Improved bounds on the Ramsey number of fans

Guantao Chen[a]*, Xiaowei Yu[b]†, Yi Zhao[a]‡

[a] Department of Mathematics and Statistics, Georgia State University,
Atlanta, GA 30303, USA

[b] Department of Mathematics and Statistics, Jiangsu Normal University,
Xuzhou, 221116, P. R. China

Abstract

For a given graph H, the Ramsey number $r(H)$ is the minimum N such that any 2-edge-coloring of the complete graph K_N yields a monochromatic copy of H. Given a positive integer n, a *fan* F_n is a graph formed by n triangles that share one common vertex. We show that $9n/2 - 5 \le r(F_n) \le 11n/2 + 6$ for any n. This improves previous best bounds $r(F_n) \le 6n$ of Lin and Li and $r(F_n) \ge 4n + 2$ of Zhang, Broersma and Chen.

Keywords: Ramsey numbers; fans; books.

1 Introduction

Let H_1 and H_2 be two graphs. The *Ramsey number* $r(H_1, H_2)$ is the minimum N such that any red-blue coloring of the edges of the complete graph K_N yields a red copy of H_1 or a blue copy of H_2. Let $r(H) = r(H, H)$ be the diagonal Ramsey number. Graph Ramsey theory is a central topic in graph theory and combinatorics. For related results, see surveys [3, 10].

In 1975, Burr, Erdős and Spencer [1] investigated Ramsey numbers for disjoint union of small graphs. Given a graph G and a positive integer n, let nG denote n vertex-disjoint copies of G. It was shown in [1] that $r(nK_3) = 5n$ for $n \ge 2$. A *book* B_n is the union of n distinct triangles having exactly one edge in common. In 1978, Rousseau and Sheehan [11] showed that the Ramsey number $r(B_n) \le 4n + 2$ for all n and the bound is tight for infinitely many values of n (*e.g.*, when $4n+1$ is a prime power). A more general *book* $B_n^{(k)}$ is the union of n distinct copies of complete graphs K_{k+1}, all sharing a common K_k (thus $B_n = B_n^{(2)}$). Conlon [2] recently proved that for every k, $r(B_n^{(k)}) = 2^k n + o_k(n)$, answering

*Email: gchen@gsu.edu, partially supported by NSF grant DMS-1855716.

†Email: xwyu@jsnu.edu.cn, partially supported by grants 11901252, 19KJB110010 and 12031018.

‡Email: yzhao6@gsu.edu, partially supported by NSF grant DMS-1700622 and Simons Collaboration Grant 710094.

a question of Erdős, Faudree, Rousseau, and Schelp [6] and asymptotically confirming a conjecture of Thomason [12]. More recently, Conlon, Fox, and Wigderson [4] provided another proof of Conlon's result.

Inspired by these old and recent results on $r(nK_3)$ and $r(B_n^{(k)})$, in this paper we study the Ramsey number of fans. A *fan* F_n is a union of n triangles sharing exactly one common vertex, named the *center*, and all other vertices are distinct. Therefore, nK_3, F_n and B_n are three graphs formed by n triangles that share zero, one, and two common vertices, respectively. Since nK_3 has more vertices than F_n and F_n has more vertices and edges than B_n, it is reasonable to believe that $r(B_n) \le r(F_n) \le r(nK_3)$ for sufficiently large n. We obtain the following bounds for $r(F_n)$ confirming $r(B_n) < r(F_n)$ for sufficiently large n.[1]

Theorem 1.1. *For every positive integer n,*

$$9n/2 - 5 \le r(F_n) \le 11n/2 + 6.$$

Theorem 1.1 improves previously best known bounds

$$4n + 2 \le r(F_n) \le 6n. \tag{1}$$

Indeed, Li and Rousseau [7] first studied off-diagonal Ramsey numbers of fans. They showed that $r(F_1, F_n) = 4n + 1$ for $n \ge 2$ and $4n + 1 \le r(F_m, F_n) \le 4n + 4m - 2$ for $n \ge m \ge 1$. Lin and Li [8] proved that $r(F_2, F_n) = 4n + 1$ for $n \ge 2$ and improved the general upper bound as

