Vertex degree sums for matchings in 3-uniform hypergraphs

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Abstract

Let n, s be positive integers such that n is sufficiently large and $s \leqslant n/3$. Suppose H is a 3-uniform hypergraph of order n without isolated vertices. If $\deg(u) + \deg(v) > 2(s-1)(n-1)$ for any two vertices u and v that are contained in some edge of H, then H contains a matching of size s. This degree sum condition is best possible and confirms a conjecture of the authors [Electron. J. Combin. 25 (3), 2018], who proved the case when $s = n/3$.

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1 Introduction

A *k-uniform hypergraph* H (in short, *k-graph*) is a pair (V, E), where V is a finite set of vertices and E is a family of k-element subsets of V. Note that a 2-graph is simply a graph. Let $V(H)$ and $E(H)$ denote the vertex set and edge set of H, respectively. A

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matching of size s in H is a family of s pairwise disjoint edges of H. If the matching covers all the vertices of H, then we call it a *perfect matching*. Given a set $S \subseteq V$, the *degree* $\deg_H(S)$ of S is the number of the edges of H containing S. We simply write $\deg(S)$ when H is obvious from the context. Further, let $\delta_\ell(H) = \min\{\deg(S) : S \subseteq V(H), |S| = \ell\}$.

Given integers $\ell < k \leqslant n$ such that k divides n, let $m_\ell(k,n)$ denote the smallest integer m such that every k-graph H on n vertices with $\delta_\ell(H) \geqslant m$ contains a perfect matching. In recent years the problem of determining $m_\ell(k,n)$ has received much attention (see [2, 5, 6, 7, 8, 9, 10, 12, 14, 17, 16, 18, 20, 21, 22]). In particular, Rödl, Ruciński and Szemerédi [18] determined $m_{k-1}(k,n)$ for all $k \geqslant 3$ and sufficiently large n. Treglown and Zhao [20, 21] determined $m_\ell(k,n)$ for all $\ell \geqslant k/2$ and sufficiently large n. More Dirac-type results on hypergraphs can be found in surveys [15, 27].

A well-known result of Ore [13] extended Dirac's theorem by determining the smallest degree sum of two non-adjacent vertices that guarantees a Hamilton cycle in graphs. Ore-type problems for hypergraphs have been studied recently. For example, Tang and Yan [19] studied the degree sum of two $(k-1)$-sets that guarantees a tight Hamilton cycle in k-graphs. Zhang and Lu [23] studied the degree sum of two $(k-1)$-sets that guarantees a perfect matching in k-graphs. Zhang, Zhao and Lu [26] determined the minimum degree sum of two adjacent vertices that guarantees a perfect matching in 3-graphs without isolated vertices, see Theorem 2 (two vertices in a hypergraph are *adjacent* if there exists an edge containing both of them). Note that one may study the minimum degree sum of two arbitrary vertices and that of two non-adjacent vertices that guarantees a perfect matching instead. In fact, it was mentioned in [26] that the former equals to $2m_1(3,n)-1$ while the latter does not exist.

Let us define (potential) extremal 3-graphs for the matching problem. For $1 \leqslant \ell \leqslant 3$, let $H^\ell_{n,s}$ denote the 3-graph of order n, whose vertex set is partitioned into two sets S and T of size $n-s\ell+1$ and $s\ell-1$, respectively, and whose edge set consists of all triples with at least ℓ vertices in T. A well-known conjecture of Erdős [3], recently verified for 3-graphs [4, 11], implies that $H^1_{n,s}$ or $H^3_{n,s}$ is the densest 3-graph on n vertices not containing a matching of size s. On the other hand, Kühn, Osthus and Treglown [10] showed that for sufficiently large n, $H^1_{n,s}$ has the largest minimum vertex degree among all 3-graphs on n vertices not containing a matching of size s.

Theorem 1. *[10] There exists $n_0 \in \mathbb{N}$ such that if H is a 3-graph of order $n \geqslant n_0$ with $\delta_1(H) > \delta_1(H^1_{n,s}) = \binom{n-1}{2} - \binom{n-s}{2}$ and $n \geqslant 3s$, then H contains a matching of size s.*

Given a 3-graph H, let $\sigma_2(H)$ denote the minimum $\deg(u)+\deg(v)$ among all adjacent vertices u and v. It is easy to see that

$$\sigma_2(H^3_{n,s}) = 2\binom{3s-2}{2}, \quad \sigma_2(H^1_{n,s}) = 2\left(\binom{n-1}{2} - \binom{n-s}{2}\right), \text{ and}$$

$$\sigma_2(H^2_{n,s}) = \binom{2s-2}{2} + (n-2s+1)\binom{2s-2}{1} + \binom{2s-1}{2} = (2s-2)(n-1).$$

The following is [26, Theorem 1], which implies that, when n is divisible by 3 and sufficiently large, $H^2_{n,n/3}$ has the largest $\sigma_2(H)$ among all n-vertex 3-graphs H containing no isolated vertex or perfect matching.

Theorem 2. *[26] There exists $n_0 \in \mathbb{N}$ such that the following holds for all integers $n \geqslant n_0$ that are divisible by 3. Let H be a 3-graph of order n without an isolated vertex. If $\sigma_2(H) > \sigma_2(H^2_{n,n/3}) = \frac{2}{3}n^2 - \frac{8}{3}n + 2$, then H contains a perfect matching.*

Zhang, Zhao and Lu [26, Conjecture 12] further conjectured that for sufficiently large n and any $s < n/3$, $H^2_{n,s}$ has the largest $\sigma_2(H)$ among all n-vertex 3-graphs H containing no isolated vertex or matching of size s. In this paper we verify this conjecture.

Theorem 3. *There exists $n_1 \in \mathbb{N}$ such that the following holds for all integers $n \geqslant n_1$ and $s \leqslant n/3$. If H is a 3-graph of order n without an isolated vertex and $\sigma_2(H) > \sigma_2(H^2_{n,s}) = 2(s-1)(n-1)$, then H contains a matching of size s.*

Since the two theorems have different extremal hypergraphs, Theorem 3 does not imply Theorem 1 (analogously Theorem 1 does not imply Erdős' matching conjecture for 3-graphs). On the other hand, one may wonder why we assume that H contains no isolated vertex in Theorem 3 (especially when $s < n/3$). In fact, as shown in the concluding remarks of [26], Theorem 3 implies another conjecture [26, Conjecture 13], which determines the largest $\sigma_2(H)$ among all 3-graphs containing no matching of size s. Note that $\sigma_2(H^2_{n,s}) \geqslant \sigma_2(H^3_{n,s})$ if and only if $s \leqslant (2n+4)/9$.

