TURÁN DENSITIES OF SOME HYPERGRAPHS RELATED TO K_{k+1}^k*

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Abstract. Let $B_i^{(k)}$ be the k-uniform hypergraph whose vertex set is of the form $S \cup T$, where $|S| = i$, $|T| = k - 1$, and $S \cap T = \emptyset$, and whose edges are the k-subsets of $S \cup T$ that contain either S or T. We derive upper and lower bounds for the Turán density of $B_i^{(k)}$ that are close to each other as $k \to \infty$. We also obtain asymptotically tight bounds for the Turán density of several other infinite families of hypergraphs. The constructions that imply the lower bounds are derived from elementary number theory by probabilistic arguments, and the upper bounds follow from some results of de Caen, Sidorenko, and Keevash.

Key words. Turán density, extremal problem, hypergraph

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1. Introduction. Given an integer $k \geq 2$ and a set V of size $|V| \geq k$, we denote by $\binom{V}{k}$ the family of all k-element subsets (in short, k-subsets) of V. A k-uniform hypergraph H is an ordered pair (V, E), where the edge set E is a subfamily of $\binom{V}{k}$. Given two k-uniform hypergraphs H and G, we call G H-*free* if G does not contain H as a subhypergraph. The *extremal function* $\mathrm{ex}(n, H)$ is defined as the maximum number of edges in an H-free k-uniform hypergraph on n vertices, while the *Turán density* of H is

$$\pi(H) = \lim_{n \to \infty} \frac{\mathrm{ex}(n, H)}{\binom{n}{k}}.$$

For $k = 2$ the Turán density of H is determined by its chromatic number, namely, $\pi(H) = 1 - \frac{1}{\chi(H)-1}$.

In the case $k \geq 3$ much less is known. Let K_r^k be the complete k-uniform hypergraph on r vertices. A classical problem of Turán [13] is to determine $\pi(K_r^k)$ when $r > k \geq 3$. Erdos [3] offered \$500 for determining $\pi(K_k^r)$ for any particular $k > r \geq 3$ and \$1000 for solving the problem for all k and r. There are several constructions which show that $\pi(K_4^3) \geq \frac{5}{9}$ (see, e.g., [7]), while the best upper bound is $\pi(K_4^3) \leq 0.561666$ obtained by Razborov [10]. (See Falgas-Ravry and Vaughan [4] for more recent results obtained using flag algebras.) For $\pi(K_{k+1}^k)$, the natural asymp-

totic question remains open: the best known bounds for $\pi(K^k_{k+1})$ [12, 8] imply that

$$(1) \qquad\qquad 1 - \Theta\left(\frac{\ln k}{k}\right) \leq \pi\left(K^k_{k+1}\right) \leq 1 - \Theta\left(\frac{1}{k}\right).$$

In this paper we consider the asymptotic Turán densities of some classes of hypergraphs related to K^k_{k+1}. Given $2 \leq i \leq k$, define a k-uniform hypergraph $B^{(k)}_i$ as follows. The vertex set of $B^{(k)}_i$ is of the form $S \cup T$, where $S \cap T = \emptyset$, $|S| = i$, and $|T| = k-1$, and the edge set consists of all k-element subsets of $S \cup T$ containing either S or T. Clearly, $B^{(k)}_2 = K^k_{k+1}$. Concerning the other extreme value of i, Bohman et al. [1] recently showed that as $k \to \infty$

$$\pi\left(B^{(k)}_k\right) = 1 - (2 + o(1))\left(\frac{\ln k}{k}\right).$$

As noted by Mubayi, the sequence of hypergraphs $B^{(k)}_2, B^{(k)}_3, \ldots, B^{(k)}_k$ starts with K^k_{k+1} and ends with the graph $B^{(k)}_k$ studied in [1]. Since this is a natural progression and the asymptotic Turán denisty of $B^{(k)}_k$ is known, progress on the Turán densities for the hypergraphs $B^{(k)}_i$ with $2 < i < k$ are a plausible route to a better understanding of the limiting behavior of $\pi(K^k_{k+1})$. In this paper we take the first tentative steps in this direction: we derive bounds resembling (1) for the hypergraphs $B^{(k)}_{k-i}$ and $B^{(k)}_{i+1}$ with i fixed and k tending to infinity.

