PARTIAL INDEPENDENT TRANSVERSALS IN MULTIPARTITE GRAPHS

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Abstract. Given integers $r > d \geq 0$ and an r-partite graph, an independent $(r-d)$-transversal or $(r-d)$-IT is an independent set of size $r-d$ that intersects each part in at most one vertex. We show that every r-partite graph with maximum degree Δ and parts of size n contains an $(r-d)$-IT if $n > 2\Delta(1-\frac{1}{q})$, provided $q = \lfloor \frac{r}{d+1} \rfloor \geq \frac{4r}{4d+5}$. This is tight when q is even and extends a classical result of Haxell in the $d = 0$ case. When $q = \lfloor \frac{r}{d+1} \rfloor \geq \frac{6r+6d+7}{6d+7}$ is odd, we show that $n > 2\Delta(1-\frac{1}{q-1})$ guarantees an $(r-d)$-IT in any r-partite graph. This is also tight and extends a result of Haxell and Szabó in the $d = 0$ case. In addition, we show that $n > 5\Delta/4$ guarantees a 5-IT in any 6-partite graph and this bound is tight, answering a question of Lo, Treglown and Zhao.

1. Introduction

Let $G = (V, E)$ be a graph with partition $V = V_1 \cup \cdots \cup V_r$. An *independent transversal* of G is an independent set with exactly one vertex in each V_i. Given $0 \leq d < r$, an *independent $(r-d)$-transversal* or *$(r-d)$-IT* is an independent set with one vertex in each of $r-d$ parts of G. We sometimes call an $(r-d)$-IT with $d > 0$ a *partial IT* and an independent transversal a *full IT*. Independent transversals have found applications in many areas, including combinatorics (e.g. [1, 2, 7, 15, 17]), groups and rings (e.g. [6, 8]) and combinatorial optimization (e.g. [3, 9]).

The existence of full independent transversals under maximum degree conditions was first studied by Bollobás, Erdős, and Szemerédi [5] in 1975 in the complementary form. This problem was extensively studied and eventually solved by Haxell [11], Haxell and Szabó [14], and Szabó and Tardos [18], see (2).

Bollobás, Erdős, and Szemerédi also considered the existence of partial independent transversals (again in the complementary form). They [5, Theorem 3.1] determined the maximum number of edges in r-partite graphs with n vertices in each part and without a copy of K_t (complete graph on t vertices) for any $t \leq r$. The corresponding problem under a minimum degree condition was studied by Lo, Treglown, and Zhao [16].

Given integers n and $2 \leq t \leq r$, let $f(n, r, t)$ denote the largest minimum degree $\delta(G)$ among all r-partite graphs G with parts of size n and without a copy of K_t. A result of Bollobás, Erdős, and Straus [4] implies that $f(n, r, 3) = \lfloor r/2 \rfloor n$ for all $r \geq 3$ while [5, Theorem 3.1] implies that $f(n, r, t+1) = (r - r/t)\, n$ whenever t divides r. The authors of [16] observed that Turán's theorem implies that

$$\left(r - \left\lceil \frac{r}{t} \right\rceil\right) n \leq f(n, r, t+1) \leq \left(r - \frac{r}{t}\right) n. \qquad (1)$$

The main result of [16] determined $f(n, r, t+1)$ when t divides $r+1$, when $r \geq (3t-1)(t-1)$, and $f(n, r, 4)$ for all $r \neq 7$.

The smallest unknown case (with respect to r) is $f(n, 6, 5)$. In this paper we resolve this and many other cases of $f(n, r, t+1)$ when $t > r/2$ by studying the complementary problem of

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finding partial independent transversals under maximum degree conditions. Instead of $f(n,r,t)$, we find it convenient to use another extremal function. For $r, D, \Delta \in \mathbb{N}$, let $n(r, D, \Delta)$ be the largest $n \in \mathbb{N}$ such that there is an r-partite graph G with at least n vertices in each part, maximum degree Δ, and without an $(r-D+1)$-IT.

The two functions $f(n,r,t)$ and $n(r,D,\Delta)$ are closely related, as shown in Lemma 2.1. Under the $n(r,D,\Delta)$ notation, the results of [11, 14, 18] are as follows: given integers $r \geq 2$ and $\Delta > 0$, we have

$$n(r,1,\Delta) = \begin{cases} \left\lfloor 2\Delta\left(1-\frac{1}{r}\right)\right\rfloor, & \text{if } r \text{ is even} \\ \left\lfloor 2\Delta\left(1-\frac{1}{r-1}\right)\right\rfloor, & \text{if } r \text{ is odd.} \end{cases} \tag{2}$$

Given an r-partite graph G, let G' be obtained from G by adding d vertex-disjoint copies of K_r with one vertex in each part. Then G has an $(r-d)$-IT if and only if G' has a full IT. Using this standard idea for proving "defect" versions, one can adapt the proof of $n(r,1,\Delta) \leq 2\Delta(1-1/r)$ (e.g. in [12, Section 2]) to obtain that

$$n(r,d+1,\Delta) \leq 2\Delta\left(1-\frac{d+1}{r}\right). \tag{3}$$

Using multiple copies of [18, Construction 3.3] of Szabó and Tardos, one can easily see that (3) is tight when $r/(d+1)$ is an even integer. This already improves (1) and extends the first case of (2).

Our first result improves (3) substantially.

Theorem 1.1. *Given integers $r, d \geq 0$, let q, k be integers such that $r = q(d+1)+k$ and $0 \leq k \leq d$. If G is an r-partite graph with maximum degree at most Δ and with vertex classes $V_1, \ldots, V_r$ of size*

$$|V_i| > \max\left\{2\Delta\left(1-\frac{4d+5}{4r}\right), 2\Delta\left(1-\frac{1}{q}\right)\right\},$$

then G has an $(r-d)$-IT. In other words,

$$n(r,d+1,\Delta) \leq \max\left\{2\Delta\left(1-\frac{4d+5}{4r}\right), 2\Delta\left(1-\frac{1}{q}\right)\right\}.$$

It is easy to see that the maximum in Theorem 1.1 is always less than or equal to $2\Delta\left(1-\frac{d+1}{r}\right)$. Thus, Theorem 1.1 is an improvement of (3). Furthermore, together with the constructions in Section 2, Theorem 1.1 gives the value of $n(r,d+1,\Delta)$ whenever $q \geq 4k$ is even (note that the maximum in Theorem 1.1 is equal to $2\Delta(1-1/q)$ if and only if $q \geq 4k$ or equivalently $q \geq \frac{4r}{4d+5}$).

Corollary 1.2. *Given integers $r, d \geq 0$, let q, k be integers such that $r = q(d+1)+k$ and $0 \leq k \leq d$. If q is even and $q \geq 4k$, then $n(r,d+1,\Delta) = \left\lfloor 2\Delta\left(1-\frac{1}{q}\right)\right\rfloor$, and equivalently, $f(n,r,r-d) = (r-1)n - \left\lceil \frac{qn}{2(q-1)} \right\rceil$.*

The $d=0$ case of Corollary 1.2 says that $n(r,1,\Delta) = \lfloor 2\Delta(1-1/r)\rfloor$ for all even r, recovering the first case of (2) given by [11, 18].

When $q = \lfloor \frac{r}{d+1} \rfloor$ is odd, we have the following results.

Theorem 1.3. *Let $r, d, q \geq 0$ be integers such that $r = q(d+1)$. If $q \geq 3$ is odd, then*

$$n(r,d+1,\Delta) \leq \max\left\{2\Delta\left(1-\frac{4d+5}{4r}\right), 2\Delta\left(1-\frac{q}{q^2-1}\right)\right\}.$$

Theorem 1.4. *Given integers $r, d \geq 0$, let q, k be integers such that $r = q(d+1)+k$ and $0 \leq k \leq d$. If $q \geq 3$ is odd, then*

$$n(r,d+1,\Delta) \leq \max\left\{2\Delta\left(1-\frac{6d+7}{6r}\right), 2\Delta\left(1-\frac{1}{q-1}\right)\right\}.$$

Together with constructions in Section 2, Theorem 1.3 determines $f(n,6,5)$, answering a specific question asked in [16]. Previously, the best bounds were

$$\frac{25}{6}n \leq f(n,6,5) \leq \frac{9}{2}n,$$

where the lower bound was given by [16, Proposition 4.1] and the upper bound follows from (1). Theorem 1.1 or (3) gives $n(6,2,\Delta) \leq \frac{4}{3}\Delta$, which implies $f(n,6,5) \leq \frac{17}{4}n$ by Lemma 2.1.

Corollary 1.5. *We have* $n(6,2,\Delta) = \lfloor \frac{5}{4}\Delta \rfloor$, *and thus,* $f(n,6,5) = \lfloor \frac{21}{5}n \rfloor$.

Let us compare the bounds in Theorems 1.1, 1.3, and 1.4. Since $1 - \frac{1}{q-1} < 1 - \frac{q}{q^2-1} < 1 - \frac{1}{q}$, the bound in Theorem 1.3 is at most the one in Theorem 1.1. When $r = q(d+1)+k$ and $q \geq 6k$, the bound in Theorem 1.4 is at most the one in Theorem 1.1 because

$$\max\left\{2 - \frac{6d+7}{3r}, 2 - \frac{2}{q-1}\right\} \leq 2 - \frac{2}{q} = \max\left\{2 - \frac{4d+5}{2r}, 2 - \frac{2}{q}\right\}.$$

Furthermore, when $k = 0$ and $q \geq 6d+7 > \sqrt{4d+5}$, we have

$$\max\left\{2 - \frac{6d+7}{3r}, 2 - \frac{2}{q-1}\right\} = 2 - \frac{2}{q-1} < 2 - \frac{2q}{q^2-1} = \max\left\{2 - \frac{4d+5}{2r}, 2 - \frac{2q}{q^2-1}\right\}.$$

and thus, the bound in Theorem 1.4 is smaller than the bound in Theorem 1.3.

Together with the constructions in Section 2, Theorem 1.4 determines the value of $n(r,d+1,\Delta)$ whenever $q \geq 6d+6k+7$ (or equivalently $q \geq \frac{6r+6d+7}{6d+7}$) is odd.

Corollary 1.6. *Let* $r, d \geq 0$ *and* q, k *be integers such that* $r = q(d+1)+k$ *and* $0 \leq k \leq d$. *If* q *is odd and* $q \geq 6d+6k+7$, *then* $n(r,d+1,\Delta) = \left\lfloor 2\Delta\left(1 - \frac{1}{q-1}\right)\right\rfloor$, *and equivalently,* $f(n,r,r-d) = (r-1)n - \left\lceil \frac{(q-1)n}{2(q-2)} \right\rceil$.

