

HAMILTONICITY IN RANDOMLY PERTURBED HYPERGRAPHS

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ABSTRACT. For integers $k \geq 3$ and $1 \leq \ell \leq k - 1$, we prove that for any $\alpha > 0$, there exist $\epsilon > 0$ and $C > 0$ such that for sufficiently large $n \in (k - \ell)\mathbb{N}$, the union of a k -uniform hypergraph with minimum vertex degree αn^{k-1} and a binomial random k -uniform hypergraph $\mathbb{G}^{(k)}(n, p)$ with $p \geq n^{-(k-\ell)-\epsilon}$ for $\ell \geq 2$ and $p \geq Cn^{-(k-1)}$ for $\ell = 1$ on the same vertex set contains a Hamiltonian ℓ -cycle with high probability. Our result is best possible up to the values of ϵ and C and answers a question of Krivelevich, Kwan and Sudakov.

1. INTRODUCTION

1.1. Hamiltonian cycles and random graphs. The study of Hamiltonicity (the existence of a spanning cycle) has been a central and fruitful area in graph theory. In particular, a celebrated result of Karp [19] states that the decision problem for Hamiltonicity in graphs is NP-complete. So it is desirable to study sufficient conditions that guarantees Hamiltonicity. Among a large variety of such results, probably the most well-known is a theorem of Dirac from 1952 [11]: every n -vertex graph ($n \geq 3$) with minimum degree at least $n/2$ is Hamiltonian.

Another well-studied object in graph theory is the random graph $\mathbb{G}(n, p)$, which contains n vertices and each pair of vertices forms an edge with probability p independently from other pairs. Pósa [27] and Korshunov [21] independently determined the threshold for Hamiltonicity in $\mathbb{G}(n, p)$, which is around $\log n/n$. This implies that almost all dense graphs are Hamiltonian. Furthermore, Bohman, Frieze and Martin [6] showed that for every $\alpha > 0$ there is $c = c(\alpha)$ such that every n -vertex graph G with minimum degree αn becomes Hamiltonian *a.a.s.* after adding cn random edges (we say that an event happens *asymptotically almost surely*, or *a.a.s.*, if it happens with probability $1 - o(1)$). This result is tight up to the value of c by considering a complete bipartite graph $K_{\alpha n, (1-\alpha)n}$. A comparison can be drawn to the notion of smoothed analysis of algorithms introduced by Spielman and Teng [34], which involves studying the performance of algorithms on randomly perturbed inputs.

1.2. Uniform hypergraphs. It is natural to study the Hamiltonicity of uniform hypergraphs. Given $k \geq 2$, a k -uniform hypergraph (in short, a k -graph) $H = (V, E)$ consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$, where every edge is a k -element subset of V . Given a k -graph H with a set S of d vertices (where $1 \leq d \leq k - 1$) we define $N_H(S)$ to be the collection of $(k - d)$ -sets T such that $S \cup T \in E(H)$, and let $\deg_H(S) := |N_H(S)|$. The *minimum d -degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d -vertex sets S in H .

In the last two decades, there has been a growing interest of extending Dirac's theorem to hypergraphs. Despite other notion of cycles in hypergraphs (e.g., Berge cycles), the following definition of cycles has become more popular recently (see surveys [29, 35]). For integers $1 \leq \ell \leq k - 1$ and $m \geq 3$, a k -graph F with $m(k - \ell)$ vertices and m edges is called an ℓ -cycle if its vertices can be ordered cyclically such that each of its edges consists of k consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly ℓ vertices. A k -graph is called ℓ -Hamiltonian if it contains an ℓ -cycle as a spanning subgraph. Extending Dirac's theorem, the minimum d -degree conditions that force ℓ -Hamiltonicity (for $1 \leq d, \ell \leq k - 1$) have been intensively studied [2, 3, 9, 10, 14–17, 20, 24, 25, 30–33]. For example, the minimum 1-degree threshold for 2-Hamiltonicity in 3-graphs was determined asymptotically [28].

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Let $\mathbb{G}^{(k)}(n, p)$ denote the binomial random k -graph on n vertices, where each k -set forms an edge independently with probability p . The thresholds for ℓ -Hamiltonicity have been studied by Dudek and Frieze [12, 13], who proved that the asymptotic threshold is $1/n^{k-\ell}$ for $\ell \geq 2$ and $\log n/n^{k-1}$ for $\ell = 1$ (they also gave a sharp threshold for $k \geq 4$ and $\ell = k - 1$).