$$r(F_m, F_n) \le 4n + 2m \quad \text{for} \quad n \ge m \ge 2. \tag{2}$$

Lin, Li and Dong [9] showed that $r(F_m, F_n) = 4n + 1$ if n is sufficiently larger than m. The latest result for $r(F_m, F_n)$ is due to Zhang, Broersma and Chen [13], who proved that $r(F_m, F_n) = 4n + 1$ if $n \ge \max\{(m^2 - m)/2, 11m/2 - 4\}$. They also showed that $r(F_n, F_m) \ge 4n + 2$ for $m \le n < (m^2 - m)/2$. This and (2) together give (1).

The lower bound given in Theorem 1.1 is obtained from constructing a regular 3-partite graph with about $3n/2$ vertices in each part such that every vertex has less than n neighbors in one of the other parts. To prove the upper bound given in Theorem 1.1, we first find a large monochromatic clique in any 2-edge-colored $K_{11n/2+6}$ and then use this clique to find the desired copy of F_n. This approach is summarized in the following two lemmas.

Lemma 1.2. *Let m, n, N be positive integers such that $N = 4n + m + \left\lfloor \frac{6n}{m} \right\rfloor + 1$. Then every 2-coloring of $E(K_N)$ yields a monochromatic copy of F_n or K_m.*

Lemma 1.3. *Let n be a positive integer. If a graph G contains a clique V_0 with $|V_0| \ge 3n/2 + 1$ such that every vertex $v \in V_0$ has at least n neighbors in $V \backslash V_0$, then G or its complement $\overline{G}$ contains a copy of F_n with center in V_0.*

We prove Lemmas 1.2 and 1.3 by using the theorems of Hall and Tutte on matchings along with a result on $r(nK_2, F_m)$ from [8]. Unfortunately our approach (of finding a large monochromatic clique) cannot prove $r(F_n) < 11n/2$ because Lemma 1.3 is tight with respect to the size of V_0, see Section 5 for details.

[1]These inequalities fail when $n = 2$ because $r(B_2) = r(2K_3) = 10$ [1, 11] while $r(F_2) = 9$ [8].

We organize our paper as follows. We give notation and preliminary results in Section 2. After proving Lemmas 1.2 and 1.3 in Section 3, we complete the proof of Theorem 1.1 in Section 4. We give concluding remarks, including a lower bound for $r(F_n, F_m)$, in the last section.

2 Notation and preliminaries

We start this section with some notation and terminologies. Given a positive integer n, let $[n] := \{1, 2, \ldots, n\}$. All graphs considered are simple and finite. Given a graph G, we denote by $V(G)$ and $E(G)$ the vertex and edge sets of G, respectively. $|G| := |V(G)|$ and $|E(G)|$ are the *order* and the *size* of G, respectively. Let $\overline{G}$ denote the complement graph of G.

Given a graph G, let v be a vertex and H be a subgraph. Denote by $N_H(v)$ the set of neighbors of v in H. For a subset $S \subseteq V(G)$, define $N_H(S) = \cup_{v\in S} N_H(v)$. The *degree* of v in H is denoted by $d_H(v)$, that is, $d_H(v) = |N_H(v)|$. When all the vertices of G have the same degree d, we call G a *d-regular graph*. The subgraph induced by the vertices of S is denoted by $G[S]$. We simply write $G[V(G)\backslash S]$ as $G - S$. A component of G is *odd* if it consists of an odd number of vertices. We denote by $o(G)$ the number of odd components of G.

Given a graph G, we denote by $\nu(G)$ the size of a largest matching of G. We will use the following defect versions of Hall's and Tutte's theorems (see, *e.g.*, [5]).