When $d = 0$ (and thus $k = 0$ and $q = r$), Corollary 1.6 recovers the second case of (2) derived from the main result of [14], which assumes that $r \geq 7$ is odd.

Recall that $f(n,r,t)$ was determined in [16] for $r = \Omega(t^2)$. Corollaries 1.2 and 1.6 together determine $n(r,d+1,\Delta)$ and $f(n,r,r-d)$ for $r = \Omega(d^2)$. Indeed, given integers $d \geq 0$ and $r \geq 12d^2 + 20d + 7$, we can write $r = q(d+1)+k$ such that $0 \leq k \leq d$ and $q \geq 6d+6k+7 \geq 4k$. Then, by Corollaries 1.2 and 1.6,

$$n(r,d+1,\Delta) = \begin{cases} \left\lfloor 2\Delta\left(1 - \frac{1}{q}\right)\right\rfloor, & \text{if } q \text{ is even} \\ \left\lfloor 2\Delta\left(1 - \frac{1}{q-1}\right)\right\rfloor, & \text{if } q \text{ is odd.} \end{cases}$$

1.1. **Notation.** Given a graph G, let $V(G)$ and $E(G)$ be the edge and vertex sets of G respectively. For $Z \subseteq E(G)$, we write $V(Z)$ to be the set of vertices incident to Z. Given $v \in V(G)$ and $I \subseteq V(G)$, let $N(v,I) = \{x \in I : \{x,v\} \in E(G)\}$ be the neighborhood of v in I. Suppose G is r-partite with parts $V_1, \cdots, V_r$. Given $I \subseteq V(G)$, let $S(I) = \{V_i \colon I \cap V_i \neq \emptyset\}$. We also let $\mathcal{G}_I$ be the multigraph obtained from the induced subgraph $G[I]$ by contracting all the vertices of $V_i \cap I$ into a single vertex denoted by V_i (thus, $V(\mathcal{G}_I) = S(I)$). For $Y \subseteq [r] := \{1, 2, \ldots, r\}$, we let G_Y be $G[\cup_{i \in Y} V_i]$.

1.2. **Organization.** We begin by considering constructions in Section 2, where we prove Proposition 2.2 and Corollary 2.3 and derive Corollaries 1.2, 1.5, and 1.6. We prove Theorem 1.1 in Section 3 by generalizing the theory of *Induced Matching Configurations* (IMCs) introduced in [14]. In Section 4 we prove similar structural results as in [14] and derive Theorems 1.3 and 1.4. We give concluding remarks in the last section.

2. Constructions and proofs of corollaries

We first show how the two extremal functions $f(n,r,t)$ and $n(r,D,\Delta)$ defined earlier are related. To do so, it will be convenient to define another extremal function $\Delta(n,r,t)$ to be the smallest $\Delta \in \mathbb{N}$ such that there is an r-partite graph with at least n vertices in each part, maximum degree Δ, and without a t-IT.

Lemma 2.1. *Suppose r, d, n, Δ are integers such that $r \geq 2$, $0 \leq d < r$, and $n, \Delta \geq 1$. Let $c \geq 1$ be a real number. Then the following are equivalent:*

(i) $n(r, d+1, \Delta) = \lfloor c\Delta \rfloor$,
(ii) $\Delta(n, r, r-d) = \lceil n/c \rceil$,
(iii) $f(n, r, r-d) = (r-1)n - \lceil n/c \rceil$.

Proof. We fix integers r, d with $r \geq 2$ and $0 \leq d < r$ throughout the proof. We first observe that for $n, \Delta \in \mathbb{N}$,

$$n > \lfloor c\Delta \rfloor \quad \text{if and only if} \quad \Delta < \lceil n/c \rceil. \tag{4}$$

Indeed, as n is an integer, $n > \lfloor c\Delta \rfloor$ implies $n \geq \lfloor c\Delta \rfloor + 1 > c\Delta$, so $\Delta < n/c \leq \lceil n/c \rceil$. On the other hand, $\Delta < \lceil n/c \rceil$ gives $\Delta \leq \lceil n/c \rceil - 1 < n/c$, so $n > c\Delta \geq \lfloor c\Delta \rfloor$.

(i) $\Longrightarrow$ (ii): We assume that $n(r, d+1, \Delta) = \lfloor c\Delta \rfloor$. Suppose $n \in \mathbb{N}$ and G is r-partite such that each class has exactly n vertices and $\Delta(G) = \Delta < \lceil n/c \rceil$. Then $n > \lfloor c\Delta \rfloor$ by (4). Since $n(r, d+1, \Delta) = \lfloor c\Delta \rfloor$, it follows that G has an $(r-d)$-IT. Thus, $\Delta(n, r, r-d) \geq \lceil n/c \rceil$. On the other hand, the assumption that $n(r, d+1, \Delta) = \lfloor c\Delta \rfloor$ also implies that, for every $\Delta \in \mathbb{N}$, there exists an r-partite graph G with maximum degree Δ and at least $\lfloor c\Delta \rfloor$ vertices in each part and without an $(r-d)$-IT. Now, given $n \in \mathbb{N}$, let $\Delta = \lceil n/c \rceil$ and G be an r-partite graph with maximum degree $\Delta = \lceil n/c \rceil$ and at least $\lfloor c\lceil n/c \rceil \rfloor$ vertices in each part, that has no $(r-d)$-IT. Note that $\lfloor c\lceil n/c \rceil \rfloor \geq n$ because $c\lceil n/c \rceil \geq c(n/c) = n$. The existence of G shows that $\Delta(n, r, r-d) \leq \lceil n/c \rceil$.

(ii) $\Longrightarrow$ (i): Assume that $\Delta(n, r, r-d) = \lceil n/c \rceil$. Suppose $\Delta \in \mathbb{N}$ and G is an r-partite graph with maximum degree Δ and parts of size $n > \lfloor c\Delta \rfloor$. We know $\Delta < \lceil n/c \rceil$ from (4). Since $\Delta(n, r, r-d) = \lceil n/c \rceil$, it follows that G has an $(r-d)$-IT, establishing $n(r, d+1, \Delta) \leq \lfloor c\Delta \rfloor$. On the other hand, the assumption $\Delta(n, r, r-d) = \lceil n/c \rceil$ implies that, for every n, there exists an r-partite graph G with parts of size at least n, maximum degree $\Delta(G) = \lceil n/c \rceil$, with no $(r-d)$-IT. Now, given $\Delta \in \mathbb{N}$, let $n = \lfloor c\Delta \rfloor$ and G be the r-partite graph with parts of size at least $n = \lfloor c\Delta \rfloor$ and maximum degree $\lceil \lfloor c\Delta \rfloor / c \rceil$ and without an $(r-d)$-IT. Since $c \geq 1$, we have

$$\Delta - 1 \leq \frac{c\Delta - 1}{c} < \frac{\lfloor c\Delta \rfloor}{c} < \frac{c\Delta}{c} = \Delta,$$

and consequently, $\lceil \lfloor c\Delta \rfloor / c \rceil = \Delta$. The existence of G shows that $n(r, d+1, \Delta) \geq \lfloor c\Delta \rfloor$.

(ii) $\Longleftrightarrow$ (iii): By considering the complements of graphs, we have

$$\Delta(n, r, r-d) + f(n, r, r-d) = (r-1)n.$$

Hence, $\Delta(n, r, r-d) = \lceil n/c \rceil$ if and only if $f(n, r, r-d) = (r-1)n - \lceil n/c \rceil$. □

Since our results in this paper assume $d+1 \leq r/2$, the assumption $c \geq 1$ in Lemma 2.1 always holds. Indeed, given $r, \Delta \in \mathbb{N}$, let G be the union of $\lfloor r/2 \rfloor$ vertex-disjoint copies of $K_{\Delta,\Delta}$ and an isolated set of Δ vertices if r is odd. Then $\Delta(G) = \Delta$ and a maximum IT of G has size $\lceil r/2 \rceil \leq r - (d+1)$ if $d+1 \leq r/2$. This implies that $n(r, d+1, \Delta) \geq \Delta$.

We now give several properties of $n(r, D, \Delta)$. It is clear that

$$n(r', D, \Delta) \leq n(r, D, \Delta) \leq n(r, D', \Delta) \quad \text{if} \quad r \geq r' \quad \text{and} \quad D \geq D'. \tag{5}$$

By building new constructions from old ones, we derive the following proposition.

Proposition 2.2. *The following holds for any positive integers m, r, D, j, l.*

(i) $n(r, D-1, \Delta) \geq n(r, D, \Delta) + \left\lfloor \frac{\Delta}{r-1} \right\rfloor$;

(ii) $n(mr, mD, \Delta) \geq n(r, D, \Delta)$;

(iii) $n(mr, (m-j)D, \Delta) \geq n(r, D, \Delta) + \left\lfloor \frac{\Delta}{(l-1)r} \right\rfloor$ *if* $m = jl$;

(iv) $n(mr, (m-j-1)D, \Delta) \geq n(r, D, \Delta) + \left\lfloor \frac{\Delta}{(l-1)r} \right\rfloor + \left\lfloor \frac{\Delta}{(m-1)r} \right\rfloor$ *if* $m = jl$.

Proof. We say that a graph is an (r, D)-*construction* if it has r classes and its maximum IT has size at most $r - D$. Suppose G is an optimal (r, D)-construction with maximum degree Δ and r classes of size $n = n(r, D, \Delta)$. We will construct a (r', D)-construction G' with maximum degree Δ and classes of size n, where $r' = r$ for (i), $r' = mr$ for (ii)-(iv), and $D' = D - 1, mD, (m-j)D$, and $(m-j-1)D$ for (i)-(iv) respectively.

We first prove (i). Let G' be the union of G with $K_r(\lfloor \frac{\Delta}{r-1} \rfloor)$ (a at most Δ-regular r-partite graph) such that each part of G' has size

$$n + \left\lfloor \frac{\Delta}{r-1} \right\rfloor = n(r, D, \Delta) + \left\lfloor \frac{\Delta}{r-1} \right\rfloor.$$

An IT of G' has at most one vertex from $K_r(\lfloor \frac{\Delta}{r-1} \rfloor)$ and thus at most $r - D + 1$ vertices in total. Thus G' is an $(r, D-1)$-construction giving (i).

