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Minimum codegree threshold for Hamilton ℓ-cycles in k-uniform hypergraphs ☆

Jie Han, Yi Zhao

Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, United States

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ABSTRACT

For $1 \leq \ell < k/2$, we show that for sufficiently large n, every k-uniform hypergraph on n vertices with minimum codegree at least $\frac{n}{2(k-\ell)}$ contains a Hamilton ℓ-cycle. This codegree condition is best possible and improves on work of Hàn and Schacht who proved an asymptotic result.

1. Introduction

A well-known result of Dirac [4] states that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamilton cycle. In recent years, researchers have worked on extending this result to hypergraphs — see recent surveys [15,18]. Given $k \geq 2$, a k-uniform hypergraph (in short, *k-graph*) consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$, where every edge is a k-element subset of V. Given a k-graph $\mathcal{H}$ with a set S of d vertices (where $1 \leq d \leq k-1$) we define $\deg_{\mathcal{H}}(S)$ to be the number of edges containing S (the subscript $\mathcal{H}$ is omitted if it is clear from the context). The *minimum*

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E-mail addresses: jhan22@gsu.edu (J. Han), yzhao6@gsu.edu (Y. Zhao).

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d-degree $\delta_d(\mathcal{H})$ of $\mathcal{H}$ is the minimum of $\deg_{\mathcal{H}}(S)$ over all d-vertex sets S in $\mathcal{H}$. We refer to $\delta_1(\mathcal{H})$ as the *minimum vertex degree* and $\delta_{k-1}(\mathcal{H})$ the *minimum codegree* of $\mathcal{H}$. For $1 \le \ell < k$, a k-graph is a called an *ℓ-cycle* if its vertices can be ordered cyclically so that each of its edges consists of k consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly ℓ vertices. In k-graphs, a $(k-1)$-cycle is often called a *tight* cycle while a 1-cycle is often called a *loose* cycle. We say that a k-graph contains a *Hamilton ℓ-cycle* if it contains an ℓ-cycle as a spanning subhypergraph. Since a k-uniform ℓ-cycle on n vertices contains exactly $n/(k-\ell)$ edges, a necessary condition for a k-graph on n vertices to contain a Hamilton ℓ-cycle is that $k-\ell$ divides n.

Confirming a conjecture of Katona and Kierstead [11], Rödl, Ruciński and Szemerédi [19,20] showed that for any fixed k, every k-graph $\mathcal{H}$ on n vertices with $\delta_{k-1}(\mathcal{H}) \ge n/2 + o(n)$ contains a tight Hamilton cycle. When $k-\ell$ divides both k and $|V|$, a $(k-1)$-cycle on V trivially contains an ℓ-cycle on V. Thus the result in [20] implies that for all $1 \le \ell < k$ such that $k-\ell$ divides k, every k-graph $\mathcal{H}$ on $n \in (k-\ell)\mathbb{N}$ vertices with $\delta_{k-1}(\mathcal{H}) \ge n/2 + o(n)$ contains a Hamilton ℓ-cycle. It is not hard to see that these results are best possible up to the $o(n)$ term — see concluding remarks in Section 4 for more discussion. With long and involved arguments, Rödl, Ruciński and Szemerédi [21] determined the minimum codegree threshold for tight Hamilton cycles in 3-graphs for sufficiently large n. (Unless stated otherwise, we assume that n is sufficiently large throughout the paper.)

Loose Hamilton cycles were first studied by Kühn and Osthus [14], who proved that every 3-graph on n vertices with $\delta_2(\mathcal{H}) \ge n/4 + o(n)$ contains a loose Hamilton cycle. This was generalized to arbitrary k and $\ell = 1$ by Keevash, Kühn, Mycroft, and Osthus [12] and to arbitrary k and arbitrary $\ell < k/2$ by Hàn and Schacht [7].

Theorem 1.1. *(See [7].) Fix integers $k \ge 3$ and $1 \le \ell < k/2$. Assume that $\gamma > 0$ and $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $\mathcal{H} = (V, E)$ is a k-graph on n vertices such that $\delta_{k-1}(\mathcal{H}) \ge (\frac{1}{2(k-\ell)} + \gamma)n$, then $\mathcal{H}$ contains a Hamilton ℓ-cycle.*

Later Kühn, Mycroft, and Osthus [13] proved that whenever $k-\ell$ does not divide k, every k-graph on n vertices with $\delta_{k-1}(\mathcal{H}) \ge \frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} + o(n)$ contains a Hamilton ℓ-cycle. This generalizes Theorem 1.1 because $\lceil k/(k-\ell) \rceil = 2$ when $\ell < k/2$. Rödl and Ruciński [18, Problem 2.9] asked for the exact minimum codegree threshold for Hamilton ℓ-cycles in k-graphs. The $k = 3$ and $\ell = 1$ case was answered by Czygrinow and Molla [3] recently. In this paper we determine this threshold for all $k \ge 3$ and $\ell < k/2$.

Theorem 1.2 *(Main result). Fix integers $k \ge 3$ and $1 \le \ell < k/2$. Assume that $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $\mathcal{H} = (V, E)$ is a k-graph on n vertices such that*

$$\delta_{k-1}(\mathcal{H}) \ge \frac{n}{2(k-\ell)}, \tag{1.1}$$

then $\mathcal{H}$ contains a Hamilton ℓ-cycle.

The following simple construction [13, Proposition 2.2] shows that Theorem 1.2 is best possible, and the aforementioned results in [7,12–14] are asymptotically best possible. Let $\mathcal{H}_0 = (V, E)$ be an n-vertex k-graph in which V is partitioned into sets A and B such that $|A| = \lceil \frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} \rceil - 1$. The edge set E consists of all k-sets that intersect A. It is easy to see that $\delta_{k-1}(\mathcal{H}_0) = |A|$. However, an ℓ-cycle on n vertices has $n/(k-\ell)$ edges and every vertex on such a cycle lies in at most $\lceil \frac{k}{k-\ell} \rceil$ edges. Since $\lceil \frac{k}{k-\ell} \rceil |A| < n/(k-\ell)$, $\mathcal{H}_0$ contains no Hamilton ℓ-cycle.

A related problem was studied by Buß, Hàn, and Schacht [1], who proved that every 3-graph $\mathcal{H}$ on n vertices with minimum vertex degree $\delta_1(\mathcal{H}) \geq (\frac{7}{16} + o(1))\binom{n}{2}$ contains a loose Hamilton cycle. Recently we [10] improved this to an exact result.

Using the typical approach of obtaining exact results, our proof of Theorem 1.2 consists of an *extremal case* and a *nonextremal case*.

Definition 1.3. *Let $\Delta > 0$, a k-graph $\mathcal{H}$ on n vertices is called Δ-extremal if there is a set $B \subset V(\mathcal{H})$, such that $|B| = \lfloor \frac{2(k-\ell)-1}{2(k-\ell)} n \rfloor$ and $e(B) \leq \Delta n^k$.*

Theorem 1.4 *(Nonextremal case). For any integer $k \geq 3$, $1 \leq \ell < k/2$ and $0 < \Delta < 1$ there exists $\gamma > 0$ such that the following holds. Suppose that $\mathcal{H}$ is a k-graph on n vertices such that $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $\mathcal{H}$ is not Δ-extremal and satisfies $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-\ell)} - \gamma)n$, then $\mathcal{H}$ contains a Hamilton ℓ-cycle.*

Theorem 1.5 *(Extremal case). For any integer $k \geq 3$, $1 \leq \ell < k/2$ there exists $\Delta > 0$ such that the following holds. Suppose $\mathcal{H}$ is a k-graph on n vertices such that $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $\mathcal{H}$ is Δ-extremal and satisfies (1.1), then $\mathcal{H}$ contains a Hamilton ℓ-cycle.*

Theorem 1.2 follows from Theorem 1.4 and 1.5 immediately by choosing Δ from Theorem 1.5.

Let us compare our proof with those in the aforementioned papers. There is no extremal case in [7,12–14] because only asymptotic results were proved. Our Theorem 1.5 is new and more general than [3, Theorem 3.1]. Following previous work [7,13,19–21], we prove Theorem 1.4 by using the absorbing method initiated by Rödl, Ruciński and Szemerédi. More precisely, we find the desired Hamilton ℓ-cycle by applying the Absorbing Lemma (Lemma 2.1), the Reservoir Lemma (Lemma 2.2), and the Path-cover Lemma (Lemma 2.3). In fact, when $\ell < k/2$, the Absorbing Lemma and the Reservoir Lemma are not very difficult and already proven in [7] (in contrast, when $\ell > k/2$, the Absorbing Lemma in [13] is more difficult to prove). Thus the main step is to prove the Path-cover Lemma. As shown in [7,13], after the Regularity Lemma is applied, it suffices to prove that the cluster k-graph $\mathcal{K}$ can be tiled almost perfectly by the k-graph $\mathcal{F}_{k,\ell}$, whose vertex set consists of disjoint sets $A_1, \dots, A_{a-1}, B$ of size $k-1$, and whose edges are all the k-sets of the form $A_i \cup \{b\}$ for $i = 1, \dots, a-1$ and all $b \in B$, where $a = \lceil \frac{k}{k-\ell} \rceil (k-\ell)$. In this paper we reduce the problem to tile $\mathcal{K}$ with a much simpler k-graph $\mathcal{Y}_{k,2\ell}$, which consists of two edges sharing 2ℓ vertices. Because of the simple structure of $\mathcal{Y}_{k,2\ell}$, we can easily find an almost perfect $\mathcal{Y}_{k,2\ell}$-tiling unless $\mathcal{K}$ is in the extremal case (thus the original k-graph $\mathcal{H}$ is in the extremal case). Interestingly $\mathcal{Y}_{3,2}$-tiling was studied in the very first paper [14] on loose Hamilton cycles but as a separate problem. Our recent paper [10] indeed used $\mathcal{Y}_{3,2}$-tiling as a tool to prove the corresponding path-cover lemma. On the other hand, the authors of [3] used a different approach (without the Regularity Lemma) to prove the Path-tiling Lemma (though they did not state such lemma explicitly).

The rest of the paper is organized as follows. We prove Theorem 1.4 in Section 2 and Theorem 1.5 in Section 3, and give concluding remarks in Section 4.

Notation. Given an integer $k \geq 0$, a k-set is a set with k elements. For a set X, we denote by $\binom{X}{k}$ the family of all k-subsets of X. Given a k-graph $\mathcal{H}$ and a set $A \subseteq V(\mathcal{H})$, we denote by $e_{\mathcal{H}}(A)$ the number of the edges of $\mathcal{H}$ in A. We often omit the subscript that represents the underlying hypergraph if it is clear from the context. Given a k-graph $\mathcal{H}$ with two vertex sets S, R such that $|S| < k$, we denote by $\deg_{\mathcal{H}}(S, R)$ the number of $(k - |S|)$-sets $T \subseteq R$ such that $S \cup T$ is an edge of $\mathcal{H}$ (in this case, T is called a *neighbor* of S). We define $\overline{\deg}_{\mathcal{H}}(S, R) = \binom{|R \setminus S|}{k-|S|} - \deg(S, R)$ as the number of *non-edges* on $S \cup R$ that contain S. When $R = V(\mathcal{H})$ (and $\mathcal{H}$ is obvious), we simply write $\deg(S)$ and $\overline{\deg}(S)$. When $S = \{v\}$, we use $\deg(v, R)$ instead of $\deg(\{v\}, R)$.

A k-graph $\mathcal{P}$ is an *ℓ-path* if there is an ordering $(v_1, \ldots, v_t)$ of its vertices such that every edge consists of k consecutive vertices and two consecutive edges intersect in exactly ℓ vertices. Note that this implies that $k-\ell$ divides $t-\ell$. In this case, we write $\mathcal{P} = v_1 \cdots v_t$ and call two ℓ-sets $\{v_1, \ldots, v_\ell\}$ and $\{v_{t-\ell+1}, \ldots, v_t\}$ *ends* of $\mathcal{P}$.

2. Proof of Theorem 1.4

In this section we prove Theorem 1.4 by following the approach in [7].

2.1. Auxiliary lemmas and proof of Theorem 1.4

We need [7, Lemma 5] and [7, Lemma 6] of Hàn and Schacht, in which only a linear codegree condition is needed. Given a k-graph $\mathcal{H}$ with an ℓ-path $\mathcal{P}$ and a vertex set $U \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ with $|U| \in (k-\ell)\mathbb{N}$, we say that $\mathcal{P}$ *absorbs* U if there exists an ℓ-path $\mathcal{Q}$ of $\mathcal{H}$ with $V(\mathcal{Q}) = V(\mathcal{P}) \cup U$ such that $\mathcal{P}$ and $\mathcal{Q}$ have exactly the same ends.

Lemma 2.1 *(Absorbing Lemma). (See [7].) For all integers $k \geq 3$ and $1 \leq \ell < k/2$ and every $\gamma_1 > 0$ there exist $\eta > 0$ and an integer n_0 such that the following holds. Let $\mathcal{H}$ be a k-graph on $n \geq n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq \gamma_1 n$. Then $\mathcal{H}$ contains an absorbing ℓ-path $\mathcal{P}$ with $|V(\mathcal{P})| \leq \gamma_1^5 n$ that can absorb any subset $U \subset V(\mathcal{H}) \setminus V(\mathcal{P})$ of size $|U| \leq \eta n$ and $|U| \in (k-\ell)\mathbb{N}$.*

Lemma 2.2 *(Reservoir Lemma). (See [7].) For all integers $k \geq 3$ and $1 \leq \ell < k/2$ and every $0 < d, \gamma_2 < 1$ there exists an n_0 such that the following holds. Let $\mathcal{H}$ be a k-graph on $n > n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq dn$, then there is a set R of size at most $\gamma_2 n$ such that for all $(k-1)$-sets $S \in \binom{V}{k-1}$ we have $\deg(S, R) \geq d\gamma_2 n/2$.*

The main step in our proof of Theorem 1.4 is the following lemma, which is stronger than [7, Lemma 7]. We defer its proof to the next subsection.