Theorem 2.1 (Hall)**.** *Let G be a bipartite graph on parts X and Y. For any non-negative integer d, $\nu(G) \geq |X| - d$ if and only if $|N_G(S)| \geq |S| - d$ for every $S \subseteq X$.*

Theorem 2.2 (Tutte)**.** *Let G be a graph on order n. For any non-negative integer d, $\nu(G) \geq (n - d)/2$ if and only if $o(G - S) \leq |S| + d$ for every subset S of $V(G)$.*

The aforementioned result $r(F_n, F_m) \leq 4n + 2m$ for $n \geq m$ follows from the following lemma, in which nK_2 is a matching of size n. Note that the $n = m$ case of this lemma was proved in the same way as our Lemma 1.2.

Lemma 2.3 (Lin and Li [8])**.** *Let m, n be two positive integers with $n \geq m$. Then $r(nK_2, F_m) = 2n + m$.*

We will use the following corollary.

Corollary 2.4. *Let G be a graph with maximum degree $\Delta(G)$. If $\Delta(G) \geq 3n$, then G or $\overline{G}$ contains a copy of F_n.*

Proof. Assume v is a vertex such that $d_G(v) \geq 3n$. By Lemma 2.3, there is a copy of nK_2 in $G[N_G(v)]$ or a copy of F_n in $\overline{G}[N_G(v)]$. So, G has a copy of F_n centered at v or $\overline{G}$ contains a copy of F_n. □

3 Proofs of Lemmas 1.2 and 1.3

Proof of Lemma 1.2. Let $c := \left\lfloor \frac{6n}{m} \right\rfloor + 1$ for convenience, and so $N = 4n + m + c$. Fix a red-blue edge coloring of K_N and let R, B be the graphs induced by red and blue edges, respectively. Assuming there is no monochromatic K_m, we will find a monochromatic F_n.

Fix a vertex w. Assume, without loss of generality, that $d_B(w) \geq \frac{N-1}{2} = 2n + \frac{m+c-1}{2}$. Let $G := B[N_B(w)]$. If $\nu(G) \geq n$, we get a blue F_n with center w. So, we assume $\nu(G) \leq n-1$. Applying Theorem 2.2 with $d := d_B(w) - 2n \geq \frac{m+c-1}{2}$, we get a subset $S \subseteq N_B(w)$ such that $o(G-S) \geq |S| + d + 1 \geq |S| + \frac{m+c+1}{2}$.

Let $C_1, C_2, \ldots, C_\ell$ be the vertex sets of the components of $G-S$. We have the following observations.

(a) $\ell \geq o(G-S) \geq |S| + \frac{m+c+1}{2}$.

(b) For any distinct $i, j \in [\ell]$, all edges between C_i and C_j are red.

We further assume that $|C_1| := \min\{|C_i| : i \in [\ell]\}$ and let $D = \cup_{i=2}^{\ell} C_i$. By (b), $\overline{G}$ contains a red K_ℓ, which in turn shows $\ell \leq m-1$.

If $d_B(w) \geq 3n$, then by Corollary 2.4, $N_B(w)$ spans a blue nK_2 or a red F_n, which in turn shows that there is a monochromatic F_n. So we assume $d_B(w) \leq 3n-1$. By the minimality of $|C_1|$, we have the following.

$$|C_1| \leq \frac{d_B(w) - |S|}{\ell} \leq \frac{3n-1}{(m+c+1)/2} < \frac{3n}{m/2} = \frac{6n}{m}.$$

Thus, $|C_1| \leq \lfloor 6n/m \rfloor$ and

$$\begin{aligned}
|D| &= d_B(w) - |S| - |C_1| \\
&\geq 2n + \frac{m+c-1}{2} - \left(\ell - \frac{m+c+1}{2}\right) - \left\lfloor \frac{6n}{m} \right\rfloor \\
&= m + 2n - \ell + 1 \quad (\text{as } c = \lfloor 6n/m \rfloor + 1) \\
&\geq 2n + 2. \qquad (3)
\end{aligned}$$

For every $i \in [\ell]$, fix an arbitrary vertex $v_i \in C_i$. Let $X = \{v_2, v_3, \ldots, v_\ell\}$. Note that $X \subseteq D$ and its vertices form a red clique, and v_1 is red-adjacent to all vertices in D.