For (ii), let G' be the union of m disjoint copies of G. Then each part of G' has $n(r, D, \Delta)$ vertices, as desired.

To prove (iii), we first show the case when $j = 1$:

$$n(mr, (m-1)D, \Delta) \geq n(r, D, \Delta) + \left\lfloor \frac{\Delta}{(m-1)r} \right\rfloor. \tag{6}$$

To see (6), we take a union of m (disjoint) copies of G and a copy of $K_m\left(r \left\lfloor \frac{\Delta}{(m-1)r} \right\rfloor\right)$ such that each copy of G is attached to an independent set of $r \left\lfloor \frac{\Delta}{(m-1)r} \right\rfloor$ vertices that are evenly distributed to r classes. The resulting graph G' has

$$n + \left\lfloor \frac{\Delta}{(m-1)r} \right\rfloor = n(r, D, \Delta) + \left\lfloor \frac{\Delta}{(m-1)r} \right\rfloor$$

vertices in each class. It remains to show that a maximum IT of G' has size at most $mr - (m-1)D$. We regard each copy of G together with $r \left\lfloor \frac{\Delta}{(m-1)r} \right\rfloor$ added vertices a *row* and call the vertices of G *large* and the other vertices *small*. By definition, an IT of G' misses at least D large classes from each row and intersects small classes from at most one row. Thus the IT misses at least $mD - D$ classes of G', confirming (6).

We now derive (iii) from (ii) and (6). Since $m - j = j(l-1)$, we have $n(mr, (m-j)D, \Delta) \geq n(lr, (l-1)D, \Delta)$ by (ii). By (6), we have $n(lr, (l-1)D, \Delta) \geq n(r, D, \Delta) + \left\lfloor \frac{\Delta}{(l-1)r} \right\rfloor$, proving (iii).

To see (iv), let G' be the (disjoint) union of the following three graphs:

- m copies of G, arranged in m *rows*, and we call their vertices *large*;
- j copies of $K_l\left(r \left\lfloor \frac{\Delta}{(l-1)r} \right\rfloor\right)$ such that each row has $r \left\lfloor \frac{\Delta}{(l-1)r} \right\rfloor$ independent vertices evenly distributed into r classes, and we call their vertices *medium*;
- one copy of $K_m(r \left\lfloor \frac{\Delta}{(m-1)r} \right\rfloor)$ such that each row has $r \left\lfloor \frac{\Delta}{(m-1)r} \right\rfloor$ independent vertices evenly distributed into r classes, and we call their vertices *small*.

The resulting graph G' has

$$n + \left\lfloor \frac{\Delta}{(l-1)r} \right\rfloor + \left\lfloor \frac{\Delta}{(m-1)r} \right\rfloor = n(r, D, \Delta) + \left\lfloor \frac{\Delta}{(l-1)r} \right\rfloor + \left\lfloor \frac{\Delta}{(m-1)r} \right\rfloor.$$

vertices in each class. It remains to show that the largest IT of G' missed at least $(m-j-1)D$ vertices. Each IT of G' has small vertices from at most one row, medium vertices from at most

j (additional) rows, and thus must use large vertices in at least $m-j-1$ rows. Since an IT of G' misses at least D vertices in each row, it thus misses at least $(m-j-1)D$ vertices in total, proving (iv). □

Corollary 2.3. *Given integers $r \geq 1$ and $d \geq 0$, let q, i, k be integers such that $q \geq 2$ is even, $1 \leq i \leq d+2$, $0 \leq k < d+i$, $r = q(d+i)+k$, and either $i=1$ or $i-1$ divides $d+i$. Then,*

$$n(r, d+1, \Delta) \geq \left\lfloor 2\Delta\left(1-\frac{1}{q}\right)\right\rfloor + \left\lfloor \frac{(i-1)\Delta}{(d+1)q} \right\rfloor.$$

Proof. Suppose $i=1$. Then we have

$$n(q(d+1)+k, d+1, \Delta) \overset{(5)}{\geq} n(q(d+1), d+1, \Delta) \overset{2.2\ (\text{ii})}{\geq} n(q, 1, \Delta) \overset{(2)}{=} \lfloor 2\Delta(1-1/q) \rfloor.$$

Now assume $i > 1$ and $d+i = (i-1)l$ for some integer l. Then,

$$\begin{aligned} n(r, d+1, \Delta) &\overset{(5)}{\geq} n(q(d+i), d+1, \Delta) \overset{2.2\ (\text{iii})}{\geq} n(q, 1, \Delta) + \left\lfloor \frac{\Delta}{(l-1)q} \right\rfloor \\ &\overset{(2)}{=} \left\lfloor 2\Delta\left(1-\frac{1}{q}\right)\right\rfloor + \left\lfloor \frac{\Delta}{\left(\frac{d+i}{i-1}-1\right)q} \right\rfloor = \left\lfloor 2\Delta\left(1-\frac{1}{q}\right)\right\rfloor + \left\lfloor \frac{(i-1)\Delta}{(d+1)q} \right\rfloor. \end{aligned}$$

□

Corollaries 1.2, 1.5, and 1.6 follow from Proposition 2.2 and Corollary 2.3 easily (assuming Theorems 1.1, 1.3, and 1.4).

Proof of Corollary 1.2. Recall that $2 - \frac{4d+5}{2r} \leq 2 - \frac{2}{q}$ precisely when $4k \leq q$. Hence

$$n(r, d+1, \Delta) \leq \left\lfloor \max\left\{2\Delta\left(1-\frac{4d+5}{4r}\right), 2\Delta\left(1-\frac{1}{q}\right)\right\}\right\rfloor = \left\lfloor 2\Delta - \frac{2\Delta}{q} \right\rfloor$$

by Theorem 1.1. On the other hand, applying Corollary 2.3 with $i=1$ gives $n(r, d+1, \Delta) \geq \lfloor 2\Delta - 2\Delta/q \rfloor$. Thus, $n(r, d+1, \Delta) = \lfloor 2\Delta(1-1/q) \rfloor$. By Lemma 2.1, we have $f(n, r, r-d) = (r-1)n - \left\lceil \frac{qn}{2(q-1)} \right\rceil$. □

Proof of Corollary 1.5. We have $n(6, 2, \Delta) \leq \lfloor 5\Delta/4 \rfloor$ by Theorem 1.3 with $r=6$ and $d=1$. On the other hand, we have

$$n(6, 2, \Delta) \geq n(2, 1, \Delta) + \lfloor \Delta/4 \rfloor = \lfloor 5\Delta/4 \rfloor$$

by Proposition 2.2 (iii) with $m=3$, $j=1$, and $l=3$ (this can also be derived from Corollary 2.3 with $q=2$ and $i=2$). Hence, $n(6, 2, \Delta) = \lfloor 5\Delta/4 \rfloor$. Consequently, $f(n, 6, 5) = 5n - \lceil 4n/5 \rceil = \lfloor 21n/5 \rfloor$ by Lemma 2.1. □

Proof of Corollary 1.6. Since $q \geq 6d+6k+7$ is odd, Theorem 1.4 gives that

$$n(r, d+1) \leq \left\lfloor \max\left\{2\Delta\left(1-\frac{6d+7}{6r}\right), 2\Delta\left(1-\frac{1}{q-1}\right)\right\}\right\rfloor = \left\lfloor 2\Delta - \frac{2\Delta}{q-1} \right\rfloor.$$

On the other hand, since $r = q(d+1)+k$, we have

$$n(r, d+1, \Delta) \overset{(5)}{\geq} n(q(d+1), d+1, \Delta) \overset{2.2\ (\text{ii})}{\geq} n(q, 1, \Delta) \overset{(2)}{=} \left\lfloor 2\Delta - \frac{2\Delta}{q-1} \right\rfloor.$$

Thus, $n(r, d+1, \Delta) = \lfloor 2\Delta - 2\Delta/(q-1) \rfloor$. By Lemma 2.1, it follows that $f(n, r, r-d) = (r-1)n - \left\lceil \frac{(q-1)n}{2(q-2)} \right\rceil$, as desired. □

The proof of Corollary 1.5 shows that Proposition 2.2 (iii) is tight. Proposition 2.2 and Corollary 1.2 together imply that, for any even $q \geq 2$ and $0 \leq k \leq q/4$,

$$\left\lfloor 2\Delta - \frac{2\Delta}{q} \right\rfloor \overset{1.2}{=} n(q(d+1)+k, d+1, \Delta) \overset{(5)}{\geq} n(q(d+1), d+1, \Delta) \overset{2.2\text{ (ii)}}{\geq} n(q,1,\Delta) \overset{(2)}{=} \left\lfloor 2\Delta - \frac{2\Delta}{q} \right\rfloor.$$

In particular, this shows that the function $n(r, D)$ is not strictly increasing in r and Proposition 2.2 (ii) is tight. (Note that, in contrast, Proposition 2.2 (i) shows that $n(r, D)$ is strictly decreasing in D.)

3. Induced Matching Configurations and proof of Theorem 1.1

Suppose G is an r-partite graph with vertex classes $V_1, \dots, V_r$. Let $\mathcal{V} = \{V_1, \dots, V_r\}$. Fix a subset $I \subseteq V(G)$. Recall that $S(I)$ is the set of classes that intersect I, and $\mathcal{G}_I$ is the multigraph formed by contracting all vertices of $G[I]$ into one vertex, which we still denote by V_i. Note that $V(\mathcal{G}_I) = S(I)$ and $\mathcal{G}_I$ has parallel edges if $G[I]$ has multiple edges between two classes V_i and V_j.

Definition 3.1. *A set of vertices $I \subseteq V(G)$ is an Induced Matching Configuration (IMC) if $G[I]$ is a perfect matching and $\mathcal{G}_I$ is a forest. We say that I is an IMC of p components if $\mathcal{G}_I$ has p components.*

If I is an IMC, then every edge of $G[I]$ corresponds to a (unique) edge of $\mathcal{G}_I$ (because $\mathcal{G}_I$ is a forest and thus has no multiple edges). Therefore, if $|I| = 2t$ and $|S(I)| = s$, then $\mathcal{G}_I$ has s vertices and t edges, thus consisting of $s - t$ components. We remark that IMCs were defined differently in [14], in which $\mathcal{G}_I$ is a tree on $\mathcal{V}$.

The forest structure of $\mathcal{G}_I$ makes it easy to find a partial IT in $G[\cup_{V_i \in S(I)} V_i]$ covering all but one class of every component of $\mathcal{G}_I$. The following lemma is essentially [14, Lemma 2.1] but revised slightly due to our definition of IMCs.