It is also natural to consider ℓ -Hamiltonicity in randomly perturbed k -graphs. In fact, Krivelevich, Kwan and Sudakov [22] extended the result of Bohman–Frieze–Martin [6] to hypergraphs.

Theorem 1.1. [22] *Let $k \in \mathbb{N}$, and let H be a k -graph on $n \in (k - 1)\mathbb{N}$ vertices with $\delta_{k-1}(H) \geq \alpha n$. There exists a function $c_k = c_k(\alpha)$ such that for $p = c_k/n^{k-1}$, $H \cup \mathbb{G}^{(k)}(n, p)$ a.a.s. is 1-Hamiltonian.*

Theorem 1.1 is tight up to the value of c_k (see the paragraph after Theorem 1.2). Similar results for the powers of Hamiltonian $(k - 1)$ -cycles were obtained by Bennett, Dudek and Frieze [5], and recently by Bedenknecht, Han, Kohayakawa and Mota [4]. In addition, Böttcher, Montgomery, Parczyk and Person [8] proved embedding results for bounded degree subgraphs in randomly perturbed graphs. Other results in randomly perturbed graphs can be found in [1, 7, 23].

Krivelevich, Kwan and Sudakov [22] asked whether Theorem 1.1 can be extended to ℓ -Hamiltonicity under minimum d -degree conditions for $1 \leq d, \ell \leq k - 1$. McDowell and Mycroft [26] found such results for $d \geq \max\{\ell, k - \ell\}$ and reiterated the question for arbitrary d and ℓ . In this paper we solve this problem completely. Since the minimum 1-degree condition is the weakest among d -degree conditions for all $d \geq 1$, we only state and prove our result with respect to the minimum 1-degree.

Theorem 1.2. *For integers $k \geq 3$, $1 \leq \ell \leq k - 1$ and $\alpha > 0$ there exist $\epsilon > 0$ and an integer $C > 0$ such that the following holds for sufficiently large $n \in (k - \ell)\mathbb{N}$. Suppose H is a k -graph on n vertices with $\delta_1(H) \geq \alpha n^{k-1}$ and*

$$p = p(n) \geq \begin{cases} n^{-(k-\ell)-\epsilon} & \text{if } \ell \geq 2, \\ Cn^{-(k-1)} & \text{if } \ell = 1. \end{cases} \quad (1.1)$$

Then $H \cup \mathbb{G}^{(k)}(n, p)$ a.a.s. is ℓ -Hamiltonian.

Theorem 1.2 is sharp up to the constants ϵ and C . Indeed, given k and ℓ , let $\alpha > 0$ be sufficiently small and $n \in (k - \ell)\mathbb{N}$ be sufficiently large. Consider a partition $A \cup B$ of a vertex set V such that $|A| = \alpha n$ and $|B| = (1 - \alpha)n$. Let H_0 be the k -graph with all k -tuples that intersect both A and B as edges. It is easy to see that $\delta_1(H_0) = \alpha n \binom{n-\alpha n-1}{k-2}$. Suppose $H_0 \cup \mathbb{G}^{(k)}(n, p)$ a.a.s. contains a Hamiltonian ℓ -cycle C . Since $|A| = \alpha n$, C contains at least $1/\alpha - 1$ consecutive vertices in B . Let $a = \lfloor (1/\alpha - 1 - \ell)/(k - \ell) \rfloor$. Since B is an independent set in H_0 , this implies that $\mathbb{G}^{(k)}(n, p)$ a.a.s. contains an ℓ -path on a edges (a k -graph with vertices $v_1, v_2, \dots, v_{a(k-\ell)+\ell}$ and edges $\{v_{i(k-\ell)+1}, \dots, v_{i(k-\ell)+k}\}$ for $i = 0, \dots, a - 1$). When $p < (1/2)^{1/a} n^{-(k-\ell)-\ell/a}$, we have $n^{\ell+a(k-\ell)} p^a < 1/2$. By Markov's inequality, with probability at least $1/2$, $\mathbb{G}^{(k)}(n, p)$ contains no ℓ -path on a edges. When $\ell = 1$, if $H_0 \cup \mathbb{G}^{(k)}(n, p)$ is a.a.s. ℓ -Hamiltonian, then $\mathbb{G}^{(k)}(n, p)$ a.a.s. contains $n/(k - 1) - 2|A| > n/k$ edges (because a 1-Hamiltonian cycle contains $n/(k - 1)$ edges and each vertex is contained in at most 2 of them). When $p < n^{-(k-1)}/(2k)$, we have $n^k p < n/(2k)$. By Markov's inequality, with probability at least $1/2$, $\mathbb{G}^{(k)}(n, p)$ contains fewer than n/k edges.