THEOREM 1. *Fix* $i \geq 1$. *As* $k \to \infty$ *we have*

$$1 - \Theta\left(\frac{\ln k}{k^i}\right) \leq \pi\left(B^{(k)}_{i+1}\right), \pi\left(B^{(k)}_{k-i}\right) \leq 1 - \Theta\left(\frac{1}{k^i}\right).$$

Actually, Theorem 1 is the special case of the following more general statement.

THEOREM 2. *For every pair of integers* i, k *satisfying* $1 \leq i \leq \ln k/(10 \ln \ln k)$, *we have*

$$1 - (1 + o(1))\frac{12i^2 \ln k}{\binom{k}{i}} \leq \pi\left(B^{(k)}_{i+1}\right), \pi\left(B^{(k)}_{k-i}\right) \leq 1 - \frac{1}{k - i + \binom{k-1}{i} - 1},$$

where the $o(1)$ *term tends to* 0 *as* $k \to \infty$, *independently of* i.

The upper bound in Theorem 2, which is actually valid for $i < k/2$ as well, immediately follows from a result of Sidorenko [11], which relates the Turán density of a hypergraph to its size. The construction we used to prove the lower bound in Theorem 2 does not use the entire structure of the excluded hypergraph; it requires only the edges containing S and a single edge containing T. Motivated by this observation, we define $\widehat{B}^{(k)}_i$ to be the k-uniform hypergraph whose vertex set is the disjoint union $S \cup T \cup \{v\}$, where $|S| = i$ and $|T| = k - 1$, and whose edge set is

$$\{T \cup \{v\}\} \cup \left\{S \cup T' : T' \in \binom{T}{k-i}\right\}.$$

In other words, we obtain $\widehat{B}^{(k)}_i$ from $B^{(k)}_i$ by removing all of the edges that contain T and adding a new vertex v and a new edge consisting of T and v. Applying to this hypergraph a result of de Caen [2] on the number of complete subhypergraphs in dense hypergraphs, we shall obtain asymptotically matching bounds for $\pi(\widehat{B}^{(k)}_{k-i})$.

THEOREM 3. *Fix $i \geq 1$. We have $\pi(\widehat{B}_{k-i}^{(k)}) = 1 - \Theta(\frac{\ln k}{k^i})$ as $k \to \infty$.*

Finally, define a k-uniform hypergraph $C^{(k)}$ with $2k - 2$ vertices and k edges as follows. The vertex set of $C^{(k)}$ has a partition into two $(k-1)$-sets: $V(C^{(k)}) = S \cup T$ with $S \cap T = \emptyset$ and $|S| = |T| = k - 1$, and the edge set of $C^{(k)}$ consists of all k-sets containing S and a single k-set containing T. By applying a result of Keevash [5] that relates the Turán density of a hypergraph to the maximal size of a forest contained in it, we shall obtain asymptotically matching bounds for $\pi(C^{(k)})$.

THEOREM 4. *For $k \to \infty$, we have $\pi(C^{(k)}) = 1 - \Theta(\frac{\ln k}{k})$.*

While our upper and lower bounds for the Turán density of $B_i^{(k)}$ are close to each other, it would be nice to know which one is closer to the truth. We suspect that it should be closer to the upper bound. It would also be interesting to find a good lower bound on the Turán density of $B_i^{(k)}$ for i near $k/2$.

In section 2 we prove the lower bounds in Theorems 2, 3, and 4, and in section 3 we prove the upper bounds.

2. Lower bounds. Kim and Roush [6] constructed a dense K_{k+1}^k-free k-uniform hypergraph $\mathcal{H}_k$; let us start by recalling this construction. Let $\ell \approx \frac{k}{2\ln k}$ be a prime number, and take an equipartition of the vertex set $[n]$ into ℓ sets $V_1, \ldots, V_\ell$. It is easy to see that almost all k-subsets of $[n]$ intersect each of the sets V_i. Define the *weight* of a subset X of $[n]$ as $w(X) = \sum_{j=1}^{\ell} j|X \cap V_j| \pmod{\ell}$. Let $i \in \mathbb{Z}_\ell$ be chosen such that the number of k-sets having weight i is minimal. The edge set of $\mathcal{H}_k$ will consist of all k-sets that intersect all $V_1, \ldots, V_\ell$ except for those with weight i. In this section we generalize this construction.