Lemma 3.2 ([14, Lemma 2.1])**.** *Suppose I is an IMC in G and I' is a subset of I such that $\mathcal{G}_{I'}$ is a tree on $S \subseteq \mathcal{V}$. Then, for any $V_i \in S$, there is an $(|S|-1)$-IT on $S \setminus \{V_i\}$. Further, if $v \in V_i \in S$ is not dominated by I', then there is an $|S|$-IT on S.*

We need the concept of feasible pairs introduced in [14] in order to prove Theorem 3.6.

Definition 3.3. *Given an r-partite graph G, a pair (I, T) with $I, T \subseteq V(G)$ is feasible if the following conditions hold:*

- *(a) T is a partial IT in G of maximum size.*
- *(b) $S(I \cap T) = S(I) \cap S(T)$, i.e. if $v \in T$ and $S(\{v\}) \in S(I)$, then $v \in I$.*
- *(c) $G[I]$ is a forest, whose components are stars with centers in $W := I \setminus T$ and at least one leaf.*
- *(d) $\mathcal{G}_I$ is a forest on $S(I)$.*
- *(e) Let $\mathcal{T}$ be the set of all partial independent transversals of G on $S(T)$. Then, for every $v \in W$, there is no $T' \in \mathcal{T}$ such that $T' \cap W = \emptyset$, $|N(v, T')| < |N(v, T)|$, and $N(w, T') = N(w, T)$ for all $w \in W \setminus \{v\}$.*

Note that if T is a partial IT in G of maximum size, then $(\emptyset, T)$ is a feasible pair.

The following algorithm allows us to construct feasible pairs that dominate all vertex classes intersecting them. Recall that $N(I)$ is the set of *neighbors* of I, in other words, the vertices that are dominated by I.

Algorithm 3.4. *Start with a feasible pair (I_0, T_0) in an r-partite graph G on $V_1 \cup \cdots \cup V_r$. Let $R = \{V_1, \dots, V_r\} \setminus S(T_0)$. Initialize $I = I_0$ and $T = T_0$. Throughout the algorithm, we maintain $W = I \setminus T$ and $\mathcal{T}$, the set of all maximum partial ITs T' on $S(T)$ such that $T' \cap W = \emptyset$, $N(v, T') = N(v, T)$ for all $v \in W$.*

Step 1 *If I dominates all the vertices in $S(I) \cup R$, then stop and return (I, T)*

Step 2 *If I dominates $V_I = \bigcup_{V_i \in S(I)} V_i$ but not all the vertices in R, go to Step 3.*

If V_I is not dominated by I, then select a vertex $w \in V_I \setminus N(I)$ and $T' \in \mathcal{T}$ such that $\deg(w, T')$ is minimal. Update I by adding $\{w\} \cup N(w, T')$, update $T = T'$, and go to Step 1.

Step 3 *Select a vertex $w \in \bigcup_{V_i \in R} V_i \setminus N(I)$ and $T' \in \mathcal{T}$ such that $\deg(w, T')$ is minimal. Update I by adding $\{w\} \cup N(w, T')$ and update $T = T'$. Go to Step 1.*

Lemma 3.5. *For any input (I_0, T_0), Algorithm 3.4 terminates and returns a feasible pair.*

As the proof of Lemma 3.5 is very similar to the corresponding part of the proof of [14, Theorem 2.2], we defer it to Appendix A. Indeed, Algorithm 3.4 is the same as the one given in the proof of [14, Theorem 2.2], except that in [14] the algorithm terminates when I dominates $S(I)$, while Algorithm 3.4 terminates when I dominates $S(I) \cup R$. As a result, the resulting $\mathcal{G}_I$ is a forest, instead of a tree as in [14].

Theorem 3.6. *Let G be an r-partite graph on $\mathcal{V} = \{V_1, \dots, V_r\}$. Suppose that the largest partial IT of G has size $r - d - 1$. Given a feasible pair (I_0, T_0), there exists a feasible pair (I, T) in G such that*

(i) *$I_0 \subseteq I$, $S(I) \geq 2$, $S(T) = S(T_0)$, and $T \cap V_i = T_0 \cap V_i$ for every $V_i \in S(I_0)$.*
(ii) *I dominates all of the vertices in $S(I) \cup R$, where $R = \mathcal{V} \setminus S(T_0)$.*
(iii) *Let $F_I = (R \cup S(I), E(\mathcal{G}_I))$ be the extension of $\mathcal{G}_I$ to $R \cup S(I)$ if $R \not\subseteq S(I)$ (otherwise $F_I = \mathcal{G}_I$). Then F_I is a forest of $d + 1$ components, where each component contains exactly one $V_i \in R$ (which may be the only V_i in that component).*
(iv) *Let $t = |R \cup S(I)|$. Then $|I| \leq 2(t - d - 1)$.*

If $n > 2\Delta\left(1 - \frac{2d+3}{2r}\right)$, then I is an IMC of G and $|I| = 2(t - d - 1)$.

Proof. Let (I, T) be the feasible pair obtained by applying Algorithm 3.4 to (I_0, T_0). We now show that (I, T) satisfies the conditions in the statement of Theorem 3.6.

Part (i) follows from the definition of feasible pairs and the algorithm.

Part (ii) follows from the algorithm immediately.

To see Part (iii), we note that each tree of $\mathcal{G}_I$ has exactly one class from R, and $|R| = d + 1$ because $|T_0| = |T| = r - d - 1$. It is possible that a class $V_i \in R$ contains no vertex of I (if all the vertices of V_i are dominated by I at some stage of the algorithm).

For Part (iv), since F_I is a forest on t vertices with $d+1$ components, we have $|E(F)| = t-d-1$ and $|I| \leq 2|E(F)| = 2(t - d - 1)$.

We now show that if $n > 2\Delta\left(1 - \frac{2d+3}{2r}\right)$, then $G[I]$ is a perfect matching, or equivalently $|I| = 2(t - d - 1)$. Suppose that $|I| \leq 2(t - d - 1) - 1$. Since I dominates $S(I) \cup R$, we have $|I|\Delta \geq |S(I) \cup R|n$, which gives $\Delta(2(t - d - 1) - 1) > 2t\Delta\left(1 - \frac{2d+3}{2r}\right)$. This implies $\Delta(2d + 3) < \Delta\frac{t}{r}(2d + 3)$, contradicting the fact that $r \geq t$. □

Since we will make use of it later in this section, we explicitly state the $d = 0$ case of Theorem 3.6, which is precisely the content of [14, Theorem 2.2]

Lemma 3.7 ([14, Theorem 2.2])**.** *Let G be an r-partite graph on $V_1 \cup \ldots \cup V_r$. Suppose that G has no IT. Then, there exists a set of class indices $S \subseteq [r]$ and a nonempty set of edges Z of the subgraph $G_S := G[\bigcup_{i \in S} V_i]$ such that $V(Z)$ dominates G_S and $|Z| \leq |S| - 1$. Moreover, $Z \cap V_s \neq \emptyset$ for every $s \in S$.*

When Algorithm 3.4 terminates, the set $T \setminus I$ must be a partial IT that is independent of I because if $w \in T \setminus I$ is adjacent to some $v \in I$, then w would have been included in I when v was added to I. The following proposition strengthens this fact by showing that such a partial IT exists outside $N_G(x)$ for any fixed $x \in V(G)$.

Lemma 3.8. *Let G be an r-partite graph with parts $V_1, \dots, V_r$ of size $n > \Delta\left(2 - \frac{2d+3}{r}\right)$. Let $I \subset V(G)$, $d + 1 \leq t \leq r$, and $U \subseteq \{V_1, \dots, V_r\}$ be such that $|U| = r - t$, $|I| \leq 2(t - d - 1)$, and*

I dominates all the classes not in U. For any vertex $x \in V(G)$, there exists an IT T' of $G[U]$ such that $T' \cap N_G(I \cup \{x\}) = \emptyset$.

Proof. Let $V_i' := V_i \setminus N_G(I \cup \{x\})$ for all i and $U' = \{V_i' : V_i \in U\}$. Note that it is possible for some $V_i' = \emptyset$. If there is an IT of $G[U']$, then we are done. Suppose there is no IT of $G[U']$. If all V_i' are nonempty, then by Lemma 3.7, there exists a subset $U'' \subseteq U'$ such that $G[U'']$ is dominated by a set $X \subseteq \bigcup_{V_i' \in U''} V_i'$ of most $2(|U''| - 1)$ vertices. If some $V_{i_0}' = \emptyset$, then let $U'' = \{V_{i_0}\}$ and $X = \emptyset$. In either case, the set $I \cup N_G(x) \cup X$ dominates $\bigcup\{V_i : V_i \notin U$ or $V_i' \in U''\}$. Since $|I| \leq 2(t - d - 1)$, we have

$$|I \cup \{x\} \cup X| \leq 2(t - d - 1) + 2(|U''| - 1) + 1 = 2(t - d + |U''| - 2) + 1.$$

Since there are $(t + |U''|)n$ vertices in $\bigcup\{V_i : V_i \notin U$ or $V_i' \in U''\}$, we derive that

$$(2(t - d + |U''| - 2) + 1)\Delta \geq n(t + |U''|) > \Delta\left(2 - \frac{2d+3}{r}\right)(t + |U''|),$$

which implies $2(-d - 2) + 1 > -\frac{2d+3}{r}(t + |U''|)$. This is a contradiction because $r = t + |U'| \geq t + |U''|$. □

Suppose $I \subseteq V(G)$ is a set that dominates all the classes of $R \cup S(I)$. For every vertex $v \in I$, we let A_v or $A_v(I)$ be the set of all vertices in the classes of $R \cup S(I)$ that are adjacent to v but no other vertex in I. For example, if $v, w \in I$ are adjacent, then $w \in A_v$ and $v \in A_w$. By definition, if $x \in \bigcup\{V_i : V_i \in R \cup S(I)\} \setminus \bigcup_{v \in I} A_v$, then x is adjacent to at least two vertices of I. In this next lemma, we bound the number of vertices in G adjacent to at least two vertices of I and the number of vertices in a given number of A_v's using similar arguments as the ones for [14, Lemma 3.1 (iii) and (iv)].