Let $D^* := D \setminus X$. Then $|D^*| = |D| - (\ell - 1) \geq m + 2n - 2\ell + 2$. We claim that D^* contains a red matching of size at least $n - \ell + 2$. Otherwise, by removing the vertices of a largest red matching in D^*, we get a blue clique Z in $G[D^*]$ with $|Z| \geq |D^*| - 2\nu(\overline{G}[D^*]) \geq m + 2n - 2\ell + 2 - 2(n - \ell + 1) = m$. So, Z induces a blue K_m, giving a contradiction. Let M be a red matching in $\overline{G}[D^*]$ with $|M| \geq n - \ell + 2$ and let $Y := D^* - V(M)$.

Recall from (b) that v_1 is red-adjacent to all vertices in D. We will show that there is a red matching of size at least n in D, which gives a red F_n with center v_1. Since $v_2, v_3, \ldots, v_\ell$ are in different components of $G-S$, every vertex in Y is red-adjacent to at least $|X|-1$ vertices in X. Hence we can greedily find a red matching M' of size at least $\min\{|Y|, |X|-1\}$ between X and Y. If $|M'| = |Y|$, then $M' \cup M$ saturates all the vertices in D^*. Since $R[X]$ is a red complete graph, the vertices in $D = D^* \cup X$ contains a red matching of size at least $\lfloor |D|/2 \rfloor \geq n$ by (3). If $|M'| \geq |X| - 1$, then $|M' \cup M| \geq |X| - 1 + (n - \ell + 2) = \ell - 2 + (n - \ell + 2) = n$. In either case, we find a red matching of size at least n in D, as desired. □

Proof of Lemma 1.3. Suppose to the contrary that neither G nor $\overline{G}$ contains a copy of F_n. We make the following observation:

For every $v \in V_0$, there is no matching M in $G[N(v)]$ such that $|V(M) \backslash V_0| \geq \left\lfloor \frac{n}{2} \right\rfloor$. (4)

Otherwise, there are $v \in V_0$ and a matching M in $G[N(v)]$ such that $|V(M)\backslash V_0| \geq \lfloor n/2 \rfloor$. Since V_0 is a clique, M can be extended to a matching M^* containing all vertices in $V(M) \cup V_0\backslash\{v\}$ if $|V(M) \cup V_0\backslash\{v\}|$ is even and all but one vertex in $V(M) \cup V_0\backslash\{v\}$ if $|V(M) \cup V_0\backslash\{v\}|$ is odd. Since $|V_0| \geq \lceil 3n/2 \rceil + 1$, it follows that M^* is a matching M in $G[N(v)]$ of size

$$\left\lfloor \frac{|V(M) \cup V_0\backslash\{v\}|}{2} \right\rfloor \geq \left\lfloor \frac{\lfloor n/2 \rfloor + \lceil 3n/2 \rceil}{2} \right\rfloor = n,$$

which in turn gives an F_n centered at v, a contradiction.

In the rest of the proof, we will find disjoint subsets $S_{v_1}, S_{v_2}, \ldots, S_{v_t}$ of $V\backslash V_0$ for some $t > 3$ and a vertex $w \in V_0$ such that $\overline{G}[\cup_{1\leq i\leq t} S_{v_i} \cup \{w\}]$ contains a subgraph isomorphic to F_n. For this goal, we first prove the following claim.

Claim 3.1. *For every vertex $v \in V_0$, there exists an independent set $S_v \subseteq N(v)\backslash V_0$ such that $|S_v| \geq |N(S_v) \cap V_0| + n/2$ and $|N(S_v) \cap V_0| \leq n/2$.*