DEFINITION 5. *Given positive integers $i, \ell,$ and k, an i-fold sum cover of $\mathbb{Z}_\ell$ is a multiset X of elements of $\mathbb{Z}_\ell$ such that every $b \in \mathbb{Z}_\ell$ can be written as a sum of exactly i distinct elements of X.*

Construction. Let

$$(2) \qquad \ell \sim \frac{1}{6i^2 \ln k}\binom{k}{i} \geq \frac{1}{12i^2 \ln k}\binom{k}{i}$$

be a prime number. Let V be a set of n vertices with an equipartition $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$, and define $\sigma : V \to \mathbb{Z}_\ell$ by $\sigma(v) = j$ when $v \in V_j$. We associate an $X \subseteq V$ with the multiset $\sigma(X) \subseteq \mathbb{Z}_\ell$ by $\sigma(X) = \{\sigma(v) : v \in X\}$ and define the weight $w(X) = \Sigma_{v \in X}\sigma(v) \pmod{\ell}$. Let φ be the weight with the fewest k-subsets having that weight. Finally, the edges of our hypergraph G_i are

$$\left\{X \in \binom{V}{k} : w(X) \neq \varphi \text{ and } \forall v \in X, \sigma(X - v) \text{ is an } i\text{-fold sum cover of } \mathbb{Z}_\ell\right\}.$$

The lower bounds in Theorems 2, 3, and 4 follow from the following two claims. Claim 6 implies that G_i does not contain $B_{i+1}^{(k)}$, $B_{k-i}^{(k)}$, $\widehat{B}_{k-i}^{(k)}$, and in particular, G_1 does not contain $C^{(k)}$.

CLAIM 6. *Let F be a k-uniform hypergraph (V, E) such that V contains two disjoint subsets S, T with $|T| = k - 1$ and $|S| = i + 1$ or $|S| = k - i$ and E contains an edge containing T and all k-sets $S \cup T'$ with $T' \in \binom{T}{k-|S|}$. Then G_i is F-free.*

Proof. We assume that G contains a copy of F. Then $V(G)$ contains two disjoint sets S and T such that $|T| = k - 1$ and T is contained in an edge of G. By the definition of G_i, the set $\sigma(T)$ is an i-fold sum cover of $\mathbb{Z}_\ell$, namely, $\{w(T') : T' \in \binom{T}{i}\}$ covers all values of $\mathbb{Z}_\ell$.

By the definition of F we have either $|S| = i+1$ or $|S| = k-i$. First assume that $|S| = i+1$. If we take all k-sets of the form $S \cup T'$, where $T' \subseteq T$ with $|T'| = k-i-1$, then the set of weights of these k-sets will range over all elements of $\mathbb{Z}_\ell$ because $w(S \cup T') = w(S) + w(T) - w(T \setminus T')$ and $w(T \setminus T')$ can take any value of $\mathbb{Z}_\ell$ ($w(S)$ and $w(T)$ are fixed). However, this is impossible since we explicitly prohibited one of the weights in the definition of G_i.

Now assume that $|S| = k-i$. If we take all of k-sets of the form $S \cup T'$, where $T' \subseteq T$ with $|T'| = i$, then the set of weights of these k-sets will again range over all the elements of $\mathbb{Z}$ because $w(S \cup T') = w(S) + w(T')$, where $w(T')$ can take any value of $\mathbb{Z}_\ell$ ($w(S)$ is fixed). However, this is impossible by the construction of G_i. $\square$

CLAIM 7.

$$e(G_i) \geq \left(1 - (1+o(1))\frac{1}{\ell}\right)\binom{n}{k} \geq \left[1 - \frac{12i^2 \ln k}{\binom{k}{i}} - o\left(\frac{\ln k}{k^i}\right)\right]\binom{n}{k}.$$

Proof. By removing the edges with weight φ, we retain at least $1 - \frac{1}{\ell}$ of the edges of $\binom{V}{k}$. On the other hand, by Lemma 8 below, with probability $1 - o(1/\ell)$ that an edge chosen uniformly at random from $\binom{V}{k}$ satisfies the property that all of its $(k-1)$-element subsets are i-fold sum covers of $\mathbb{Z}_\ell$. It follows that the graph will have edge density at least $1 - (1+o(1))(1/\ell)$. $\square$