Lemma 3.9. *Let G be an r-partite graph with parts $V_1, \dots, V_r$ of size n. Suppose $I \subseteq V(G)$ is a set of size at most $2(t - d - 1)$ that dominates all the classes of $R \cup S(I)$ with $t = |R \cup S(I)|$. Then*

(i) *The number of vertices $x \in V(G)$ such that $|N(x) \cap I| \geq 2$ is at most $2\Delta(t - d - 1) - tn$,*
(ii) *For any subset $Y \subseteq I$, we have $|\bigcup_{v \in Y} A_v| \geq (|Y| + 4d + 4 - 4t)\Delta + 2tn$,*
(iii) *If $n > 2\Delta\left(1 - \frac{4d+5}{4r}\right)$, then for any $Y \subseteq I$, we have $|\cup_{v \in Y} A_v| > (|Y| - 1)\Delta$.*

Proof. Let $V' = \bigcup\{V_i : V_i \in S(I) \cup R\}$.

Part (i): Let D be the set of vertices $x \in V(G)$ such that $|N(x) \cap I| \geq 2$. Then $V' \setminus D = \bigcup_{v \in I} A_v$ consists of the vertices of V' that are dominated by I exactly once. Hence

$$2\Delta(t - d - 1) \geq \Delta|I| \geq 2|D| + (tn - |D|) \geq tn + |D|,$$

which implies that $|D| \leq 2\Delta(t - d - 1) - tn$.

Part (ii): We know $\bigcup_{v \in Y} A_v = V' \setminus (D \cup \bigcup_{v \in I \setminus Y} A_v)$. Applying Part (i), we obtain that

$$\begin{aligned}|\bigcup_{v \in Y} A_v| &\geq tn - (2\Delta(t - d - 1) - tn) - (2(t - d - 1) - |Y|)\Delta \\ &= (|Y| + 4d + 4 - 4t)\Delta + 2tn.\end{aligned}$$

Part (iii): By Part (ii), we have

$$\begin{aligned}|\bigcup_{v \in Y} A_v| &> (|Y| + 4d + 4 - 4t)\Delta + 4t\Delta\left(1 - \frac{4d+5}{4r}\right) \\ &= \left(|Y| + 4d + 4 - \frac{t(4d+5)}{r}\right)\Delta \geq (|Y| - 1)\Delta\end{aligned}$$

as $t \leq r$. □

For convenience, we introduce the following setup for the next few lemmas. Part 3.10 (i) includes all the properties of an IMC returned by Theorem 3.6 while Part (ii) has a stronger lower bound for n required by Lemma 3.9 (iii) (iii).

Setup 3.10. *We have the following two setups:*

(i) *Let G be an r-partite graph on $V_1 \cup \cdots \cup V_r$. Let T be a maximum IT of G of size $r-d-1$. Assume that $\Delta(G) \leq \Delta$ and $|V_i| \geq n > 2\Delta\left(1 - \frac{2d+3}{2r}\right)$ for all i. Let I be an IMC of G returned by Theorem 3.6, which has the following properties:*
   - *I dominates all the vertices in $R \cup S(I)$, where $R = \{V_1, \dots, V_r\} \setminus S(T)$;*
   - *$F_I = (R \cup S(I), E(\mathcal{G}_I))$ is a forest of $d+1$ components, where each component C is a subset of $R \cup S(I)$ such that $F_I[C]$ is a tree.*
   - *$|R \cup S(I)| = t$ and $|I| = 2(t-d-1)$ for some $d+1 \leq t \leq r$.*

(ii) *Let G be as in Setup 3.10 (i) with the additional assumption $n > 2\Delta\left(1 - \frac{4d+5}{4r}\right)$.*

Suppose G is as in Setup 3.10 (i). If a vertex $x \in V_i$ and $V_i \in C$ for some component C of F_I, then we may simply say x is in C. In other words, we partition the vertices G in $R \cup S(I)$ based on the components of F_I. In particular, we say that *I is an IMC of $d+1$ components.* For example, any two adjacent vertices $v, w \in I$ belong to the same component.

In next few lemmas we may replace the IMC I in Setup 3.10 by another IMC I' of G. To make sure that I' satisfies the same conditions in Setup 3.10, we say that I' is *similar* to I if $S(I') = S(I)$, I' also dominates $R \cup S(I)$, and $F_{I'}$ has the same components as F_I. It follows that $|I'| = |I| = 2(t-d-1)$, and, in particular, that I' satisfies the assumptions of Lemma 3.8 and Lemma 3.9 (iii).

In the following lemma, Parts (ii) and (iii) resemble [14, Lemma 3.1 (i) and (ii)], respectively. Their proofs are similar to the ones in [14] but require more work because our IMC has $d+1$ components and only dominates $t \leq r$ classes (in contrast, IMCs in [14] have only one component and dominate the vertices in all r classes of G). Furthermore, in Lemma 3.11 (iii), we need to show that the new IMC is similar to I.

Lemma 3.11. *Let G be as in Setup 3.10 (i). Suppose $v, w \in I$ are adjacent and contained in the component C of F_I. Then the following holds.*

(i) *All vertices in $A_v \cup A_w$ are in C.*
(ii) *$G[A_v, A_w]$ is a complete bipartite graph.*
(iii) *For any $a \in A_v$ and $b \in A_w$, $(I \setminus \{v, w\}) \cup \{a, b\}$ is an IMC in G similar to I.*

Proof. For a vertex $x \in V(G)$, let $V(x)$ denote the class $V_i \ni x$.

(i) Suppose some vertex $a \in A_v$ is contained in C' for some component $C' \neq C$ of F_I. By Lemma 3.2, in every component of F_I, we can find a subset of I forming a partial IT covering all but one arbitrary class. We pick a partial IT in C omitting the class of v, a partial IT in C' omitting the class of a, and partial IT's omitting an arbitrary class in the other components. Let T_1 be the union of these partial IT's. By Lemma 3.8, there is an IT T_2 of the classes not dominated by I such that $T_2 \cap N_G(I \cup \{a\} = \emptyset$. Then $T_1 \cup T_2 \cup \{a\}$ is an $(r-d)$-IT of G because $N(a) \cap (I \setminus \{v\}) = \emptyset$.

(ii) Consider $a \in A_v$ and $b \in A_w$. Part (i) shows that a, b are both in C. Let $F_I - vw$ be the forest obtained by removing the edge vw from F_I, and let C_v, C_w be two components of $F_I - vw$ that contain v, w, respectively. We claim that

$$V(a) \in C_w, \quad \text{and} \quad V(b) \in C_v. \tag{7}$$

Indeed, suppose that $V(a) \notin C_w$. Then $V(a) \in C_v$. Let $I_v = (I \setminus \{v\}) \cap \bigcup_{V_i \in C_v} V_i$. Then I_v is an IMC of $G[\bigcup_{V_i \in C_v} V_i]$. By Lemma 3.2, we can find a set $T_1 \subset I \cap C_v$ as a partial IT avoiding the class of a. Similarly we find a partial IT T_2 in C_w avoiding the class of w. The union $T_1 \cup T_2 \cup \{a, w\}$ is a full *IT* of C because $N(a) \cap (I \setminus \{v\}) = \emptyset$. We then extend it to an $(r-d)$-IT of G by Lemma 3.8, which contradicts our assumption. The same arguments show that $V(b) \in C_v$.

Now suppose that $ab \notin E(G)$. Then, we can obtain partial ITs $T_1 \subseteq I_w$ and $T_2 \subseteq I_v$ such that T_1 covers all classes of C_w except for the class of a and T_2 covers all classes of C_v except for the class of b. Since $ab \notin E(G)$, $N(a) \cap (I \setminus \{v\}) = \emptyset$, and $N(b) \cap (I \setminus \{w\}) = \emptyset$, the union $T_1 \cup T_2 \cup \{a, b\}$ is a full IT of C, which gives an IT of size $r - d$ in G after being extended to U by Lemma 3.8 , a contradiction. This shows that $G[A_v, A_w]$ is a complete bipartite graph.

(iii) We first show that $I' := (I \cup \{a\}) \setminus \{w\}$ is an IMC of G similar to I. Indeed, since I is an IMC and $N(a) \cap I' = \{v\}$, I' is an induced matching in G. Recall that $V(a) \in C_w$ and $V(v) \in C_v$ by (7), where C_v, C_w are the two components of $F_I - vw$ that contain v, w, respectively. Since the edge av connects C_v and C_w, the graph $\mathcal{G}_{I'}$ is a forest with the same components as $\mathcal{G}_I$. We claim that I' dominates all the classes of $R \cup S(I)$. Indeed, let $x \in \bigcup_{V_i \in R \cup S(I)} V_i$. If x has a neighbor in $I \setminus \{w\}$, then x is dominated by I'. Otherwise x is only adjacent to w in I, giving $x \in A_w$ and thus $ax \in E(G)$ by Part (ii). Hence, x is dominated by I' as desired.

We now show that $S(I') = S(I)$. To see $S(I') \subseteq S(I)$, it suffices to show that $V(a) \in S(I)$. By (i), a is in C so $V(a) \cap I \neq \emptyset$. To see $S(I) \subseteq S(I')$, it suffices to have $V(w) \in S(I')$. If this is not the case, then $V(w) \neq V(a)$ and $V(w)$ is a leaf in F_I. Then $V(a) \notin C_w = \{V(w)\}$, contradicting (7).

For $u \in I'$, recall $A_u(I') = \{x \in \bigcup_{V_i \in R \cup S(I)} V_i : N(x) \cap I' = \{u\}\}$. We have $b \in A_a(I')$ because $N(b) \cap I = \{w\}$ and $ab \in E(G)$. The arguments in the previous paragraph show that $I' \cup \{b\} \setminus \{v\} = (I \setminus \{v, w\}) \cup \{a, b\}$ is an IMC in G similar to I. This completes the proof of (iii). □

Suppose that I is an IMC on $2(t - d + 1)$ vertices. Let H be the $2(t - d - 1)$-partite graph on $\{A_v : v \in I\}$ obtained from G by removing edges in $G[A_v]$ for all $v \in I$ and edges between A_v and A_w for all $vw \in E(G[I])$. *An IT of H* is an independent set of H with one vertex from each A_v (not from original classes V_i of G), for example, I is an IT of H.

The following lemma and proof are similar to [14, Lemma 3.3] and its proof.