Proof. Let v be a vertex in V_0 and M_v be a largest matching in $G[N(v) \setminus V_0]$. Let $m := |M_v|$. Then $N(v) \setminus (V_0 \cup V(M_v))$ is an independent set. Since v has at least n neighbors in $V\backslash V_0$, we have $|N(v) \setminus (V_0 \cup V(M_v))| \geq n - 2m$. Let $Z_v \subseteq N(v) \setminus (V_0 \cup V(M_v))$ with $|Z_v| = n-2m$. If there is a matching M' between Z_v and $V_0\backslash\{v\}$ with $|M'| \geq \lfloor n/2 \rfloor - 2m$, then $M := M' \cup M_v$ is a matching with $|V(M)\backslash V_0| \geq \lfloor n/2 \rfloor$, contradicting (4). Thus there is no matching of size $\lfloor n/2 \rfloor - 2m = |Z_v| - \lceil n/2 \rceil$ between Z_v and $V_0\backslash\{v\}$. Applying Theorem 2.1 on $G\left[Z_v, V_0\backslash\{v\}\right]$ by taking

$$X := Z_v, \quad Y := V_0\backslash\{v\} \quad \text{and} \quad d := \lceil n/2 \rceil,$$

we get a subset $S_v \subseteq Z_v$ (thus S_v is independent) such that

$$|N(S_v) \cap V_0\backslash\{v\}| \leq |S_v| - d - 1.$$

This implies that $|S_v| \geq |N(S_v) \cap V_0\backslash\{v\}| + 1 + d \geq |N(S_v) \cap V_0| + n/2$ and

$$|N(S_v) \cap V_0| = |N(S_v) \cap V_0\backslash\{v\}| + 1 \leq |S_v| - d \leq |Z_v| - d \leq n/2.$$

This proves the claim. $\square$

For every $v \in V_0$, let S_v be the subset of $N(v)\backslash V_0$ defined in Claim 3.1.

- Let $v_1 \in V_0$ such that $|N(S_{v_1}) \cap V_0|$ is the maximum among all vertices in V_0. Let $V_1 := V_0\backslash N(S_{v_1})$. By definition, every vertex in V_1 is not adjacent to any vertex in S_{v_1}.

- For each $i \geq 1$, if $V_{i-1}\backslash N(S_{v_i}) \neq \emptyset$, then define $V_i := V_{i-1}\backslash N(S_{v_i})$ and choose $v_{i+1} \in V_i$ such that $|N(S_{v_{i+1}}) \cap V_i|$ is the maximum among all vertices in V_i. Note that $N(S_{v_{i+1}}) \cap V_i \neq \emptyset$ because $v_{i+1} \in N(S_{v_{i+1}}) \cap V_i$. Together with the choice of v_i, we derive that

$$0 < |N(S_{v_{i+1}}) \cap V_i| \leq |N(S_{v_{i+1}}) \cap V_{i-1}| \leq |N(S_{v_i}) \cap V_{i-1}|. \tag{5}$$

For simplicity, let $N'(S_{v_{i+1}}) := N(S_{v_{i+1}}) \cap V_i$. By definition, $N'(S_{v_1})$, $N'(S_{v_2})$, $\ldots$ are nonempty and pairwise disjoint. Suppose the above process stops when $i = t$ due to $V_{t-1} \backslash N(S_{v_t}) = \emptyset$. Then

$$\bigcup_{1\le i\le t} N'(S_{v_i}) = V_0 \quad \text{and} \quad \bigcup_{1\le i<t} N'(S_{v_i}) \subsetneq V_0. \tag{6}$$

By Claim 3.1, (5), and the choice of v_i, we have

(i) $|N'(S_{v_t})| \le |N'(S_{v_{t-1}})| \le \cdots \le |N'(S_{v_1})| \le n/2$;

(ii) $S_{v_1}, S_{v_2}, \ldots, S_{v_t}$ are disjoint independent sets such that $|S_{v_i}| \ge |N'(S_{v_i})| + n/2$ for all $i \in [t]$;

(iii) every vertex in V_i is not adjacent to any vertex in $\bigcup_{1\le j\le i} S_{v_j}$ for all $i \in [t]$.