Lemma 2.3 *(Path-cover Lemma). For all integers $k \geq 3$, $1 \leq \ell < k/2$, and every $\gamma_3, \alpha > 0$ there exist integers p and n_0 such that the following holds. Let $\mathcal{H}$ be a k-graph on $n > n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-\ell)} - \gamma_3)n$, then there is a family of at most p vertex disjoint ℓ-paths that together cover all but at most αn vertices of $\mathcal{H}$, or $\mathcal{H}$ is $14\gamma_3$-extremal.*

We can now prove Theorem 1.4 in a similar way as in [7].

Proof of Theorem 1.4. Given $k \geq 3$, $1 \leq \ell < k/2$ and $0 < \Delta < 1$, let $\gamma = \min\{\frac{\Delta}{43}, \frac{1}{4k^2}\}$ and $n \in (k-\ell)\mathbb{N}$ be sufficiently large. Suppose that $\mathcal{H} = (V, E)$ is a k-graph on n vertices with $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-\ell)} - \gamma)n$. Since $\frac{1}{2(k-\ell)} - \gamma > \gamma$, we can apply Lemma 2.1 with $\gamma_1 = \gamma$ and obtain $\eta > 0$ and an absorbing path $\mathcal{P}_0$ with ends S_0, T_0 such that $|V(\mathcal{P}_0)| \leq \gamma^5 n$ and $\mathcal{P}_0$ can absorb any u vertices outside $\mathcal{P}_0$ if $u \leq \eta n$ and $u \in (k-\ell)\mathbb{N}$.

Let $V_1 = (V \setminus V(\mathcal{P}_0)) \cup S_0 \cup T_0$ and $\mathcal{H}_1 = \mathcal{H}[V_1]$. Note that $|V(\mathcal{P}_0)| \leq \gamma^5 n$ implies that $\delta_{k-1}(\mathcal{H}_1) \geq (\frac{1}{2(k-\ell)} - \gamma)n - \gamma^5 n \geq \frac{1}{2k}n$ as $\gamma < \frac{1}{4k^2}$ and $\ell \geq 1$. We next apply Lemma 2.2 with $d = \frac{1}{2k}$ and $\gamma_2 = \min\{\eta/2, \gamma\}$ to $\mathcal{H}_1$ and get a reservoir $R \subset V_1$ with $|R| \leq \gamma_2 |V(\mathcal{H}_1)| \leq \gamma_2 n$ such that for any $(k-1)$-set $S \subset V_1$, we have

$$\deg(S, R) \geq d\gamma_2 |V_1|/2 \geq d\gamma_2 n/4. \tag{2.1}$$

Let $V_2 = V \setminus (V(\mathcal{P}_0) \cup R)$, $n_2 = |V_2|$, and $\mathcal{H}_2 = \mathcal{H}[V_2]$. Note that $|V(\mathcal{P}_0) \cup R| \leq \gamma_1^5 n + \gamma_2 n \leq 2\gamma n$, so

$$\delta_{k-1}(\mathcal{H}_2) \geq \left(\frac{1}{2(k-\ell)} - \gamma\right)n - 2\gamma n \geq \left(\frac{1}{2(k-\ell)} - 3\gamma\right)n_2.$$

Applying Lemma 2.3 to $\mathcal{H}_2$ with $\gamma_3 = 3\gamma$ and $\alpha = \eta/2$, we obtain at most p vertex disjoint ℓ-paths that cover all but at most αn_2 vertices of $\mathcal{H}_2$, unless $\mathcal{H}_2$ is $14\gamma_3$-extremal. In the latter case, there exists $B' \subseteq V_2$ such that $|B'| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n_2 \rfloor$ and $e(B') \leq 42\gamma n_2^k$. Then we add at most $n - n_2 \leq 2\gamma n$ vertices from $V \setminus B'$ to B' and obtain a vertex set $B \subseteq V(\mathcal{H})$ such that $|B| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \rfloor$ and

$$e(B) \leq 42\gamma n_2^k + 2\gamma n \cdot \binom{n-1}{k-1} \leq 42\gamma n^k + \gamma n^k \leq \Delta n^k,$$

which means that $\mathcal{H}$ is Δ-extremal, a contradiction. In the former case, denote these ℓ-paths by $\{\mathcal{P}_i\}_{i \in [p']}$ for some $p' \leq p$, and their ends by $\{S_i, T_i\}_{i \in [p']}$. Note that both S_i and T_i are ℓ-sets for $\ell < k/2$. We arbitrarily pick disjoint $(k-2\ell-1)$-sets $X_0, X_1, \dots, X_{p'} \subset R \setminus (S_0 \cup T_0)$ (note that $k-2\ell-1 \geq 0$). Let $T_{p'+1} = T_0$. By (2.1), as $d\gamma_2 n/4 \geq k(p'+1)$, we may find $p'+1$ vertices $v_0, v_1, \dots, v_{p'} \in R$ such that $S_i \cup T_{i+1} \cup X_i \cup \{v_i\} \in E(\mathcal{H})$ for $0 \leq i \leq p'$. We thus connect $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{p'}$ together and obtain an ℓ-cycle $\mathcal{C}$. Note that

$$\left|V(\mathcal{H}) \setminus V(\mathcal{C})\right| \leq |R| + \alpha n_2 \leq \gamma_2 n + \alpha n \leq \eta n$$

and $k-\ell$ divides $|V \setminus V(\mathcal{C})|$ because $k-\ell$ divides both n and $|V(\mathcal{C})|$. So we can use $\mathcal{P}_0$ to absorb all unused vertices in R and uncovered vertices in V_2 thus obtaining a Hamilton ℓ-cycle in $\mathcal{H}$. □

The rest of this section is devoted to the proof of Lemma 2.3.

2.2. *Proof of Lemma 2.3*

Following the approach in [7], we use the Weak Regularity Lemma, which is a straightforward extension of Szemerédi's regularity lemma for graphs [22].

Let $\mathcal{H} = (V, E)$ be a k-graph and let $A_1, \dots, A_k$ be mutually disjoint non-empty subsets of V. We define $e(A_1, \dots, A_k)$ to be the number of *crossing* edges, namely, those with one vertex in each A_i, $i \in [k]$, and the density of $\mathcal{H}$ with respect to $(A_1, \dots, A_k)$ as

$$d(A_1, \dots, A_k) = \frac{e(A_1, \dots, A_k)}{|A_1| \cdots |A_k|}.$$

We say a k-tuple $(V_1, \dots, V_k)$ of mutually disjoint subsets $V_1, \dots, V_k \subseteq V$ is (ϵ, d)-*regular*, for $\epsilon > 0$ and $d \geq 0$, if

$$\left|d(A_1, \dots, A_k) - d\right| \leq \epsilon$$

for all k-tuples of subsets $A_i \subseteq V_i$, $i \in [k]$, satisfying $|A_i| \geq \epsilon |V_i|$. We say $(V_1, \dots, V_k)$ is ϵ-*regular* if it is (ϵ, d)-regular for some $d \geq 0$. It is immediate from the definition that in an (ϵ, d)-regular k-tuple $(V_1, \dots, V_k)$, if $V_i' \subset V_i$ has size $|V_i'| \geq c|V_i|$ for some $c \geq \epsilon$, then $(V_1', \dots, V_k')$ is $(\epsilon/c, d)$-regular.

Theorem 2.4 *(Weak Regularity Lemma). Given $t_0 \geq 0$ and $\epsilon > 0$, there exist $T_0 = T_0(t_0, \epsilon)$ and $n_0 = n_0(t_0, \epsilon)$ so that for every k-graph $\mathcal{H} = (V, E)$ on $n > n_0$ vertices, there exists a partition $V = V_0 \cup V_1 \cup \dots \cup V_t$ such that*

(i) *$t_0 \leq t \leq T_0$,*
(ii) *$|V_1| = |V_2| = \dots = |V_t|$ and $|V_0| \leq \epsilon n$,*
(iii) *for all but at most $\epsilon \binom{t}{k}$ k-subsets $\{i_1, \dots, i_k\} \subset [t]$, the k-tuple $(V_{i_1}, \dots, V_{i_k})$ is ϵ-regular.*

The partition given in Theorem 2.4 is called an *ϵ-regular partition* of $\mathcal{H}$. Given an ϵ-regular partition of $\mathcal{H}$ and $d \geq 0$, we refer to V_i, $i \in [t]$, as *clusters* and define the *cluster hypergraph* $\mathcal{K} = \mathcal{K}(\epsilon, d)$ with vertex set $[t]$ and $\{i_1, \dots, i_k\} \subset [t]$ is an edge if and only if $(V_{i_1}, \dots, V_{i_k})$ is ϵ-regular and $d(V_{i_1}, \dots, V_{i_k}) \geq d$.

We combine Theorem 2.4 and [7, Proposition 16] into the following corollary, which shows that the cluster hypergraph almost inherits the minimum degree of the original hypergraph. Its proof is standard and similar as the one of [7, Proposition 16] so we omit it.[1]

Corollary 2.5. *(See [7].) Given $c, \epsilon, d > 0$, integers $k \geq 3$ and t_0, there exist T_0 and n_0 such that the following holds. Let $\mathcal{H}$ be a k-graph on $n > n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq cn$. Then $\mathcal{H}$ has an ϵ-regular partition $V_0 \cup V_1 \cup \dots \cup V_t$ with $t_0 \leq t \leq T_0$, and in the cluster hypergraph $\mathcal{K} = \mathcal{K}(\epsilon, d)$, all but at most $\sqrt{\epsilon} t^{k-1}$ $(k-1)$-subsets S of $[t]$ satisfy $\deg_{\mathcal{K}}(S) \geq (c - d - \sqrt{\epsilon})t - (k-1)$.*

Let $\mathcal{H}$ be a k-partite k-graph with partition classes $V_1, \dots, V_k$. Given $1 \leq \ell < k/2$, we call an ℓ-path $\mathcal{P}$ with edges $\{e_1, \dots, e_q\}$ *canonical* with respect to $(V_1, \dots, V_k)$ if

$$e_i \cap e_{i+1} \subseteq \bigcup_{j \in [\ell]} V_j \quad \text{or} \quad e_i \cap e_{i+1} \subseteq \bigcup_{j \in [2\ell] \setminus [\ell]} V_j$$

for $i \in [q-1]$. When $j > 2\ell$, all $e_1 \cap V_j, \dots, e_q \cap V_j$ are distinct and thus $|V(\mathcal{P}) \cap V_j| = |(e_1 \cup \dots \cup e_q) \cap V_j| = q$. When $j \leq 2\ell$, exactly one of $e_{i-1} \cap e_i$ and $e_i \cap e_{i+1}$ intersects V_j. Thus $|V(\mathcal{P}) \cap V_j| = \frac{q+1}{2}$ if q is odd.

We need the following proposition from [7].

Proposition 2.6. *[7, Proposition 19] Suppose that $1 \leq \ell < k/2$ and $\mathcal{H}$ is a k-partite, k-graph with partition classes $V_1, \dots, V_k$ such that $|V_i| = m$ for all $i \in [k]$, and $|E(\mathcal{H})| \geq dm^k$. Then there exists a canonical ℓ-path in $\mathcal{H}$ with $t > \frac{dm}{2(k-\ell)}$ edges.*

In [7] the authors used Proposition 2.6 to cover an (ϵ, d)-regular tuple $(V_1, \dots, V_k)$ of sizes $|V_1| = \dots = |V_{k-1}| = (2k - 2\ell - 1)m$ and $|V_k| = (k-1)m$ with vertex disjoint ℓ-paths. Our next lemma shows that an (ϵ, d)-regular tuple $(V_1, \dots, V_k)$ of sizes $|V_1| = \dots = |V_{2\ell}| = m$ and $|V_i| = 2m$ for $i > 2\ell$ can be covered with ℓ-paths.

Lemma 2.7. *Fix $k \geq 3$, $1 \leq \ell < k/2$ and $\epsilon, d > 0$ such that $d > 2\epsilon$. Let $m > \frac{k^2}{\epsilon^2(d-\epsilon)}$. Suppose $\mathcal{V} = (V_1, V_2, \dots, V_k)$ is an (ϵ, d)-regular k-tuple with*

$$|V_1| = \dots = |V_{2\ell}| = m \quad \text{and} \quad |V_{2\ell+1}| = \dots = |V_k| = 2m. \tag{2.2}$$

[1] Roughly speaking, the lower bound for $\deg_{\mathcal{K}}(S)$ contains $-d$ because when forming $\mathcal{K}$, we discard all k-tuple $(V_{i_1}, \dots, V_{i_k})$ of density less than d, contains $-\sqrt{\epsilon}$ because at most $\epsilon\binom{t}{k}$ k-tuple are not regular, and contains $-(k-1)$ because we discard all non-crossing edges of $\mathcal{H}$.