Lemma 3.12. *Let G be as in Setup 3.10 (i). Every IT T of H is an IMC in G similar to I.*

Proof. Suppose that $I = \{v_i, w_i : 1 \leq i \leq t - d - 1\}$ with $v_i w_i \in E(G)$ and $T = \{a_i, b_i : 1 \leq i \leq t - d - 1\}$ with $a_i \in A_{v_i}$ and $b_i \in A_{w_i}$. For $1 \leq j \leq t - d - 1$, we claim that

$$I_j := \left(I \setminus \bigcup_{i \leq j} \{v_i, w_i\} \right) \cup \bigcup_{i \leq j} \{a_i, b_i\}$$

is an IMC in G similar to I. In particular, $T = I_{t-d-1}$ is similar to I, as desired.

We prove the claim by the induction on j. The $j = 1$ case is Lemma 3.11 (iii). Suppose $j \geq 1$ and I_j is similar to I. We note that $I_{j+1} = (I_j \setminus \{v_{j+1}, w_{j+1}\}) \cup \{a_{j+1}, b_{j+1}\}$. For simplicity, we write v, w, a, b by omitting subscripts $j + 1$. We know $a \in A_v(I)$ and thus $N(a) \cap I = \{v\}$. Since T is an independent set, it follows that $N(a) \cap I_j = \{v\}$. Thus $a \in A_v(I_j)$. Similarly $b \in A_w(I_j)$. We can thus apply Lemma 3.11 (iii) and conclude that I_{j+1} is an IMC in G similar to I. □

Lemma 3.13. *Suppose that G is as in Setup 3.10 (ii) and let $Y \subseteq I$. If $A'_v \subseteq A_v$ for $v \in Y$ are subsets such that $A'_v \neq \emptyset$ for all $v \in Y$ and $\sum_{v \in Y} |A_v \setminus A'_v| \leq \Delta$, there exists an IT of $\{A'_v : v \in Y\}$ in H.*

Proof. Suppose that $\{A'_v : v \in Y\}$ contains no IT in H. Then, by Lemma 3.7, there exists a set $S \subseteq Y$ and an nonempty set Z of edges on $V_S := \{A'_v : v \in S\}$ such that (1) $X := V(Z)$ dominates V_S in H, (2) $|X| \leq 2(|S| - 1)$, and (3) $X \cap A'_v \neq \emptyset$ for every $v \in S$.

Since X dominates V_S in H and $\sum_{v \in Y} |A_v \setminus A'_v| \leq \Delta$, we have

$$\Delta |X| - \sum_{x \in X} \deg_{G \setminus H}(x) \geq \sum_{x \in X} \deg_H(x) \geq \sum_{v \in S} |A'_v| \geq \sum_{v \in S} |A_v| - \Delta. \tag{8}$$

For every $x \in X$, by the definition of H and Lemma 3.11, we have $\deg_{G\setminus H}(x) \geq |A_{w_x}|$, where w_x is the neighbor of $v \in I$ in $G[I]$ such that $x \in A_v$. Since $X \cap A'_v \neq \emptyset$ for every $v \in S$, there exists $X' \subseteq X$ such that $|X' \cap A'_v| = 1$ for every $v \in S$. Since each $v \in S$ is adjacent to a unique $w \in I$, we have

$$\sum_{x \in X} \deg_{G\setminus H}(x) \geq \sum_{x \in X} |A_{w_x}| \geq \sum_{x \in X'} |A_{w_x}| > (|S|-1)\Delta$$

by Lemma 3.9 (iii). Together with (8) and the assumption $|X| \leq 2(|S|-1)$, we obtain that

$$\Delta(2|S|-1) \geq \Delta(|X|+1)| \ \geq \sum_{x \in X} |A_{w_x}| + \sum_{v \in S} |A_v| > 2(|S|-1)\Delta \tag{9}$$

by applying Lemma 3.9 (iii) again.

If $|X| < 2(|S|-1)$ then $\Delta(|X|+1) \leq (2|S|-2)\Delta$, giving a contradiction. We thus assume that $|X| = 2(|S|-1)$. Let T be the tree obtained from (V_S, Z) by contracting each A'_v into a vertex. We proceed by cases on $q := |\{v \in S \colon |A_v \cap X| \geq 2\}$.

Suppose $q = 0$. Then, $|Z| = 1$ and $|S| = |X| = 2$. Suppose $S = \{v_0, v_1\}$. We must have $v_0, v_1 \notin E(G[I])$ as otherwise there would be no edges connecting A_{v_0} and A_{v_1} in H. Thus v_0, w_0, v_1, w_1 are all distinct and we may calculate

$$\sum_{v \in S} |A_v| + \sum_{x \in X} |A_{w_x}| = |A_{v_0}| + |A_{w_0}| + |A_{v_1}| + |A_{w_1}| > 3\Delta,$$

where the last inequality follows from Lemma 3.9 (iii).

Now consider the case $q = 1$. Then T is a star with at least two edges. Moreover, let $s = |S| \geq 3$. Let v_1 be the distinct vertex in S with $|A_v \cap X| \geq 2$. Suppose $x_1, x_2, \ldots, x_{s_1} \in X$. Then $Z = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{s_1}, y_{s-1}\}\}$, where $y_1, \ldots, y_{s_1}$ are in distinct members of $\{A_{v_2}, A_{v_3}, \ldots, A_{v_s}\}$. As before, v_1 is not adjacent to any other v_i in I as this would imply that there are no edges between A_{v_1} and A_{v_i} in H. So,

$$\begin{aligned}
\sum_{v \in S} |A_v| + \sum_{x \in X} |A_{w_x}| &= \sum_{v \in S} |A_v| + \left(\sum_{w \in V(I) \colon N(w) \cap S \neq \emptyset} |A_w| \right) + |A_{w_1}|(s-2) \\
&> \sum_{v \in S \cup \{w_1\}} |A_v| + \sum_{w \in V(I) \colon N(w) \cap S \neq \emptyset} |A_w| \\
&> s\Delta + (s-1)\Delta,
\end{aligned}$$

where again the last inequality follows from Lemma 3.9 (iii).

Finally, let $q \geq 2$. Let v_1, v_2 be distinct vertices in S such that $|A_{v_i} \cap X| \geq 2$ for $i = 1, 2$. Then,

$$\begin{aligned}
\sum_{v \in S} |A_v| + \sum_{x \in X} |A_{w_x}| &\geq \sum_{v \in S} |A_v| + \left(\sum_{w \in V(I) \colon N(w) \cap S \neq \emptyset} |A_w| \right) + |A_{w_1}| + |A_{w_2}| \\
&> (s-1)\Delta + (s-1)\Delta + \Delta.
\end{aligned}$$

□

In each case we derived that $\sum_{v \in S} |A_v| + \sum_{x \in X} |A_{w_x}| > (2|S|-1)\Delta$, which contradicts (9).

We now derive a key property on the structure of G.

Lemma 3.14. *Let G be as in Setup 3.10 (ii). Then, every vertex $x \in \bigcup_{V_i \in R \cup S(I)} V_i$ is completely joined to some A_v where $v \in I$ is in the same component as x.*

Proof. Fix $x \in \bigcup_{V_i \in R \cup S(I)} V_i$. If $x \in A_w$ for some $w \in I$ such that $vw \in E(G[I])$, then by Lemma 3.11, x is completely joined to A_v and all the vertices in A_v are the same component as x.

We thus assume that $x \notin \bigcup_{v \in I} A_v$. We first show that x is completely joined to some A_v. Suppose, to the contrary, that $A_v \setminus N(x) \neq \emptyset$ for all $v \in I$. Applying Lemma 3.13 with $Y = I$

and $A'_v = A_v \setminus N(x)$, we obtain an IT $I' \subset \bigcup_{v\in I} A'_v$ of H that is independent of x. Now, Lemma 3.12 implies that I' is an IMC in G that dominates all the vertices of $R \cup S(I)$, including x. This contradicts the fact that $I' \cap N(x) = \emptyset$.

Suppose x is completely joined to A_{v_1} for some $v_1 \in I$ but x and A_{v_1} are in different components. If x is completely joined to $A_{v'}$ for some $v' \in I \setminus \{v_1\}$, then, by Lemma 3.9 (iii), $\deg_G(x) \geq |A_{v_1}| + |A_{v'}| > \Delta$, a contradiction. Thus, $A_v \setminus N(x) \neq \emptyset$ for all $v \in I \setminus \{v_1\}$. Applying Lemma 3.13 with $A'_{v_1} = A_{v_1}$ and $A'_v = A_v - N(x)$ for $v \in I \setminus v_1$ (thus $A'_v \neq \emptyset$), we obtain an IT $I' \subset \bigcup_{v\in I} A'_v$ of H. By Lemma 3.12, I' is an IMC in G that dominates all the vertices in $R \cup S(I)$ and has the same components as I. Suppose $\{u\} = I' \cap A'_{v_1}$. Then $x \in A_u(I')$ because $N(x) \cap I' = \{u\}$. However, x and u are in different components, contradicting Lemma 3.11 (i). □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose G has no $(r-d)$-IT. Add edges if necessary so that the largest partial IT of G has size $r-d-1$. Let I be the IMC given by Theorem 3.6. Let J be the smallest component of F_I with $j \leq q$ classes. By Lemma 3.14, all the vertices in $V_J := \bigcup_{V_i \in J} V_i$ are dominated by some vertex in $I \cap V_J$. Since $F_I[J]$ is a tree on j vertices and I is an IMC, it follows that $|I \cap V_J| = 2(j-1)$. However, since

$$jn - 2\Delta(j-1) > 2j\Delta\left(1 - \frac{1}{q}\right) - 2\Delta(j-1) = 2\Delta\left(1 - \frac{j}{q}\right) \geq 0,$$

$2(j-1)$ vertices cannot dominate jn vertices in G, giving the desired contradiction. □

4. Proofs of Theorems 1.3 and 1.4

Let G be an r-partite graph, where $r = q(d+1)+k$, $0 \leq k \leq d$, and q is odd. We have a new setup due to the new lower bounds for n in Theorems 1.3 and 1.4.

Setup 4.1. *We have the following two assumptions:*

(i) *Let G be as in Setup 3.10 (ii) with the additional assumption that* $n > 2\Delta\left(1 - \frac{1}{q-1}\right)$.

(ii) *Let G be as in Setup 4.1 (i) with the additional assumption* $n > \Delta\left(2 - \frac{6d+7}{3r}\right)$.

Since $2 - \frac{6d+7}{3r} > 2 - \frac{4d+5}{2r}$, Setup 4.1 (ii) has the largest bound for n. Furthermore, when G is as in Setup 4.1, we can use all the lemmas in Section 3.

We first show that if G is as in Setup 4.1 (i), then every component of F_I has at least q vertices.