1.3. Proof ideas. The proof of Theorem 1.2 follows the *absorbing method* introduced by Rödl, Ruciński, and Szemerédi in [31]. Let us define *absorbers* for our problem. Given an ℓ -path P , we call the first and last ℓ vertices two ℓ -ends of P . Let H be a k -graph and S be a set of $k - \ell$ vertices in $V(H)$. We call an ℓ -path P an S -*absorber* if $V(P) \cap S = \emptyset$ and $V(P) \cup S$ spans an ℓ -path with the same ℓ -ends as P .

Below is a typical procedure for finding a Hamilton ℓ -cycle in H by the absorbing method.

- (1) We show that every $(k - \ell)$ -subset of $V(H)$ has many absorbers (of the same fixed length). This enables us to obtain a path P_{abs} of linear length such that every $(k - \ell)$ -set has many absorbers on P_{abs} .
- (2) We cover most vertices of $V \setminus V(P_{abs})$ by short paths and then connect them together with P_{abs} into a cycle C .

- (3) The vertices not covered by C are arbitrarily partitioned into $(k - \ell)$ -sets and absorbed by P_{abs} greedily.

The proof thus has three main components:

- an *absorbing lemma*, which provides a family \mathcal{A} of vertex-disjoint short paths such that every $(k - \ell)$ -set has many absorbers in \mathcal{A} ;
- a *path cover lemma*, which allows us to cover most vertices of $V(H)$ by vertex-disjoint paths; and
- a *connecting lemma*, which allows us to connect \mathcal{A} into a single path P_{abs} and connect the paths from the path cover lemma together.

Let $\mathbb{G}^{(k)}(n, p) \cup H$ be the underlying k -graph on the same vertex set V . Using Janson's inequality, one can derive the path cover lemma by using the edges of $\mathbb{G}^{(k)}(n, p)$. If we have $\delta_{k-\ell}(H) \geq \alpha \binom{n}{\ell}$, then every $(k - \ell)$ -set of V has many neighbors and it is not difficult to prove the absorbing lemma. If we have $\delta_\ell(H) \geq \alpha \binom{n}{k-\ell}$, then every ℓ -set of V has many neighbors and it is easy to prove the connecting lemma. However, our Theorem 1.2 only assumes $\delta_1(H) \geq \alpha n^{k-1}$. In order to prove Theorem 1.2, we “shave” H by removing all the edges of H that contain an ℓ -set of low degree. This results in a k -graph H' in which every ℓ -subset of V either has a high degree or a zero degree. Our connecting lemma only connects two ℓ -sets with high degree. To overcome the difficulty in absorbing, an earlier version of this paper used the hypergraph regularity method. Following the suggestion of a referee, we now give a simpler absorbing lemma without the regularity method. Note that the shaving process creates a small number of vertices that cannot be absorbed and we will cover these vertices by the path cover lemma.

The rest of the paper is organized as follows. We state and prove our lemmas in Sections 2 and 3 and prove Theorem 1.2 in Section 4.

Notation. Given positive integers $n \geq b$, let $[n] := \{1, 2, \dots, n\}$ and $(n)_b := n(n-1) \cdots (n-b+1) = n!/(n-b)!$. Given a k -graph H , we use v_H and e_H to denote the order and size of H , respectively. For two (hyper)graphs G and H , let $G \cap H$ (or $G \cup H$) denote the (hyper)graph with vertex set $V(G) \cap V(H)$ (or $V(G) \cup V(H)$) and edge set $E(G) \cap E(H)$ (or $E(G) \cup E(H)$). Given a set X , $\binom{X}{k}$ denotes the family of all k -subsets of X . A k -graph (V, E) is *complete* if $E = \binom{V}{k}$. Given $1 \leq \ell \leq k$, the ℓ -*shadow* of a k -graph H , denoted by $\partial_\ell H$, is the collection of all ℓ -subsets $S \subset V(H)$ that are contained in some edges of H .