Then there are at most $\frac{2k}{(d-\epsilon)\epsilon}$ *vertex-disjoint* ℓ*-paths that together cover all but at most* $2k\epsilon m$ *vertices of* $\mathcal{V}$*.*

Proof. We greedily find vertex-disjoint canonical ℓ-paths of odd length by Proposition 2.6 in $\mathcal{V}$ until less than ϵm vertices are uncovered in V_1 as follows. Suppose that we have obtained ℓ-paths $\mathcal{P}_1, \dots, \mathcal{P}_p$ for some $p \geq 0$. Let $q = \sum_{j=1}^{p} e(\mathcal{P}_j)$. Assume that for all j, $\mathcal{P}_j$ is canonical with respect to $\mathcal{V}$ and $e(\mathcal{P}_j)$ is odd. Then $\bigcup_{j=1}^{p} \mathcal{P}_i$ contains $\frac{q+p}{2}$ vertices of V_i for $i \in [2\ell]$ and q vertices of V_i for $i > 2\ell$. For $i \in [k]$, let U_i be the set of uncovered vertices of V_i and assume that $|U_1| \geq \epsilon m$. Using (2.2), we derive that $|U_1| = \cdots = |U_{2\ell}| \geq \epsilon m$ and

$$|U_{2\ell+1}| = \cdots = |U_k| = 2|U_1| + p. \tag{2.3}$$

We now consider a k-partite subhypergraph $\mathcal{V}'$ with arbitrary $|U_1|$ vertices in each U_i for $i \in [k]$. By regularity, $\mathcal{V}'$ contains at least $(d-\epsilon)|U_1|^k$ edges, so we can apply Proposition 2.6 and find an ℓ-path of odd length at least $\frac{(d-\epsilon)\epsilon m}{2(k-\ell)} - 1 \geq \frac{(d-\epsilon)\epsilon m}{2k}$ (dismiss one edge if needed). We continue this process until $|U_1| < \epsilon m$. Let $\mathcal{P}_1, \dots, \mathcal{P}_p$ be the ℓ-paths obtained in $\mathcal{V}$ after the iteration stops. Since $|V_1 \cap V(\mathcal{P}_j)| \geq \frac{(d-\epsilon)\epsilon m}{2k}$ for every j, we have

$$p \leq \frac{m}{\frac{(d-\epsilon)\epsilon m}{2k}} = \frac{2k}{(d-\epsilon)\epsilon}.$$

Since $m > \frac{k^2}{\epsilon^2(d-\epsilon)}$, it follows that $p(k-2\ell) < \frac{2k^2}{(d-\epsilon)\epsilon} < 2\epsilon m$. By (2.3), the total number of uncovered vertices in $\mathcal{V}$ is

$$\begin{aligned}\sum_{i=1}^{k} |U_i| &= |U_1| 2\ell + \big(2|U_1| + p\big)(k-2\ell) = 2(k-\ell)|U_1| + p(k-2\ell) \\ &< 2(k-1)\epsilon m + 2\epsilon m = 2k\epsilon m. \qquad \square\end{aligned}$$

Given $k \geq 3$ and $0 \leq b < k$, let $\mathcal{Y}_{k,b}$ be a k-graph with two edges that share exactly b vertices. In general, given two (hyper)graphs $\mathcal{G}$ and $\mathcal{H}$, a $\mathcal{G}$*-tiling* is a sub(hyper)graph of $\mathcal{H}$ that consists of vertex-disjoint copies of $\mathcal{G}$. A $\mathcal{G}$-tiling is *perfect* if it is a spanning sub(hyper)graph of $\mathcal{H}$. The following lemma is the main step in our proof of Lemma 2.3 and we prove it in the next subsection. Note that it generalizes [2, Lemma 3.1] of Czygrinow, DeBiasio, and Nagle.

Lemma 2.8 *($\mathcal{Y}_{k,b}$-tiling lemma). Given integers* $k \geq 3$, $1 \leq b < k$ *and constants* $\gamma, \beta > 0$*, there exist* $0 < \epsilon' < \gamma\beta$ *and an integer* n' *such that the following holds. Suppose* $\mathcal{H}$ *is a* k*-graph on* $n > n'$ *vertices with* $\deg(S) \geq (\frac{1}{2k-b} - \gamma)n$ *for all but at most* $\epsilon' n^{k-1}$ *sets* $S \in \binom{V}{k-1}$*, then there is a* $\mathcal{Y}_{k,b}$*-tiling that covers all but at most* βn *vertices of* $\mathcal{H}$ *unless* $\mathcal{H}$ *contains a vertex set* B *such that* $|B| = \lfloor \frac{2k-b-1}{2k-b} n \rfloor$ *and* $e(B) < 6\gamma n^k$*.*

Now we are ready to prove Lemma 2.3.

Proof of Lemma 2.3. Fix integers k, ℓ, $0 < \gamma_3, \alpha < 1$. Let ϵ', n' be the constants returned from Lemma 2.8 with $b = 2\ell$, $\gamma = 2\gamma_3$, and $\beta = \alpha/2$. Thus $\epsilon' < \gamma\beta = \gamma_3\alpha$. Let T_0 be the constant returned from Corollary 2.5 with $c = \frac{1}{2(k-\ell)} - \gamma_3$, $\epsilon = (\epsilon')^2/16$, $d = \gamma_3/2$ and $t_0 > \max\{n', 4k/\gamma_3\}$. Furthermore, let $p = \frac{2T_0}{(d-2\epsilon)\epsilon}$.

Let n be sufficiently large and let $\mathcal{H}$ be a k-graph on n vertices with $\delta_{k-1}(\mathcal{H}) \ge (\frac{1}{2(k-\ell)} - \gamma_3)n$. Applying Corollary 2.5 with the constants chosen above, we obtain an ϵ-regular partition and a cluster hypergraph $\mathcal{K} = \mathcal{K}(\epsilon, d)$ on $[t]$ such that for all but at most $\sqrt{\epsilon}t^{k-1}$ $(k-1)$-sets $S \in \binom{[t]}{k-1}$,

$$\deg_{\mathcal{K}}(S) \ge \left(\frac{1}{2(k-\ell)} - \gamma_3 - d - \sqrt{\epsilon}\right)t - (k-1) \ge \left(\frac{1}{2(k-\ell)} - 2\gamma_3\right)t,$$

because $d = \gamma_3/2$, $\sqrt{\epsilon} = \epsilon'/4 < \gamma_3/4$ and $k - 1 < \gamma_3 t_0/4 \le \gamma_3 t/4$. Let m be the size of clusters, then $(1-\epsilon)\frac{n}{t} \le m \le \frac{n}{t}$. Applying Lemma 2.8 with the constants chosen above, we derive that either there is a $\mathcal{Y}_{k,2\ell}$-tiling $\mathscr{Y}$ of $\mathcal{K}$ which covers all but at most βt vertices of $\mathcal{K}$ or there exists a set $B \subseteq V(\mathcal{K})$, such that $|B| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} t \rfloor$ and $e_{\mathcal{K}}(B) \le 12\gamma_3 t^k$. In the latter case, let $B' \subseteq V(\mathcal{H})$ be the union of the clusters in B. By regularity,

$$e_{\mathcal{H}}(B') \le e_{\mathcal{K}}(B) \cdot m^k + \binom{t}{k} \cdot d \cdot m^k + \epsilon \cdot \binom{t}{k} \cdot m^k + t\binom{m}{2}\binom{n}{k-2},$$

where the right-hand side bounds the number of edges from regular k-tuples with high density, edges from regular k-tuples with low density, edges from irregular k-tuples and edges that lie in at most $k-1$ clusters. Since $m \le \frac{n}{t}$, $\epsilon < \gamma_3/16$, $d = \gamma_3/2$, and $t^{-1} < t_0^{-1} < \gamma_3/(4k)$, we obtain that

$$\begin{aligned} e_{\mathcal{H}}(B') &\le 12\gamma_3 t^k \cdot \left(\frac{n}{t}\right)^k + \binom{t}{k}\frac{\gamma_3}{2}\left(\frac{n}{t}\right)^k + \frac{\gamma_3}{16}\binom{t}{k}\left(\frac{n}{t}\right)^k + t\binom{n/t}{2}\binom{n}{k-2} \\ &< 13\gamma_3 n^k. \end{aligned}$$

Note that $|B'| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} t \rfloor m \le \frac{2k-2\ell-1}{2(k-\ell)} t \cdot \frac{n}{t} = \frac{2k-2\ell-1}{2(k-\ell)} n$, and consequently $|B'| \le \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \rfloor$. On the other hand,

$$\begin{aligned} |B'| &= \left\lfloor \frac{2k-2\ell-1}{2(k-\ell)} t \right\rfloor m \ge \left(\frac{2k-2\ell-1}{2(k-\ell)} t - 1\right)(1-\epsilon)\frac{n}{t} \\ &\ge \left(\frac{2k-2\ell-1}{2(k-\ell)} t - \epsilon\frac{2k-2\ell-1}{2(k-\ell)} t - 1\right)\frac{n}{t} \\ &\ge \left(\frac{2k-2\ell-1}{2(k-\ell)} t - \epsilon t\right)\frac{n}{t} = \frac{2k-2\ell-1}{2(k-\ell)} n - \epsilon n. \end{aligned}$$

By adding at most ϵn vertices from $V \setminus B'$ to B', we get a set $B'' \subseteq V(\mathcal{H})$ of size exactly $\lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \rfloor$, with $e(B'') \le e(B') + \epsilon n \cdot n^{k-1} < 14\gamma_3 n^k$. Hence $\mathcal{H}$ is $14\gamma_3$-extremal.

In the former case, let $m' = \lfloor m/2 \rfloor$. If m is odd, we throw away one vertex from each cluster covered by $\mathscr{Y}$ (we do nothing if m is even). Thus, the union of the clusters covered by $\mathscr{Y}$ contains all but at most $\beta t m + |V_0| + t \le \alpha n/2 + 2\epsilon n$ vertices of $\mathcal{H}$. We take the following procedure to each member $\mathcal{Y}' \in \mathscr{Y}$. Suppose that $\mathcal{Y}'$ has the vertex set $[2k-2\ell]$ with edges $\{1, \dots, k\}$ and $\{k-2\ell+1, \dots, 2k-2\ell\}$. For $i \in [2k-2\ell]$, let W_i denote the corresponding cluster in $\mathcal{H}$. We split each W_i, $i = k-2\ell+1, \dots, k$, into two disjoint sets W_i^1 and W_i^2 of equal size. Then each of the k-tuples $(W_{k-2\ell+1}^1, \dots, W_k^1, W_1, \dots, W_{k-2\ell})$ and $(W_{k-2\ell+1}^2, \dots, W_k^2, W_{k+1}, \dots, W_{2k-2\ell})$ is $(2\epsilon, d')$-regular for some $d' \ge d$ and of sizes $m', \dots, m', 2m', \dots, 2m'$. Applying Lemma 2.7 to these two k-tuples, we find a family of at most $\frac{2k}{(d'-2\epsilon)2\epsilon} \le \frac{k}{(d-2\epsilon)\epsilon}$ disjoint loose paths in each k-tuple covering all but at most $2k(2\epsilon)m' \le 2k\epsilon m$ vertices. Since $|\mathscr{Y}| \le \frac{t}{2k-2\ell}$, we thus obtain a path-tiling that consists of at most $2\frac{t}{2k-2\ell}\frac{k}{(d-2\epsilon)\epsilon} \le \frac{2T_0}{(d-2\epsilon)\epsilon} = p$ paths and covers all but at most

$$2 \cdot 2k\epsilon m \cdot \frac{t}{2k-2\ell} + \alpha n/2 + 2\epsilon n < 6\epsilon n + \alpha n/2 < \alpha n$$

vertices of $\mathcal{H}$, where we use $2k - 2\ell > k$ and $\epsilon = (\epsilon')^2/16 < (\gamma_3\alpha)^2/16 < \alpha/12$. This completes the proof. □

2.3. Proof of Lemma 2.8

We first give an upper bound on the size of k-graphs containing no copy of $\mathcal{Y}_{k,b}$. In its proof, we use the concept of *link (hyper)graph*: given a k-graph $\mathcal{H}$ with a set S of at most $k-1$ vertices, the *link graph* of S is the $(k-|S|)$-graph with vertex set $V(\mathcal{H}) \setminus S$ and edge set $\{e \setminus S : e \in E(\mathcal{H}), S \subseteq e\}$. Throughout the rest of the paper, we frequently use the simple identity $\binom{m}{b}\binom{m-b}{k-b} = \binom{m}{k}\binom{k}{b}$, which holds for all integers $0 \le b \le k \le m$.

Fact 2.9. *Let $0 \le b < k$ and $m \ge 2k - b$. If $\mathcal{H}$ is a k-graph on m vertices containing no copy of $\mathcal{Y}_{k,b}$, then $e(\mathcal{H}) < \binom{m}{k-1}$.*

Proof. Fix any b-set $S \subseteq V(\mathcal{H})$ ($S = \emptyset$ if $b = 0$) and consider its link graph L_S. Since $\mathcal{H}$ contains no copy of $\mathcal{Y}_{k,b}$, any two edges of L_S intersect. Since $m \ge 2k - b$, the Erdős–Ko–Rado Theorem [5] implies that $|L_S| \le \binom{m-b-1}{k-b-1}$. Thus,

$$e(\mathcal{H}) \le \frac{1}{\binom{k}{b}}\binom{m}{b} \cdot \binom{m-b-1}{k-b-1} = \frac{1}{\binom{k}{b}}\binom{m}{b}\binom{m-b}{k-b}\frac{k-b}{m-b} = \binom{m}{k}\frac{k-b}{m-b}$$
$$= \binom{m}{k-1}\frac{k-b}{k}\,\frac{m-k+1}{m-b} < \binom{m}{k-1}. \qquad \square$$

Proof of Lemma 2.8. Given $\gamma, \beta > 0$, let $\epsilon' = \frac{\gamma\beta^{k-1}}{(k-1)!}$ and let $n \in \mathbb{N}$ be sufficiently large. Let $\mathcal{H}$ be a k-graph on n vertices that satisfies $\deg(S) \ge (\frac{1}{2k-b} - \gamma)n$ for all but at most

$\epsilon' n^{k-1}$ $(k-1)$-sets S. Let $\mathscr{Y} = \{\mathcal{Y}_1, \dots, \mathcal{Y}_m\}$ be a largest $\mathcal{Y}_{k,b}$-tiling in $\mathcal{H}$ (with respect to m) and write $V_i = V(\mathcal{Y}_i)$ for $i \in [m]$. Let $V' = \bigcup_{i\in[m]} V_i$ and $U = V(\mathcal{H}) \setminus V'$. Assume that $|U| > \beta n$ — otherwise we are done.

