ON SUBGRAPHS OF TRIPARTITE GRAPHS

ABHIJEET BHALKIKAR AND YI ZHAO

Abstract. Bollobás, Erdős, and Szemerédi [Discrete Math 13 (1975), 97–107] investigated a tripartite generalization of the Zarankiewicz problem: what minimum degree forces a tripartite graph with n vertices in each part to contain an octahedral graph $K_3(2)$? They proved that $n + 2^{-1/2}n^{3/4}$ suffices and suggested it could be weakened to $n + cn^{1/2}$ for some constant $c > 0$. In this note we show that their method only gives $n + (1 + o(1))n^{11/12}$ and provide many constructions that show that if true, $n + cn^{1/2}$ is best possible.

1. Introduction

Let K_t denote the complete graph on t vertices. As a foundation stone of extremal graph theory, Turán's theorem in 1941 [10] determines the maximum number of edges in graphs of a given order not containing K_t as a subgraph (the $t = 3$ case was proven by Mantel in 1907 [6]). In 1975 Bollobás, Erdős, and Szemerédi [1] investigated the following Turán-type problem for multipartite graphs.

Problem 1. *Given integers n and $3 \le t \le r$, what is the largest minimum degree $\delta(G)$ among all r-partite graphs G with parts of size n and which do not contain a copy of K_t?*

The $r = t$ case of Problem 1 had been a central topic in Combinatorics until it was finally settled by Haxell and Szabó [4], and Szabó and Tardos [8]. Recently Lo, Treglown, and Zhao [9] solved many $r > t$ cases of the problem, including when $r \equiv -1 \pmod{t-1}$ and $r = \Omega(t^2)$.

For simplicity, let $G_r(n)$ denote an (arbitrary) r-partite graph with parts of size n. Let $K_r(s)$ denote the complete r-partite graph with parts of size s. In particular, $K_2(2) = K_{2,2}$ is a 4-cycle C_4 and $K_3(2) = K_{2,2,2}$ is known as the *octahedral graph*. In the same paper Bollobás, Erdős, and Szemerédi [1] also asked the following question.

Problem 2. *Given a tripartite graph $G = G_3(n)$, what $\delta(G)$ guarantees a copy of $K_3(2)$?*

Problem 2 is a natural generalization of the well-known Zarankiewicz problem [11], whose symmetric version asks for the largest number of edges in a bipartite graph $G_2(n)$ that contains no $K_{s,s}$ as a subgraph (in other words, *$K_{s,s}$-free*).

In [1, Corollary 2.7] the authors stated that $\delta(G) \ge n + 2^{-1/2}n^{3/4}$ guarantees a copy of $K_3(2)$. This follows from [1, Theorem 2.6], which handles the general case of $K_3(s)$ for arbitrary s. Unfortunately, there is a miscalculation in the proof of [1, Theorem 2.6] and thus the bound $\delta(G) \ge n + 2^{-1/2}n^{3/4}$ is unjustified. We follow the approach of [1, Theorem 2.6] and obtain the following result.

Theorem 3. *Given an integer $s \ge 2$ and $\varepsilon > 0$, let n be sufficiently large. If $G = G_3(n)$ satisfies $\delta(G) \ge n + (1+\varepsilon)(s-1)^{1/(3s^2)}n^{1-1/(3s^2)}$, then G contains a copy of $K_3(s)$.*

In particular, Theorem 3 implies that every $G = G_3(n)$ with $\delta(G) \ge n + (1 + o(1))n^{11/12}$ contains a copy of $K_3(2)$. Using a result of Erdős on hypergraphs [3], we give a different proof of Theorem 3 under a slightly stronger condition $\delta(G) \ge n + (3n)^{1-1/(3s^2)}$. Thus $c\,n^{11/12}$ is a natural additive term for Problem 2 under typical approaches for extremal problems.

On the other hand, the authors of [1] conjectured that $\delta(G) \ge n + cn^{1/2}$ suffices for Problem 2. Although not explained in [1], they probably thought of Construction 10, a natural construction based on the one for the Zarankiewicz problem. We indeed find many non-isomorphic constructions, Construction 11, with the same minimum degree.

Proposition 4. *For any $n = q^2 + q + 1$ where q is a prime power, there are many tripartite graphs $G = G_3(n)$ such that $\delta(G) \ge n + n^{1/2}$ and G contains no $K_3(2)$.*

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Theorem 3 and Proposition 4 together show that the answer for Problem 2 lies between $n + n^{1/2}$ and $n + n^{11/12}$. The truth may be closer to the lower bound. If this is the case, then verifying it may be hard given the presence of many non-isomorphic constructions.