Corollary 4. *There exists $n_2 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph of order $n \geqslant n_2$ and $2 \leqslant s \leqslant n/3$. If $\sigma_2(H) > \max\{\sigma_2(H^2_{n,s}), \sigma_2(H^3_{n,s})\}$, then H contains a matching of size s.*

Let us explain our approach towards Theorem 3. The case when $s \leqslant n/13$ was already solved by Zhang and Lu [24] in a stronger form. Note that $\sigma_2(H^2_{n,s}) > \sigma_2(H^1_{n,s})$. The following theorem shows that, when $n \geqslant 13s$, not only is $H^2_{n,s}$ the (unique) 3-graph with the largest $\sigma_2(H)$ among all H containing no isolated vertex or a matching of size s, but also $H^1_{n,s}$ is the sub-extremal 3-graph for this problem. (In fact, Zhang and Lu [24] conjectured that Theorem 5 holds for *all* $n \geqslant 3s$. If true, this strengthens Theorem 1 and actually provides a link between Ore's and Dirac's problems.)

Theorem 5. *[25] Let n, s be positive integers and H be a 3-graph of order $n \geqslant 13s$ without an isolated vertex. If $\sigma_2(H) > \sigma_2(H^1_{n,s}) = 2\left(\binom{n-1}{2} - \binom{n-s}{2}\right)$, then either H contains a matching of size s or H is a subgraph of $H^2_{n,s}$.*

Therefore it suffices to prove Theorem 3 for reasonably large s. For such s, we actually prove a (stronger) stability theorem.

Theorem 6. *Given $0 < \varepsilon \ll \tau \ll 1$, let n be sufficiently large and $\tau n < s \leqslant n/3$. If H is a 3-graph of order n without an isolated vertex such that $\sigma_2(H) > 2sn - \varepsilon n^2$, then either H is a subgraph of $H^2_{n,s}$ or H contains a matching of size s.*

Theorem 3 follows from Theorem 6 immediately. Indeed, if $\sigma_2(H) > \sigma_2(H^2_{n,s})$, then it is easy to see that H is not a subgraph of $H^2_{n,s}$.[1] Suppose instead, that $V(H)$ can be

[1]Unfortunately σ_2 is not a monotone function: for example, adding an edge to $H^2_{n,s}$ indeed reduces the value of σ_2 because two vertices in S now become adjacent and their degree sum is smaller than $\sigma_2(H^2_{n,s})$.

partitioned $S \cup T$ such that $|S| = n - 2s + 1$, $|T| = 2s - 1$, and every edge of H contains at least two vertices of T. Since H contains no isolated vertices, every vertex of S is adjacent to some vertex of T. Thus $\sigma_2(H) \leqslant \deg(u) + \deg(v)$ for some $u \in S$ and $v \in T$. Consequently $\sigma_2(H) \leqslant \sigma_2(H^2_{n,s})$, a contradiction. We therefore apply Theorem 6 to derive that H contains a matching of size s. Furthermore, Theorem 6 implies that $H^2_{n,s}$ is the unique extremal 3-graph for Theorem 3 because all proper subgraphs H of $H^2_{n,s}$ satisfy $\sigma_2(H) < \sigma_2(H^2_{n,s})$.

In order to prove Theorem 6, we follow the same approach as in [26]: using the condition on $\sigma_2(H)$, we greedily extend a matching of H until it has s edges. An important intermediate step is finding a matching that covers a certain number of low-degree vertices (see Lemma 7). Nevertheless, the proof of Theorem 6 does require new ideas: in particular, the meaning of an *optimal* matching is more complicated (see Definition 8); we proceed differently depending on whether the number of low-degree vertices in the optimal matching is at the threshold. In one case we reduce the problem to that of finding a perfect matching in a subgraph of H and apply the main result of [26] (see Theorem 9).

This paper is organized as follows. In Section 2, we give an outline of the proof along with some preliminary results. We prove Lemma 7 in Section 3 and complete the proof in Section 4.

Notation: Given a graph G and a vertex u in G, $N_G(u)$ is the set of neighbors of u in G. Suppose H is a 3-uniform hypergraph. For $u \neq v \in V(H)$, let $N_H(u, v) = \{w \in V(H) : \{u, v, w\} \in E(H)\}$ (the subscript is often omitted when H is clear from the context). Given three subsets V_1, V_2, V_3 of $V(H)$, we say that an edge $\{v_1, v_2, v_3\} \in E(H)$ is a type of $V_1V_2V_3$ if $v_i \in V_i$ for $1 \leqslant i \leqslant 3$. Given a vertex $v \in V(H)$ and a subset $A \subseteq V(H)$, we define *the link* $L_v(A) = \{uw : u, w \in A \text{ and } \{u, v, w\} \in E(H)\}$. When A and B are two disjoint subsets of $V(H)$, we let $L_v(A, B) = \{uw : u \in A,\ w \in B \text{ and } \{u, v, w\} \in E(H)\}$.

We write $0 < a_1 \ll a_2 \ll a_3$ if we can choose the constants a_1, a_2, a_3 from right to left. More precisely there are increasing functions f and g such that given a_3, whenever we choose some $a_2 \leqslant f(a_3)$ and $a_1 \leqslant g(a_2)$, all calculations needed in our proof are valid.

2 Outline of the proof and preliminaries

Let n be sufficiently large and $\tau n < s \leqslant n/3$. Suppose H is a 3-graph of order n without an isolated vertex and $\sigma_2(H) > 2sn - \varepsilon n^2$. Let $U = \{u \in V(H) : \deg(u) > sn - \frac{\varepsilon}{2} n^2\}$ and $W = V \setminus U$. Then any two vertices of W are not adjacent – otherwise $\sigma_2(H) \leqslant 2sn - \varepsilon n^2$, a contradiction. If $|U| \leqslant 2s - 1$, then H is a subgraph of $H^2_{n,s}$ and we are done. We thus assume that $|U| \geqslant 2s$.

Throughout the proof we use small constants

$$0 < \varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \eta_1 \ll \eta_2 \ll \gamma \ll \gamma' \ll \tau \ll 1. \tag{1}$$

We first prove the following lemma, which is an extension of [26, Lemma 4].

Lemma 7. *Given $0 < \varepsilon \ll \tau \ll 1$, let n be sufficiently large and $\tau n < s \leqslant n/3$. Suppose H is a 3-graph of order n without an isolated vertex and $\sigma_2(H) > 2sn - \varepsilon n^2$. Let*

$U = \{u \in V(H) : \deg(u) > sn - \varepsilon n^2/2\}$ and $W = V \setminus U$. If $2s \leqslant |U| \leqslant 3s$, then H contains a matching of size $3s - |U|$, each of which contains exactly one vertex of W.

Definition 8. We call a matching M *optimal* if (i) M contains a submatching $M_1 = \{e \in M : e \cap W \neq \emptyset\}$ of size at least $3s - |U|$; (ii) subject to (i), $|M|$ is as large as possible; (iii) subject to (i) and (ii), $|M_1|$ is as large as possible.