Let C be the set of vertices $v \in V'$ such that $\deg(v, U) \ge (2k-b)^2\binom{|U|}{k-2}$. We will show that $|C| \le \frac{n}{2k-b}$ and C covers almost all the edges of $\mathcal{H}$, which implies that $\mathcal{H}[V \setminus C]$ is sparse and $\mathcal{H}$ is in the extremal case. We first observe that every $\mathcal{Y}_i \in \mathscr{Y}$ contains at most one vertex in C. Suppose instead, two vertices $x, y \in V_i$ are both in C. Since $\deg(x, U) \ge (2k-b)^2\binom{|U|}{k-2} > \binom{|U|}{k-2}$, by Fact 2.9, there is a copy of $\mathcal{Y}_{k-1,b-1}$ in the link graph of x on U, which gives rise to $\mathcal{Y}'$, a copy of $\mathcal{Y}_{k,b}$ on $\{x\} \cup U$. Since the link graph of y on $U \setminus V(\mathcal{Y}')$ has at least

$$(2k-b)^2\binom{|U|}{k-2} - (2k-b-1)\binom{|U|}{k-2} > \binom{|U \setminus V(\mathcal{Y}')|}{k-2}$$

edges, we can find another copy of $\mathcal{Y}_{k,b}$ on $\{y\} \cup (U \setminus V(\mathcal{Y}'))$ by Fact 2.9. Replacing $\mathcal{Y}_i$ in $\mathscr{Y}$ with these two copies of $\mathcal{Y}_{k,b}$ creates a $\mathcal{Y}_{k,b}$-tiling larger than $\mathscr{Y}$, contradiction. Consequently,

$$\begin{aligned}
\sum_{S\in\binom{U}{k-1}} \deg(S, V') &\le |C|\binom{|U|}{k-1} + \big|V' \setminus C\big|(2k-b)^2\binom{|U|}{k-2} \\
&< |C|\binom{|U|}{k-1} + (2k-b)^2 n\binom{|U|}{k-2} \quad \text{because } \big|V' \setminus C\big| < n \\
&= \binom{|U|}{k-1}\left(|C| + \frac{(2k-b)^2 n(k-1)}{|U|-k+2}\right). \qquad (2.4)
\end{aligned}$$

Second, by Fact 2.9, $e(U) \le \binom{|U|}{k-1}$ since $\mathcal{H}[U]$ contains no copy of $\mathcal{Y}_{k,b}$, which implies

$$\sum_{S\in\binom{U}{k-1}} \deg(S, U) \le k\binom{|U|}{k-1}. \qquad (2.5)$$

By the definition of ϵ', we have

$$\epsilon' n^{k-1} = \frac{\gamma\beta^{k-1}}{(k-1)!} n^{k-1} < \frac{\gamma|U|^{k-1}}{(k-1)!} < 2\gamma\binom{|U|}{k-1}$$

as $|U|$ is large enough. At last, by the degree condition, we have

$$\begin{aligned}
\sum_{S\in\binom{U}{k-1}} \deg(S) &\ge \left(\binom{|U|}{k-1} - \epsilon' n^{k-1}\right)\left(\frac{1}{2k-b} - \gamma\right)n \\
&> (1-2\gamma)\binom{|U|}{k-1}\left(\frac{1}{2k-b} - \gamma\right)n. \qquad (2.6)
\end{aligned}$$

Since $\deg(S) = \deg(S, U) + \deg(S, V')$, we combine (2.4), (2.5) and (2.6) and get

$$|C| > (1-2\gamma)\left(\frac{1}{2k-b} - \gamma\right)n - k - \frac{(2k-b)^2 n(k-1)}{|U|-k+2}.$$

Since $|U| > 16k^3/\gamma$, we get

$$\frac{(2k-b)^2 n(k-1)}{|U|-k+2} < \frac{4k^3 n}{|U|/2} < \gamma n/2.$$

Since $2\gamma^2 n > k$ and $2k - b \geq 4$, it follows that $|C| > (\frac{1}{2k-b} - 2\gamma)n$.

Let I_C be the set of all $i \in [m]$ such that $V_i \cap C \neq \emptyset$. Since each V_i, $i \in I_C$, contains one vertex of C, we have

$$|I_C| = |C| \geq \left(\frac{1}{2k-b} - 2\gamma\right)n \geq m - 2\gamma n. \tag{2.7}$$

Let $A = (\bigcup_{i \in I_C} V_i \setminus C) \cup U$.

Claim 2.10. *$\mathcal{H}[A]$ contains no copy of $\mathcal{Y}_{k,b}$, thus $e(A) < \binom{n}{k-1}$.*

Proof. The first half of the claim implies the second half by Fact 2.9. Suppose instead, $\mathcal{H}[A]$ contains a copy of $\mathcal{Y}_{k,b}$, denoted by $\mathcal{Y}_0$. Note that $V(\mathcal{Y}_0) \not\subseteq U$ because $\mathcal{H}[U]$ contains no copy of $\mathcal{Y}_{k,b}$. Without loss of generality, suppose that $V_1, \ldots, V_j$ contain the vertices of $\mathcal{Y}_0$ for some $j \leq 2k - b$. For $i \in [j]$, let c_i denote the unique vertex in $V_i \cap C$. We greedily construct vertex-disjoint copies of $\mathcal{Y}_{k,b}$ on $\{c_i\} \cup U$, $i \in [j]$ as follows. Suppose we have found $\mathcal{Y}'_1, \ldots, \mathcal{Y}'_i$ (copies of $\mathcal{Y}_{k,b}$) for some $i < j$. Let U_0 denote the set of the vertices of U covered by $\mathcal{Y}_0, \mathcal{Y}'_1, \ldots, \mathcal{Y}'_i$. Then $|U_0| \leq (i+1)(2k-b-1) \leq (2k-b)(2k-b-1)$. Since $\deg(c_{i+1}, U) \geq (2k-b)^2\binom{|U|}{k-2}$, the link graph of c_{i+1} on $U \setminus U_0$ has at least

$$(2k-b)^2\binom{|U|}{k-2} - |U_0|\binom{|U|}{k-2} > \binom{|U|}{k-2}$$

edges. By Fact 2.9, there is a copy of $\mathcal{Y}_{k,b}$ on $\{c_{i+1}\} \cup (U \setminus U_0)$. Let $\mathcal{Y}'_1, \ldots, \mathcal{Y}'_j$ denote the copies of $\mathcal{Y}_{k,b}$ constructed in this way. Replacing $\mathcal{Y}_1, \ldots, \mathcal{Y}_j$ in $\mathscr{Y}$ with $\mathcal{Y}_0, \mathcal{Y}'_1, \ldots, \mathcal{Y}'_j$ gives a $\mathcal{Y}_{k,b}$-tiling larger than $\mathscr{Y}$, contradiction. $\square$

Note that the edges not incident to C are either contained in A or intersect some V_i, $i \notin I_C$. By (2.7) and Claim 2.10,

$$\begin{aligned} e(V \setminus C) &\leq e(A) + (2k-b) \cdot 2\gamma n\binom{n-1}{k-1} < \binom{n}{k-1} + (4k-2b)\gamma n\binom{n}{k-1} \\ &< 4k\gamma n\binom{n}{k-1} < \frac{4k}{(k-1)!}\gamma n^k \leq 6\gamma n^k, \end{aligned}$$

where the last inequality follows from $k \geq 3$. Since $|C| \leq \frac{n}{2k-b}$, we can pick a set $B \subseteq V \setminus C$ of order $\lfloor \frac{2k-b-1}{2k-b} n \rfloor$ such that $e(B) < 6\gamma n^k$. □

3. The extremal theorem

In this section we prove Theorem 1.5. Assume that $k \geq 3$, $1 \leq \ell < k/2$ and $0 < \Delta \ll 1$. Let $n \in (k-\ell)\mathbb{N}$ be sufficiently large. Let $\mathcal{H}$ be a k-graph on V of n vertices such that $\delta_{k-1}(\mathcal{H}) \geq \frac{n}{2(k-\ell)}$. Furthermore, assume that $\mathcal{H}$ is Δ-extremal, namely, there is a set $B \subseteq V(\mathcal{H})$, such that $|B| = \lfloor \frac{(2k-2\ell-1)n}{2(k-\ell)} \rfloor$ and $e(B) \leq \Delta n^k$. Let $A = V \setminus B$. Then $|A| = \lceil \frac{n}{2(k-\ell)} \rceil$.

The following is an outline of the proof. We denote by A' and B' the sets of the vertices of $\mathcal{H}$ that behave as typical vertices of A and B, respectively. Let $V_0 = V \setminus (A' \cup B')$. It is not hard to show that $A' \approx A$, $B' \approx B$, and thus $V_0 \approx \emptyset$. In the ideal case, when $V_0 = \emptyset$ and $|B'| = (2k-2\ell-1)|A'|$, we assign a cyclic order to the vertices of A', construct $|A'|$ copies of $\mathcal{Y}_{k,\ell}$ such that each copy contains one vertex of A' and $2k-\ell-1$ vertices of B', and any two consecutive copies of $\mathcal{Y}_{k,\ell}$ share exactly ℓ vertices of B'. This gives rise to the desired Hamilton ℓ-cycle of $\mathcal{H}$. In the general case, we first construct an ℓ-path $\mathcal{Q}$ with ends L_0 and L_1 such that $V_0 \subseteq V(\mathcal{Q})$ and $|B_1| = (2k-2\ell-1)|A_1| + \ell$, where $A_1 = A' \setminus V(\mathcal{Q})$ and $B_1 = (B \setminus V(\mathcal{Q})) \cup L_0 \cup L_1$. Next we complete the Hamilton ℓ-cycle by constructing an ℓ-path on $A_1 \cup B_1$ with ends L_0 and L_1.

For the convenience of later calculations, we let $\epsilon_0 = 2k!e\Delta \ll 1$ and claim that $e(B) \leq \epsilon_0 \binom{|B|}{k}$. Indeed, since $2(k-\ell)-1 \geq k$, we have

$$\frac{1}{e} \leq \left(1 - \frac{1}{2(k-\ell)}\right)^{2(k-\ell)-1} \leq \left(1 - \frac{1}{2(k-\ell)}\right)^{k}.$$

Thus we get

$$e(B) \leq \frac{\epsilon_0}{2k!e} n^k \leq \epsilon_0 \left(1 - \frac{1}{2(k-\ell)}\right)^{k} \frac{n^k}{2k!} \leq \epsilon_0 \binom{|B|}{k}. \tag{3.1}$$

In general, given two disjoint vertex sets X and Y and two integers $i, j \geq 0$, a set $S \subset X \cup Y$ is called an X^iY^j-set if $|S \cap X| = i$ and $|S \cap Y| = j$. When X, Y are two disjoint subsets of $V(\mathcal{H})$ and $i + j = k$, we denote by $\mathcal{H}(X^iY^j)$ the family of all edges of $\mathcal{H}$ that are X^iY^j-sets, and let $e_{\mathcal{H}}(X^iY^j) = |\mathcal{H}(X^iY^j)|$ (the subscript may be omitted if it is clear from the context). We use $\bar{e}_{\mathcal{H}}(X^iY^{k-i})$ to denote the number of non-edges among X^iY^{k-i}-sets. Given a set $L \subseteq X \cup Y$ with $|L \cap X| = l_1 \leq i$ and $|L \cap Y| = l_2 \leq k-i$, we define $\deg(L, X^iY^{k-i})$ as the number of edges in $\mathcal{H}(X^iY^{k-i})$ that contain L, and $\overline{\deg}(L, X^iY^{k-i}) = \binom{|X|-l_1}{i-l_1}\binom{|Y|-l_2}{k-i-l_2} - \deg(L, X^iY^{k-i})$. Our earlier notation $\deg(S, R)$ may be viewed as $\deg(S, S^{|S|}(R \setminus S)^{k-|S|})$.

3.1. Classification of vertices

Let $\epsilon_1 = {\epsilon_0}^{1/3}$ and $\epsilon_2 = 2\epsilon_1^2$. Assume that the partition $V(\mathcal{H}) = A \cup B$ satisfies that $|B| = \lfloor \frac{(2k-2\ell-1)n}{2(k-\ell)} \rfloor$ and (3.1). In addition, assume that $e(B)$ is the smallest among all such partitions. We now define

$$A' := \left\{ v \in V : \deg(v, B) \geq (1-\epsilon_1)\binom{|B|}{k-1} \right\},$$
$$B' := \left\{ v \in V : \deg(v, B) \leq \epsilon_1 \binom{|B|}{k-1} \right\},$$
$$V_0 := V \setminus \left(A' \cup B'\right).$$

Claim 3.1. *$A \cap B' \neq \emptyset$ implies that $B \subseteq B'$, and $B \cap A' \neq \emptyset$ implies that $A \subseteq A'$.*

Proof. First, assume that $A \cap B' \neq \emptyset$. Then there is some $u \in A$ such that $\deg(u, B) \leq \epsilon_1 \binom{|B|}{k-1}$. If there exists some $v \in B \setminus B'$, namely, $\deg(v, B) > \epsilon_1 \binom{|B|}{k-1}$, then we can switch u and v and form a new partition $A'' \cup B''$ such that $|B''| = |B|$ and $e(B'') < e(B)$, which contradicts the minimality of $e(B)$.

Second, assume that $B \cap A' \neq \emptyset$. Then some $u \in B$ satisfies that $\deg(u, B) \geq (1-\epsilon_1)\binom{|B|}{k-1}$. Similarly, by the minimality of $e(B)$, we get that for any vertex $v \in A$, $\deg(v, B) \geq (1-\epsilon_1)\binom{|B|}{k-1}$, which implies that $A \subseteq A'$. $\square$

Claim 3.2. $\{|A \setminus A'|, |B \setminus B'|, |A' \setminus A|, |B' \setminus B|\} \leq \epsilon_2 |B|$ *and* $|V_0| \leq 2\epsilon_2 |B|$.

Proof. First assume that $|B \setminus B'| > \epsilon_2 |B|$. By the definition of B', we get that

$$e(B) > \frac{1}{k}\epsilon_1 \binom{|B|}{k-1} \cdot \epsilon_2 |B| > 2\epsilon_0 \binom{|B|}{k},$$

which contradicts (3.1).

Second, assume that $|A \setminus A'| > \epsilon_2 |B|$. Then by the definition of A', for any vertex $v \notin A'$, we have that $\overline{\deg}(v, B) > \epsilon_1 \binom{|B|}{k-1}$. So we get

$$\bar{e}\left(AB^{k-1}\right) > \epsilon_2 |B| \cdot \epsilon_1 \binom{|B|}{k-1} = 2\epsilon_0 |B| \binom{|B|}{k-1}.$$

Together with (3.1), this implies that

$$\begin{aligned}\sum_{S \in \binom{B}{k-1}} \overline{\deg}(S) &= k\bar{e}(B) + \bar{e}\left(AB^{k-1}\right) \\ &> k(1-\epsilon_0)\binom{|B|}{k} + 2\epsilon_0 |B| \binom{|B|}{k-1} \\ &= \left((1-\epsilon_0)\left(|B| - k + 1\right) + 2\epsilon_0 |B|\right)\binom{|B|}{k-1} > |B| \binom{|B|}{k-1}.\end{aligned}$$

where the last inequality holds because n is large enough. By the pigeonhole principle, there exists a set $S \in \binom{B}{k-1}$, such that $\overline{\deg}(S) > |B| = \lfloor \frac{(2k-2\ell-1)n}{2(k-\ell)} \rfloor$, contradicting (1.1).

