FORBIDDING HAMILTON CYCLES IN UNIFORM HYPERGRAPHS

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Abstract. For $1 \le d \le \ell < k$, we give a new lower bound for the minimum d-degree threshold that guarantees a Hamilton ℓ-cycle in k-uniform hypergraphs. When $k \ge 4$ and $d < \ell = k-1$, this bound is larger than the conjectured minimum d-degree threshold for perfect matchings and thus disproves a well-known conjecture of Rödl and Ruciński. Our (simple) construction generalizes a construction of Katona and Kierstead and the space barrier for Hamilton cycles.

1. Introduction

The study of Hamilton cycles is an important topic in graph theory. A classical result of Dirac [4] states that every graph on $n \ge 3$ vertices with minimum degree $n/2$ contains a Hamilton cycle. In recent years, researchers have worked on extending this theorem to hypergraphs – see recent surveys [16, 18, 26].

To define Hamilton cycles in hypergraphs, we need the following definitions. Given $k \ge 2$, a k-uniform hypergraph (in short, *k-graph*) consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$, where every edge is a k-element subset of V. Given a k-graph H with a set S of d vertices (where $1 \le d \le k-1$) we define $\deg_H(S)$ to be the number of edges containing S (the subscript H is omitted if it is clear from the context). The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-vertex sets S in H. For $1 \le \ell \le k-1$, a k-graph is a called an *ℓ-cycle* if its vertices can be ordered cyclically such that each of its edges consists of k consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly ℓ vertices. In k-graphs, a $(k-1)$-cycle is often called a *tight* cycle. We say that a k-graph contains a *Hamilton ℓ-cycle* if it contains an ℓ-cycle as a spanning subhypergraph. Note that a Hamilton ℓ-cycle of a k-graph on n vertices contains exactly $n/(k-\ell)$ edges, implying that $k-\ell$ divides n.

Let $1 \le d, \ell \le k-1$. For $n \in (k-\ell)\mathbb{N}$, we define $h_d^\ell(k,n)$ to be the smallest integer h such that every n-vertex k-graph H satisfying $\delta_d(H) \ge h$ contains a Hamilton ℓ-cycle. Note that whenever we write $h_d^\ell(k,n)$, we always assume that $1 \le d \le k-1$. Moreover, we often write $h_d(k,n)$ instead of $h_d^{k-1}(k,n)$ for simplicity. Similarly, for $n \in k\mathbb{N}$, we define $m_d(k,n)$ to be the smallest integer m such that every n-vertex k-graph H satisfying $\delta_d(H) \ge m$ contains a perfect matching. The problem of determining $m_d(k,n)$ has attracted much attention recently and the asymptotic value of $m_d(k,n)$ is conjectured as follows. Note that the $o(1)$ term refers to a function that tends to 0 as $n \to \infty$ throughout the paper.

Conjecture 1.1. [6, 15] *For $1 \le d \le k-1$ and $k \mid n$,*

$$m_d(k,n) = \left(\max\left\{ \frac{1}{2}, 1 - \left(1 - \frac{1}{k}\right)^{k-d} \right\} + o(1) \right) \binom{n-d}{k-d}.$$

Conjecture 1.1 has been confirmed [1, 17] for $\min\{k-4, k/2\} \le d \le k-1$ (the exact values of $m_d(k,n)$ are also known in some cases, e.g., [23, 25]). On the other hand, $h_d^\ell(k,n)$ has also been extensively studied [2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 19, 20, 22, 24]. In particular, Rödl, Ruciński and Szemerédi [20, 22] showed that $h_{k-1}(k,n) = (1/2 + o(1))n$. The same authors proved in [21] that $m_{k-1}(k,n) = (1/2 + o(1))n$ (later they determined $m_{k-1}(k,n)$ exactly [23]). This suggests that the values of $h_d(k,n)$ and $m_d(k,n)$ are closely related and inspires Rödl and Ruciński to make the following conjecture.