Lemma 7 shows that H contains an optimal matching M. We separate the cases when $|M_1| = 3s - |U|$ and when $|M_1| > 3s - |U|$. When $|M_1| = 3s - |U|$, we first consider the case when $s \leqslant n/3 - \eta_1 n$. If no vertex of $U_3 := U \setminus V(M)$ is adjacent to any vertex of $W_2 := W \setminus V(M)$, then the assumption $|M_1| = 3s - |U|$ forces $\sum_{i=1}^{3} \deg(u_i)$ to be smaller than $3sn - \frac{3}{2}\varepsilon n^2$ for any three vertices $u_1, u_2, u_3 \in U_3$. If some vertex $u_1 \in U_3$ is adjacent to $v_1 \in W_2$, then the fact $v_1 \in W$ reduces $\sum_{i=1}^{2} \deg(u_i) + \deg(v_1)$ to a number less than $3sn - \frac{3}{2}\varepsilon n^2$ (where u_2 is another vertex of U_3). When $s > n/3 - \eta_1 n$, we consider $H' = H[V \setminus W_2]$. Since $|W_2| = n - 3s$ is very small, we deduce that $\sigma_2(H')$ is greater than $2sn - \eta_2 n^2$. This allows us to apply the following theorem from [26] to obtain a perfect matching of H', which is also a matching of size s of H.

Theorem 9. *[26] There exist $\eta_2 > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all integers $n \geqslant n_0$ that are divisible by 3. Suppose that H is a 3-graph of order n without an isolated vertex and $\sigma_2(H) > 2n^2/3 - \eta_2 n^2$, then either H is a subgraph of $H^2_{n,n/3}$ or H contains a perfect matching.*

Now consider the case when $|M_1| > 3s - |U|$. Let $W' := \{v \in W : \deg(v) \leqslant sn - s^2/2 + \gamma' n^2\}$. If $|W'|$ is very small, then we can find a matching of size s in $H[V \setminus W']$ by Theorem 1. When $|W'|$ is not small, we consider $u_1, u_2, u_3 \in U_3$. If one of u_1, u_2, u_3 is adjacent to one vertex from W', then $\sum_{i=1}^{3} \deg(u_i)$ becomes much larger than $3sn$; otherwise we show that $\sum_{i=1}^{3} \deg(u_i) < 3sn - \frac{3}{2}\varepsilon n^2$ by proceeding with the cases when $|W' \cap W_1| > \gamma n/2$ and when $|W' \cap W_2| > \gamma n/2$ separately.

In the proof we need several (simple) extremal results on (hyper)graphs. Lemma 10 is Observation 1.8 of Aharoni and Howard [1]. Lemmas 11 and 12 are from [26]. A k-graph H is called *k-partite* if $V(H)$ can be partitioned into $V_1, \dots, V_k$, such that each edge of H meets every V_i in precisely one vertex. If all parts are of the same size n, we call H *n-balanced*.

Lemma 10. *[1] Let F be the edge set of an n-balanced k-partite k-graph. If F does not contain s disjoint edges, then $|F| \leqslant (s-1)n^{k-1}$.*

Lemma 11. *[26] Let G_1, G_2, G_3 be three graphs on the same set V of $n \geqslant 4$ vertices such that every edge of G_1 intersects every edge of G_i for both $i = 2, 3$. Then $\sum_{i=1}^{3} \sum_{v \in A} \deg_{G_i}(v) \leqslant 6(n-1)$ for any set $A \subset V$ of size 3.*

Lemma 12. *[26] Let G_1, G_2, G_3 be three graphs on the same set V of $n \geqslant 5$ vertices such that for any $i \neq j$, every edge of G_i intersects every edge of G_j. Then $\sum_{i=1}^{3} \sum_{v \in A} \deg_{G_i}(v) \leqslant 3(n+1)$ for any set $A \subset V$ of size 3.*

Following the same proof of Lemmas 11 and 12 from [26], we obtain another lemma and omit its proof.

Lemma 13. *Let $G_1, \cdots, G_k$ be k graphs on the same set V of $n \geqslant 4$ vertices such that for any $1 \leqslant i < j \leqslant k$, every edge of G_i intersects every edge of G_j. Then $\sum_{i=1}^{k} \sum_{v \in A} \deg_{G_i}(v) \leqslant kn$ for any set $A \subset V$ of size 2.* □

The following lemma needs slightly more work so we include a proof.

Lemma 14. *Given two disjoint vertex sets $A = \{u_1, u_2, \dots, u_a\}$ and $B = \{v_1, v_2, \dots, v_b\}$ with $a \geqslant 3$ and $b \geqslant 1$. Let G_i, $i = 1, 2, 3$, be graphs on $A \cup B$ such that every vertex of B is an isolated vertex in G_1, and every edge of G_i ($i = 2, 3$) contains at least one vertex of A. If there are no two disjoint edges (i) one from G_1 and the other from G_2 or G_3; or (ii) one from G_2 and the other from G_3, and at least one of them contains a vertex from B, then*

$$\sum_{i=1}^{3} \left(\sum_{j=1}^{2} \deg_{G_i}(u_j) + \deg_{G_i}(v_1) \right) \leqslant \max\{4a + 7, 3a + 2b + 5\}.$$

Proof. For convenience, let $s_i = \sum_{j=1}^{2} \deg_{G_i}(u_j) + \deg_{G_i}(v_1)$ for $i = 1, 2, 3$ and $y = s_1 + s_2 + s_3$. Below we show that $y \leqslant \max\{4a + 7, 3a + 2b + 5\}$.

We first observe that if $\deg_{G_i}(v_1) \geqslant 3$ for some $i \in \{2, 3\}$, then $E(G_1) = \emptyset$ and $G_{i'}$ is a star centered at v_1, where $i' = 5 - i$. Indeed, if G_1 or $G_{i'}$ contains an edge e not incident to v_1, then e is disjoint from some edge of G_i that is incident to v_1 – this contradicts our assumption. The observation implies that if $\deg_{G_i}(v_1) \geqslant 3$ for both $i = 2, 3$, then $E(G_1) = \emptyset$ and both G_2 and G_3 are stars centered at v_1. In this case, $s_i \leqslant a + 2$ for $i = 2, 3$ and thus $y \leqslant 2(a + 2)$. If $\deg_{G_2}(v_1) \geqslant 3$ and $\deg_{G_3}(v_1) \leqslant 2$, then $E(G_1) = \emptyset$ and G_3 consists of at most two edges incident to v_1. In this case, $s_1 \leqslant 2(a + b - 1) + a$, $s_2 \leqslant 4$ and thus $y \leqslant 3a + 2b + 2$. The case when $\deg_{G_2}(v_1) \leqslant 2$ and $\deg_{G_3}(v_1) \geqslant 3$ is analogous. We thus assume that

$$\deg_{G_i}(v_1) \leqslant 2 \quad \text{for } i = 2, 3 \tag{2}$$

for the rest of the proof.