In this paper, unless stated otherwise, we assume that the vertex sets of paths and related hypergraphs are *ordered*. When A and B are ordered sets, let AB denote their concatenation. Given positive integers k, ℓ, a such that $\ell < k$, let P_a denote a k -uniform ℓ -path of length a , that is, a k -graph on vertices $v_1, v_2, \dots, v_{a(k-\ell)+\ell}$ with edges $\{v_{i(k-\ell)+1}, \dots, v_{i(k-\ell)+k}\}$ for $i = 0, \dots, a-1$. In general, given a k -graph F on $\{x_1, \dots, x_s\}$ and a k -graph H , we say that an *ordered subset* (v_1, \dots, v_s) of $V(H)$ *spans a (labeled) copy of F* if $\{v_{i_1}, \dots, v_{i_k}\} \in E(H)$ whenever $\{x_{i_1}, \dots, x_{i_k}\} \in E(F)$. Given integers $a \geq 1$ and $x \geq 0$, let $P_{a,x}$ denote a k -graph on $a(k-\ell) + \ell + 2x$ vertices with an order such that the first and last x vertices are isolated and the middle $a(k-\ell) + \ell$ vertices span a copy of P_a .

Throughout the rest of the paper, we write $\alpha \ll \beta \ll \gamma$ to mean that we can choose the positive constants α, β, γ from right to left. More precisely, there are increasing functions f and g such that, given γ , whenever $\beta \leq f(\gamma)$ and $\alpha \leq g(\beta)$, the subsequent statement holds. Hierarchies of other lengths are defined similarly.

Throughout the paper we omit floor and ceiling functions when they are not crucial.

2. SUBGRAPHS IN RANDOM HYPERGRAPHS

In this section we introduce some results related to binomial random k -graphs (similar ones can be found in [4]). Our main tools are Janson's inequality (see, e.g., [18, Theorem 2.14]) and Chebyshev's inequality.

We first recall Janson's inequality. Let Γ be a finite set and let Γ_p be a random subset of Γ such that each element of Γ is included independently with probability p . Let \mathcal{S} be a family of non-empty subsets of Γ and for each $S \in \mathcal{S}$, let I_S be the indicator random variable for the event $S \subseteq \Gamma_p$. Thus each I_S is a Bernoulli random variable $\text{Be}(p^{|S|})$. Let $X := \sum_{S \in \mathcal{S}} I_S$ and $\lambda = \mathbb{E}(X)$. Let $\Delta_X := \sum_{S \cap T \neq \emptyset} \mathbb{E}(I_S I_T)$, where the sum is

over not necessarily distinct ordered pairs $S, T \in \mathcal{S}$. Then Janson's inequality says that for any $0 \leq t \leq \lambda$,

$$\mathbb{P}(X \leq \lambda - t) \leq \exp\left(-\frac{t^2}{2\Delta_X}\right). \quad (2.1)$$

Next note that $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \leq \Delta_X$. Then by Chebyshev's inequality,

$$\mathbb{P}(X \geq 2\lambda) \leq \frac{\text{Var}(X)}{\lambda^2} \leq \frac{\Delta_X}{\lambda^2}. \quad (2.2)$$

Consider the random k -graph $\mathbb{G}^{(k)}(n, p)$ on an n -vertex set V . Note that we can view $\mathbb{G}^{(k)}(n, p)$ as Γ_p with $\Gamma = \binom{V}{k}$. Let $\Phi_F = \Phi_F(n, p) = \min\{n^{v_H} p^{e_H} : H \subseteq F, e_H > 0\}$. The following simple proposition is useful.

Proposition 2.1. *Let F be a k -graph with s vertices and f edges and let $G := \mathbb{G}^{(k)}(n, p)$ on V . Given a family \mathcal{A} of ordered s -subsets of V , let $X_{\mathcal{A}} = \sum_{A \in \mathcal{A}} I_A$, where I_A is the Bernoulli random variable for the event that A spans a labeled copy of F in G . Then $\Delta_{X_{\mathcal{A}}} \leq 2^s s! n^{2s} p^{2f} / \Phi_F$.*

Proof. Fix $1 \leq i \leq s$. There are $\binom{s}{i} (s)_i$ ways that two labeled s -sets share exactly i vertices. Fixing two such s -sets, there are $\binom{n}{2s-i}$ ways mapping their $2s-i$ vertices into V . Let f_i denote the maximum number of edges of an i -vertex subgraph of F . We have

$$\Delta_{X_{\mathcal{A}}} \leq \sum_{i=1}^s \binom{s}{i} (s)_i (n)_{2s-i} p^{2f-f_i} \leq \sum_{i=1}^s \binom{s}{i} s! n^{2s-i} p^{2f-f_i} \leq 2^s s! n^{2s} p^{2f} / \Phi_F. \quad \square$$

The next two lemmas gather all the properties of $\mathbb{G}^{(k)}(n, p)$ that we will use.