LEMMA 8. *Let $k \geq i \geq 1$ be integers and ℓ be a prime number satisfying* (2). *Let $y_1, y_2, \ldots, y_k$ be chosen uniformly and independently at random from $\mathbb{Z}_\ell$ (with replacement). Let Y be the multiset $\{y_1, \ldots, y_k\}$. With probability $1 - o(1/\ell)$ every $(k-1)$-element subset of Y is an i-fold sum cover of $\mathbb{Z}_\ell$.*

Proof. We first consider the case when $i = 1$. In this case $\ell \sim \frac{k}{3 \ln k}$ suffices. For $x \in \mathbb{Z}_\ell$ and $Y' \in \binom{Y}{k-1}$, let $\mathcal{M}_{x,Y'}$ be the event that $x \neq y$ for any $y \in Y'$. We have

$$\Pr(\mathcal{M}_{x,Y'}) = \left(1 - \frac{1}{\ell}\right)^{k-1} \leq e^{-(k-1)/\ell} = e^{-(3-o(1))\ln k} = o(\ell^{-2}k^{-1}).$$

Thus the probability that there exists a $(k-1)$-subset of Y which is not a 1-fold sum cover of $\mathbb{Z}_\ell$ is at most $\sum_{x,Y'} \Pr(\mathcal{M}_{x,Y'}) = o(\ell^{-2}k^{-1}) \cdot \ell k = o(1/\ell)$.

We thus assume that $i \geq 2$ for the rest of our proof. We begin by studying the distribution of $(i-1)$-fold sums of elements of Y. For an arbitrary set $A \subseteq [k]$ we use the notation $\sum A$ for the sum $\sum_{a \in A} y_a$. For each $x \in \mathbb{Z}_\ell$ let $\mathcal{F}_x$ be the collection of sets $A \in \binom{[k]}{i-1}$ such that $\sum A = x$. Let $\mathcal{E}$ be the event that there exists an $x \in \mathbb{Z}_\ell$ such that $|\mathcal{F}_x| \geq 2i+1$. We show that $\Pr(\mathcal{E}) = o(1/\ell)$. Note that this is a simple observation for $i = 2$.

Let $i \geq 3$. For $x \in \mathbb{Z}_\ell$, define $\mathcal{E}_x$ to be the event that $|\mathcal{F}_x| \geq 2i+1$. We further partition $\mathcal{E}_x$ into two cases.

Case 1. There exist $2i+1$ $(i-1)$-subsets $A_1, \ldots, A_{2i+1}$ of $[k]$ such that $\sum A_1 = \cdots = \sum A_{2i+1} = x$ and $A_1, \ldots, A_{2i+1}$ has a system of distinct representatives (SDR).

Case 2. There exist $2i+1$ $(i-1)$-subsets $A_1, \ldots, A_{2i+1}$ of $[k]$ such that $\sum A_1 = \cdots = \sum A_{2i+1} = x$ and $A_1, \ldots, A_{2i+1}$ do not have an SDR.

We bound the probability of each case by an application of the union bound. First, if $A_1, \ldots, A_t$ are distinct $(i-1)$-subsets of $[k]$ with an SDR, then

$$(3) \qquad \Pr\left(\sum A_1 = \sum A_2 = \cdots = \sum A_t = x\right) = \left(\frac{1}{\ell}\right)^t.$$

To see this, assume that $a_1, \ldots, a_t$ are representatives of $A_1, \ldots, A_t$. We rewrite

$$\Pr\left(\sum A_1 = \cdots = \sum A_t = x\right) = \prod_{j=1}^{t} \Pr(E_j | E_1 \ldots E_{j-1}),$$

where E_j is the event that $\sum A_j = x$. Each term in this product equals $1/\ell$ because each E_j sets the value of y_{a_j}. Using the fact $\binom{m}{n} \le \left(\frac{em}{n}\right)^n$ for all positive integers $m \ge n$ and (2), the probability of Case 1 is at most

$$\binom{\binom{k}{i-1}}{2i+1} \left(\frac{1}{\ell}\right)^{2i+1} \le \left(\frac{e\binom{k}{i-1}}{(2i+1)\ell}\right)^{2i+1} \le \left(\frac{6e\, i^2 \ln k}{k-i+1}\right)^{2i+1} = o\left(\frac{1}{\ell^2}\right),$$

where the last equality uses the assumption $i \le \ln k/(10 \ln \ln k)$.