Conjecture 1.2. [18, Conjecture 2.18] *Let $k \ge 3$ and $1 \le d \le k-2$. Then*

$$h_d(k,n) = m_d(k,n) + o(n^{k-d}).$$

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By using the value of $m_d(k,n)$ from Conjecture 1.1, Kühn and Osthus stated this conjecture explicitly for the case $d=1$.

Conjecture 1.3. [16, Conjecture 5.3] *Let $k \ge 3$. Then*

$$h_1(k,n) = \left(1 - \left(1 - \frac{1}{k}\right)^{k-1} + o(1)\right)\binom{n-1}{k-1}.$$

In this note we provide new lower bounds for $h_d^\ell(k,n)$ when $d \le \ell$.

Theorem 1.4. *Let $1 \le d \le k-1$ and $t = k-d$, then*

$$h_d(k,n) \ge \left(1 - \binom{t}{\lfloor t/2 \rfloor}\frac{\lceil t/2\rceil^{\lceil t/2\rceil}(\lfloor t/2\rfloor+1)^{\lfloor t/2\rfloor}}{(t+1)^t} + o(1)\right)\binom{n}{t}.$$

Theorem 1.5. *Let $1 \le d \le \ell \le k-1$ and $t = k-d$. Then*

$$h_d^\ell(k,n) \ge \left(1 - b_{t,k-\ell}2^{-t} + o(1)\right)\binom{n}{t},$$

where $b_{t,k-\ell}$ equals to the largest sum of $k-\ell$ consecutive binomial coefficients taken from $\{\binom{t}{0},\dots,\binom{t}{t}\}$.

Theorem 1.4 disproves both Conjectures 1.2 and 1.3.

Corollary 1.6. *For all k,*

$$h_{k-2}(k,n) \ge \left(\frac{5}{9} + o(1)\right)\binom{n}{2},\ h_{k-3}(k,n) \ge \left(\frac{5}{8} + o(1)\right)\binom{n}{3},\ h_{k-4}(k,n) \ge \left(\frac{409}{625} + o(1)\right)\binom{n}{4}$$

and in general, for any $1 \le d \le k-1$,

$$h_d(k,n) > \left(1 - \frac{1}{\sqrt{3(k-d)/2+1}}\right)\binom{n}{k-d}. \tag{1.1}$$

These bounds imply that Conjecture 1.2 is false when $k \ge 4$ and $\min\{k-4, k/2\} \le d \le k-2$, and Conjecture 1.3 is false whenever $k \ge 4$.

We will prove Theorem 1.4, Theorem 1.5, and Corollary 1.6 in the next section.

We believe that Conjecture 1.2 is false whenever $k \ge 4$ but due to our limited knowledge on $m_d(k,n)$, we can only disprove Conjecture 1.2 for the cases when $m_d(k,n)$ is known.

This bound $h_{k-2}(k,n) \ge (\frac{5}{9} + o(1))\binom{n}{2}$ coincides with the value of $m_1(3,n)$ – it was shown in [6] that $m_1(3,n) = (5/9 + o(1))\binom{n}{2}$, and it was widely believed that $h_1(3,n) = (5/9+o(1))\binom{n}{2}$, e.g., see [19]. On the other hand, it is known [17] that $m_2(4,n) = (\frac{1}{2} + o(1))\binom{n}{2}$, which is smaller than $\frac{5}{9}\binom{n}{2}$. Therefore $k=4$ and $d=2$ is the smallest case when Theorem 1.4 disproves Conjecture 1.2. More importantly, (1.1) shows that $h_d(k,n)/\binom{n}{k-d}$ tends to one as $k-d$ tends to ∞. For example, as k becomes sufficiently large, $h_{k-\ln k}(k,n)$ is close to $\binom{n-d}{k-d}$, the trivial upper bound. In contrast, Conjecture 1.1 suggests that there exists $c > 0$ independent of k and d ($c = 1/e$, where $e = 2.718...$, if Conjecture 1.1 is true) such that $m_d(k,n) \le (1-c)\binom{n-d}{k-d}$.