Lemma 2.2. *Let $k, \ell, a, x \in \mathbb{Z}$ such that $k \geq 3, 1 \leq \ell \leq k-1, a \geq 1$, and $0 \leq x \leq k$. Write $b = b(x) = 2x + \ell + (k-\ell)a$. Suppose $0 < \epsilon \leq \ell/(3a)$ and $1/n \ll 1/C \ll \gamma, 1/a, 1/k$. Let $G = \mathbb{G}^{(k)}(n, p)$ be a random k -graph with vertex set V , where p satisfies (1.1). Then the following properties hold.*

- (1) *Let L be a family of ℓ -sets in $V(G)$ and in addition assume $a \geq \ell/(k-\ell)$. Then for every $R, V^* \subseteq V(G)$ such that $|V^*| \geq \gamma n$ and $|L \cap \binom{R}{\ell}| \geq \gamma n^\ell$, with probability at least $1 - \exp(-3n)$, G contains a copy of P_a whose ℓ -ends are in $L \cap \binom{R}{\ell}$ and whose other vertices are from V^* . Moreover, this property holds for all choices of R and V^* simultaneously with probability $1 - o(1)$.*
- (2) *With probability at least $1 - o(1)$, at most $2p^a n^b$ ordered b -subsets of $V(G)$ span copies of $P_{a,x}$.*
- (3) *With probability at least $1 - o(1)$, G contains at most $4b^2 n^{2b-1} p^{2a}$ pairs of overlapping (i.e., not vertex-disjoint) copies of $P_{a,x}$.*

Proof. Note that if H is a subgraph of $P_{a,x}$, then $v_H \geq \ell + (k-\ell)e_H$. Thus,

$$\Phi_{P_{a,x}} = \min_{1 \leq e_H \leq a} n^{v_H} p^{e_H} \geq \min_{1 \leq e_H \leq a} n^{\ell + (k-\ell)e_H} p^{e_H} = n^\ell \min_{1 \leq e_H \leq a} (n^{k-\ell} p)^{e_H} \geq \begin{cases} n^{\ell-a\epsilon} & \text{if } \ell \geq 2, \\ Cn & \text{if } \ell = 1, \end{cases} \quad (2.3)$$

where we used (1.1) in the last inequality. Since $\epsilon \leq \ell/(3a)$, $\Phi_{P_{a,x}} \geq Cn$ holds for all ℓ .

Given a family \mathcal{A} of ordered b -sets of vertices in V , let \mathcal{S} consist of the edge sets of the labeled copies of $P_{a,x}$ spanned on A in the complete k -graph on V for all $A \in \mathcal{A}$. Let $X_{\mathcal{A}} = \sum_{S \in \mathcal{S}} I_S$, where I_S is the indicator variable for the event $S \subseteq E(G)$ (thus $X_{\mathcal{A}}$ counts the number of $A \in \mathcal{A}$ that spans a copy of $P_{a,x}$ in G). Since $\Phi_{P_{a,x}} \geq Cn$, Proposition 2.1 implies that

$$\Delta_{X_{\mathcal{A}}} \leq 2^b b! n^{2b} p^{2a} / \Phi_{P_{a,x}} \leq (2^b b! / C) n^{2b-1} p^{2a} \leq (\gamma^{2b} / 24) n^{2b-1} p^{2a} \quad (2.4)$$

because $1/C \ll \gamma, 1/a, 1/k$.

For (1), fix such a choice for R and V^* and let $x = 0$ and $b = \ell + (k-\ell)a$. Let \mathcal{A} be the family of all ordered $(\ell + (k-\ell)a)$ -sets in $V(G)$ whose first and last ℓ vertices are in $L \cap \binom{R}{\ell}$ and all other vertices are from V^* . Then $|\mathcal{A}| \geq (\gamma n^\ell)^2 (\gamma n)^{(k-\ell)a-\ell} / 2 \geq (\gamma n)^b / 2$. Recall that $X_{\mathcal{A}}$ counts the number of $A \in \mathcal{A}$ that spans a copy of P_a in G . Then $(\gamma n)^b p^a / 2 \leq \mathbb{E}(X_{\mathcal{A}}) \leq n^b p^a$. By (2.1) and (2.4), we have

$$\mathbb{P}(X_{\mathcal{A}} = 0) \leq \exp\left(-\frac{\mathbb{E}(X_{\mathcal{A}})^2}{2\Delta_{X_{\mathcal{A}}}}\right) \leq \exp\left(-\frac{(\gamma n)^{2b} p^{2a} / 4}{(\gamma^{2b} / 12) n^{2b-1} p^{2a}}\right) = \exp(-3n).$$

The second part of (1) follows from the union bound because there are at most 2^n choices for each of R and V^* and $2^n \cdot 2^n \cdot \exp(-3n) \leq \exp(-n)$.