Consequently,

$$\begin{aligned}
|A' \setminus A| &= |A' \cap B| \le |B \setminus B'| \le \epsilon_2 |B|, \\
|B' \setminus B| &= |A \cap B'| \le |A \setminus A'| \le \epsilon_2 |B|, \\
|V_0| &= |A \setminus A'| + |B \setminus B'| \le \epsilon_2 |B| + \epsilon_2 |B| = 2\epsilon_2 |B|. \qquad \square
\end{aligned}$$

3.2. Classification of ℓ-sets in B'

In order to construct our Hamilton ℓ-cycle, we need to connect two ℓ-paths. To make this possible, we want the ends of our ℓ-paths to be ℓ-sets in B' that have high degree in $\mathcal{H}[A'B'^{k-1}]$. Formally, we call an ℓ-set $L \subset V$ *typical* if $\deg(L, B) \le \epsilon_1 \binom{|B|}{k-\ell}$, otherwise *atypical*. We prove several properties related to typical ℓ-sets in this subsection.

Claim 3.3. *The number of atypical ℓ-sets in B is at most $\epsilon_2 \binom{|B|}{\ell}$.*

Proof. Let m be the number of atypical ℓ-sets in B. By (3.1), we have

$$\frac{m\epsilon_1 \binom{|B|}{k-\ell}}{\binom{k}{\ell}} \le e(B) \le \epsilon_0 \binom{|B|}{k},$$

which gives that $m \le \frac{\epsilon_0 \binom{k}{l}\binom{|B|}{k}}{\epsilon_1 \binom{|B|}{k-\ell}} = \frac{\epsilon_2}{2}\binom{|B|-k+\ell}{\ell} < \epsilon_2 \binom{|B|}{\ell}$. $\square$

Claim 3.4. *Every typical ℓ-set $L \subset B'$ satisfies $\overline{\deg}(L, A'B'^{k-1}) \le 4k\epsilon_1 \binom{|B'|-\ell}{k-\ell-1}|A'|$.*

Proof. Fix a typical ℓ-set $L \subset B'$ and consider the following sum,

$$\sum_{L \subset D \subset B', |D|=k-1} \overline{\deg}(D) = \sum_{L \subset D \subset B', |D|=k-1} \big(\overline{\deg}(D, A') + \overline{\deg}(D, B') + \overline{\deg}(D, V_0)\big).$$

By (1.1), the left-hand side is at least $\binom{|B'|-\ell}{k-\ell-1}|A|$. On the other hand,

$$\sum_{L \subset D \subset B', |D|=k-1} \big(\overline{\deg}(D, B') + \overline{\deg}(D, V_0)\big) \le (k-\ell)\deg(L, B') + \binom{|B'|-\ell}{k-\ell-1}|V_0|.$$

Since L is typical and $|B' \setminus B| \le \epsilon_2 |B|$ (Claim 3.2), we have

$$\begin{aligned}
\deg(L, B') &\le \deg(L, B) + |B' \setminus B| \binom{|B'|-1}{k-\ell-1} \\
&\le \epsilon_1 \binom{|B|}{k-\ell} + \epsilon_2 |B| \binom{|B'|-1}{k-\ell-1}.
\end{aligned}$$

Since $\epsilon_2 \ll \epsilon_1$ and $||B| - |B'|| \le \epsilon_2 |B|$, it follows that

$$
\begin{aligned}
(k-\ell)\deg(L, B') &\le \epsilon_1 |B| \binom{|B|-1}{k-\ell-1} + (k-\ell)\epsilon_2 |B| \binom{|B'|-1}{k-\ell-1} \\
&\le 2\epsilon_1 |B| \binom{|B'|-\ell}{k-\ell-1}.
\end{aligned}
$$

Putting these together and using Claim 3.2, we obtain that

$$
\begin{aligned}
\sum_{L \subset D \subset B', |D|=k-1} \deg(D, A') &\ge \binom{|B'|-\ell}{k-\ell-1}\big(|A| - |V_0|\big) - 2\epsilon_1 |B| \binom{|B'|-\ell}{k-\ell-1} \\
&\ge \binom{|B'|-\ell}{k-\ell-1}\big(|A'| - 3\epsilon_2 |B| - 2\epsilon_1 |B|\big).
\end{aligned}
$$

Note that $\deg(L, A'{B'}^{k-1}) = \sum_{L \subset D \subset B', |D|=k-1} \deg(D, A')$. Since $|B| \le (2k - 2\ell - 1)|A| \le (2k - 2\ell)|A'|$, we finally derive that

$$
\begin{aligned}
\deg\big(L, A'{B'}^{k-1}\big) &\ge \binom{|B'|-\ell}{k-\ell-1}\big(1 - (2k-2\ell)(3\epsilon_2 + 2\epsilon_1)\big)|A'| \\
&\ge (1 - 4k\epsilon_1)\binom{|B'|-\ell}{k-\ell-1}|A'|.
\end{aligned}
$$

as desired. $\square$

We next show that we can connect any two disjoint typical ℓ-sets of B' with an ℓ-path of length two while avoiding any given set of $\frac{n}{4(k-\ell)}$ vertices of V.

Claim 3.5. *Given two disjoint typical ℓ-sets L_1, L_2 in B' and a vertex set $U \subseteq V$ with $|U| \le \frac{n}{4(k-\ell)}$, there exist a vertex $a \in A' \setminus U$ and a $(2k - 3\ell - 1)$-set $C \subset B' \setminus U$ such that $L_1 \cup L_2 \cup \{a\} \cup C$ spans an ℓ-path (of length two) ended at L_1, L_2.*

Proof. Fix two disjoint typical ℓ-sets L_1, L_2 in B'. Using Claim 3.2, we obtain that $|U| \le \frac{n}{4(k-\ell)} \le \frac{|A|}{2} < \frac{2}{3}|A'|$ and

$$
\frac{n}{4(k-\ell)} \le \frac{|B|+1}{2(2k-2\ell-1)} \le \frac{(1+2\epsilon_2)|B'|}{2k} < \frac{|B'|}{k}.
$$

Thus $|A' \setminus U| > \frac{|A'|}{3}$ and $|B' \setminus U| > \frac{k-1}{k}|B'|$. Consider a $(k-\ell)$-graph $\mathcal{G}$ on $(A' \cup B') \setminus U$ such that an $A'{B'}^{k-\ell-1}$-set T is an edge of $\mathcal{G}$ if and only if $T \cap U = \emptyset$ and T is a common neighbor of L_1 and L_2 in $\mathcal{H}$. By Claim 3.4, we have

$$\begin{aligned}\bar{e}(\mathcal{G}) &\le 2\cdot 4k\epsilon_1\binom{|B'|-\ell}{k-\ell-1}|A'| < 8k\epsilon_1\binom{\frac{k}{k-1}|B'\setminus U|}{k-\ell-1}\cdot 3|A'\setminus U|\\ &\le 24k\epsilon_1\left(\frac{k}{k-1}\right)^{k-1}\binom{|B'\setminus U|}{k-\ell-1}|A'\setminus U|.\end{aligned}$$

Consequently, $e(\mathcal{G}) > \frac{1}{2}\binom{|B'\setminus U|}{k-\ell-1}|A'\setminus U|$. Hence there exists a vertex $a\in A'\setminus U$ such that $\deg_{\mathcal{G}}(a) > \frac{1}{2}\binom{|B'\setminus U|}{k-\ell-1} > \binom{|B'\setminus U|}{k-\ell-2}$. By Fact 2.9, the link graph of a contains a copy of $\mathcal{Y}_{k-\ell-1,\ell-1}$ (two edges of the link graph sharing $\ell-1$ vertices). In other words, there exists a $(2k-3\ell-1)$-set $C\subset B'\setminus U$ such that $C\cup\{a\}$ contains two edges of $\mathcal{G}$ sharing ℓ vertices. Together with L_1, L_2, this gives rise to the desired ℓ-path (in $\mathcal{H}$) of length two ended at L_1, L_2. $\square$

The following claim shows that we can always extend a typical ℓ-set to an edge of $\mathcal{H}$ by adding one vertex from A' and $k-\ell-1$ vertices from B' such that every ℓ-set of these $k-\ell-1$ vertices is typical. This can be done even when at most $\frac{n}{4(k-\ell)}$ vertices of V are not available.

Claim 3.6. *Given a typical ℓ-set $L\subseteq B'$ and a set $U\subseteq V$ with $|U|\le\frac{n}{4(k-\ell)}$, there exists an $A'B'^{\,k-\ell-1}$-set $C\subset V\setminus U$ such that $L\cup C$ is an edge of $\mathcal{H}$ and every ℓ-subset of $C\cap B'$ is typical.*

Proof. First, since L is typical in B', by Claim 3.4, $\overline{\deg}(L, A'B'^{\,k-1}) \le 4k\epsilon_1\binom{|B'|-\ell}{k-\ell-1}|A'|$. Second, note that a vertex in A' is contained in $\binom{|B'|}{k-\ell-1}$ $A'B'^{\,k-\ell-1}$-sets, while a vertex in B' is contained in $|A'|\binom{|B'|-1}{k-\ell-2}$ $A'B'^{\,k-\ell-1}$-sets. It is easy to see that $|A'|\binom{|B'|-1}{k-\ell-2} < \binom{|B'|}{k-\ell-1}$ (as $|A'|\approx\frac{n}{2k-2\ell}$ and $|B'|\approx\frac{2k-2\ell-1}{2k-2\ell}n$). We thus derive that at most

$$|U|\binom{|B'|}{k-\ell-1} \le \frac{n}{4(k-\ell)}\binom{|B'|}{k-\ell-1}$$

$A'B'^{\,k-\ell-1}$-sets intersect U. Finally, by Claim 3.3, the number of atypical ℓ-sets in B is at most $\epsilon_2\binom{|B|}{\ell}$. Using Claim 3.2, we derive that the number of atypical ℓ-sets in B' is at most

$$\epsilon_2\binom{|B|}{\ell} + |B'\setminus B|\binom{|B'|-1}{\ell-1} \le 2\epsilon_2\binom{|B'|}{\ell} + \epsilon_2|B|\binom{|B'|-1}{\ell-1} < 3\ell\epsilon_2\binom{|B'|}{\ell}.$$

Hence at most $3\ell\epsilon_2\binom{|B'|}{\ell}|A'|\binom{|B'|-\ell}{k-2\ell-1}$ $A'B'^{\,k-\ell-1}$-sets contain an atypical ℓ-set. In summary, at most

$$4k\epsilon_1\binom{|B'|-\ell}{k-\ell-1}|A'| + \frac{n}{4(k-\ell)}\binom{|B'|}{k-\ell-1} + 3\ell\epsilon_2\binom{|B'|}{\ell}\binom{|B'|-\ell}{k-2\ell-1}|A'|$$

$A'B'^{\,k-\ell-1}$-sets fail some of the desired properties. Since $\epsilon_1,\epsilon_2 \ll 1$ and $|A'|\approx\frac{n}{2(k-\ell)}$, the desired $A'B'^{\,k-\ell-1}$-set always exists. $\square$

3.3. Building a short path $\mathcal{Q}$

First, by the definition of B, for any vertex $b \in B'$, we have

$$\begin{aligned}\deg\bigl(b, B'\bigr) &\le \deg(b, B) + \bigl|B' \setminus B\bigr| \binom{|B'|-1}{k-2} \\ &\le \epsilon_1 \binom{|B|}{k-1} + \epsilon_2 |B| \binom{|B'|-1}{k-2} < 2\epsilon_1 \binom{|B|}{k-1}. \end{aligned} \tag{3.2}$$

The following claim is the only place where we used the exact codegree condition (1.1).

Claim 3.7. *Suppose that $|A \cap B'| = q > 0$. Then there exists a family $\mathcal{P}_1$ of $2q$ vertex-disjoint edges in B', each of which contains two disjoint typical ℓ-sets.*

Proof. Let $|A \cap B'| = q > 0$. Since $A \cap B' \neq \emptyset$, by Claim 3.1, we have $B \subseteq B'$, and consequently $|B'| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \rfloor + q$. By Claim 3.2, we have $q \le |A \setminus A'| \le \epsilon_2 |B|$.

Let $\mathcal{B}$ denote the family of the edges in B' that contain two disjoint typical ℓ-sets. We derive a lower bound for $|\mathcal{B}|$ as follows. We first pick a $(k-1)$-subset of B (recall that $B \subseteq B'$) that contains no atypical ℓ-subset. Since $2\ell \le k-1$, such a $(k-1)$-set contains two disjoint typical ℓ-sets. By Claim 3.3, there are at most $\epsilon_2 \binom{|B|}{\ell}$ atypical ℓ-sets in $B \cap B' = B$ and in turn, there are at most $\epsilon_2 \binom{|B|}{\ell} \binom{|B|-\ell}{k-\ell-1}$ $(k-1)$-subsets of B that contain an atypical ℓ-subset. Thus there are at least

$$\binom{|B|}{k-1} - \epsilon_2 \binom{|B|}{\ell} \binom{|B|-\ell}{k-\ell-1} = \left(1 - \binom{k-1}{\ell} \epsilon_2\right) \binom{|B|}{k-1}$$

$(k-1)$-subsets of B that contain no atypical ℓ-subset. After picking such a $(k-1)$-set $S \subset B$, we find a neighbor of S by the codegree condition. Since $|B'| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \rfloor + q$, by (1.1), we have $\deg(S, B') \ge q$. We thus derive that

$$|\mathcal{B}| \ge \left(1 - \binom{k-1}{\ell} \epsilon_2\right) \binom{|B|}{k-1} \frac{q}{k},$$

in which we divide by k because every edge of $\mathcal{B}$ is counted at most k times.