Now consider Case 2. Assume that $A_1, \ldots, A_{2i+1}$ are $(i-1)$-subsets of $[k]$ such that $\sum A_1 = \cdots = \sum A_{2i+1} = x$ and $A_1, \ldots, A_{2i+1}$ do not have an SDR. Let $B_1, \ldots, B_b, B_{b+1}$ be a minimal subfamily of $A_1, \ldots, A_{2i+1}$ such that $|\cup_{j=1}^{b+1} B_j| \le b$. (Hall's theorem guarantees the existence of such a subfamily.) Then $|\cup_{j=1}^{b} B_j| \le |\cup_{j=1}^{b+1} B_j| \le b$, and on the other hand, $|\cup_{j=1}^{b} B_j| \ge b$ by the minimality of $B_1, \ldots, B_{b+1}$. Thus $|\cup_{j=1}^{b} B_j| = b$. Since every proper subfamily of $B_1, \ldots, B_{b+1}$ satisfies Hall's condition, $B_1, \ldots, B_b$ has an SDR. Clearly $b \le 2i$. Furthermore, we claim that $b \ge i+1$. In fact, since $A_1, \ldots, A_t$ are $(i-1)$-sets, we have $b \ge |\cup_{\ell=1}^{b+1} B_\ell| \ge i-1 \ge 1$. Consequently $b \ge |B_1 \cup B_2| \ge i$. Since an i-set has only i distinct $(i-1)$-subsets, we have $b \ge |B_1 \cup \cdots \cup B_{i+1}| \ge i+1$, as desired.

Therefore the probability of Case 2 is at most the probability that there exists b $(i-1)$-subsets $B_1, \ldots, B_b$ of $[k]$ for some $i+1 \le b \le 2i$ such that

$$|\cup_{j=1}^{b} B_j| = b, \ \sum B_1 = \cdots = \sum B_b = x, \ \text{and } B_1, \ldots, B_b \text{ have an SDR.}$$

By (3), this probability is at most

$$\sum_{b=i+1}^{2i} \binom{k}{b} \binom{\binom{b}{i-1}}{b} \left(\frac{1}{\ell}\right)^b \le \sum_{b=i+1}^{2i} \left(\frac{e^2 k \binom{b}{i-1}}{b^2 \ell}\right)^b \le i \left(\frac{e^2 k \binom{2i}{i-1}}{(i+1)^2 \ell}\right)^{i+1} = o\left(\frac{1}{\ell^2}\right),$$

where the last equality uses the assumption that $3 \le i \le \ln k/(10 \ln \ln k)$.

Adding the probabilities in the two cases together, we have $\Pr(\mathcal{E}_x) = o\left(\frac{1}{\ell^2}\right)$ and $\Pr(\mathcal{E}) \le \sum_{x \in \mathbb{Z}_\ell} \Pr(\mathcal{E}_x) = o\left(\frac{1}{\ell}\right)$.

Next, let us consider the i-fold sums of a fixed $(k-1)$-element subset of Y. We may assume that the elements of our $(k-1)$-element subset are $y_1, y_2, \ldots, y_{k-1}$. Let S_j be the set of elements of $\mathbb{Z}_\ell$ covered by $(i-1)$-fold sums of $\{y_1, \ldots, y_j\}$. Note that if the event $\mathcal{E}$ does not hold, then $|S_j| \ge \frac{1}{2i}\binom{j}{i-1}$. For $x \in \mathbb{Z}_\ell$ let $\mathcal{M}_x$ be the event that x does not appear as an i-fold sum of $y_1, \ldots, y_{k-1}$. We have

$$\Pr(\mathcal{M}_x \cap \overline{\mathcal{E}}) = \prod_{j=i}^{k-1} \Pr\left(\forall A \in \binom{[j-1]}{i-1}, y_j + \sum A \ne x\right) = \prod_{j=i}^{k-1} \left(1 - \frac{|S_{j-1}|}{\ell}\right)$$

$$\le \exp\left\{-\sum_{j=i}^{k-1} \frac{1}{2i}\binom{j-1}{i-1}\frac{1}{\ell}\right\} = \exp\left\{-\frac{1}{2i} \cdot \frac{1}{\ell} \cdot \binom{k-1}{i}\right\}$$

$$= \exp\left\{-(3-o(1))i \ln k\right\} \le o(\ell^{-2} k^{-1}).$$

Summing over all $x \in \mathbb{Z}_\ell$ and all $(k-1)$-element subsets of Y gives the desired result. $\square$