Next, we observe that if $|N_{G_i}(u_j) \cap B| \geqslant 2$ for some $i \in \{2, 3\}$ and some $j \in \{1, 2\}$, then $G_{i'}$ is a star centered at u_j for $i' \in \{1, 2, 3\} \setminus \{i\}$. This is again due to our assumption on G_1, G_2 and G_3. The observation implies that if $|N_{G_i}(u_j) \cap B| \geqslant 2$ for both $j = 1, 2$, then $E(G_{i'}) \subseteq \{u_1 u_2\}$ and consequently, $s_{i'} \leqslant 2$ for $i' \in \{1, 2, 3\} \setminus \{i\}$. By (2), we have $s_i \leqslant 2(a + b - 1) + 2$. Therefore, $y \leqslant 2(a + b - 1) + 2 + 4 = 2a + 2b + 4$. The observation also implies that if $|N_{G_i}(u_j) \cap B| \geqslant 2$ for both $i = 2, 3$, then G_1, G_2, G_3 are all stars centered at u_j. In this case, $s_1 \leqslant a$ and $s_i \leqslant a + b + 1$ for $i = 2, 3$, which implies that $y \leqslant a + 2(a + b + 1) = 3a + 2b + 2$. We now consider the case when $|N_{G_2}(u_1) \cap B| \geqslant 2$, $|N_{G_2}(u_2) \cap B| \leqslant 1$, and $|N_{G_3}(u_1) \cap B| \leqslant 1$. Thus G_3 is a star (centered at u_1) of size at most a, which yields $s_3 \leqslant a + 2$. Now suppose $N_{G_2}(u_2) \cap B \subseteq \{v_p\}$ for some p. Let $A' := A \cup \{v_p\}$ (note that $|A'| = a + 1 \geqslant 4$). Since every edge of G_1

intersects every edge of G_2, we can apply Lemma 13 to $G_1[A']$ and $G_2[A']$ and obtain that $\sum_{i=1}^2\sum_{j=1}^2 \deg_{G_i[A']}(u_j) \leqslant 2a+2$. Since $|N_{G_2}(u_1)\cap(B\setminus\{v_p\})| \leqslant b-1$ and $\deg_{G_2}(v_1) \leqslant 2$, it follows that $s_1+s_2 \leqslant 2a+2+b-1+2 = 2a+b+3$ and $y \leqslant 2a+b+3+a+2 = 3a+b+5$.

We thus assume that $|N_{G_i}(u_j)\cap B| \leqslant 1$ for $i=2,3$ and $j=1,2$. Suppose $N_{G_2}(u_2)\cap B \subseteq \{v_p\}$ for some p and let $A' := A\cup\{v_p\}$. We apply Lemma 13 to $G_1[A']$ and $G_2[A']$ and obtain that $\sum_{i=1}^2\sum_{j=1}^2 \deg_{G_i[A']}(u_j) \leqslant 2a+2$. Since $|N_{G_2}(u_1)\cap B| \leqslant 1$ and $\deg_{G_2}(v_1) \leqslant 2$, it follows that $s_1+s_2 \leqslant 2a+2+1+2$. On the other hand, we have $s_3 \leqslant 2a+2$ because $\deg_{G_3}(u_j) \leqslant a$ for $j=1,2$ and $\deg_{G_3}(v_1) \leqslant 2$. Thus $y \leqslant 2a+5+2a+2 = 4a+7$. ☐

3 Proof of Lemma 7

The proof is similar to that of [26, Lemma 4]. Let M be a largest matching of H such that each edge of M contains (exactly) one vertex of W. To the contrary, assume $|M| \leqslant 3s-|U|-1$. Let $U_1 = V(M)\cap U$, $U_2 = U\setminus U_1$, $W_1 = V(M)\cap W$ and $W_2 = W\setminus W_1$. Since $|U| \geqslant 2s$, we have $|U_2| = |U|-2|M| \geqslant 2$. Since $|W_2| = |W|-|M|$ and $|W| \geqslant 3s-|U|$, it follows that $W_2 \neq \emptyset$.

Below is a sketch of the proof. We first assume $|U| < 2s+\varepsilon' n$. In this case every vertex in U is adjacent to some vertex in W. If $|M|$ is not close to s, then we easily obtain a contradiction because U_2 is not small. When $|M|$ is close to s, we consider three vertices $u_1 \neq u_2 \in U_2$ and $v_0 \in W_2$, and derive a contradiction on $\deg(u_1)+\deg(u_2)+\deg(v_0)$. Next we assume $|U| \geqslant 2s+\varepsilon' n$. In this case U_2 is not small. If no vertex of W_2 is adjacent to any vertex of U_2, then consider two adjacent vertices $v_0 \in W_2$ and $u_0 \in U_1$. We have $\deg(v_0) \leqslant \binom{2|M|}{2}$, which eventually yields that $\deg(v_0)+\deg(u_0) < 2sn-\varepsilon n^2$. Now assume $v_0 \in W_2$ is adjacent to some vertex $u_0 \in U_2$. In this case we define M' consisting of all $e \in M$ that contains a vertex $u' \in U$ such that $|N(v_0,u')\cap U_2| \geqslant 3$. We show that if $|M'|$ is small, then $\deg(v_0)$ is small; otherwise $\deg(u_0)$ is small. In either case we derive that $\deg(v_0)+\deg(u_0) < 2sn-\varepsilon n^2$.

We now give the details of the proof.

Case 1. $2s \leqslant |U| < 2s+\varepsilon' n$.

In this case we have the following two claims.

Claim 15. *$|M| \geqslant s-\varepsilon'' n$.*

Proof. To the contrary, assume that $|M| < s-\varepsilon'' n$. Fix $v_0 \in W_2$. Then $\deg(v_0) \leqslant \binom{|U|}{2} - \binom{|U_2|}{2}$ because there is no edge of type $U_2U_2W_2$. Since v_0 is not an isolated vertex, v_0 is adjacent to some vertex $u \in U$. Trivially $\deg(u) \leqslant \binom{|U|-1}{2} + (|U|-1)|W|$. Thus

$$\begin{aligned}\deg(v_0)+\deg(u) &\leqslant \binom{|U|-1}{2} + (|U|-1)|W| + \binom{|U|}{2} - \binom{|U_2|}{2}\\ &= (n-1)(|U|-1) - \binom{|U_2|}{2}.\end{aligned}$$

Since $|U| \geqslant 2s$ and $|M| < s-\varepsilon'' n$, it follows that $|U_2| = |U|-2|M| > 2\varepsilon'' n$. As a result,

$$\deg(u)+\deg(v_0) \leqslant (n-1)(2s+\varepsilon' n-1) - \binom{2\varepsilon'' n}{2},$$

which contradicts the condition that $\deg(u) + \deg(v_0) > 2sn - \varepsilon n^2$ because $\varepsilon \ll \varepsilon' \ll \varepsilon''$. □