By (6) and (i), we have

$$\frac{\sum_{i=1}^{t-1} |N'(S_{v_i})|}{t-1} \ge \frac{\sum_{i=1}^{t} |N'(S_{v_i})|}{t} = \frac{|V_0|}{t} \quad \text{and} \quad t \ge \frac{|V_0|}{|N'(S_{v_1})|} > \frac{3n/2}{n/2} = 3.$$

It follows that

$$\sum_{i=1}^{t-1} |N'(S_{v_i})| \ge |V_0| \cdot \frac{t-1}{t} \ge \frac{3n}{2} \cdot \frac{2}{3} = n.$$

By (ii) and the fact that $t \ge 3$, we have

$$\sum_{i=1}^{t-1} |S_{v_i}| \ge \sum_{i=1}^{t-1} \left(|N'(S_{v_i})| + \frac{n}{2} \right) \ge n + \frac{n}{2} \cdot 2 = 2n.$$

Since all S_{v_i} are independent sets, we obtain a matching M' of size n in $\overline{G}\left[\bigcup_{i=1}^{t-1} S_{v_i}\right]$. Since $\bigcup_{i=1}^{t-1} N(S_{v_i}) \subsetneq V_0$, there is a vertex $w \in V_0 \setminus \bigcup_{i=1}^{t-1} N(S_{v_i})$. By (iii), w is not adjacent to any vertex in $\bigcup_{i=1}^{t-1} S_{v_i}$. Therefore, $V(M') \cup \{w\}$ spans a fan F_n in $\overline{G}$. $\square$

4 Proof of Theorem 1.1

4.1 Lower bound

Let n be a positive integer and let t be the largest even number less than $3n/2$. Thus $t \ge 3n/2 - 2$. We construct a graph $G = (V, E)$ on $3t$ vertices as follows. Let $V_1 \cup V_2 \cup V_3$ be a partition of V such that $|V_1| = |V_2| = |V_3| = t$ and all $G[V_i]$ are complete graphs. For each $i \in [3]$, further partition V_i into two subsets X_i and Y_i with $|X_i| = |Y_i| = t/2$, and add edges between X_i and Y_{i+1} such that $G[X_i, Y_{i+1}]$ is an $\lceil \frac{n}{2} \rceil$-regular bipartite graph, where we assume $Y_4 = Y_1$. The graph G is depicted in Figure 1.

Observe that G does not contain a copy of F_n because every vertex has degree $\lceil n/2 \rceil + t - 1 < 2n$. To see that $\overline{G}$ contains no copy of F_n, we note that $\overline{G}$ is 3-partite because V_1, V_2, V_3 induce cliques in G. Thus $\overline{G}$ induces a bipartite graph on $N_{\overline{G}}(v)$ for every vertex $v \in V$. Furthermore, two parts of this bipartite graph have sizes t and $t - \lceil n/2 \rceil < n$ and thus there is no matching of size n in $\overline{G}[N_{\overline{G}}(v)]$. Consequently $\overline{G}$ contains no copy of F_n.

Since neither G nor $\overline{G}$ contains a copy of F_n, we have $r(F_n) \ge |V| + 1 = 3t + 1 \ge 9n/2 - 5$.

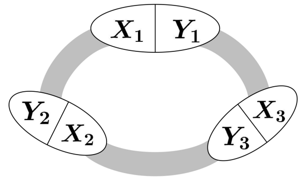

Figure 1: Illustration of G

4.2 Upper bound

Given a red-blue edge coloring of a complete graph on $N = \lceil 11n/2 \rceil + 5$, let R, B be the graphs induced by the red and blue edges, respectively. If there is a vertex v with $|N_R(v)| \geq 3n$ or $|N_B(v)| \geq 3n$, then there is a monochromatic F_n by Corollary 2.4. We thus assume that $|N_R(v)| \leq 3n-1$ and $|N_B(v)| \leq 3n-1$ for all vertices v. Because R and B are complementary to each other, it follows that $d_R(v), d_B(v) \geq (N-1)-(3n-1) = N-3n$. Define $m := N - 4n - 4 = \lceil 3n/2 \rceil + 1$. Since

$$\frac{6n}{m} = \frac{6n}{\lceil 3n/2 \rceil + 1} < \frac{6n}{3n/2} = 4,$$

we have $\left\lfloor \frac{6n}{m} \right\rfloor \leq 3$. So $4n+m+\left\lfloor \frac{6n}{m} \right\rfloor+1 \leq N$. By Lemma 1.2, there exists a monochromatic F_n or a monochromatic K_m in K_N. If there exists a monochromatic F_n, we are done. Otherwise, assume there is a monochromatic K_m. Without loss of generality, suppose that K_m is blue. Let V_0 be the blue clique of order m. For every $v \in V_0$, v has at least $d_B(v) - (m-1) \geq (N-3n) - (N-4n-5) = n+5 > n$ neighbors in $V(B) \backslash V_0$. Applying Lemma 1.3 with $G := B$, we get a monochromatic F_n. Thus $r(F_n) \leq N \leq 11n/2 + 6$. $\square$