3. Upper bounds.

Proof of the upper bound in Theorem 2. We shall apply a result of Sidorenko [11] that for a k-uniform hypergraph F with f edges,

$$\pi(F) \leq 1 - \frac{1}{f-1}.$$

Therefore, for $2 \leq i \leq k$,

$$\pi\big(B_i^{(k)}\big) \leq 1 - \frac{1}{i + \binom{k-1}{i-1} - 1}.$$

For fixed i and $k \to \infty$, it follows that

$$\pi\big(B_{i+1}^{(k)}\big) \leq 1 - \frac{1}{i + \binom{k-1}{i}} \leq 1 - \frac{1}{k - i + \binom{k-1}{i} - 1},$$

$$\pi\big(B_{k-i}^{(k)}\big) \leq 1 - \frac{1}{k - i + \binom{k-1}{i} - 1}. \qquad \square$$

In order to prove the upper bound of Theorem 3, we shall count the number of complete subhypergraphs of a dense hypergraph. Fix a k-uniform hypergraph G of order n. Let m_i denote the number of i-cliques K_i^k for $i \geq k$ in G, for example, m_k is the number of hyperedges of G. In addition, define $m_{k-1} = \binom{n}{k-1}$. Adapting the techniques previously used by Moon and Moser, de Caen [2] provided the following recursive lower bound on m_i.

THEOREM 9. *If $m_{i-1} > 0$, then*

$$m_{i+1} \geq \frac{i^2 m_i}{(i - k + 1)(i + 1)} \left(\frac{m_i}{m_{i-1}} - \frac{(k-1)(n-i) + i}{i^2} \right).$$

This implies the following lower bound on m_ℓ for $\ell > k$.

COROLLARY 10. *Let $0 \leq \alpha \leq 1$, and let G be a k-uniform hypergraph with at least $(1 - \alpha)\binom{n}{k}\frac{n}{n-k+1}$ edges. Then for any $\ell \geq k$,*

$$m_\ell \geq \binom{n}{\ell} \prod_{i=k}^{\ell} \left(1 - \binom{i-1}{k-1}\alpha \right).$$

Proof. The $k = 2$ case can be found in Lovász [9, Problem 10.40]. The proof of the general case is almost the same. Let $a_i = m_i/m_{i-1}$. We claim that for $i \geq k$,

$$(4) \qquad a_i \geq \frac{n}{i} \left(1 - \binom{i-1}{k-1}\alpha \right).$$

Let us prove (4) by induction. By the definition of m_k and m_{k-1}, we have

$$a_k \geq (1 - \alpha)\binom{n}{k}\frac{n}{n-k+1}\frac{1}{\binom{n}{k-1}} = (1 - \alpha)\frac{n}{k}.$$

For $i > k$, by Theorem 9 and the induction hypothesis,

$$
\begin{aligned}
a_{i+1} &\geq \frac{i^2}{(i-k+1)(i+1)}\left(a_i - \frac{(k-1)(n-i)+i}{i^2}\right) \\
&\geq \frac{i^2}{(i-k+1)(i+1)}\left(\frac{n}{i}\left(1 - \binom{i-1}{k-1}\alpha\right) - \frac{n}{i^2}(k-1)\right) \\
&= \frac{n}{(i-k+1)(i+1)}\left(i\left(1 - \binom{i-1}{k-1}\alpha\right) - k+1\right) \\
&= \frac{n}{i+1}\left(1 - \binom{i}{k-1}\alpha\right).
\end{aligned}
$$

Let $\ell \geq k$. By applying (4) to $i = k, \ldots, \ell$, we obtain

$$
\begin{aligned}
m_\ell = m_{k-1}\prod_{i=k}^{\ell} a_i &\geq \binom{n}{k-1}\prod_{i=k}^{\ell}\frac{n}{i}\left(1 - \binom{i-1}{k-1}\alpha\right) \\
&\geq \binom{n}{\ell}\prod_{i=k}^{\ell}\left(1 - \binom{i-1}{k-1}\alpha\right). \qquad \square
\end{aligned}
$$