Claim 16. *Every vertex in U is adjacent to one vertex in W.*

Proof. To the contrary, assume that $u \in U$ is not adjacent to any vertex in W. Then

$$\deg(u) \leqslant \binom{|U|-1}{2} < \binom{2s+\varepsilon' n}{2},$$

which contradicts the condition that $\deg(u) > sn - \frac{1}{2}\varepsilon n^2$ because $\tau n < s \leqslant n/3$ and $\varepsilon \ll \varepsilon' \ll \tau$. □

Fix $u_1 \neq u_2 \in U_2$ and $v_0 \in W_2$. Trivially $\deg(w) \leqslant \binom{|U|}{2}$ for any vertex $w \in W$ and $\deg(u) \leqslant \binom{|U|-1}{2} + |W|(|U|-1)$ for any vertex $u \in U$. Furthermore, for any two distinct edges $e_1, e_2 \in M$, we observe that at least one triple of type UUW with one vertex in e_1, one vertex in e_2 and one vertex in $\{u_1, u_2, v_0\}$ is *not* an edge by the choice of M. By Claim 15, $|M| \geqslant s - \varepsilon'' n$. Thus,

$$\deg(u_1) + \deg(u_2) + \deg(v_0) \leqslant 2\left(\binom{|U|-1}{2} + |W|(|U|-1)\right) + \binom{|U|}{2} - \binom{s-\varepsilon'' n}{2}.$$

On the other hand, Claim 16 implies that u_i is adjacent to some vertex in W for $i = 1, 2$. We know that v_0 is adjacent to some vertex in U. Therefore, $\deg(u_i) > (2sn - \varepsilon n^2) - \binom{|U|}{2}$ for $i = 1, 2$, and $\deg(v_0) > (2sn - \varepsilon n^2) - \left(\binom{|U|-1}{2} + |W|(|U|-1)\right)$. It follows that

$$\deg(u_1) + \deg(u_2) + \deg(v_0) > 3\left(2sn - \varepsilon n^2\right) - 2\binom{|U|}{2} - \binom{|U|-1}{2} - |W|(|U|-1).$$

The upper and lower bounds for $\deg(u_1) + \deg(u_2) + \deg(v_0)$ together imply that

$$3\left(\binom{|U|-1}{2} + |W|(|U|-1) + \binom{|U|}{2}\right) - \binom{s-\varepsilon'' n}{2} > 3\left(2sn - \varepsilon n^2\right),$$
$$\text{or} \quad (|U|-1)(n-1) - \frac{1}{3}\binom{s-\varepsilon'' n}{2} > 2sn - \varepsilon n^2,$$

which is impossible because $|U| < 2s + \varepsilon' n$, $\tau n < s \leqslant n/3$, and $\varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \tau$.
Case 2. $2s + \varepsilon' n \leqslant |U| \leqslant 3s$.

We consider the following two subcases.
Subcase 2.1. No vertex in U_2 is adjacent to any vertex in W_2.

Fix $v_0 \in W_2$. Then $\deg(v_0) \leqslant \binom{|U_1|}{2} = \binom{2|M|}{2}$. Since v_0 is not an isolated vertex, v_0 is adjacent to some vertex $u_0 \in U_1$. We know that $\deg(u_0) \leqslant \binom{|U|-1}{2} + (|U|-1)|W| - |U_2||W_2|$

because no vertex in U_2 is adjacent to any vertex in W_2. Since $|W| = n - |U|$, $|U_2| = |U| - 2|M|$ and $|W_2| = n - |U| - |M|$, we derive that

$$\begin{aligned}\sigma_2(H) &\leqslant \deg(v_0) + \deg(u_0)\\ &\leqslant \binom{2|M|}{2} + \binom{|U|-1}{2} + (|U|-1)(n-|U|) - (|U|-2|M|)(n-|U|-|M|)\\ &\leqslant (2n-|U|)|M| + \frac{|U|^2}{2}.\end{aligned}$$

Since $|M| < 3s - |U|$, it follows that

$$\sigma_2(H) < (2n-|U|)(3s-|U|) + \frac{|U|^2}{2} = 6sn - (3s+2n)|U| + \frac{3}{2}|U|^2.$$

Note that the quadratic function $\frac{3}{2}x^2 - (3s+2n)x$ is minimized at $x = s + \frac{2}{3}n$. Since $2s + \varepsilon' n \leqslant |U| \leqslant 3s \leqslant s + \frac{2}{3}n$, we derive that

$$\begin{aligned}\sigma_2(H) &\leqslant 6sn - (3s+2n)(2s+\varepsilon' n) + \frac{3}{2}(2s+\varepsilon' n)^2\\ &= 2sn - 2\varepsilon' n^2 + 3s\varepsilon' n + \frac{3}{2}\varepsilon'^2 n^2 \leqslant 2sn - \varepsilon' n^2 + \frac{3}{2}\varepsilon'^2 n^2\end{aligned}$$

because $s \leqslant n/3$. Since $\varepsilon \ll \varepsilon'$, this contradicts the assumption that $\sigma_2(H) > 2sn - \varepsilon n$.

Subcase 2.2. Two vertices $u_0 \in U_2$ and $v_0 \in W_2$ are adjacent.

Let $M' = \{e \in M : \exists\, u' \in e, |N(v_0, u') \cap U_2| \geqslant 3\}$. Assume $\{u_1, u_2, v_1\} \in M'$ such that $u_1, u_2 \in U_1$, $v_1 \in W_1$ and $|N(v_0, u_1) \cap U_2| \geqslant 3$. We claim that

$$N(u_0, v_1) \cap U_2 = \emptyset. \tag{3}$$

Indeed, if $\{u_0, v_1, u_3\} \in E(H)$ for some $u_3 \in U_2$, then we can find $u_4 \in U_2 \setminus \{u_0, u_3\}$ such that $\{v_0, u_1, u_4\} \in E(H)$. Replacing $\{u_1, u_2, v_1\}$ by $\{u_0, v_1, u_3\}$ and $\{v_0, u_1, u_4\}$ gives a larger matching than M, a contradiction.

By the definition of M', we have

$$\deg(v_0) \leqslant \binom{|U_1|}{2} + 2|M'||U_2| + 2(|U_1| - 2|M'|) = \binom{|U_1|}{2} + 2|U_1| + |M'|(2|U_2| - 4).$$

By (3), we have

$$\deg(u_0) \leqslant \binom{|U|-1}{2} + |U_1||W| + (|U_2|-1)(|W_1| - |M'|)$$

and consequently

$$\deg(v_0) + \deg(u_0) \leqslant \binom{|U_1|}{2} + \binom{|U|-1}{2} + |U_1|(|W|+2) + (|U_2|-1)|W_1| + |M'|(|U_2|-3).$$

Since $|M'| \leqslant |M| = |W_1| = \frac{|U_1|}{2}$, it follows that

$$\begin{aligned}\deg(v_0) + \deg(u_0) &\leqslant \binom{|U_1|}{2} + \binom{|U| - 1}{2} + |U_1|(|W| + 2) + (|U_2| - 2)|U_1| \\ &= \binom{|U|}{2} - \binom{|U_2|}{2} + \binom{|U| - 1}{2} + |U_1||W| \\ &= (|U| - 1)^2 - \binom{|U_2|}{2} + 2|M|\,(n - |U|)\,.\end{aligned}$$