We know less about the minimum degree of $G_3(n)$ that forces a copy of $K_3(s)$. Theorem 3 shows that $\delta(G_3(n)) \geq n + cn^{1-1/(3s^2)}$ suffices. As shown in Remark 12, if there is a $K_{s,s}$-free bipartite graph $B = G_2(n)$ with $\delta(B) = \Omega(n^{1-1/s})$, then our constructions for Proposition 4 provide a tripartite $K_3(s)$-free graph $G = G_3(n)$ with $\delta(G) = n + \Omega(n^{1-1/s})$.

2. Proof of Theorem 3

In order to prove Theorem 3, we need the following results from [1].

Lemma 5. [1, Theorem 2.3] *Suppose every vertex of $G = G_3(n)$ has degree at least $n+t$ for some integer $t \leq n$. Then there are at least t^3 triangles in G.*

Lemma 6. [1, Lemma 2.4] *Let $X = \{1, \dots, N\}$ and $Y = \{1, \dots, p\}$. Suppose $A_1, \dots, A_p$ are subsets of X such that $\sum_{i=1}^{p} |A_i| \geq pwN$ and $(1-\alpha)wp \geq q$, where $0 < \alpha < 1$ and N, p and q are natural numbers. Then there are q subsets $A_{i_1}, \dots, A_{i_q}$ such that $\left|\bigcap_{j=1}^{q} A_{i_j}\right| \geq N(\alpha w)^q$.*

Let $z(n, s)$ denote the largest number of edges in a bipartite $K_{s,s}$-free graph with n vertices in each part. Kővári, Sós, and Turán [5] gave the following upper bound for $z(n, s)$.[1]

Lemma 7. [5] $z(n, s) \leq (s-1)^{1/s} n^{2-1/s} + sn$.

We are ready to prove Theorem 3.

Proof of Theorem 3. Let G be a tripartite graph with three parts V_1, V_2, V_3 of size n each. Suppose $\delta(G) \geq n+t$, where $t = (1+\varepsilon)(s-1)^{\frac{1}{3s^2}} n^{1-\frac{1}{3s^2}} > n^{1-\frac{1}{3s^2}}$. By Lemma 5, G contains at least t^3 triangles.

We apply Lemma 6 in the following setting. Let $Y = V_1 = \{1, \dots, n\}$ and $X = V_2 \times V_3$ be the set of n^2 pairs (x, y), $x \in V_2, y \in V_3$. For $1 \leq i \leq n$, let A_i be the set of pairs $(x, y) \in X$ for which $\{i, x, y\}$ spans a triangle of G. Then $\sum_{i=1}^{n} |A_i|$ is the number of triangles in G so $\sum_{i=1}^{n} |A_i| \geq t^3$. Let $N = n^2$, $p = n$, $q = s$, $w = t^3/n^3$, and $\alpha = 1/(1+\varepsilon)$. The assumptions of Lemma 6 hold because $pwN = t^3$ and

$$(1-\alpha)wp = \frac{\varepsilon}{1+\varepsilon}\left(\frac{t}{n}\right)^3 n > \frac{\varepsilon}{1+\varepsilon} n^{-1/s^2} n > s$$

as n is sufficiently large. By Lemma 6, there are $i_1, \dots, i_s \in V_1$ such that

$$\left|\bigcap_{j=1}^{s} A_{i_j}\right| \geq N(\alpha w)^q = n^2 \left(\frac{t^3}{(1+\varepsilon).n^3}\right)^s$$

Since

$$t > (1+\varepsilon)^{\frac{2}{3}} (s-1)^{\frac{1}{3s^2}} n^{1-\frac{1}{3s^2}} \quad \text{and} \quad \frac{t^3}{(1+\varepsilon)n^3} > (1+\varepsilon)(s-1)^{1/s^2} n^{-1/s^2},$$

we have

$$\left|\bigcap_{j=1}^{s} A_{i_j}\right| > (1+\varepsilon)^s (s-1)^{1/s} n^{2-1/s} \geq (s-1)^{1/s} n^{2-1/s} + sn. \tag{1}$$