Similarly, by Theorem 1.5, if $k - \ell = o(\sqrt{t})$, $h_d^\ell(k,n)/\binom{n}{t}$ tends to one as t tends to ∞ because

$$1 - b_{t,k-\ell}2^{-t} \ge 1 - \frac{k-\ell}{2^t}\binom{t}{\lfloor t/2\rfloor} \approx 1 - \frac{o(\sqrt{t})}{\sqrt{\pi t/2}}.$$

Theorem 1.5 also implies the following special case: suppose k is odd and $\ell = d = k-2$. Then $t=2$ and $b_{t,k-\ell} = b_{2,2} = 3$, and consequently $h_{k-2}^{k-2}(k,n) \ge \left(\frac{1}{4} + o(1)\right)\binom{n}{2}$. Previously it was only known that $h_{k-2}^{k-2}(k,n) \ge (1 - (\frac{k}{k+1})^2 + o(1))\binom{n}{2}$ by (2.1) with $a = \lceil k/(k-\ell)\rceil = (k+1)/2$ from Section 2. When k is large, the bound provided by Theorem 1.5 is much better.

Finally, we do not know if Theorems 1.4 and 1.5 are best possible. Glebov, Person, and Weps [5] gave a general upper bound (far away from our lower bounds)

$$h_d^\ell(k,n) \le \left(1 - \frac{1}{ck^{3k-3}}\right)\binom{n-d}{k-d},$$

where c is a constant independent of d, ℓ, k, n.

2. The proofs

Before proving our results, it is instructive to recall the so-called *space barrier*. Throughout the paper, we write $X\dot\cup Y$ for $X\cup Y$ when sets X, Y are disjoint.

Proposition 2.1. [13] *Let $H=(V,E)$ be an n-vertex k-graph such that $V=X\dot\cup Y$ and $E=\{e\in\binom{V}{k}: e\cap X\neq\emptyset\}$. Suppose $|X|<\frac{1}{a(k-\ell)}n$, where $a:=\lceil k/(k-\ell)\rceil$, then H does not contain a Hamilton ℓ-cycle.*

A proof of Proposition 2.1 can be found in [13, Proposition 2.2] and is actually included in our proof of Proposition 2.2 below. It is not hard to see that Proposition 2.1 shows that

$$h_d^\ell(k,n)\geq\left(1-\left(1-\frac{1}{a(k-\ell)}\right)^{k-d}+o(1)\right)\binom{n-d}{k-d}. \tag{2.1}$$

Now we state our construction for Hamilton cycles – it generalizes the one given by Katona and Kierstead [11, Theorem 3] (where $j=\lfloor k/2\rfloor$) and the space barrier (where $j=\ell+1-k$) simultaneously. The special case with $k=3, \ell=2, j=1$, and $|X|=n/3$ appears in [19, Construction 2].

Proposition 2.2. *Given an integer j such that $\ell+1-k\leq j\leq k$, let $H=(V,E)$ be an n-vertex k-graph such that $V=X\dot\cup Y$ and $E=\{e\in\binom{V}{k}:|e\cap X|\notin\{j,j+1,\dots,j+k-\ell-1\}$. Suppose $\frac{j-1}{a'(k-\ell)}n<|X|<\frac{j+k-\ell}{a(k-\ell)}n$, where $a':=\lfloor k/(k-\ell)\rfloor$ and $a:=\lceil k/(k-\ell)\rceil$, then H does not contain a Hamilton ℓ-cycle.*

Proof. Suppose instead, that H contains a Hamilton ℓ-cycle C. Then all edges e of C satisfy $|e\cap X|\notin\{j,j+1,\dots,j+k-\ell-1\}$. We claim that either all edges e of C satisfy $|e\cap X|\leq j-1$ or all edges e of C satisfy $|e\cap X|\geq j+k-\ell$. Otherwise, there must be two consecutive edges e_1,e_2 in C such that $|e_1\cap X|\leq j-1$ and $|e_2\cap X|\geq j+k-\ell$. However, since $|e_1\cap e_2|=\ell$, we have $||e_1\cap X|-|e_2\cap X||\leq k-\ell$, a contradiction.