5 Concluding remarks

Theorem 1.1 contains upper and lower bounds for $r(F_n)$ that differ by about n. We do not have a conjecture on the value of $r(F_n)$ but speculate that the lower bound is closer to the truth.

As mentioned in Section 1, we believe that $r(F_n) \leq r(nK_3) = 5n$. Although we are unable to verify this, there is some evidence for this assertion. First, $r(F_2) = 9 < 10 = r(2K_3)$. Second, let t, n be positive integers such that t divides n. One way of proving $r(F_n) \leq r(nK_3)$ is showing that $r(\frac{n}{t}F_t) \leq r(nK_3)$ for all such t. Indeed, Burr, Erdős and Spencer [1] proved the following theorem.

Theorem 5.1. ([1, Theorem 1]) *Let n be a positive integer and G be a graph of order k and independence number i. Then there exists a constant $C = C_G$ such that*

$$(2k-i)n - 1 \leq r(nG) \leq (2k-i)n + C.$$

We can apply Theorem 5.1 with $G = F_t$ (thus $k = 2t+1$ and $i = t$) and obtain that $(3t+2)\frac{n}{t} - 1 \leq r(\frac{n}{t}F_t) \leq (3t+2)\frac{n}{t} + C$ for some C depending only on F_t. For fixed $t \geq 2$, this implies that $r(\frac{n}{t}F_t) = \left(3 + \frac{2}{t}\right)n + O(1)$, much smaller than $r(nK_3)$.[2]

[2]The proof of [1, Theorem 1] shows that C is double exponential in t and thus $r(\frac{n}{t}F_t) = \left(3 + \frac{2}{t}\right)n + o(n)$ whenever $t = o(\log\log n)$.

We now give a construction that shows Lemma 1.3 is best possible with respect to $|V_0|$. Suppose n is even. Let $G = (V, E)$ be a graph on $9n/2 - 2$ vertices that contains a clique V_0 of order $3n/2$, and V_0 is partitioned into $V_1 \cup V_2 \cup V_3$ such that $|V_1| = |V_2| = |V_3| = n/2$. The set $V \backslash V_0$ is independent and is partitioned into $U_1 \cup U_2 \cup U_3 \cup \{x_0\}$ with $|U_1| = |U_2| = |U_3| = n - 1$. For every $i \in [3]$, $G[V_i, U_i]$ is complete but $G[V_i, U_j]$ is empty for distinct $i, j \in [3]$. In addition, all the vertices of V_0 are adjacent to x_0. Then each $v \in V_0$ has exactly n neighbors in $V \setminus V_0$. But neither G or $\overline{G}$ contains an F_n centered in V_0 (there are copies of F_n whose centers are outside V_0 in $\overline{G}$). Indeed, for $v \in V_0$, every matching M in $G[N_G(v)]$ contains at most $n/2$ vertices in $V \setminus V_0$ and thus $|V(M)| \leq |V_0| - 1 + n/2 < 2n$. In $\overline{G}$, every $v \in V_0$ has exactly $2n - 2$ neighbors so there is no matching of order $2n$ in $\overline{G}[N_{\overline{G}}(v)]$.

We can generalize the construction that gives the lower bound of Theorem 1.1 and obtain a new lower bound for $r(F_n, F_m)$. When $m \leq n < 3m/2 - 7$, our bound is better than $r(F_n, F_m) \geq 4n + 2$ given in [13].