Since $|M| \leqslant 3s - |U|$ and $|U_2| = |U| - 2|M| \geqslant 3|U| - 6s$, we have

$$\begin{aligned}\deg(v_0) + \deg(u_0) &\leqslant (|U| - 1)^2 - \binom{3|U| - 6s}{2} + 2(3s - |U|)\,(n - |U|) \\ &= -\frac{3}{2}|U|^2 + \left(12s - 2n - \frac{1}{2}\right)|U| + 6sn - 18s^2 - 3s + 1 \\ &\leqslant -\frac{3}{2}|U|^2 + (12s - 2n)\,|U| + 6sn - 18s^2.\end{aligned}$$

Note that the quadratic function $-\frac{3}{2}x^2 + (12s - 2n)x$ is maximized at $x = 4s - \frac{2}{3}n$. Since $3s \geqslant |U| \geqslant 2s + \varepsilon' n \geqslant 4s - \frac{2}{3}n$, we have

$$\begin{aligned}\sigma_2(H) \leqslant \deg(v_0) + \deg(u_0) &\leqslant -\frac{3}{2}(2s + \varepsilon' n)^2 + (12s - 2n)\,(2s + \varepsilon' n) + 6sn - 18s^2 \\ &= 2sn - 2\varepsilon' n^2 + 6\varepsilon' sn - \frac{3}{2}\varepsilon'^2 n^2 \leqslant 2sn - \frac{3}{2}\varepsilon'^2 n^2\end{aligned}$$

because $s \leqslant n/3$. Since $\varepsilon \ll \varepsilon'$, this contradicts the assumption that $\sigma_2(H) > 2sn - \varepsilon n$.

4 Proof of Theorem 6

Suppose H is a 3-graph of order n without an isolated vertex and $\sigma_2(H) > 2sn - \varepsilon n^2$. Let $U = \{u \in V(H) : \deg(u) > sn - \varepsilon n^2/2\}$ and $W = V \setminus U$. We know that no two vertices in W are adjacent and $|U| \geqslant 2s$. Let M be an optimal matching as in Definition 8. By Lemma 7, such M exists. Let $M_2 = M \setminus M_1$, $U_1 = V(M_1) \cap U$, $U_2 = V(M_2)$, $U_3 = U \setminus V(M)$, $W_1 = V(M_1) \cap W$ and $W_2 = W \setminus W_1$. Since M is optimal, no edge of H is of type $W_2U_3U_3$ or $W_2U_2U_3$. In addition, for any $e \in M_1$, there are no two disjoint edges $e_1, e_2 \in e \cup W_2 \cup U_3$ such that $(e_1 \cup e_2) \cap W_2 \neq \emptyset$.

Suppose to the contrary, that $|M| \leqslant s - 1$. We know that $|U_3| = |U| + |M_1| - 3|M| \geqslant 3 + |M_1| - (3s - |U|) \geqslant 3$. Let $u_1, u_2, u_3 \in U_3$. Since $u_i \in U$ for $i = 1, 2, 3$, we have

$$\sum_{i=1}^{3} \deg(u_i) > 3sn - \frac{3}{2}\varepsilon n^2. \tag{4}$$

On the other hand, if u_1 is adjacent to some $v_1 \in W_2$, then

$$\sum_{i=1}^{2} \deg(u_i) + \deg(v_1) \geqslant \sigma_2(H) + \deg(u_2) > 3sn - \frac{3}{2}\varepsilon n^2. \tag{5}$$

Claim 17. *For any two distinct edges e_1, e_2 from M, we have* $\sum_{i=1}^{3} |L_{u_i}(e_1, e_2)| \leqslant 18$ *and* $\sum_{i=1}^{2} |L_{u_i}(e_1, e_2)| + |L_{v_1}(e_1, e_2)| \leqslant 18$.

Proof. Let H_1 be the 3-partite subgraph of H induced on three parts $\{u_1, u_2, u_3\}$, e_1, and e_2. We observe that H_1 does not contain a perfect matching by the choice of M. By Lemma 10, we have $|E(H_1)| = \sum_{i=1}^{3} |L_{u_i}(e_1, e_2)| \leqslant 18$. The same argument shows that $\sum_{i=1}^{2} |L_{u_i}(e_1, e_2)| + |L_{v_1}(e_1, e_2)| \leqslant 18$. □

We proceed in two cases.

Case 1. $|M_1| = 3s - |U|$.

In this case, we have $|M_2| = |M| + |U| - 3s$, $|U_3| = 3s - 3|M|$ and $|W_2| = n - 3s$.

Claim 18. *For any $e \in M_1$, we have*
(i) $\sum_{i=1}^{2} |L_{u_i}(e, U_3 \cup W_2)| + |L_{v_1}(e, U_3 \cup W_2)| \leqslant \max\{4|U_3| + 7, 3|U_3| + 2|W_2| + 5\}$*, where $v_1 \in W_2$;*
(ii) $\sum_{i=1}^{3} |L_{u_i}(e, U_3)| \leqslant 6|U_3|$.

Proof. Assume $e = \{u_1', u_2', u_3'\} \in M_1$ with $u_1' \in W_1$ and $u_2', u_3' \in U_1$.

(i) Let $A = U_3$, $B = W_2$, and $E(G_i) = L_{u_i'}(U_3 \cup W_2)$ for $i = 1, 2, 3$. By the choice of M, there are not two disjoint edges, one from G_1 and the other from G_2 or G_3; or one from G_2 and the other from G_3, and at least one of them contains one vertex from B. Furthermore, it is easy to see that

$$\sum_{i=1}^{2} |L_{u_i}(e, U_3 \cup W_2)| + |L_{v_1}(e, U_3 \cup W_2)| = \sum_{i=1}^{3} \left(\sum_{j=1}^{2} \deg_{G_i}(u_j) + \deg_{G_i}(v_1) \right).$$

The desired inequality thus follows from Lemma 14.

(ii) For $i = 1, 2, 3$, let G_i be the graph obtained from $L_{u_i'}(U_3)$ after adding an isolated vertex u^*. Then $|V(G_i)| = |U_3| + 1 \geqslant 4$. By the choice of M, every edge of G_1 intersects every edge of G_2 and G_3. The desired inequality thus follows from Lemma 11. □

Claim 19. *For any $e \in M_2$, we have*
(i) $\sum_{i=1}^{3} |L_{u_i}(e, U_3)| \leqslant 3(|U_3| + 3)$*;*
(ii) $\sum_{i=1}^{2} |L_{u_i}(e, U_3)| \leqslant 3(|U_3| + 1)$.

Proof. Assume $e = \{u_1', u_2', u_3'\} \in M_2$ with $u_1', u_2', u_3' \in U_2$.