Let B denote the bipartite graph between V_2 and V_3 with $E(B) = \bigcap_{j=1}^{s} A_{i_j}$. By (1) and Lemma 7, B contains a copy of $K_{s,s}$. Since every edge of B forms a triangle with each of $i_1, \dots, i_s \in V_1$, this copy of $K_{s,s}$ together with $i_1, \dots, i_s$ span a desired copy of $K_3(s)$ in G. $\square$

We now give another proof of Theorem 3 with slightly larger $\delta(G)$ by a classical result of Erdős on hypergraphs [3]. An r-uniform hypergraph or r-graph is a hypergraph such that all its edges contain exactly r vertices. Let $K_r^r(s)$ denote the complete r-partite r-graph with s vertices in each part, namely, its vertex set consists of disjoint parts $V_1, \dots, V_r$ of size s, and edges set consists of all r-sets $\{v_1, \dots, v_r\}$ with $v_i \in V_i$ for all i.

Lemma 8. [3, Theorem 1] *Given integers $r, s \geq 2$, let n be sufficiently large. Then every r-graph on n vertices with at least $n^{r-s^{1-r}}$ edges contains a copy of $K_r^r(s)$.*

[1] In [1] the authors instead used the Turán number $\mathrm{ex}(2n, K_{s,s})$, which gives a slightly worse constant here.

Proposition 9. *Let $s \geq 2$ and n be sufficiently large. Every tripartite graph $G = G_3(n)$ with $\delta(G) \geq n + (3n)^{1-1/(3s^2)}$ contains a copy of $K_3(s)$.*

Proof. Suppose $G = G_3(n)$ satisfies $\delta(G) \geq n+(3n)^{1-1/(3s^2)}$. By Lemma 5, G contains at least $(3n)^{3-1/s^2}$ triangles. Let H be the 3-graph on $V(G)$, whose edges are triangles of G. Then H has $3n$ vertices and at least $(3n)^{3-s^{-2}}$ edges. By Lemma 8 with $r = 3$, H contains a copy of $K_3^3(s)$, which gives a copy of $K_3(s)$ in G. □

3. Proof of Proposition 4

In this section we prove Proposition 4 by constructing many tripartite $K_3(2)$-free graphs $G_3(n)$ with $\delta(G_3(n)) \geq n + n^{1/2}$.

One main building block is a bipartite $K_{2,2}$-free graph $G_0 = G_2(n)$ with $\delta(G_0) \geq \sqrt{n}$. First shown in [7], such a graph exists when $n = q^2 + q + 1$ and a projective plane of order q exists. Indeed, two parts of $V(G)$ correspond to the points and lines of the projective plane and a point is adjacent to a line if and only if the point lies on the line. It is easy to see that such graph contains no $K_{2,2}$ and is regular with degree $q + 1 > \sqrt{n}$.

Construction 10. *Suppose $G = G_3(n)$ has parts V_1, V_2 and V_3 each of size n. Let the bipartite graphs between V_1 and V_2 and between V_1 and V_3 be complete, while the bipartite graph between V_2 and V_3 is G_0 defined above.*

Since $\deg_{G_0}(v) \geq \sqrt{n}$ for $v \in V_2 \cup V_3$, we have $\delta(G) \geq n + \sqrt{n}$. Furthermore, G contains no $K_3(2)$ because by the definition of G_0, there is no $K_2(2)$ between V_2 and V_3.

We now provide a family of constructions with the same properties.

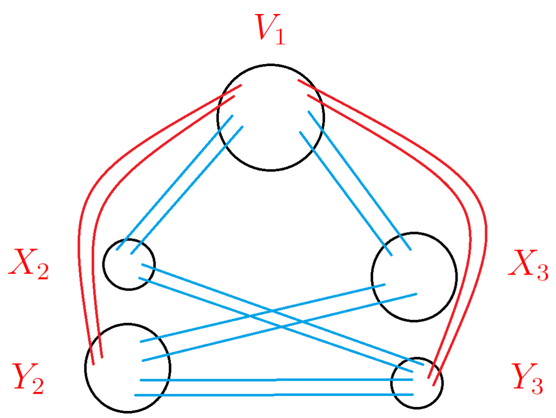

Figure 1. Graph from Construction 11

Construction 11. *Let $G = G_3(n)$ be a tripartite graph with parts V_1, V_2, and V_3 of size n each. Partition $V_2 = X_2 \cup Y_2$ arbitrarily such that $\alpha n \leq |X_2| \leq |Y_2|$ for some $\alpha \in (0, 1/2)$. Partition $V_3 = X_3 \cup Y_3$ arbitrarily such that $|X_3| = |Y_2|$ and $|Y_3| = |X_2|$.*

The bipartite graphs (V_1, X_2), (X_2, Y_3), (Y_3, Y_2), (Y_2, X_3), and (X_3, V_1) are complete, in other words, V_1, X_2, Y_3, Y_2, X_3 form a blowup of C_5. Let the bipartite graph between V_1 and $Y_2 \cup Y_3$ be isomorphic to G_0 (note that $|X_2| + |Y_2| = |X_3| + |Y_3| = n$).