Theorem 5.2. *Let m, n be positive integers with $m \leq n \leq \frac{3m}{2} - 3$. We have*

$$r(F_n, F_m) \geq \frac{3m}{2} + 3n - 5.$$

Proof. We construct a graph $G = (V, E)$ on $3t$ vertices, where t is the largest even number less than $\frac{m}{2} + n$. Thus $t \geq \frac{m}{2} + n - 2$. Our goal is to show that neither G contains F_n nor $\overline{G}$ contains F_m. This will imply that $r(F_n, F_m) \geq 3t + 1 \geq 3m/2 + 3n - 5$ as desired.

Let $V_1 \cup V_2 \cup V_3$ be a partition of V such that $|V_1| = |V_2| = |V_3| = t$ and all $G[V_i]$ are complete graphs. For every $i \in [3]$, partition V_i into two subsets X_i and Y_i with $|X_i| = |Y_i| = t/2$. Observe that

$$\frac{t}{2} - \left\lceil n - \frac{m}{2} \right\rceil \geq \frac{m}{4} + \frac{n}{2} - 1 - \left(n - \frac{m}{2} + \frac{1}{2} \right) = \frac{3m}{4} - \frac{n}{2} - \frac{3}{2} \geq 0 \quad \text{as } n \leq \tfrac{3m}{2} - 3.$$

For every $i \in [3]$, we add edges between X_i and Y_{i+1} (assuming $Y_4 = Y_1$) such that $G[X_i, Y_{i+1}]$ is an $\lceil n - \frac{m}{2} \rceil$-regular bipartite graph.

The graph G contains no F_n because for every vertex $v \in V$,

$$d_G(v) \leq t - 1 + \left\lceil n - \frac{m}{2} \right\rceil < \frac{m}{2} + n - 1 + n - \frac{m}{2} + \frac{1}{2} < 2n \quad \text{as } t < \frac{m}{2} + n.$$

For every $v \in V$, $\overline{G}$ induces a bipartite graph on $N_{\overline{G}}(v)$ with one part of size

$$t - \left\lceil n - \frac{m}{2} \right\rceil < \frac{m}{2} + n - \left(n - \frac{m}{2} \right) = m.$$

It follows that $\overline{G}$ contains no F_m. $\square$

Acknowledgement

We thank anonymous referees for their comments that improved the presentation of this paper.

References

[1] S. A. Burr, P. Erdős, and J. H. Spencer. Ramsey theorems for multiple copies of graphs. *Trans. Amer. Math. Soc.*, 209:87–99, 1975.

[2] D. Conlon. The Ramsey number of books. *Adv. Comb.*, pages Paper No. 3, 12, 2019.

[3] D. Conlon, J. Fox, and B. Sudakov. Recent developments in graph Ramsey theory. In *Surveys in combinatorics 2015*, volume 424 of *London Math. Soc. Lecture Note Ser.*, pages 49–118. Cambridge Univ. Press, Cambridge, 2015.

[4] D. Conlon, J. Fox, and Y. Wigderson. Ramsey numbers of books and quasirandomness. arXiv:2001.00407v2, 2020.

[5] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2017.

[6] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp. The size Ramsey number. *Period. Math. Hungar.*, 9(1-2):145–161, 1978.

[7] Y. Li and C. C. Rousseau. Fan-complete graph Ramsey numbers. *J. Graph Theory*, 23(4):413–420, 1996.

[8] Q. Lin and Y. Li. On Ramsey numbers of fans. *Discrete Appl. Math.*, 157(1):191–194, 2009.

[9] Q. Lin, Y. Li, and L. Dong. Ramsey goodness and generalized stars. *European J. Combin.*, 31(5):1228–1234, 2010.

[10] S. P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, 1:Dynamic Survey 1, 30, 1994.

[11] C. C. Rousseau and J. Sheehan. On Ramsey numbers for books. *J. Graph Theory*, 2(1):77–87, 1978.

[12] A. Thomason. On finite Ramsey numbers. *European J. Combin.*, 3(3):263–273, 1982.

[13] Y. Zhang, H. Broersma, and Y. Chen. A note on Ramsey numbers for fans. *Bull. Aust. Math. Soc.*, 92(1):19–23, 2015.