For (2), let X_2 be the random variable that counts the number of labeled copies of $P_{a,x}$ in G . Then $\mathbb{E}(X_2) = \binom{n}{b} p^a$. By (2.2) and (2.4), we have

$$\mathbb{P}(X_2 \geq 2p^a n^b) \leq \mathbb{P}(X_2 \geq 2\mathbb{E}(X_2)) \leq \frac{\Delta_{X_2}}{\mathbb{E}(X_2)^2} \leq \frac{(\gamma^{2b}/24)n^{2b-1}p^{2a}}{\left(\binom{n}{b} p^a\right)^2} = o(1).$$

For (3), let \mathcal{Q} consist of edge sets of all overlapping pairs of $P_{a,x}$ in the complete k -graph on V . Let $Y = \sum_{Q \in \mathcal{Q}} I_Q$, where I_Q is the indicator variable for the event $Q \subseteq E(G)$. We first estimate $\mathbb{E}(Y)$. For X_2 defined above, we have $\Delta_{X_2} = \mathbb{E}(\sum_Q I_Q)$, where the sum is over all $Q \in \mathcal{Q}$ whose two copies of $P_{a,x}$ share at least one edge. As shown in the proof of Proposition 2.1, for $1 \leq i \leq b$, there are $\binom{n}{2b-i} \binom{b}{i}$ members of \mathcal{Q} whose two copies of $P_{a,x}$ share exactly i vertices. Hence $\mathbb{E}(Y) \geq \binom{n}{2b-1} b^2 p^{2a} \geq n^{2b-1} p^{2a} b^2 / 2$. Since $\sum_{2 \leq i \leq b} \binom{n}{2b-i} \binom{b}{i}^2 \leq n^{2b-1} / 2$, using (2.4), we derive that

$$\mathbb{E}(Y) \leq \binom{n}{2b-1} b^2 p^{2a} + (n^{2b-1}/2) p^{2a} + \Delta_{X_2} \leq 2b^2 n^{2b-1} p^{2a}.$$

We next compute $\text{Var}(Y)$. For each $Q \in \mathcal{Q}$, let S_Q denote the k -graph induced by Q (thus S_Q is the union of two overlapping copies of $P_{a,x}$). Fix two $Q, R \in \mathcal{Q}$ such that $Q \cap R \neq \emptyset$. We write $S_Q = T_1 \cup T_2$ and $S_R = T_3 \cup T_4$, where T_i 's are copies of $P_{a,x}$ such that $E(T_1) \cap E(T_3) \neq \emptyset$. Define $H_1 := T_1 \cap T_2$, $H_2 := (T_1 \cup T_2) \cap T_3$ and $H_3 := (T_1 \cup T_2 \cup T_3) \cap T_4$. Since $V(T_1) \cap V(T_2) \neq \emptyset$, $V(T_3) \cap V(T_4) \neq \emptyset$, and $E(T_1) \cap E(T_3) \neq \emptyset$, it follows that $v_{H_i} \geq 1$ for $i = 1, 2, 3$. We claim that $n^{v_{H_i}} p^{e_{H_i}} \geq n$ for $i = 1, 2, 3$. Indeed, since each H_i is a subgraph of $P_{a,x}$, if $e_{H_i} \geq 1$, then by (2.4), $n^{v_{H_i}} p^{e_{H_i}} \geq \Phi_{P_{a,x}} \geq Cn$; otherwise $e_{H_i} = 0$ and then we have $n^{v_{H_i}} p^{e_{H_i}} = n^{v_{H_i}} \geq n^1 = n$. Consequently,

$$n^{v_{H_1}} p^{e_{H_1}} \cdot n^{v_{H_2}} p^{e_{H_2}} \cdot n^{v_{H_3}} p^{e_{H_3}} \geq n^3. \quad (2.5)$$

