocate the centers of these two stars, and consequently $|A'_1| = |B'_2| = ks$. Otherwise, say, there exists no s-star from A'_1 to B'_1. This implies that $G[A'_1, B'_2]$ is complete. Since $\delta(G_i) \ge s - 1$ for $i = 1, 2$, starting from an $(s-1)$-star with center $x \in A'_2$ and leaves in B'_2, we can find an $(s-1)$-star from $N(x, B'_1)$ to A'_1. The union of these two stars induces a copy of $K_{s,s}$ in G because $G[A'_1, B'_2]$ is complete. We then relocate the centers of the two stars and are done.

We thus complete the proof of Theorem 5.1. □

We now prove Lemma 5.2 by using the following simple fact.

FACT 5.3. *Let $F = (A, B; E)$ be a bipartite graph with $\delta(A, B) = \delta$ and $\Delta(B, A) = \Delta$. Then F contains f_h vertex-disjoint h-stars from A to B, and g_h vertex-disjoint h-stars from B to A (the stars from A to B and those from B to A need not be disjoint), where*

$$f_h \ge \frac{|A|(\delta - h + 1)}{h\Delta + \delta - h + 1}, \quad g_h \ge \frac{\delta|A| - (h-1)|B|}{\Delta - h + 1 + \delta h}.$$

Proof. We first bound f_h, the size of a maximum family $\mathcal{S}_1$ of vertex-disjoint stars from A to B. Denote the sets of the centers and the leaves of the stars in $\mathcal{S}_1$ by A_0 and B_0, respectively. For each $x \in A - A_0$, we have $\deg(x, B - B_0) \le h - 1$ (otherwise $\mathcal{S}_1$ is not maximal). Consequently $\deg(x, B_0) \ge \delta - h + 1$. The desired bound of f_h now follows from

$$(\delta - h + 1)(|A| - f_h) \le e(A - A_0, B_0) \le |B_0|\Delta(B, A) = f_h h\Delta,$$

or $(\delta - h + 1)|A| \le f_h(\delta - h + 1 + \Delta h)$.

We next bound g_h, the size of a maximum family $\mathcal{S}_2$ of vertex-disjoint stars from B to A. Denote the sets of the centers and the leaves of the stars in $\mathcal{S}_2$ by B_0 and A_0, respectively. For each $y \in B - B_0$, we have $\deg(y, A - A_0) \le h - 1$. For $y \in B_0$, we have $\deg(y, A - A_0) \le \Delta$. The desired bound of g_h now follows from

$$\delta(|A| - hg_h) \le e(B, A - A_0) \le (|B| - g_h)(h - 1) + g_h\Delta,$$

or $\delta|A| - |B|(h-1) \le g_h(\Delta - h + 1 + \delta h)$. □

Proof of Lemma 5.2. Since $c < \frac{1}{6h+7}$, we have $\frac{1-c}{hc+c} > 6$. We apply Fact 5.3 with $A = V_1$ and $B = V_2$ and obtain

$$f_h \ge \frac{|V_1|(\delta - h + 1)}{h\Delta + \delta - h + 1} \ge \frac{(1-c)M}{hcM + cM}(\delta - h + 1) > 6(\delta - h + 1)$$

vertex-disjoint h-stars from V_2 to V_1. Let $\mathcal{S}_1$ be the family of $4(\delta-h+1)$ such stars and S be the set of their centers. We now apply Fact 5.3 with $A=V_1-S$ and $B=V_2$, and the number of vertex-disjoint h-stars from V_2 to V_1-S is

$$\begin{aligned} g_h &\geq \frac{\delta|V_1-S|-(h-1)|V_2|}{\Delta-h+1+\delta h} \geq \frac{\delta(M-cM-4(\delta-h+1))-(h-1)(1+c)M}{cM+hcM} \\ &= \frac{\delta(1-c)-(h-1)(1+c)}{c+hc} - \frac{4\delta(\delta-h+1)}{cM(1+h)} \\ &\geq (\delta-h+1)\frac{1-c}{c+hc} - \frac{2c(h-1)}{c+hc} - \frac{4}{1+h}(\delta-h+1) \\ &> 6(\delta-h+1)-2-2(\delta-h+1) \geq 2(\delta-h+1). \end{aligned}$$

Let $\mathcal{S}_2$ consist of $2(\delta-h+1)$ h-stars from V_2 to V_1-S. The centers of the stars in $\mathcal{S}_2$ overlap with at most $2(\delta-h+1)$ leaves of the stars in $\mathcal{S}_1$. Removing from $\mathcal{S}_1$ the stars containing these leaves, we obtain a family of vertex-disjoint h-stars with $2(\delta-h+1)$ of them centered at V_1 and $2(\delta-h+1)$ of them centered at V_2. ∎

6. Concluding remarks.

Given a bipartite graph H of order $h\geq 2$, let $\delta_2(n,H)$ denote the smallest integer k such that every balanced bipartite graph G whose order $2n$ is divisible by h and with $\delta(G)\geq k$ contains an H-factor. Assume that n is sufficiently large. In this paper we determine $\delta_2(n,K_{s,s})$ exactly for $s\geq 2$. Recently Hladký and Schacht [11] determined $\delta_2(n,K_{s,t})$ for $s<t$ by applying the methods and results of this paper.

Now consider $\delta_2(n,H)$ for arbitrary bipartite H. Since $K_{h,h}$ trivially contains an H-factor, Theorem 1.2 implies that

$$\delta_2(n,H) \leq \delta_2(n,K_{h,h}) \leq \frac{n}{2}+\frac{3h}{2}-2. \tag{6.1}$$

On the other hand, it is easy to see that $\delta_2(n,H)>n/2-1$ for connected H. For example, when $n=mh$ for some even integer m, we let G be the disjoint union of $K_{n/2,n/2-1}$ and $K_{n/2,n/2+1}$. Then G contains no H-factor though $\delta(G)=n/2-1$. We thus know that $\delta_2(n,H)=\frac{n}{2}+O(1)$ for any connected H. For disconnected H, the $n/2$ in (6.1) may not be tight. Here we recall the minimum degree threshold $\delta(n,H)$ for tiling H perfectly in arbitrary graphs of order n. Kühn and Osthus [16] determined $\delta(n,H)$ for all graphs H up to an additive constant, in particular $\delta(n,H)=\frac{n}{2}+O(1)$ for connected bipartite H. It is interesting to see if $\delta_2(n,H)=\delta(n,H)+O(1)$ for all bipartite H.

Instead of H-factors, we may study the minimum degree condition for G containing an H-tiling of size $(1-o(1))v(G)$ (an *approximate* H-factor). It was shown in [12, 21] that a graph G contains an approximate H-factor if $\delta(G)\geq(1-1/\chi_{cr}(H))v(G)$, where $\chi_{cr}(H)$ is the so-called *critical chromatic number* satisfying $\chi(H)-1<\chi_{cr}(H)\leq\chi(H)$. It is interesting to prove a similar result for bipartite tiling.

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