Observe that every vertex of H is contained in either a or a' edges of C and C contains $\frac{n}{k-\ell}$ edges. This implies that

$$a'|X|\leq\sum_{e\in C}|e\cap X|\leq a|X|.$$

On the other hand, we have $\sum_{e\in C}|e\cap X|<(j-1)\frac{n}{k-\ell}$ or $\sum_{e\in C}|e\cap X|>(j+k-\ell)\frac{n}{k-\ell}$. In either case, we get a contradiction with the assumption $\frac{j-1}{a'(k-\ell)}n<|X|<\frac{j+k-\ell}{a(k-\ell)}n$. □

Note that by reducing the lower and upper bounds for $|X|$ by small constants, we can conclude that H actually contains no *Hamilton ℓ-path.*

In the proofs of Theorems 1.4 and 1.5, we will consider binomial coefficients $\binom{p}{q}$ with $q<0$ – in this case $\binom{p}{q}=0$. We will conveniently write $|X|=xn$, where $0<x<1$, instead of $|X|=\lfloor xn\rfloor$ – this does not affect our calculations as n is sufficiently large.

In the proofs of Theorem 1.4 we apply Proposition 2.2 with j and x such that $\frac{j-1}{k}<x<\frac{j+1}{k}$. Note that (when $j\leq k$, $1\leq d\leq k-1$, and $t=k-d$) $\frac{j-1}{k}<x<\frac{j+1}{k}$ implies that $\frac{j-d}{t+1}<x<\frac{j+1}{t+1}$, in particular,

$$j-d\leq\lfloor x(t+1)\rfloor\leq j. \tag{2.2}$$

Proof of Theorem 1.4. Let $x=\lceil t/2\rceil/(t+1)$.[1] Since $\bigcup_{j=1}^{k-1}(\frac{j-1}{k},\frac{j+1}{k})=(0,1)$ and $1/3\leq\frac{\lceil t/2\rceil}{t+1}\leq 1/2$, there exists an integer $j\in[k-1]$ such that $\frac{j-1}{k}<\frac{\lceil t/2\rceil}{t+1}<\frac{j+1}{k}$. Let $H=(V,E)$ be an n-vertex k-graph such that $V=X\dot\cup Y$, $|X|=xn$ and $E=\{e\in\binom{V}{k}:|e\cap X|\neq j\}$. Since $\frac{j-1}{k}n<|X|<\frac{j+1}{k}n$, H contains no tight Hamilton cycle by Proposition 2.2 with $\ell=k-1$.

Now let us compute $\delta_d(H)$. For $0\leq i\leq d$, let S_i be any d-vertex subset of V that contains exactly i vertices in X. By the definition of H,

$$\deg_H(S_i)=\binom{n-d}{t}-\binom{|X|-i}{j-i}\binom{|Y|-(d-i)}{t-j+i}.$$

[1] It can be shown that among all $0<x<1$ and $0\leq j\leq k$ such that $\frac{j-1}{k}<x<\frac{j+1}{k}$, our choice of x and j minimizes $\max_{j-d\leq i\leq j}\binom{t}{i}x^i(1-x)^{t-i}$, and thus maximizes $\delta_d(H)$ in (2.3).

Note that this holds for $i > j$ or $i < j - t$ trivially. So we have

$$\delta_d(H) = \min_{0\le i\le d}\left\{\binom{n-d}{t} - \binom{|X|-i}{j-i}\binom{|Y|-(d-i)}{t-j+i}\right\}$$
$$= \binom{n}{t} - \max_{j-d\le i'\le j}\left\{\binom{|X|}{i'}\binom{|Y|}{t-i'}\right\} + o(n^t).$$

Write $|X| = xn$ and $|Y| = yn$. When $0 \le i' \le t$, we have

$$\binom{|X|}{i'}\binom{|Y|}{t-i'} = \frac{(xn)^{i'}(yn)^{t-i'}}{i'!(t-i')!} + o(n^t) = \binom{t}{i'}x^{i'}y^{t-i'}\binom{n}{t} + o(n^t).$$