We claim that $\mathcal{B}$ contains $2q$ disjoint edges. Suppose instead, a maximum matching in $\mathcal{B}$ has $i < 2q$ edges. By (3.2), at most $2qk \cdot 2\epsilon_1 \binom{|B|}{k-1}$ edges of B' intersect the i edges in the matching. Hence, the number of edges of $\mathcal{B}$ that are disjoint from these i edges is at least

$$\frac{q}{k} \left(1 - \binom{k-1}{\ell} \epsilon_2\right) \binom{|B|}{k-1} - 4k\epsilon_1 q \binom{|B|}{k-1} \ge \left(\frac{1}{k} - (4k+1)\epsilon_1\right) q \binom{|B|}{k-1} > 0,$$

as $\epsilon_2 \ll \epsilon_1 \ll 1$. We may thus obtain a matching of size $i+1$, a contradiction. $\square$

Claim 3.8. *There exists a non-empty ℓ-path $\mathcal{Q}$ in $\mathcal{H}$ with the following properties:*

- $V_0 \subseteq V(\mathcal{Q})$,
- $|V(\mathcal{Q})| \le 10k\epsilon_2|B|$,
- *the two ends L_0, L_1 of $\mathcal{Q}$ are typical ℓ-sets in B',*
- $|B_1| = (2k-2\ell-1)|A_1| + \ell$, *where $A_1 = A' \setminus V(\mathcal{Q})$ and $B_1 = (B' \setminus V(\mathcal{Q})) \cup L_0 \cup L_1$.*

Proof. We split into two cases here.

Case 1. $A \cap B' \neq \emptyset$.

By Claim 3.1, $A \cap B' \neq \emptyset$ implies that $B \subseteq B'$. Let $q = |A \cap B'|$. We first apply Claim 3.7 and find a family $\mathcal{P}_1$ of vertex-disjoint $2q$ edges in B'. Next we associate each vertex of V_0 with $2k-\ell-1$ vertices of B (so in B') forming an ℓ-path of length two such that these $|V_0|$ paths are pairwise vertex-disjoint, and also vertex-disjoint from the paths in $\mathcal{P}_1$, and all these paths have typical ends. To see it, let $V_0 = \{x_1, \dots, x_{|V_0|}\}$. Suppose that we have found such ℓ-paths for $x_1, \dots, x_{i-1}$ with $i \le |V_0|$. Since $B \subseteq B'$, it follows that $A \setminus A' = (A \cap B') \cup V_0$. Hence $|V_0| + q = |A \setminus A'| \le \epsilon_2|B|$ by Claim 3.2. Therefore

$$(2k-\ell-1)(i-1) + \left|V(\mathcal{P}_1)\right| < 2k|V_0| + 2kq \le 2k\epsilon_2|B|$$

and consequently at most $2k\epsilon_2|B|\binom{|B|-1}{k-2} < 2k^2\epsilon_2\binom{|B|}{k-1}$ $(k-1)$-sets of B intersect the existing paths (including $\mathcal{P}_1$). By the definition of V_0, $\deg(x_i, B) > \epsilon_1\binom{|B|}{k-1}$. Let $\mathcal{G}_{x_i}$ be the $(k-1)$-graph on B such that $e \in \mathcal{G}_{x_i}$ if

- $\{x_i\} \cup e \in E(\mathcal{H})$,
- e does not contain any vertex from the existing paths,
- e does not contain any atypical ℓ-set.

By Claim 3.3, the number of $(k-1)$-sets in B containing at least one atypical ℓ-set is at most $\epsilon_2\binom{|B|}{\ell}\binom{|B|-\ell}{k-\ell-1} = \epsilon_2\binom{k-1}{\ell}\binom{|B|}{k-1}$. Thus, we have

$$e(\mathcal{G}_{x_i}) \ge \epsilon_1\binom{|B|}{k-1} - 2k^2\epsilon_2\binom{|B|}{k-1} - \epsilon_2\binom{k-1}{\ell}\binom{|B|}{k-1} > \frac{\epsilon_1}{2}\binom{|B|}{k-1} > \binom{|B|}{k-2},$$

because $\epsilon_2 \ll \epsilon_1$ and $|B|$ is sufficiently large. By Fact 2.9, $\mathcal{G}_{x_i}$ contains a copy of $\mathcal{Y}_{k-1,\ell-1}$, which gives the desired ℓ-path of length two containing x_i.

Denote by $\mathcal{P}_2$ the family of ℓ-paths we obtained so far. Now we need to connect paths of $\mathcal{P}_2$ together to a single ℓ-path. For this purpose, we apply Claim 3.5 repeatedly to connect the ends of two ℓ-paths while avoiding previously used vertices. This is possible because $|V(\mathcal{P}_2)| = (2k-\ell)|V_0| + 2kq$ and $(2k-3\ell)(|V_0| + 2q - 1)$ vertices are needed to connect all the paths in $\mathcal{P}_2$ — the set U (when we apply Claim 3.5) thus satisfies

$$|U| \le (4k-4\ell)|V_0| + (6k-6\ell)q - 2k + 3\ell \le 6(k-\ell)\epsilon_2|B| - 2k + 3\ell.$$

Let $\mathcal{P}$ denote the resulting ℓ-path. We have $|V(\mathcal{P}) \cap A'| = |V_0| + 2q - 1$ and

$$\begin{aligned}\left|V(\mathcal{P}) \cap B'\right| &= k \cdot 2q + (2k - \ell - 1)|V_0| + (2k - 3\ell - 1)\left(|V_0| + 2q - 1\right) \\ &= 2(2k - 2\ell - 1)|V_0| + 2(3k - 3\ell - 1)q - (2k - 3\ell - 1).\end{aligned}$$

Let $s = (2k - 2\ell - 1)|A' \setminus V(\mathcal{P})| - |B' \setminus V(\mathcal{P})|$. We have

$$\begin{aligned}s &= (2k - 2\ell - 1)\left(\left|A'\right| - |V_0| - 2q + 1\right) - \left|B'\right| + 2(2k - 2\ell - 1)|V_0| \\ &\quad + 2(3k - 3\ell - 1)q - (2k - 3\ell - 1) \\ &= (2k - 2\ell - 1)\left|A'\right| - \left|B'\right| + (2k - 2\ell - 1)|V_0| + (2k - 2\ell)q + \ell.\end{aligned}$$

Since $|A'| + |B'| + |V_0| = n$, we have

$$s = (2k - 2\ell)\left(\left|A'\right| + |V_0| + q\right) - n + \ell. \tag{3.3}$$

Note that $|A'| + |V_0| + q = |A|$ and

$$(2k - 2\ell)|A| - n = \begin{cases} 0, & \text{if } \frac{n}{k-\ell} \text{ is even,} \\ k - \ell, & \text{if } \frac{n}{k-\ell} \text{ is odd.} \end{cases} \tag{3.4}$$

Thus $s = \ell$ or $s = k$. If $s = k$, then we extend $\mathcal{P}$ to an ℓ-path $\mathcal{Q}$ by applying Claim 3.6, otherwise let $\mathcal{Q} = \mathcal{P}$. Then

$$\left|V(\mathcal{Q})\right| \le \left|V(\mathcal{P})\right| + (k - \ell) \le 6k\epsilon_2|B|,$$

and $\mathcal{Q}$ has two typical ends $L_0, L_1 \subset B'$. We claim that

$$(2k - 2\ell - 1)\left|A' \setminus V(\mathcal{Q})\right| - \left|B' \setminus V(\mathcal{Q})\right| = \ell. \tag{3.5}$$

Indeed, when $s = \ell$, this is obvious; when $s = k$, $V(\mathcal{Q}) \setminus V(\mathcal{P})$ contains one vertex of A' and $k - \ell - 1$ vertices of B' and thus

$$(2k - 2\ell - 1)\left|A' \setminus V(\mathcal{Q})\right| - \left|B' \setminus V(\mathcal{Q})\right| = s - (2k - 2\ell - 1) + (k - \ell - 1) = \ell.$$

Let $A_1 = A' \setminus V(\mathcal{Q})$ and $B_1 = (B' \setminus V(\mathcal{Q})) \cup L_0 \cup L_1$. We derive that $|B_1| = (2k - 2\ell - 1)|A_1| + \ell$ from (3.5).

Case 2. $A \cap B' = \emptyset$.

Note that $A \cap B' = \emptyset$ means that $B' \subseteq B$. Then we have

$$\left|A'\right| + |V_0| = \left|V \setminus B'\right| = |A| + \left|B \setminus B'\right|. \tag{3.6}$$

If $V_0 \neq \emptyset$, we handle this case similarly as in Case 1 except that we do not need to construct $\mathcal{P}_1$. By Claim 3.2, $|B \setminus B'| \le \epsilon_2|B|$ and thus for any vertex $x \in V_0$,

$$\begin{aligned}\deg\bigl(x,B'\bigr) &\ge \deg(x,B) - \bigl|B\setminus B'\bigr|\cdot\binom{|B|-1}{k-2}\\ &\ge \epsilon_1\binom{|B|}{k-1} - (k-1)\epsilon_2\binom{|B|}{k-1} > \frac{\epsilon_1}{2}\binom{|B'|}{k-1}.\end{aligned} \tag{3.7}$$

As in Case 1, we let $V_0 = \{x_1, \ldots, x_{|V_0|}\}$ and cover them with vertex-disjoint ℓ-paths of length two. Indeed, for each $i \le |V_0|$, we construct $\mathcal{G}_x$ as before and show that $e(\mathcal{G}_{x_i}) \ge \frac{\epsilon_1}{4}\binom{|B'|}{k-1}$. We then apply Fact 2.9 to $\mathcal{G}_{x_i}$ obtaining a copy of $\mathcal{Y}_{k-1,\ell-1}$, which gives an ℓ-path of length two containing x_i. As in Case 1, we connect these paths to a single ℓ-path $\mathcal{P}$ by applying Claim 3.5 repeatedly. Then $|V(\mathcal{P})| = (2k-\ell)|V_0| + (2k-3\ell)(|V_0|-1)$. Define s as in Case 1. Thus (3.3) holds with $q = 0$. Applying (3.6) and (3.4), we derive that

$$\begin{aligned}s &= 2(k-\ell)\bigl(|A| + \bigl|B\setminus B'\bigr|\bigr) - n + \ell\\ &= \begin{cases}\ell + 2(k-\ell)|B\setminus B'|, & \text{if } \frac{n}{k-\ell} \text{ is even},\\ k + 2(k-\ell)|B\setminus B'|, & \text{if } \frac{n}{k-\ell} \text{ is odd},\end{cases}\end{aligned} \tag{3.8}$$

which implies that $s \equiv \ell \bmod (k-\ell)$. We extend $\mathcal{P}$ to an ℓ-path $\mathcal{Q}$ by applying Claim 3.6 $\frac{s-\ell}{k-\ell}$ times. Then

$$\begin{aligned}\bigl|V(\mathcal{Q})\bigr| &= \bigl|V(\mathcal{P})\bigr| + s - \ell \le (4k-4\ell)|V_0| - 2k + 3\ell + k - \ell + 2(k-\ell)\bigl|B\setminus B'\bigr|\\ &\le 10k\epsilon_2|B|\end{aligned}$$

by Claim 3.2. Note that $\mathcal{Q}$ has two typical ends $L_0, L_1 \subset B'$. Since $V(\mathcal{Q})\setminus V(\mathcal{P})$ contains $\frac{s-\ell}{k-\ell}$ vertices of A' and $\frac{s-\ell}{k-\ell}(k-\ell-1)$ vertices of B', we have

$$\begin{aligned}&(2k-2\ell-1)\bigl|A'\setminus V(\mathcal{Q})\bigr| - \bigl|B'\setminus V(\mathcal{Q})\bigr|\\ &\quad = s - \frac{s-\ell}{k-\ell}(2k-2\ell-1) + \frac{s-\ell}{k-\ell}(k-\ell-1) = \ell.\end{aligned}$$

We define A_1 and B_1 in the same way and similarly we have $|B_1| = (2k-2\ell-1)|A_1| + \ell$.

When $V_0 = \emptyset$, we pick an arbitrary vertex $v \in A'$ and form an ℓ-path $\mathcal{P}$ of length two with typical ends such that v is in the intersection of the two edges. This is possible by the definition of A'. Define s as in Case 1. It is easy to see that (3.8) still holds. We then extend $\mathcal{P}$ to $\mathcal{Q}$ by applying Claim 3.6 $\frac{s-\ell}{k-\ell}$ times. Then $|V(\mathcal{Q})| = 2k - \ell + s - \ell \le 2k\epsilon_2|B|$ because of (3.8). The rest is the same as in the previous case. □

Claim 3.9. *The A_1, B_1 and L_0, L_1 defined in Claim 3.8 satisfy the following properties:*

(1) $|B_1| \ge (1-\epsilon_1)|B|$,
(2) *for any vertex* $v \in A_1$, $\overline{\deg}(v, B_1) < 3\epsilon_1\binom{|B_1|}{k-1}$,
(3) *for any vertex* $v \in B_1$, $\overline{\deg}(v, A_1B_1^{k-1}) \le 3k\epsilon_1\binom{|B_1|}{k-1}$,
(4) $\overline{\deg}(L_0, A_1B_1^{k-1}) \le 5k\epsilon_1\binom{|B_1|}{k-\ell}$, $\overline{\deg}(L_1, A_1B_1^{k-1}) \le 5k\epsilon_1\binom{|B_1|}{k-\ell}$.

Proof. Part (1): By Claim 3.2, we have $|B_1 \setminus B| \le |B' \setminus B| \le \epsilon_2|B|$. Furthermore,

$$|B_1| \ge \left|B'\right| - \left|V(\mathcal{Q})\right| \ge |B| - \epsilon_2|B| - 10k\epsilon_2|B| \ge (1-\epsilon_1)|B|.$$

Part (2): For a vertex $v \in A_1$, since $\overline{\deg}(v, B) \le \epsilon_1\binom{|B|}{k-1}$, we have

$$\begin{aligned}\overline{\deg}(v, B_1) &\le \overline{\deg}(v, B) + |B_1 \setminus B|\binom{|B_1|-1}{k-2} \\ &\le \epsilon_1\binom{|B|}{k-1} + \epsilon_2|B|\binom{|B_1|-1}{k-2} \\ &< \epsilon_1\binom{|B|}{k-1} + \epsilon_1\binom{|B_1|}{k-1} < 3\epsilon_1\binom{|B_1|}{k-1},\end{aligned}$$

where the last inequality follows from Part (1).