Lemma 4.2. *Let G be as in Setup 4.1 (i). Then the forest F_I has at least $q(d+1)$ vertices and exactly $d+1$ components, each of which has at least q vertices.*

Proof. Since F_I has $d+1$ components, it suffices to show that every component J of F_I has at least q vertices. Suppose that J has b vertices. Then $|I \cap J| = 2(b-1)$ because $\mathcal{G}_I[J]$ is a tree. By Lemma 3.14, every vertex of G in J is dominated by a vertex in $I \cap J$. Thus,

$$2\Delta(b-1) \geq bn > b2\Delta\left(1 - \frac{1}{q-1}\right),$$

which implies that $b > q-1$. Since b is an integer, it follows that $b \geq q$. □

We can now prove Theorem 1.3. The main idea is to find a component of F_I with q vertices and apply (2) to find a full IT in this component by using the vertices of $\bigcup_{v\in I} A_v$.

Proof of Theorem 1.3. Suppose to the contrary, that G has no $(r-d)$-IT. Since $n > \Delta(2 - \frac{4d+5}{2r})$, by Theorem 3.6, G has an IMC I of $d+1$ components that dominates $S(I) \cup R$. Since $n > 2\Delta(1 - \frac{q}{q^2-1}) > 2\Delta(1 - \frac{1}{q-1})$, Lemma 4.2 says that F_I has at least $q(d+1)$ vertices and every component of F_I has at least q classes. Since $r = q(d+1)$, F_I has exactly $r = q(d+1)$ vertices, and every component of F_I has q classes.

Let D denote the set of vertices in $V(G)$ that are not in any A_v. By Lemma 3.9 (i), $|D| \leq 2\Delta(r-d-1) - rn$. We know that some component J of I contains at most $|D|/(d+1)$ vertices of D. Let $G'_J = G[V_J \setminus D]$ be the induced subgraph of G formed by removing the vertices of D from the vertex classes of J. Then each class of G_J has size at least n' with

$$\begin{aligned} n' &\geq n - \frac{|D|}{d+1} \geq n - \frac{2\Delta(r-d-1) - rn}{d+1} \\ &= n - \frac{2\Delta(q-1)(d+1) - q(d+1)n}{d+1} = n(q+1) - 2\Delta(q-1) \\ &> 2\Delta(q+1)\left(1 - \frac{q}{q^2-1}\right) - 2\Delta(q-1) = 2\Delta\left(1 - \frac{1}{q-1}\right). \end{aligned}$$

By (2), since q is odd, the graph G_J has a full IT T_J consisting only of vertices of $V_J \setminus D$. We know $V_J \setminus D = \bigcup_{v \in J} A_v$ from Lemma 3.11. Hence, T_J is independent of $I \setminus J$. Since for each component $J' \neq J$, we can find in J' an IT of all except one class of J', these together with T_J form an $(r-d)$-IT of G. This contradiction shows that G indeed has an $(r-d)$-IT, and completes the proof. □

In order to prove Theorem 1.4, we follow the approach of [14] and obtain the structure of G. To this end, we prove analogues of [14, Lemma 3.5, Theorem 3.6, Theorem 3.7] under Setup 4.1 (ii), in which we assume $n > \Delta\left(2 - \frac{6d+7}{3r}\right)$.

Lemma 4.3. [14, Lemma 3.5] *Let G be as in Setup 4.1 (ii) and $V' = \bigcup\{V_i : V_i \in S(I) \cup R\}$. Suppose $a, b \in V'$ are completely connected to A_w and A_v respectively, where $vw \in E(G[I])$. Then $ab \in E(G)$.*

Proof. Suppose that a is not adjacent to b. Then, for $u \in I - \{v, w\}$, let $A'_u = A_u \setminus N(a, b)$. If there exists $u \in I - \{v, w\}$ such that A'_u is empty, then $\{a, b\}$ dominates A_v, A_w, and A_u, which contains more than 2Δ vertices by Lemma 3.9 (iii), contradicting $\Delta(G) \leq \Delta$.

Now, since $\{a, b\}$ dominates $A_v \cup A_w$, which has size more than Δ by Lemma 3.9 (iii), we can see that $\sum_{u \in (I - \{v,w\})} |A_u \setminus A'_u| \leq \Delta$ and thus an IT I'_0 of $\{A'_u \colon u \in I - \{v, w\}\}$ exists by Lemma 3.13. Then $I' := I'_0 \cup \{v, w\}$ is an IT of $\{A'_u \colon u \in I\}$, which is an IMC of G similar to I by Lemma 3.12.

By definition, we have $N(a, b) \cap I' = \{v, w\}$. Also, neither a nor b can individually dominate both $A_v(I')$ and $A_w(I')$ as they have size more than Δ combined by Lemma 3.9 (iii). We now claim that there exist $v' \in A_v(I')$ and $w' \in A_w(I')$ such that v' is not adjacent to a and w' is not adjacent to b. Then, $I'' := I' - \{v, w\} \cup \{v', w'\}$ is an IMC similar to I by Lemma 3.12 such that $a \in A_{w'}(I'')$ and $b \in A_{v'}(I'')$ and therefore $ab \in E(G)$ by Lemma 3.11 (ii).

Suppose that this is not possible. Assume a dominates $A_v(I')$. By definitions, $A_v(I') \cap A_w(I) = \emptyset$, and the vertices in $A_v(I) \setminus A_v(I')$ are dominated twice in I'. Hence, we have

$$\begin{aligned} \Delta &\geq \deg(a) \geq |A_w| + |A_v(I')| \geq |A_w| + |A_v| - |A_v \setminus A_v(I')| \\ &\geq 2tn - \Delta(4t - 4d - 6) - |A_v \setminus A_v(I')| && \text{by Lemma 3.9 (ii)} \\ &\geq 3tn - \Delta(6t - 6d - 8) && \text{by Lemma 3.9 (i).} \end{aligned}$$

This implies $n \leq 2\Delta - \frac{6d+7}{3t}\Delta \leq 2\Delta - \frac{6d+7}{3r}\Delta$, a contradiction. □

Lemma 4.4. [14, Lemma 3.6] *Let G be a graph with a vertex partition $V_1 \cup \cdots \cup V_r$ such that $|V_i| > 2\Delta(1 - \frac{2d+3}{2r})$. Suppose G has no $(r-d)$-IT but $G - e$ has an $(r-d)$-IT for some edge $e \in E(G)$. Then e lies in an IMC of G returned by Theorem 3.6.*

Proof. Let $e = \{v_1, v_2\}$. Then, by assumption there exists a transversal $T' = \{v_1, v_2, \ldots, v_{r-d}\}$ such that $T' - \{v_j\}$ is independent for $j = 1, 2$. Let $I_0 = \{v_1, v_2\}$ and $T_0 = T' - \{v_1\}$. It is easy to check that (I_0, T_0) is a feasible pair in G. Indeed, Conditions (1)–(4) of Definition 3.3 are trivially satisfied, and Condition (5) follows from the observation that $W = \{v_1\}$ so any T contradicting Condition (5) is an $(r-d)$-IT in G. Then, Theorem 3.6 with the same (I_0, T_0) gives an IMC I in G which contains e by Theorem 3.6 (i). □

Lemma 4.5. [14, Lemma 3.7] *Let G be as in Setup 4.1 (ii). Moreover, assume that $G - e$ has an $(r-d)$-IT for every $e \in G$. Let $V' = \bigcup\{V_i : V_i \in S(I) \cup R\}$. Then $G[V']$ is a union of $t-d-1$ vertex-dsijoint complete bipartite graphs.*

Proof. We label $I = \{v_i, w_i : 1 \le i \le t-d-1\}$ with $v_i w_i \in E(G)$. By Lemma 3.14, every vertex $x \in V'$ is completely joined to some A_v. By Lemma 3.9 (iii), such A_v is unique (otherwise $\deg(x) > \Delta$). By Lemma 4.3, we can write $G[V']$ as a disjoint union $A_1 \cup \cdots A_{t-d-1} \cup B_1 \cup \cdots \cup B_{t-d-1}$ such that $G[A_i, B_i]$ is complete, $A_{v_i} \subseteq A_i$, and $A_{w_i} \subseteq B_i$ for all i. Thus, it remains to show that there are no edges outside these bipartite subgraphs.

Suppose $e = xy \notin \bigcup_{1 \le i \le t-d-1} E(G[A_i, B_i])$. By Lemma 4.4, there is an IMC I' in G returned by Theorem 3.6 containing e. Without loss of generality, let us write $x \in A_1$. Then by assumption, $y \notin B_1$. Suppose that $y \in A_1$. Then B_1 is dominated by $\{x, y\} \subseteq I'$. Thus $|B_1| \le 2\Delta(t-d-1) - tn$ by applying Lemma 3.9 (i) with I'. On the other hand, by applying Lemma 3.9 (ii) with I, we get $|B_1| \ge 2tn - \Delta(4t - 4d - 5)$. Together, these two inequalities give $n \le 2\Delta - \frac{6d+7}{3t}\Delta \le 2\Delta - \Delta\frac{6d+7}{3r}$, a contradiction.

Thus we may assume $y \notin A_1 \cup B_1$. Without loss of generality, assume $y \in A_2$. Considering the IMC I', we have that $x \in A_y(I')$ and $y \in A_x(I')$. Suppose that $A_y(I') \cap B_2$ or $A_x(I') \cap B_1$ is empty, for example, $A_y(I') \cap B_2 = \emptyset$. Since B_2 is dominated by y, it follows that B_2 is dominated at least twice in I' and the argument in the previous paragraph gives the same contradiction.

We are left with the case that both $A_y(I') \cap B_2$ and $A_x(I') \cap B_1$ are nonempty. Let $u \in A_y(I') \cap B_2$ and $w \in A_x(I') \cap B_1$. By Lemma 3.11, we know $A_x(I') \subseteq N(u)$ and $A_y(I') \subseteq N(w)$. Consequently,

$$\begin{aligned}
2\Delta &\ge \deg(w) + \deg(u) \ge (|A_1| + |A_y(I') \cap B_2|) + (|A_2| + |A_x(I') \cap B_1|) \\
&\ge (|A_1| + |A_y(I')| + |B_2| - |A_y(I') \cup B_2|) + (|A_2| + |A_x(I')| + |B_1| - |A_x(I') \cup B_1|) \\
&\ge (|A_1| + |A_2| + |B_1| + |B_2|) + (|A_x(I')| + |A_y(I')|) - (|A_y(I') \cup B_2| + |A_x(I') \cup B_1|) \\
&> 3\Delta + \Delta - (|A_y(I') \cup B_2| + |A_x(I') \cup B_1|) \qquad \text{(by Lemma 3.9 (iii)).}
\end{aligned}$$

Since $A_x(I') \cup B_1$ and $A_y(I') \cup B_2$ are dominated by x and y respectively, each has size at most Δ. Thus, $2\Delta > 4\Delta - \Delta - \Delta = 2\Delta$, a contradiction. □

We are ready to prove Theorem 1.4. The main idea is, due to Lemmas 3.11 and 4.5, the components of F_I give rise to a partition of $G[V']$ into $d+1$ components that are independent of each other. Once we apply (2) and find a full IT in one of the components, we immediately obtain an $(r-d)$-IT in G.