When $i' < 0$ or $i' > t$, we have $\binom{|X|}{i'}\binom{|Y|}{t-i'} = 0 = \binom{t}{i'}x^{i'}y^{t-i'}\binom{n}{t}$. In all cases, we have

$$\delta_d(H) = \binom{n}{t} - \max_{j-d\le i'\le j}\left\{\binom{t}{i'}x^{i'}y^{t-i'}\right\}\binom{n}{t} + o(n^t). \tag{2.3}$$

Let $a_i := \binom{t}{i}x^i y^{t-i}$. Since $x = \lceil t/2\rceil/(t+1)$ and $y = 1 - x$, it is easy to see that $\max_{0\le i\le t} a_i = a_{\lceil t/2\rceil}$ (e.g., by observing $\frac{a_i}{a_{i+1}} = \frac{y}{x}\cdot\frac{i+1}{t-i}$ for $0 \le i < t$). Moreover, by (2.2), we have $j - d \le \lceil t/2\rceil \le j$. Therefore,

$$\max_{j-d\le i\le j}\{a_i\} = a_{\lceil t/2\rceil} = \binom{t}{\lceil t/2\rceil}\frac{\lceil t/2\rceil^{\lceil t/2\rceil}(\lfloor t/2\rfloor+1)^{\lfloor t/2\rfloor}}{(t+1)^t},$$

and we complete the proof by substituting it into (2.3). □

Now we turn to the proof of Theorem 1.5, in which we assume that $|X| = n/2$, though a further improvement of the lower bound may be possible by considering other values of $|X|$.

Proof of Theorem 1.5. The proof is similar to the one of Theorem 1.4. Let $H = (V, E)$ be an n-vertex k-graph such that $V = X\dot\cup Y$, $|X| = n/2$ and $E = \{e \in \binom{V}{k} : |e \cap X| \notin \{\lceil \ell/2\rceil, \dots, \lceil \ell/2\rceil + k - \ell - 1\}\}$. Note that

$$a'(k-\ell) = \left\lfloor\frac{k}{k-\ell}\right\rfloor(k-\ell) \ge k - (k-\ell-1) = \ell+1 > 2(\lceil \ell/2\rceil - 1), \text{ and}$$
$$a(k-\ell) = \left\lceil\frac{k}{k-\ell}\right\rceil(k-\ell) \le k + (k-\ell-1) < 2(k - \lfloor \ell/2\rfloor) = 2(\lceil \ell/2\rceil + k - \ell).$$

So we have

$$\frac{\lceil \ell/2\rceil - 1}{a'(k-\ell)}n < |X| = \frac{n}{2} < \frac{\lceil \ell/2\rceil + k - \ell}{a(k-\ell)}n.$$

Thus, H contains no Hamilton ℓ-cycle by Proposition 2.2.

Fix $1 \le d \le k-1$ and let $t = k - d$. Now we compute $\delta_d(H)$. For $0 \le i \le d$, let S_i be any d-vertex subset of V that contains exactly i vertices in X. It is easy to see that

$$\deg_H(S_i) = \binom{n}{t} - \sum_{p=i'}^{i'+k-\ell-1}\binom{|X|}{p}\binom{|Y|}{t-p} + o(n^t),$$

where $i' = \lceil \ell/2\rceil - i$. Using $|X| = |Y| = n/2$ and the similar calculations in the proof of Theorem 1.4, we get

$$\deg_H(S_i) = \binom{n}{t} - \sum_{p=i'}^{i'+k-\ell-1}\binom{t}{p}\frac{1}{2^t}\binom{n}{t} + o(n^t).$$

By the definition of $b_{t,k-\ell}$, we have

$$\delta_d(H) = \min_{0\le i\le d}\deg_H(S_i) \ge \binom{n}{t} - b_{t,k-\ell}2^{-t}\binom{n}{t} + o(n^t).$$ □

Corollary 1.6 follows from Theorem 1.4 via simple calculations.