Let $D = D(b, k)$ be the number of choices for H_1, H_2, H_3 . Fix some H_1, H_2, H_3 . Let $\Delta_{H_1, H_2, H_3} = \sum_{Q, R} \mathbb{E}(I_Q I_R)$, where the sum is over all $Q, R \in \mathcal{Q}$ with $Q \cap R \neq \emptyset$ that generate the given H_1, H_2, H_3 . It is easy to see that the sum contains at most

$$\binom{b}{v_{H_1}} \binom{b}{v_{H_1}} \binom{b}{v_{H_2}} (2b - v_{H_1}) v_{H_2} \binom{b}{v_{H_3}} (3b - v_{H_1} - v_{H_2}) v_{H_3} \binom{n}{4b - v_{H_1} - v_{H_2} - v_{H_3}} \leq 2^{3b} (3b)! n^{4b - (v_{H_1} + v_{H_2} + v_{H_3})}$$

terms. Together with (2.5), we obtain that

$$\Delta_{H_1, H_2, H_3} = \sum_{Q, R} \mathbb{E}(I_Q I_R) \leq 2^{3b} (3b)! n^{4b - (v_{H_1} + v_{H_2} + v_{H_3})} p^{4a - (e_{H_1} + e_{H_2} + e_{H_3})} \leq 2^{3b} (3b)! n^{4b-3} p^{4a}.$$

Consequently,

$$\Delta_Y = \sum_{H_1, H_2, H_3} \Delta_{H_1, H_2, H_3} \leq D 2^{3b} (3b)! n^{4b-3} p^{4a}.$$

By (2.2), we derive that

$$\mathbb{P}(Y \geq 4b^2 n^{2b-1} p^{2a}) \leq \frac{\Delta_Y}{\mathbb{E}(Y)^2} \leq \frac{D 2^{3b} (3b)! n^{4b-3} p^{4a}}{(n^{2b-1} p^{2a} b / 2)^2} = o(1).$$

This confirms (3). \square

In Lemma 2.2 we assume that p satisfies (1.1) and obtain that $\Phi_{P_{a,x}} \geq Cn$. This is necessary for Part (1), in which we use the union bound on 2^n events. When there are only polynomially many events, it suffices to have $\Phi_{P_{a,x}} \geq n^c$ for some $0 < c < 1$, which occurs when $p \geq n^{-(k-\ell)-\epsilon}$ (for all $\ell \geq 1$) and $\epsilon < \ell/a$. We use this weaker condition on p in the following lemma because we only have this condition in the proof of Lemma 3.5.

Lemma 2.3. *Let $k, \ell, a, x \in \mathbb{Z}$ such that $k \geq 3, 1 \leq \ell \leq k-1, a \geq 1$, and $0 \leq x \leq k$. Write $b = b(x) = 2x + \ell + (k-\ell)a$. Suppose $0 < \epsilon \leq \ell/(2a)$ and $1/n \ll \gamma, 1/a, 1/k$. Let V be an n -vertex set, and let $\mathcal{F}_1, \dots, \mathcal{F}_t$ be $t \leq n^{2k}$ families of γn^b ordered b -sets on V . Suppose $G = \mathbb{G}^{(k)}(n, p)$ with $p \geq n^{-(k-\ell)-\epsilon}$, then with probability at least $1 - \exp(-n^{1/3})$, for all $i \in [t]$, at least $(\gamma/2)p^a n^b$ members of \mathcal{F}_i span copies of $P_{a,x}$.*

Proof. By (2.3) and $\epsilon \leq \ell/(2a)$, we have $\Phi_{P_{a,x}} \geq n^{\ell-a\epsilon} \geq \sqrt{n}$. Fix $i \in [t]$ and let $X_{\mathcal{F}_i}$ be the random variable that counts the number of the members of \mathcal{F}_i that span copies of $P_{a,x}$. By (2.4), we have $\Delta_{X_{\mathcal{F}_i}} \leq 2^{bb!} n^{2b} p^{2a} / \sqrt{n}$ and note that $\mathbb{E}(X_{\mathcal{F}_i}) = \gamma n^b p^a$. By (2.1), we have

$$\mathbb{P}(X_{\mathcal{F}_i} \leq (\gamma/2)n^b p^a) \leq \exp\left(-\frac{(\mathbb{E}(X_{\mathcal{F}_i})/2)^2}{2\Delta_{X_{\mathcal{F}_i}}}\right) \leq \exp\left(-\frac{(\gamma/2)^2 n^{2b} p^{2a}}{2^{bb!} n^{2b} p^{2a} / \sqrt{n}}\right) \leq \exp(-2n^{1/3}).$$

Since $n^{2k} \exp(-2n^{1/3}) \leq \exp(-n^{1/3})$, the result follows from the union bound. \square

3. LEMMAS

In this section we prove all the lemmas that are needed for the proof of Theorem 1.2.