(i) For $i = 1, 2, 3$, let G_i be the graph obtained from $L_{u_i'}(U_3)$ after adding two isolated vertices u' and u''. Then $|V(G_i)| = |U_3| + 2 \geqslant 5$. Since M is optimal, the desired inequality follows from Lemma 12.

(ii) For $i = 1, 2, 3$, let G_i be the graph obtained from $L_{u_i'}(U_3)$ after adding an isolated vertex u^*. Then $|V(G_i)| = |U_3| + 1 \geqslant 4$. Since M is optimal, the desired inequality follows from Lemma 13. □

Claim 20. *$s > n/3 - \eta_1 n$.*

Proof. Suppose $s \leqslant n/3 - \eta_1 n$. We first consider the case that u_1, u_2, u_3 are not adjacent to any vertex of W_2.

Following Claim 17, we have

$$\sum_{i=1}^{3} \deg(u_i) \leqslant 18\binom{|M|}{2} + 9|M| + \sum_{i=1}^{3} |L_{u_i}(V(M_1), U_3)| + \sum_{i=1}^{3} |L_{u_i}(V(M_2), U_3)|. \quad (6)$$

Furthermore, by Claims 18 (ii) and 19 (i), we obtain that

$$\begin{aligned}\sum_{i=1}^{3} \deg(u_i) &\leqslant 18\binom{|M|}{2} + 9|M| + 6|M_1||U_3| + 3|M_2|(|U_3| + 3)\\ &= 18\binom{|M|}{2} + 9|M| + 6\,(3s - |U|)\,(3s - 3|M|)\\ &\quad + 3(|M| + |U| - 3s)(3s - 3|M| + 3)\\ &= (9|U| - 18s + 9)|M| + (3s - |U|)(9s - 9).\end{aligned}$$

Since $|M| \leqslant s - 1$, it follows that

$$\sum_{i=1}^{3} \deg(u_i) \leqslant (9|U| - 18s + 9)(s - 1) + (3s - |U|)(9s - 9) = 9s^2 - 9.$$

Since $\tau n < s \leqslant n/3 - \eta_1 n$ and $\eta_1 < \tau$, we know that

$$3s^2 - sn = s(3s - n) \leqslant \max\{-\eta_1 n(n - 3\eta_1 n), -\tau n(n - 3\tau n)\} = -\eta_1 n(n - 3\eta_1 n). \quad (7)$$

Consequently, $\sum_{i=1}^{3} \deg(u_i) < 9s^2 \leqslant 3sn - 3\eta_1 n(n - 3\eta_1 n)$. Since $\varepsilon \ll \eta_1$, this contradicts (4).

Now we assume, without loss of generality, that u_1 is adjacent to v_1. The choice of M implies that $L_v(e, U_3) = L_u(e, W_2) = \emptyset$ for any $v \in W_2$, $u \in U_3$ and $e \in M_2$. By Claim 17, we have

$$\begin{aligned}\sum_{i=1}^{2} \deg(u_i) + \deg(v_1) \leqslant 18\binom{|M|}{2} + 9|M| &+ \sum_{i=1}^{2} |L_{u_i}(V(M_1), U_3 \cup W_2)|\\ &+ |L_{v_1}(V(M_1), U_3)| + \sum_{i=1}^{2} |L_{u_i}(V(M_2), U_3)|. \quad (8)\end{aligned}$$

We know that $4|U_3|+7 \geqslant 3|U_3|+2|W_2|+5$ if and only if $|U_3| \geqslant 2|W_2|-2$. If $|U_3| \geqslant 2|W_2|-2$, then by (8), Claim 18 (i) and Claim 19 (ii), we have

$$\begin{aligned}\sum_{i=1}^{2} \deg(u_i) + \deg(v_1) &\leqslant 18\binom{|M|}{2} + 9|M| + |M_1|(4|U_3| + 7) + 3|M_2|(|U_3| + 1)\\ &= 18\binom{|M|}{2} + 9|M| + (3s - |U|)(4(3s - 3|M|) + 7)\\ &\quad + 3(|M| + |U| - 3s)(3s - 3|M| + 1)\\ &= (3|U| + 3)|M| - 3s|U| - 4|U| + 9s^2 + 12s.\end{aligned}$$

Since $|M| \leqslant s - 1$ and $|U| \geqslant 2s$, it follows that

$$\begin{aligned}\sum_{i=1}^{2} \deg(u_i) + \deg(v_1) &\leqslant (3|U| + 3)(s - 1) - 3s|U| - 4|U| + 9s^2 + 12s\\ &= -7|U| + 9s^2 + 15s - 3 \leqslant 9s^2 + s - 3.\end{aligned}$$

Following (7), we have $\sum_{i=1}^{2} \deg(u_i) + \deg(v_1) < 3sn - 3\eta_1 n(n - 3\eta_1 n) + n/3 - 3$. Since $\varepsilon \ll \eta_1$ and n is sufficiently large, this contradicts (5).

If $|U_3| < 2|W_2| - 2$, by (8), Claim 18 (i) and Claim 19 (ii), we have

$$\begin{aligned}\sum_{i=1}^{2} \deg(u_i) + \deg(v_1) &\leqslant 18\binom{|M|}{2} + 9|M| + |M_1|\,(3|U_3| + 2|W_2| + 5) + 3|M_2|(|U_3| + 1)\\ &= (9s + 3)|M| + (-2n + 6s - 2)|U| + 6sn - 18s^2 + 6s.\end{aligned}$$

Since $|M| \leqslant s - 1$ and $|U| \geqslant 2s$, it follows that

$$\begin{aligned}\sum_{i=1}^{2} \deg(u_i) + \deg(v_1) &\leqslant (9s + 3)(s - 1) + (-2n + 6s - 2)(2s) + 6sn - 18s^2 + 6s\\ &= 2sn + 3s^2 - 4s - 3.\end{aligned}$$

Applying (7), we have $\sum_{i=1}^{2} \deg(u_i) + \deg(v_1) < 3sn - \eta_1 n(n - 3\eta_1 n)$, which contradicts (5) because $\varepsilon \ll \eta_1$. □

By Claim 20, we have $|W_2| = n - 3s < 3\eta_1 n$. Let $H' = H[V \setminus W_2]$. We claim that $\sigma_2(H') > 2n^2/3 - \eta_2 n^2$. Indeed, recall that $\deg_H(u) + \deg_H(v) \geqslant 2n^2/3 - \varepsilon n^2$ for any two adjacent vertices u and v of H'. Since $|W_2| < 3\eta_1 n$ and $\varepsilon \ll \eta_1 \ll \eta_2$, it follows that

$$\deg_{H'}(u) + \deg_{H'}(v) \geqslant 2n^2/3 - \varepsilon n^2 - 2|W_2|n > 2n^2/3 - \eta_2 n^2.$$

Since $\eta_2 \ll 1$, we may apply Theorem 9 and conclude that either H' is a subgraph of $H^2_{3s,s}$ or H' contains a perfect matching. In the former case, there is a partition of $V(H')$ into two sets $|T| = 2s - 1$ and $|S| = s + 1$ such that for every vertex $u \in S$,

$$\deg_{H'}(u) \leqslant \binom{|T|}{2} = \binom{2s-1}{2} \leqslant \binom{2n/3 - 1}{2} < \frac{2}{9}n^2.$$

On the other hand, since $U \subseteq V(H')$ and $|U| \geqslant 2s$, there exists a vertex $u \in U \cap S$ such that

$$\begin{aligned}\deg_{H'}(u) &\geqslant \deg_H(u) - |W_2|n \geqslant sn - \frac{\varepsilon}{2}n^2 - |W_2|n \\ &\geqslant \left(\frac{n}{3} - \eta_1 n\right) n - \frac{\varepsilon}{2}n^2 - 3\eta_1 n^2 > \frac{2}{9}n^2,\end{aligned}$$

which is a contradiction. Therefore H' must contain a perfect matching, which is a matching of size s in H.