Proof of Corollary 1.6. Let $t = k - d$ and

$$f(t) := \binom{t}{\lfloor t/2 \rfloor} \frac{\lceil t/2 \rceil^{\lceil t/2 \rceil} (\lfloor t/2 \rfloor + 1)^{\lfloor t/2 \rfloor}}{(t+1)^t}.$$

Theorem 1.4 states that $h_{k-t}(k, n) \geq (1 - f(t) + o(1))\binom{n}{t}$ for any $1 \leq t \leq k-1$. Since

$$f(2) = \frac{4}{9}, \quad f(3) = \frac{3}{8}, \quad \text{and} \quad f(4) = \frac{216}{625},$$

the bounds for $h_{k-t}(k, n)$, $t = 2, 3, 4$, are immediate. To see (1.1), it suffices to show that for $t \geq 1$,

$$1 - f(t) > 1 - \frac{1}{\sqrt{3t/2 + 1}}. \tag{2.4}$$

When t is odd, $\frac{\lceil t/2 \rceil^{\lceil t/2 \rceil} (\lfloor t/2 \rfloor + 1)^{\lfloor t/2 \rfloor}}{(t+1)^t} = 1/2^t$; when t is even, $\lceil t/2 \rceil^{\lceil t/2 \rceil} (\lfloor t/2 \rfloor + 1)^{\lfloor t/2 \rfloor} < (\frac{t+1}{2})^t$. Thus, for all t, we have

$$f(t) \leq \binom{t}{\lfloor t/2 \rfloor} \frac{1}{2^t},$$

where a strict inequality holds for all even t. Now we use the fact $\binom{2m}{m} \leq 2^{2m}/\sqrt{3m+1}$, which holds for all integers $m \geq 1$. Thus, for all even t, we have $f(t) \leq 1/\sqrt{3t/2+1}$; for all odd t,

$$f(t) \leq \binom{t}{\lfloor t/2 \rfloor} \frac{1}{2^t} = \frac{1}{2} \binom{t+1}{\lfloor t/2 \rfloor + 1} \frac{1}{2^t} \leq \frac{1}{\sqrt{3(t+1)/2 + 1}} < \frac{1}{\sqrt{3t/2+1}}.$$

Hence $f(t) \leq 1/\sqrt{3t/2+1}$ for all $t \geq 1$. Moreover, by the computation above, regardless of the parity of t, the strict inequality always holds and thus (2.4) is proved.

We next show that whenever $k \geq 4$ and $2 \leq t \leq k-1$,

$$1 - f(t) > \max\left\{\frac{1}{2}, 1 - \left(1 - \frac{1}{k}\right)^t\right\}.$$

This implies that Conjecture 1.3 fails for $k \geq 4$, and Conjecture 1.2 fails for $k \geq 4$ and $\min\{k-4, k/2\} \leq d \leq k-2$ (because $m_d(k,n)/\binom{n}{k-d} = \max\left\{\frac{1}{2}, 1 - \left(1 - \frac{1}{k}\right)^{k-d}\right\} + o(1)$ in this case). It suffices to show that for $k \geq 4$ and $2 \leq t \leq k-1$,

$$f(t) < 1/2 \text{ and } f(t) < \left(1 - \frac{1}{k}\right)^t.$$

The first inequality immediately follows from (2.4) and $1/\sqrt{3t/2+1} \leq 1/2$. For the second inequality, note that

$$f(t) < \frac{1}{\sqrt{3t/2+1}} < \frac{1}{e} < \left(1 - \frac{1}{k}\right)^{k-1} \leq \left(1 - \frac{1}{k}\right)^t$$

for all $t \geq 5$. For $t = 2, 3$ and all $k \geq 4$, one can verify $f(t) < (3/4)^t \leq \left(1 - \frac{1}{k}\right)^t$ easily. Also, for $t = 4$ and all $k \geq 5$, we have $f(4) < (4/5)^4 \leq (1 - \frac{1}{k})^4$. □

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