Part (3): Consider the sum $\sum \deg(S \cup \{v\})$ taken over all $S \in \binom{B'\setminus\{v\}}{k-2}$. Since $\delta_{k-1}(\mathcal{H}) \ge |A|$, we have $\sum \deg(S \cup \{v\}) \ge \binom{|B'|-1}{k-2}|A|$. On the other hand,

$$\sum \deg\bigl(S \cup \{v\}\bigr) = \deg\bigl(v, A'B'^{\,k-1}\bigr) + \deg\bigl(v, V_0B'^{\,k-1}\bigr) + (k-1)\deg\bigl(v, B'\bigr).$$

We thus derive that

$$\deg\bigl(v, A'B'^{\,k-1}\bigr) \ge \binom{|B'|-1}{k-2}|A| - \deg\bigl(v, V_0B'^{\,k-1}\bigr) - (k-1)\deg\bigl(v, B'\bigr).$$

By Claim 3.2 and (3.2), it follows that

$$\begin{aligned}\deg\bigl(v, A'B'^{\,k-1}\bigr) &\ge \binom{|B'|-1}{k-2}\bigl(\left|A'\right| - \epsilon_2|B|\bigr) - 2\epsilon_2|B|\binom{|B'|-1}{k-2} - 2(k-1)\epsilon_1\binom{|B|}{k-1} \\ &\ge \binom{|B'|-1}{k-2}\left|A'\right| - 2k\epsilon_1\binom{|B|}{k-1}.\end{aligned}$$

By Part (1), we now have

$$\overline{\deg}\bigl(v, A_1B_1^{k-1}\bigr) \le \overline{\deg}\bigl(v, A'B'^{\,k-1}\bigr) \le 2k\epsilon_1\binom{|B|}{k-1} \le 3k\epsilon_1\binom{|B_1|}{k-1}.$$

Part (4): By Claim 3.4, for any typical $L \subseteq B'$, we have $\overline{\deg}(L, A'B'^{\,k-1}) \le 4k\epsilon_1\binom{|B'|-\ell}{k-\ell-1}|A'|$. Thus,

$$\overline{\deg}\bigl(L_0, A_1B_1^{k-1}\bigr) \le \overline{\deg}\bigl(L_0, A'B'^{\,k-1}\bigr) \le 4k\epsilon_1\binom{|B'|-\ell}{k-\ell-1}\left|A'\right| \le 5k\epsilon_1\binom{|B_1|}{k-\ell},$$

where the last inequality holds because $|B'| \le |B_1| + |V(\mathcal{Q})| \le (1+\epsilon_1)|B_1|$. The same holds for L_1. □

3.4. Completing the Hamilton cycle

We finally complete the proof of Theorem 1.5 by applying the following lemma with $X = A_1$, $Y = B_1$, $\rho = 5k\epsilon_1$, and L_0, L_1.

Lemma 3.10. *Fix $1 \le \ell < k/2$. Let $0 < \rho \ll 1$ and n be sufficiently large. Suppose that $\mathcal{H}$ is a k-graph with a partition $V(\mathcal{H}) = X \cup Y$ and the following properties:*

- $|Y| = (2k - 2\ell - 1)|X| + \ell$,
- *for every vertex $v \in X$, $\overline{\deg}(v, Y) \le \rho\binom{|Y|}{k-1}$ and for every vertex $v \in Y$, $\overline{\deg}(v, XY^{k-1}) \le \rho\binom{|Y|}{k-1}$,*
- *there are two disjoint ℓ-sets $L_0, L_1 \subset Y$ such that*

$$\overline{\deg}\big(L_0, XY^{k-1}\big), \overline{\deg}\big(L_1, XY^{k-1}\big) \le \rho\binom{|Y|}{k-\ell}. \tag{3.9}$$

Then $\mathcal{H}$ contains a Hamilton ℓ-path with L_0 and L_1 as ends.

In order to prove Lemma 3.10, we apply two results of Glebov, Person, and Weps [6]. Given $1 \le j \le k-1$ and $0 \le \rho \le 1$, an ordered set $(x_1, \ldots, x_j)$ is ρ-*typical* in a k-graph $\mathcal{G}$ if for every $i \in [j]$,

$$\overline{\deg}_{\mathcal{G}}\big(\{x_1, \ldots, x_i\}\big) \le \rho^{k-i}\binom{|V(\mathcal{G})| - i}{k-i}.$$

It was shown in [6] that every k-graph $\mathcal{G}$ with very large minimum vertex degree contains a tight Hamilton cycle. The proof of [6, Theorem 2] actually shows that we can obtain a tight Hamilton cycle by extending any fixed tight path of constant length with two typical ends. This implies the following theorem that we will use.

Theorem 3.11. *(See [6].) Given $1 \le j \le k$ and $0 < \alpha \ll 1$, there exists an m_0 such that the following holds. Suppose that $\mathcal{G}$ is a k-graph on V with $|V| = m \ge m_0$ and $\delta_1(\mathcal{G}) \ge (1-\alpha)\binom{m-1}{k-1}$. Then given any two disjoint $(22\alpha)^{\frac{1}{k-1}}$-typical ordered j-sets $(x_1, \ldots, x_j)$ and $(y_1, \ldots, y_j)$, there exists a tight Hamilton path $\mathcal{P} = x_j x_{j-1} \cdots x_1 \cdot \cdots \cdot y_1 y_2 \cdots y_j$ in $\mathcal{G}$.*

We also use [6, Lemma 3], in which V^{2k-2} denotes the set of all $(2k-2)$-tuples $(v_1, \ldots, v_{2k-2})$ such that $v_i \in V$ (v_i's are not necessarily distinct).

Lemma 3.12. *(See [6].) Let $\mathcal{G}$ be the k-graph given in Lemma 3.11. Suppose that $(x_1, \ldots, x_{2k-2})$ is selected uniformly at random from V^{2k-2}. Then the probability that all x_i's are pairwise distinct and $(x_1, \ldots, x_{k-1}), (x_k, \ldots, x_{2k-2})$ are $(22\alpha)^{\frac{1}{k-1}}$-typical is at least $\frac{8}{11}$.*

Proof of Lemma 3.10. In this proof we often write the union $A \cup B \cup \{x\}$ as ABx, where A, B are sets and x is an element.

Let $t = |X|$. Our goal is to write X as $\{x_1, \dots, x_t\}$ and partition Y as $\{L_i, R_i, S_i, R'_i : i \in [t]\}$ with $|L_i| = \ell$, $|R_i| = |R'_i| = k - 2\ell$, and $|S_i| = \ell - 1$ such that

$$L_i R_i S_i x_i,\ S_i x_i R'_i L_{i+1} \in E(\mathcal{H}) \tag{3.10}$$

for all $i \in [t]$, where $L_{t+1} = L_0$. Consequently,

$$L_1\, R_1\, S_1\, x_1\, R'_1\, L_2\, R_2\, S_2\, x_2\, R'_2\, \cdots\, L_t\, R_t\, S_t\, x_t\, R'_t\, L_{t+1}$$

is the desired Hamilton ℓ-path of $\mathcal{H}$.

Let $\mathcal{G}$ be the $(k-1)$-graph on Y whose edges are all $(k-1)$-sets $S \subseteq Y$ such that $\deg_{\mathcal{H}}(S, X) > (1 - \sqrt{\rho})t$. The following is an outline of our proof. We first find a small subset $Y_0 \subset Y$ with a partition $\{L_i, R_i, S_i, R'_i : i \in [t_0]\}$ such that for every $x \in X$, we have $L_i R_i S_i x, S_i x R'_i L_{i+1} \in E(\mathcal{H})$ for many $i \in [t_0]$. Next we apply Theorem 3.11 to $\mathcal{G}[Y \setminus Y_0]$ and obtain a tight Hamilton path, which, in particular, partitions $Y \setminus Y_0$ into $\{L_i, R_i, S_i, R'_i : t_0 < i \le t\}$ such that $L_i R_i S_i, S_i R'_i L_{i+1} \in E(\mathcal{G})$ for $t_0 < i \le t$. Finally we apply the Marriage Theorem to find a perfect matching between X and $[t]$ such that (3.10) holds for all matched x_i and i.

We now give details of the proof. First we claim that

$$\delta_1(\mathcal{G}) \ge (1 - 2\sqrt{\rho}) \binom{|Y|-1}{k-2}, \tag{3.11}$$

and consequently,

$$\bar{e}(\mathcal{G}) \le 2\sqrt{\rho} \binom{|Y|}{k-1}. \tag{3.12}$$

Suppose instead, some vertex $v \in Y$ satisfies $\overline{\deg}_{\mathcal{G}}(v) > 2\sqrt{\rho}\binom{|Y|-1}{k-2}$. Since every non-neighbor S' of v in $\mathcal{G}$ satisfies $\overline{\deg}_{\mathcal{H}}(S'v, X) \ge \sqrt{\rho}t$, we have $\overline{\deg}_{\mathcal{H}}(v, XY^{k-1}) > 2\sqrt{\rho}\binom{|Y|-1}{k-2}\sqrt{\rho}t$. Since $|Y| = (2k - 2\ell - 1)t + \ell$, we have

$$\overline{\deg}_{\mathcal{H}}\big(v, XY^{k-1}\big) > 2\rho \frac{|Y| - \ell}{2k - 2\ell - 1} \binom{|Y|-1}{k-2} > \rho \frac{|Y|}{k-1} \binom{|Y|-1}{k-2} = \rho \binom{|Y|}{k-1},$$

contradicting our assumption (the second inequality holds because $|Y|$ is sufficiently large).

Let Q be a $(2k-\ell-1)$-subset of Y. We call Q *good* (otherwise *bad*) if every $(k-1)$-subset of Q is an edge of $\mathcal{G}$ and every ℓ-set $L \subset Q$ satisfies

$$\overline{\deg}_{\mathcal{G}}(L) \le \rho^{1/4} \binom{|Y| - \ell}{k - \ell - 1}. \tag{3.13}$$

Furthermore, we say Q is *suitable* for a vertex $x \in X$ if $x \cup T \in E(\mathcal{H})$ for every $(k-1)$-set $T \subset Q$. Note that if a $(2k-\ell-1)$-set is good, by the definition of $\mathcal{G}$, it is suitable for at least $(1 - \binom{2k-\ell-1}{k-1}\sqrt{\rho})t$ vertices of X. Let $Y' = Y \setminus (L_0 \cup L_1)$.

Claim 3.13. *For any $x \in X$, at least $(1-\rho^{1/5})\binom{|Y|}{2k-\ell-1}$ $(2k-\ell-1)$-subsets of Y' are good and suitable for x.*

Proof. Since $\rho + \rho^{1/2} + 3\binom{2k-\ell-1}{\ell}\rho^{1/4} \le \rho^{1/5}$, the claim follows from the following three assertions:

- At most $2\ell\binom{|Y|-1}{2k-\ell-2} \le \rho\binom{|Y|}{2k-\ell-1}$ $(2k-\ell-1)$-subsets of Y are not subsets of Y'.
- Given $x \in X$, at most $\rho^{1/2}\binom{|Y|}{2k-\ell-1}$ $(2k-\ell-1)$-sets in Y are not suitable for x.
- At most $3\binom{2k-\ell-1}{\ell}\rho^{1/4}\binom{|Y|}{2k-\ell-1}$ $(2k-\ell-1)$-sets in Y are bad.

The first assertion holds because $|Y \setminus Y'| = 2\ell$. The second assertion follows from the degree condition of $\mathcal{H}$, namely, for any $x \in X$, the number of $(2k-\ell-1)$-sets in Y that are not suitable for x is at most $\rho\binom{|Y|}{k-1}\binom{|Y|-k+1}{k-\ell} \le \sqrt{\rho}\binom{|Y|}{2k-\ell-1}$.

To see the third one, let m be the number of ℓ-sets $L \subseteq Y$ that fail (3.13). By (3.12),

$$m\frac{\rho^{1/4}\binom{|Y|-\ell}{k-\ell-1}}{\binom{k-1}{\ell}} \le \bar{e}(\mathcal{G}) \le 2\sqrt{\rho}\binom{|Y|}{k-1},$$

which implies that $m \le 2\rho^{1/4}\binom{|Y|}{\ell}$. Thus at most

$$2\rho^{1/4}\binom{|Y|}{\ell} \cdot \binom{|Y|-\ell}{2k-2\ell-1}$$

$(2k-\ell-1)$-subsets of Y contain an ℓ-set L that fails (3.13). On the other hand, by (3.12), at most

$$\bar{e}(\mathcal{G})\binom{|Y|-k+1}{k-\ell} \le 2\sqrt{\rho}\binom{|Y|}{k-1}\binom{|Y|-k+1}{k-\ell}$$

$(2k-\ell-1)$-subsets of Y contain a non-edge of $\mathcal{G}$. Putting these together, the number of bad $(2k-\ell-1)$-sets in Y is at most

$$\begin{aligned} &2\rho^{1/4}\binom{|Y|}{\ell}\binom{|Y|-\ell}{2k-2\ell-1} + 2\sqrt{\rho}\binom{|Y|}{k-1}\binom{|Y|-k+1}{k-\ell} \\ &\quad \le 3\binom{2k-\ell-1}{\ell}\rho^{1/4}\binom{|Y|}{2k-\ell-1}, \end{aligned}$$

as $\rho \ll 1$. $\square$

Let $\mathcal{F}_0$ be the set of good $(2k-\ell-1)$-sets in Y'. We will pick a family of disjoint good $(2k-\ell-1)$-sets in Y' such that for any $x \in X$, many members of this family are suitable for x. To achieve this, we pick a family $\mathcal{F}$ by selecting each member of $\mathcal{F}_0$ randomly and independently with probability $p = 6\sqrt{\rho}|Y|/\binom{|Y|}{2k-\ell-1}$. Then $|\mathcal{F}|$ follows the binomial distribution $B(|\mathcal{F}_0|, p)$ with expectation $\mathbb{E}(|\mathcal{F}|) = p|\mathcal{F}_0| \le p\binom{|Y|}{2k-\ell-1}$. Furthermore, for every $x \in X$, let $f(x)$ denote the number of members of $\mathcal{F}$ that are suitable for x. Then $f(x)$ follows the binomial distribution $B(N, p)$ with $N \ge (1-\rho^{1/5})\binom{|Y|}{2k-\ell-1}$ by Claim 3.13. Hence $\mathbb{E}(f(x)) \ge p(1-\rho^{1/5})\binom{|Y|}{2k-\ell-1}$. Since there are at most $\binom{|Y|}{2k-\ell-1} \cdot (2k-\ell-1) \cdot \binom{|Y|-1}{2k-\ell-2}$ pairs of intersecting $(2k-\ell-1)$-sets in Y, the expected number of intersecting pairs of $(2k-\ell-1)$-sets in $\mathcal{F}$ is at most

$$p^2\binom{|Y|}{2k-\ell-1} \cdot (2k-\ell-1) \cdot \binom{|Y|-1}{2k-\ell-2} = 36(2k-\ell-1)^2\rho|Y|.$$

By Chernoff's bound (the first two properties) and Markov's bound (the last one), we can find a family $\mathcal{F}$ of good $(2k-\ell-1)$-subsets of Y' that satisfies

- $|\mathcal{F}| \le 2p\binom{|Y'|}{2k-\ell-1} \le 12\sqrt{\rho}|Y|$,
- for any vertex $x \in X$, at least $\frac{p}{2}(1-\rho^{1/5})\binom{|Y|}{2k-\ell-1} \ge 2\sqrt{\rho}|Y|$ members of $\mathcal{F}$ are suitable for x.
- the number of intersecting pairs of $(2k-\ell-1)$-sets in $\mathcal{F}$ is at most $72(2k-\ell-1)^2\rho|Y|$.