Case 2. $|M_1| > 3s - |U|$.

The difference from Case 1 is that, for any edge $e \in M$, we cannot find two disjoint edges e_1, e_2 from $e \cup U_3 \cup W_2$ – otherwise we can replace M by $M \backslash \{e\} \cup \{e_1, e_2\}$ contradicting the assumption that M is an optimal matching.

Note that $|U_3| = |U| + |M_1| - 3|M| \geqslant 3s + 1 - 3|M| \geqslant 4$.

Claim 21. *For any* $e \in M$*,* $\sum_{i=1}^{3} |L_{u_i}(e, U_3 \cup W_2)| \leqslant 3(|U_3| + |W_2| + 2)$.

Proof. Assume $e = \{u_1', u_2', u_3'\} \in M$. For $i = 1, 2, 3$, let G_i be the graph obtained from $L_{u_i'}(U_3 \cup W_2)$ after adding an isolated vertex u^*. Then $|V(G_i)| = |U_3| + |W_2| + 1 \geqslant 5$. Since H contains no two disjoint edges e_1, e_2 from $e \cup U_3 \cup W_2$, we know that for any $i \neq j$, every edge of G_i intersects every edge of G_j. The desired inequality thus follows from Lemma 12. □

By Claims 17 and 21, we obtain that

$$\begin{aligned}\sum_{i=1}^{3} \deg(u_i) &\leqslant 18\binom{|M|}{2} + 9|M| + \sum_{i=1}^{3} |L_{u_i}(V(M), U_3 \cup W_2)| \\ &\leqslant 18\binom{|M|}{2} + 9|M| + 3|M|\left(|U_3| + |W_2| + 2\right) \\ &= (3n+6)|M| \leqslant 3sn + 6s. \end{aligned} \tag{9}$$

Let $W' = \{v \in W : \deg(v) \leqslant sn - s^2/2 + \gamma' n^2\}$. If $|W'| \leqslant \gamma n$, then we let $H' := H[V \setminus W']$. By the definition of W', $\deg_H(u) > sn - s^2/2 + \gamma' n^2$ for every $u \in V(H') \cap W$. For any $u \in V(H') \cap U$, $\deg_H(u) > sn - \varepsilon n^2/2 > sn - s^2/2 + \gamma' n^2$ because $s > \tau n$ and $\varepsilon \ll \gamma' \ll \tau$. Therefore every vertex $u \in V(H')$ satisfies

$$\deg_{H'}(u) \geqslant \deg_H(u) - n|W'| > sn - \frac{s^2}{2} + \gamma' n^2 - \gamma n^2 > \binom{n-1}{2} - \binom{n-s}{2} + 1,$$

because $|W'| \leqslant \gamma n$, $\gamma \ll \gamma'$, and n is sufficiently large. By Theorem 1, H' contains a matching of size s.

We thus assume that $|W'| > \gamma n$ for the rest of the proof. If one of u_1, u_2, u_3 is adjacent to a vertex of W', then

$$\sum_{i=1}^{3} \deg(u_i) > 4\left(sn - \frac{\varepsilon}{2}n^2\right) - \left(sn - \frac{s^2}{2} + \gamma' n^2\right) = 3sn + \frac{s^2}{2} - 2\varepsilon n^2 - \gamma' n^2,$$

which contradicts (9) because $s > \tau n$ is sufficiently large and $\varepsilon \ll \gamma' \ll \tau$.

If none of u_1, u_2, u_3 is adjacent to a vertex of W', then we distinguish the following two subcases.

Subcase 2.1. $|W' \cap W_1| > \gamma n/2$.

Let $M' = \{e \in M : e \cap W' \neq \emptyset\}$, thus $|M'| > \gamma n/2$. Since u_1, u_2, u_3 are not adjacent to any vertex in $W' \cap W_1$, then for any distinct e_1, e_2 from M', we have

$$\sum_{i=1}^{3} |L_{u_i}(e_1, e_2)| \leqslant 12. \tag{10}$$

By Claims 17, 21 and (10), we have

$$\begin{aligned}\sum_{i=1}^{3} \deg(u_i) &\leqslant \left(18\binom{|M|}{2} - 6\binom{|M'|}{2}\right) + 9|M| + 3|M|\,(n - 3|M| + 2) \\ &\leqslant (3n+6)|M| - 6\binom{|M'|}{2}.\end{aligned}$$

Since $|M'| > \gamma n/2$, it follows that

$$\sum_{i=1}^{3} \deg(u_i) \leqslant (3n+6)(s-1) - 6\binom{\gamma n/2}{2},$$

which contradicts (4) because $s \leqslant n/3$ and $\varepsilon \ll \gamma$.

Subcase 2.2. $|W' \cap W_1| \leqslant \gamma n/2$.

Since $|W'| > \gamma n$, we have $|W' \cap W_2| > \gamma n/2$. Let $W_2^* = W_2 \backslash W'$. Then $W_2 \backslash W_2^* = W' \cap W_2$. By Claim 21, we obtain that $\sum_{i=1}^{3} |L_{u_i}(V(M), U_3 \cup W_2^*)| \leqslant 3|M|\,(|U_3| + |W_2^*| + 2)$. Therefore,

$$\begin{aligned}\sum_{i=1}^{3} \deg(u_i) &\leqslant 18\binom{|M|}{2} + 9|M| + \sum_{i=1}^{3} |L_{u_i}(V(M), U_3 \cup W_2^*)| \\ &\leqslant 18\binom{|M|}{2} + 9|M| + 3|M|\,(|U_3| + |W_2^*| + 2) \\ &= 18\binom{|M|}{2} + 9|M| + 3|M|\,(|U_3| + |W_2| + 2) - 3|M||W_2 \setminus W_2^*| \\ &= \left(3n + 6 - \frac{3}{2}\gamma n\right)|M|,\end{aligned}$$

which contradicts (4) because $|M| \leqslant s$, $\tau n < s$, and $\varepsilon \ll \gamma \ll \tau$. This completes the proof of Theorem 6.

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