For any vertex $v \in X_2$, $\deg(v) = |V_1|+|Y_3| \geq n+\alpha n$. The vertices $v \in X_3$ satisfy $\deg(v) = |V_1|+|Y_2| \geq n + n/2$. For any $v \in Y_2$, $\deg(v) \geq |V_3| + \delta(G_0) \geq n + \sqrt{n}$. The same holds for the vertices of Y_3. At last, every vertex $v \in V_1$ satisfies $\deg(v) \geq |X_2| + |X_3| + \delta(G_0) \geq n + \sqrt{n}$. These together show that $\delta(G) \geq n + \sqrt{n}$.

Suppose G contains a copy of $K_3(2)$ with vertex set S. Then $|S \cap V_i| = 2$ for $i = 1, 2, 3$. Since there is no edge between X_2 and X_3, either $S \cap X_2 = \emptyset$ or $S \cap X_3 = \emptyset$. Suppose, say, $S \cap X_2 = \emptyset$, which forces $|S \cap Y_2| = 2$. Hence $S \cap Y_2$ and $S \cap V_1$ span a copy of $K_{2,2}$, contradicting the definition of G_0.

If letting $X_2 = \emptyset = Y_3$ in Construction 11, then we obtain Construction 10. Nevertheless, we prefer viewing Constructions 10 and 11 as different constructions because after removing $o(n^2)$ edges, Construction 11 contains many 5-cycles while Construction 10 does not.

Remark 12. *If we replace* G_0 *by a* $K_{s,s}$*-free bipartite graph with* n *vertices in each part in Constructions 10 and 11, then we obtain a* $K_3(s)$*-free tripartite graph* $G_3(n)$*. It has been conjectured that there exist a* $K_{s,s}$*-free bipartite graph with* n *vertices in each part and* $\Omega(n^{2-1/s})$ *edges (this is known for* $s = 2, 3$ [2, 7]*). If there exists such a bipartite graph which is regular, then (revised) Constructions 10 and 11 provide a* $K_3(s)$*-free tripartite graph* $G = G_3(n)$ *with* $\delta(G) = n + \Omega(n^{1-1/s})$.

References

[1] B. Bollobás, P. Erdős, A. Szemerédi. On complete subgraphs of r-chromatic graphs. *Discrete Math*, 13:97-107, 1975.
[2] W. G. Brown. On graphs that do not contain a Thomsen graph. *Canada Math. Bull.* 9, 281-285, 1966.
[3] P. Erdős. On extremal problems of graphs and generalized graphs. *Israel Journal of Mathematics*, 2, 3:183-190, 1964.
[4] P. Haxell and T. Szabó. Odd independent transversals are odd. *Combin. Probab. Comput.*, 15(1-2):193-211, 2006.
[5] T. Kővári, V. Sós, P. Turán. On a problem of K. Zarankiewicz, *Colloquium Math.*, 3:50-57, 1954.
[6] W. Mantel. Problem 28 (Solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W. A. Wythoff), *Wiskundige Opgaven*, 10:60-61, 1907.
[7] I. Rieman. Über ein Problem von K. Zarankiewicz. *Acta Math. Acad. Sci. Hungar.* 9, 3-4:269-273, 1958.
[8] T. Szabó and G. Tardos. Extremal problems for transversals in graphs with bounded degree. *Combinatorica*, 26(3):333-351, 2006.
[9] A. Lo, A. Treglown and Y. Zhao. Complete subgraphs in a multipartite graph *Combin. Probab. Comput.*, to appear.
[10] P. Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, 48:436-452, 1941.
[11] K. Zarankiewicz. Problem P 101. *Colloq. Math.*, 2:301, 1951.

Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303
E-mail address: abhalkikar1@student.gsu.edu

Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303
E-mail address: yzhao6@gsu.edu