After deleting one $(2k-\ell-1)$-set from each of the intersecting pairs from $\mathcal{F}$, we obtain a family $\mathcal{F}' \subseteq \mathcal{F}$ consisting of at most $12\sqrt{\rho}|Y|$ disjoint good $(2k-\ell-1)$-subsets of Y' and for each $x \in X$, at least

$$2\sqrt{\rho}|Y| - 72(2k-\ell-1)^2\rho|Y| \ge \frac{3}{2}\sqrt{\rho}|Y| \tag{3.14}$$

members of $\mathcal{F}'$ are suitable for x.

Denote $\mathcal{F}'$ by $\{Q_2, Q_4, \dots, Q_{2q}\}$ for some $q \le 12\sqrt{\rho}|Y|$. We arbitrarily partition each Q_{2i} into $L_{2i} \cup P_{2i} \cup L_{2i+1}$ such that $|L_{2i}| = |L_{2i+1}| = \ell$ and $|P_{2i}| = 2k-3\ell-1$. Since Q_{2i} is good, both L_{2i} and L_{2i+1} satisfy (3.13). We claim that L_0 and L_1 satisfy (3.13) as well. Let us show this for L_0. By the definition of $\mathcal{G}$, the number of $XY^{k-\ell-1}$-sets T such that $T \cup L_0 \notin E(\mathcal{H})$ is at least $\overline{\deg}_{\mathcal{G}}(L_0)\sqrt{\rho}t$. Using (3.9), we derive that $\overline{\deg}_{\mathcal{G}}(L_0)\sqrt{\rho}t \le \rho\binom{|Y|}{k-\ell}$. Since $|Y| \le (2k-2\ell)t$, it follows that $\overline{\deg}_{\mathcal{G}}(L_0) \le 2\sqrt{\rho}\binom{|Y|-1}{k-\ell-1} \le \rho^{1/4}\binom{|Y|-\ell}{k-\ell-1}$.

Next we greedily find disjoint $(2k-3\ell-1)$-sets $P_1, P_3, \dots, P_{2q-1}$ from $Y' \setminus \bigcup_{i=1}^q Q_{2i}$ such that for each $i \in [q]$, every $(k-\ell-1)$-subset of P_{2i-1} is a common neighbor of L_{2i-1} and L_{2i} in $\mathcal{G}$. Suppose that we have found $P_1, P_3, \dots, P_{2i-1}$ for some $i < q$. Since both L_{2i-1} and L_{2i} satisfy (3.13), at most

$$2 \cdot \rho^{1/4}\binom{|Y|-\ell}{k-\ell-1}\binom{|Y|-k+1}{k-2\ell}$$

$(2k-3\ell-1)$-subsets of Y contain a non-neighbor of L_{2i-1} or L_{2i}. Thus, the number of $(2k-3\ell-1)$-sets that can be chosen as P_{2i+1} is at least

$$\binom{|Y'|-(2k-2\ell-1)2q}{2k-3\ell-1} - 2\cdot\rho^{1/4}\binom{|Y|-\ell}{k-\ell-1}\binom{|Y|-k+1}{k-2\ell} > 0,$$

as $q \le 12\sqrt{\rho}|Y|$ and $\rho \ll 1$.

Let $Y_1 = Y' \setminus \bigcup_{i=1}^{q}(P_{2i-1} \cup Q_{2i})$ and $\mathcal{G}' = \mathcal{G}[Y_1]$. Then $|Y_1| = |Y'| - (2k-2\ell-1)2q$. Since $\overline{\deg}_{\mathcal{G}'}(v) \le \overline{\deg}_{\mathcal{G}}(v)$ for every $v \in Y_1$, we have, by (3.11),

$$\delta_1(\mathcal{G}') \ge \binom{|Y_1|-1}{k-2} - 2\sqrt{\rho}\binom{|Y|-1}{k-2} \ge (1-3\sqrt{\rho})\binom{|Y_1|-1}{k-2}.$$

Let $\alpha = 3\sqrt{\rho}$ and $\rho_0 = (22\alpha)^{\frac{1}{k-1}}$. We want to find two disjoint ρ_0-typical ordered $(k-\ell-1)$-subsets $(x_1, \dots, x_{k-\ell-1})$ and $(y_1, \dots, y_{k-\ell-1})$ of Y_1 such that

$$L_{2q+1} \cup \{x_1, \dots, x_{k-\ell-1}\},\ L_0 \cup \{y_1, \dots, y_{k-\ell-1}\} \in E(\mathcal{G}). \tag{3.15}$$

To achieve this, we choose $(x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1})$ from ${Y_1}^{2k-2}$ uniformly at random. By Lemma 3.12, with probability at least $\frac{8}{11}$, $(x_1, \dots, x_{k-\ell-1})$ and $(y_1, \dots, y_{k-\ell-1})$ are two disjoint ordered ρ_0-typical $(k-\ell-1)$-sets. Since L_0 satisfies (3.13), at most $(k-\ell-1)!\rho^{1/4}\binom{|Y|-\ell}{k-\ell-1}$ ordered $(k-\ell-1)$-subsets of Y are not neighbors of L_0 (the same holds for L_{2q+1}). Thus (3.15) fails with probability at most $2(k-\ell-1)!\rho^{1/4}$, provided that $x_1, \dots, x_{k-\ell-1}, y_1, \dots, y_{k-\ell-1}$ are all distinct. Therefore the desired $(x_1, \dots, x_{k-\ell-1})$ and $(y_1, \dots, y_{k-\ell-1})$ exist.

Next we apply Theorem 3.11 to $\mathcal{G}'$ and obtain a tight Hamilton path

$$\mathcal{P} = x_{k-\ell-1}x_{k-\ell-2}\cdots x_1 \cdots\cdots y_1 y_2 \cdots y_{k-\ell-1}.$$

Following the order of $\mathcal{P}$, we partition Y_1 into

$$R_{2q+1}, S_{2q+1}, R'_{2q+1}, L_{2q+2}, \dots, L_t, R_t, S_t, R'_t$$

such that $|L_i| = \ell$, $|R_i| = |R'_i| = k-2\ell$, and $|S_i| = \ell-1$. Since $\mathcal{P}$ is a tight path in $\mathcal{G}$, we have

$$L_i R_i S_i,\ S_i R'_i L_{i+1} \in E(\mathcal{G}) \tag{3.16}$$

for $2q+2 \le i \le t-1$. Letting $L_{t+1} = L_0$, by (3.15), we also have (3.16) for $i = 2q+1$ and $i = t$.

We now arbitrarily partition P_i, $1 \le i \le 2q$ into $R_i \cup S_i \cup R'_i$ such that $|R_i| = |R'_i| = k-2\ell$, and $|S_i| = \ell-1$. By the choice of P_i, (3.16) holds for $1 \le i \le 2q$.

Consider the bipartite graph Γ between X and $Z := \{z_1, z_2, \dots, z_t\}$ such that $x \in X$ and $z_i \in Z$ are adjacent if and only if $L_i R_i S_i x, x S_i R'_i L_{i+1} \in E(\mathcal{H})$. For every $i \in [t]$, since (3.16) holds, we have $\deg_\Gamma(z_i) \ge (1 - 2\sqrt{\rho})t$ by the definition of $\mathcal{G}$. Let $Z' = \{z_{2q+1}, \dots, z_t\}$ and X_0 be the set of $x \in X$ such that $\deg_\Gamma(x, Z') \le |Z'|/2$. Then

$$|X_0|\frac{|Z'|}{2} \le \sum_{x\in X} \overline{\deg}_\Gamma(x, Z') \le 2\sqrt{\rho}t \cdot |Z'|,$$

which implies that $|X_0| \le 4\sqrt{\rho}t = 4\sqrt{\rho}\frac{|Y|-\ell}{2k-2\ell-1} \le \frac{4}{3}\sqrt{\rho}|Y|$ (note that $2k-2\ell-1 \ge k \ge 3$).

We now find a perfect matching between X and Z as follows.

Step 1: Each $x \in X_0$ is matched to some z_{2i}, $i \in [q]$ such that the corresponding $Q_{2i} \in \mathcal{F}'$ is suitable for x (thus x and z_{2i} are adjacent in Γ) — this is possible because of (3.14) and $|X_0| \le \frac{4}{3}\sqrt{\rho}|Y|$.

Step 2: Each of the unused z_i, $i \in [2q]$ is matched to a vertex in $X \setminus X_0$ — this is possible because $\deg_\Gamma(z_i) \ge (1 - 2\sqrt{\rho})t \ge |X_0| + 2q$.

Step 3: Let X' be the set of the remaining vertices in X. Then $|X'| = t - 2q = |Z'|$. Now consider the induced subgraph Γ' of Γ on $X' \cup Z'$. Since $\delta(\Gamma') \ge |X'|/2$, the Marriage Theorem provides a perfect matching in Γ'.

The perfect matching between X and Z gives rise to the desired Hamilton path of $\mathcal{H}$. □

4. Concluding remarks

Let $h_d^\ell(k, n)$ denote the minimum integer m such that every k-graph $\mathcal{H}$ on n vertices with minimum d-degree $\delta_d(\mathcal{H}) \ge m$ contains a Hamilton ℓ-cycle (provided that $k - \ell$ divides n). In this paper we determined $h_{k-1}^\ell(k, n)$ for all $\ell < k/2$ and sufficiently large n. Unfortunately our proof does not give $h_{k-1}^\ell(k, n)$ for all k, ℓ such that $k - \ell$ does not divide k even though we believe that $h_{k-1}^\ell(k, n) = \frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)}$. In fact, when $k - \ell$ does not divide k, if we can prove a path-cover lemma similar to Lemma 2.3, then we can follow the proof in [13] to solve the nonextremal case. When $\ell \ge k/2$, we cannot define $\mathcal{Y}_{k,2\ell}$ so the current proof of Lemma 2.3 fails. In addition, when $\ell \ge k/2$, the extremal case becomes complicated as well.

The situation is quite different when $k - \ell$ divides k. When k divides n, one can easily construct a k-graph $\mathcal{H}$ such that $\delta_{k-1}(\mathcal{H}) \ge \frac{n}{2} - k$ and yet $\mathcal{H}$ contains no perfect matching and consequently no Hamilton ℓ-cycle for any ℓ such that $k - \ell$ divides k. A construction in [16] actually shows that $h_{k-1}^\ell(k, n) \ge \frac{n}{2} - k$ whenever $k - \ell$ divides k, even when k does not divide n. The exact value of $h_d^\ell(k, n)$, when $k - \ell$ divides k, is not known except for $h_2^2(3, n) = \lfloor n/2 \rfloor$ given in [21]. In the forthcoming paper [9], we determine $h_d^{k/2}(k, n)$ exactly for even k and any $d \ge k/2$.

Let $t_d(n, F)$ denote the minimum integer m such that every k-graph $\mathcal{H}$ on n vertices with minimum d-degree $\delta_d(\mathcal{H}) \ge m$ contains a perfect F-tiling. One of the first results on hypergraph tiling was $t_2(n, \mathcal{Y}_{3,2}) = n/4 + o(n)$ given by Kühn and Osthus [14]. The exact value of $t_2(n, \mathcal{Y}_{3,2})$ was determined recently by Czygrinow, DeBiasio, and Nagle

[2]. We [8] determined $t_1(n, \mathcal{Y}_{3,2})$ very recently. The key lemma in our proof, Lemma 2.8, shows that every k-graph $\mathcal{H}$ on n vertices with $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2k-b} - o(1))n$ either contains an almost perfect $\mathcal{Y}_{k,b}$-tiling or is in the extremal case. Naturally this raises a question: what is $t_{k-1}(n, \mathcal{Y}_{k,b})$? Mycroft [17] recently proved a general result on tiling k-partite k-graphs, which implies that $t_{k-1}(n, \mathcal{Y}_{k,b}) = \frac{n}{2k-b} + o(n)$. The lower bound comes from the following construction. Let $\mathcal{H}_0$ be the k-graph on $n \in (2k-b)\mathbb{N}$ vertices such that $V(\mathcal{H}_0) = A \cup B$ with $|A| = \frac{n}{2k-b} - 1$, and $E(\mathcal{H}_0)$ consists of all k-sets intersecting A and some k-subsets of B such that $\mathcal{H}_0[B]$ contains no copy of $\mathcal{Y}_{k,b}$. Thus, $\delta_{k-1}(\mathcal{H}_0) \geq \frac{n}{2k-b} - 1$. Since every copy of $\mathcal{Y}_{k,b}$ contains at least one vertex in A, there is no perfect $\mathcal{Y}_{k,b}$-tiling in $\mathcal{H}_0$. We believe that one can find a matching upper bound by the absorbing method (similar to the proof in [2]). In fact, since we already proved Lemma 2.8, it suffices to prove an absorbing lemma and the extremal case.

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