Proof of the upper bound in Theorem 3. Suppose that G is a $\widehat{B}_{k-i}^{(k)}$-free k-uniform hypergraph on n vertices with at least $(1-\alpha)\frac{n}{n-k+1}\binom{n}{k}$ edges, where

$$
(5) \qquad \alpha = \frac{\ln\left[2\binom{k-1}{i}\right] - \ln\ln\left[2\binom{k-1}{i}\right]}{2\binom{k-1}{i}} \approx \frac{i!}{2k^i}\left(i\ln k - \ln\ln k\right).
$$

By standard averaging arguments, there exists a subset A of size $k-i$ so that the neighborhood of A (i.e., the collection of i-sets whose union with A is an edge of G) is an i-uniform hypergraph on $n-(k-i)$ vertices with at least $(1-\alpha)\frac{n}{n-k+1}\binom{n-(k-i)}{i} > (1-\alpha)\frac{n-(k-i)}{n-k+1}\binom{n-(k-i)}{i}$ edges. From Corollary 10 it follows that the neighborhood of A has at least $f(\alpha)\binom{n-(k-i)}{k-1}$ $(k-1)$-cliques, where

$$
f(\alpha) := \prod_{j=i}^{k-1}\left(1 - \binom{j-1}{i-1}\alpha\right).
$$

No such clique can be extended to an edge of G by adding a vertex outside A – otherwise we obtain a copy of $\widehat{B}_{k-i}^{(k)}$. The number of nonedges in G is at most

$$
(6) \qquad \left(\alpha - \frac{k-1}{n-k+1}(1-\alpha)\right)\binom{n}{k} \geq f(\alpha)\binom{n-(k-i)}{k-1}\frac{n-j-(k-1)}{k},
$$

where the term $n-j-(k-1)$ counts the number of remaining vertexes outside A and a $(k-1)$-clique, and thus each nonedge may be counted at most k times. As $n \to \infty$, (6) becomes

$$
(\alpha + o(1))\binom{n}{k} \geq (f(\alpha) + o(1))\binom{n}{k},
$$

which implies that $\alpha \geq f(\alpha)$. It is easy to see that $1 - x \geq e^{-2x}$ for $0 \leq x \leq \frac{\ln 2}{2}$. By (5), we have $\binom{k-2}{i-1}\alpha \to 0$ as $k \to \infty$, and so

$$f(\alpha) \geq \prod_{j=i}^{k-1} \exp\left(-2\binom{j-1}{i-1}\alpha\right) = \exp\left(-2\binom{k-1}{i}\alpha\right).$$

Consequently,

$$\alpha \geq f(\alpha) \geq \exp\left(-2\binom{k-1}{i}\alpha\right) = \frac{\ln\left[2\binom{k-1}{i}\right]}{2\binom{k-1}{i}},$$

contradicting (5). □

Proof of the upper bound in Theorem 4. A *forest* is a hypergraph whose edges can be ordered as $E_1, E_2, \ldots, E_m$ such that for each $1 \leq i \leq m$, we have $E_i \cap (E_1 \cup \cdots \cup E_{i-1}) \subseteq E_j$ for some $j < i$. Based on the work of Sidorenko [11], Keevash [5] showed that if a k-uniform hypergraph H has f edges and the maximal number of edges of H forming a forest is t, then its Turán density $\pi = \pi(H)$ satisfies

$$\pi^{t-1} - (f - t)(1 - \pi) \leq 0.$$

Notice that $C^{(k)}$ has k edges, $k - 1$ of which form a forest, so $t = k - 1$ and $f - t = 1$. Hence the Turán density of $C^{(k)}$ satisfies

$$\pi^{k-2} - (1 - \pi) \leq 0.$$

Letting $\pi = 1 - \alpha$ this becomes

$$(1 - \alpha)^{k-2} \leq \alpha.$$

As seen in the proof of the upper bound in Theorem 3, this implies $\alpha \geq e^{-2(k-2)\alpha}$, which further implies that

$$\alpha \geq \frac{\ln[2(k-2)] - \ln\ln[2(k-2)]}{2(k-2)}.$$

Hence $\pi = 1 - \alpha \leq 1 - \frac{\ln k}{2k} + O(\frac{\ln\ln k}{2k})$, as desired. □

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