Proof of Theorem 1.4. Suppose that G has no $(r-d)$-IT. After removing edges from G if necessary, we further assume that $G - e$ has an $(r-d)$-IT for every $e \in E(G)$.

Let I be an IMC of $t-d+1$ edges given by Theorem 3.6. Let $V' = \bigcup\{V_i : V_i \in S(I) \cup R\}$. By Lemma 4.5, $G[V']$ is the union of $t-d-1$ vertex-disjoint complete bipartite graphs on $A_i \cup B_i$ for $i \le t-d-1$. Since $G[I]$ is an induced matching of size $t-d+1$, each $A_i \cup B_i$ contains exactly one edge of I. Since there is no edge of G between $A_i \cup B_i$ and $A_j \cup B_j$ for $i \ne j$, we have $A_v = A_i$ and $A_w = B_i$ if $v \in I \cap B_i$ and $w \in I \cap A_i$.

By Lemma 3.11 (i), all the vertices of $A_i \cup B_i$ lie in the same component of the forest F_I. Therefore, each component of F_I is independent of other components because there are no edges between $A_i \cup B_i$ and $A_j \cup B_j$ for $i \ne j$.

By Lemma 4.2, every component of F_I has at least q classes of G. Since $q = \lfloor r/(d+1) \rfloor$, there exists a component J of F_I with exactly q vertices. As q is odd and $n > 2\Delta\left(1 - \frac{1}{q-1}\right)$, by (2), there is a full IT in J. We can find by Lemma 3.2 an IT in every other component of F_I missing a vertex from at one class in that component. Since each component is independent of the other components, combining these ITs gives an IT T_0 of size $t-d$ in G contained in $S(I) \cup R$. By Lemma 3.8, there exists an IT in G of size $(t-d)+(r-t)=r-d$ containing T_0, contradicting G having no $(r-d)$-IT. □

5. Concluding Remarks

Let $r, d \geq 0$ and $q, k \geq 0$ be integers such that $r = q(d+1)+k$, where $k \leq d$. In this paper we have completely determined $n(r, d+1, \Delta)$ when $q \geq 4k$ is even and when $q \geq 6d+6k+7$ is odd. We have also shown that $n(6,2,\Delta) = \lfloor 5\Delta/4 \rfloor$, answering a specific question of [16]. It is interesting to know the value of $n(r, d+1, \Delta)$ in the remaining cases:

(1) q is even and $q < 4k$, such as $r = 7$ and $d = 2$,
(2) q is odd and $q < 6d+6k+7$, such as $r = 7$ or 10 and $d = 1$.

The aforementioned results of [16] determined $n(r, d+1, \Delta)$ in many cases when $d > r/2 - 1$ (equivalently, $q = 1$) but there are still unknown ones such as $r = 7$ and $d = 3$.

One may also ask in an r-partite graph whether, given an $(r-d)$-IT T, there exists an $(r-d')$-IT $T' \supset T$ (provided $d' < d < r$). A similar proof to that of [13, Corollary 15], which uses topological methods, shows that if G is an r-partite graph with n vertices in each part, $0 \leq d < r$, and S is a k-IT in G with $k < r-d$, then there exists an $(r-d)$-IT containing S if

$$n > 2\Delta\left(1 - \frac{d+1-(k/2)}{r-k}\right).$$

It would be interesting to know if any of the methods used in this paper, in particular, the results on IMCs developed in Sections 3 and 4, can be used to improve this bound.

Finally, it was shown in [10] that for any Δ, there exists an algorithm that, given a multipartite graph G with at least $2\Delta + 1$ vertices in each part, returns an independent transversal in polynomial time of $|V(G)|$. One may ask if there is a similar polynomial-time algorithm that returns $(r-d)$-ITs under a weaker condition on the size of parts.

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Appendix A. Proof of Lemma 3.5

We prove Lemma 3.5 in this section using similar arguments to those used in the proof of [14, Theorem 2.2].

Proof of Lemma 3.5. We will first show that Algorithm 3.4 maintains a feasible pair. Let w, I, T, and T' be as defined in Step 2 or Step 3 of the algorithm. We will show that $(I \cup \{w\} \cup N(w, T'), T')$ is feasible. For convenience, let $I' = I \cup \{w\} \cup N(w, T')$. Also, we say that a class V_i is *active* in I if $V_i \in S(I)$, i.e. $V_i \cap I \neq \emptyset$, and we refer to $S(I)$ as the set of active classes of I. It follows from the algorithm that $T \cap (\cup_{V_i \in S(I)} V_i) = T' \cap (\cup_{V_i \in S(I)} V_i)$, so that T and T' agree on active classes of I.

Case 1: We are at Step 2.

Then (a) is satisfied since $T' \in \mathcal{T}$.

For (b) suppose that $v \in T'$ and $S(\{v\}) \in S(I') = S(I) \cup S(N(w, T'))$. If $S(\{v\}) \in S(I)$ then $v \in T$ as T and T' agree on active classes of I. Then since $v \in T$ and $S(\{v\}) \in S(I)$, we have $v \in I \subseteq I'$ because (I, T) satisfies (b). Now assume that $S(\{v\}) \in S(N(w, T'))$. Then we must have $v \in N(v, T') \subseteq I'$ as T' is an IT.

We will now verify (c). By definition of $\mathcal{T}$, we have $W' = I' \backslash T' = W \cup \{w\}$ and $\{v\} \cup N(v, T') = \{v\} \cup N(v, T)$ for all $v \in W$. These stars $\{v\} \cup N(v, T')$ for $v \in W$ are all disjoint and of order at least two as (I, T) is feasible. Also, $(\{w\} \cup N(w, T')) \cap (\{v\} \cup N(v, T)) = \emptyset$ for all $v \in W$. Hence it only remains to be shown that $\{w\} \cup N(w, T')$ is a star with at least one leaf, i.e. $N(w, T') \neq \emptyset$. Suppose for the sake of contradiction that $N(w, T') = \emptyset$. Then T' contains a vertex u in the class of w as if not, $T' \cup \{w\}$ is an IT in G of size $|T'| + 1$. But, since the class containing w and u intersects I, we have since T and T' agree on classes active in I, $u \in T$. Then $u \in I$ as well by (b) applied to (I, T). Also, u has exactly one neighbor in I by (c) applied to (I, T) as $u \notin W$. Let $v_0 \in W$ be its neighbor. Then $T'' = T' \cup \{w\} \setminus \{u\}$ is a partial IT on $S(T_0)$ such that $T'' \cap W = \emptyset$. We claim that the existence of T'' contradicts (I, T) satisfying (e), so $N(w, T') \neq \emptyset$ and (I', T') satisfies (c). We have $|N(v_0, T'')| = |N(v_0, T') \setminus \{u\}| = |N(v_0, T')| - 1 = |N(v_0, T)| - 1$ and $N(v, T'') = N(v, T') = N(v, T)$ for $v \in W \setminus \{v_0\}$, giving us the desired contradiction.

Observe that $w \in V_I$ and $N(w, T') \cap V_I = \emptyset$. Hence $\mathcal{G}_{I'}$ is $\mathcal{G}_I$ with classes containing vertices in $N(w, T')$ added as leaves and is thus also a forest, so (I', T') satisfies (d).

To show that (I', T') satisfies (e), suppose for the sake of contradiction that there exists $v_0 \in W \cup \{w\}$ and $T'' \in \mathcal{T}'$ such that $T'' \cap W' = \emptyset$, $|N(v_0, T'')| < |N(v_0, T')|$, and $N(v, T'') = N(v, T')$ for all $v \in W' \setminus \{v_0\}$. Suppose that $v_0 \in W$. Then, since $N(v, T') = N(v, T)$ for all $v \in W$, (I, T) would not satisfy (e), a contradiction. Hence we may assume $v_0 = w$. But then $T'' \in \mathcal{T}$ and $\deg(w, T'') < \deg(w, T')$, which contradicts our choice of T'.

Case 2: We are at Step 3. In this case, we have $S(\{w\}) \notin S(T)$.

We have that (a) is satisfied since $T' \in \mathcal{T}$.

Suppose that $v \in T'$ and $S(\{v\}) \in S(I') = S(I) \cup S(\{w\}) \cup S(N(w, T'))$. Since $w \notin T'$ we have $S(\{v\}) \in S(I) \cup S(N(w, T'))$, and the same argument used in Case 1 above shows that (I', T') satisfies (b).

We will now verify (c). By definition of $\mathcal{T}$, we have $W' = I' \setminus T' = W \cup \{w\}$ and $\{v\} \cup N(v, T') = \{v\} \cup N(v, T)$ for all $v \in W$. These stars $\{v\} \cup N(v, T')$ for $v \in W$ are all disjoint and of order at least two as (I, T) is feasible. Also, $(\{w\} \cup N(w, T')) \cap (\{v\} \cup N(v, T)) = \emptyset$ for all $v \in W$. Hence it only remains to be shown that $N(w, T') \neq \emptyset$. Suppose for the sake of contradiction that $N(w, T') = \emptyset$. But since $w \notin S(T)$, this would imply that $\{w\} \cup T'$ is an IT in G of size larger than T, a contradiction.

We have that $\mathcal{G}_{I'}$ is $\mathcal{G}_I$ with possibly a new vertex added for the class containing w (if $S(\{w\}) \notin S(I)$), and leaves corresponding to the classes containing vertices in $N(w, T')$ added. So, since $\mathcal{G}_I$ is a forest, so is $\mathcal{G}_{I'}$, and (I', T') hence satisfies (d).

The same argument made in Case 1 above shows that (I', T') satisfies (e).

We have thus showed that the algorithm maintains a feasible pair throughout. Since the size of I increases throughout the algorithm, the algorithm must eventually terminate and output a feasible pair. □

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