Since we assume $\delta_1(H) \geq \alpha n^{k-1}$, unless $\ell = 1$, the k -graph H may contain some ℓ -sets S whose degree is too low to be used for connection. To overcome this, we simply delete all edges that contain S . The following lemma reflects this “shaving” process.

Lemma 3.1. *Let $0 < \eta \leq \alpha, 1/k$. Let H be an n -vertex k -graph with $\delta_1(H) \geq \alpha n^{k-1}$. Then there exists a spanning subgraph H' of H , satisfying the following properties.*

- (1) $e(H') \geq \alpha n^k / (2k)$.
- (2) $\deg_{H'}(v) \geq 2\alpha n^{k-1} / 3$ for all but at most $3k\eta^2 n / \alpha$ vertices of H .
- (3) For every ℓ -set S of $V(H)$, either $\deg_{H'}(S) = 0$ or $\deg_{H'}(S) \geq \eta^2 n^{k-\ell}$.

Proof. Starting from H , we iteratively do the following. If the current k -graph contains an ℓ -set S whose degree is less than $\eta^2 n^{k-\ell}$, then we delete all the edges containing S . Clearly the iteration lasts at most $\binom{n}{\ell}$ steps. Let H' be the resulting k -graph, then (3) holds. Since we deleted at most $\eta^2 n^{k-\ell}$ edges in each step, we have $e(H) - e(H') \leq \binom{n}{\ell} \eta^2 n^{k-\ell} \leq \alpha n^k / (2k)$. Together with $e(H) \geq (n/k)\alpha n^{k-1}$, (1) follows. For (2), let V_0 be the set of vertices v in H' such that $\deg_{H'}(v) \leq 2\alpha n^{k-1} / 3$, then since $\delta_1(H) \geq \alpha n^{k-1}$, we have

$$|V_0| \cdot \frac{1}{3} \alpha n^{k-1} \leq k(e(H) - e(H')) \leq k \binom{n}{\ell} \eta^2 n^{k-\ell} \leq k\eta^2 n^k.$$

Thus $|V_0| \leq 3k\eta^2 n / \alpha$ and (2) holds. \square

We recall the following Chernoff’s inequality (see, e.g., [18]). For $x > 0$ and a binomial random variable $X = \text{Bin}(n, \zeta)$, it holds that

$$\mathbb{P}(X \geq n\zeta + x) < e^{-x^2/(2n\zeta+x/3)} \quad \text{and} \quad \mathbb{P}(X \leq n\zeta - x) < e^{-x^2/(2n\zeta)}. \quad (3.1)$$

The following lemma helps us to build connectors and absorbers.

Lemma 3.2. *Let $k, \ell, a, x, b, \epsilon$ be as in Lemma 2.2. Suppose $1/n \ll 1/C \ll \beta, 1/b$. Let V be an n -vertex set, and let $\mathcal{F}_1, \dots, \mathcal{F}_t$ be $t \leq n^{2k}$ families of $24\beta n^b$ ordered b -sets on V . Suppose $G = \mathbb{G}^{(k)}(n, p)$ on V and p satisfies (1.1). Then a.a.s. there exists a family $\mathcal{F} \subseteq \bigcup_{i \in [t]} \mathcal{F}_i$ of at most βn disjoint ordered b -sets such that $|\mathcal{F}_i \cap \mathcal{F}| \geq \beta^2 n / b^2$ for each $i \in [t]$, and each member of \mathcal{F} spans a labeled copy of $P_{a,x}$ in G .*

Proof. In $G = \mathbb{G}^{(k)}(n, p)$, let \mathcal{T} be the set of all ordered b -sets on V that span copies of $P_{a,x}$. By Lemma 2.2 (2) and (3), Lemma 2.3 and the union bound, a.a.s. the following properties hold simultaneously.

- $|\mathcal{F}_i \cap \mathcal{T}| \geq 12\beta p^a n^b$ for all $i \in [t]$;
- $|\mathcal{T}| \leq 2p^a n^b$;
- there are at most $4b^2 p^{2a} n^{2b-1}$ pairs of overlapping members of \mathcal{T} .

Next we select a random set $\mathcal{F}' \subset \mathcal{T}$ by including each member of \mathcal{T} independently with probability $q := \beta / (2b^2 n^{b-1} p^a)$. Because of (3.1) (for (i) and (ii) below) and Markov’s inequality (for (iii)), there exists such a family \mathcal{F}' satisfying the following properties:

- (i) $|\mathcal{F}_i \cap \mathcal{F}'| \geq 12\beta(q/2)p^a n^b = 3\beta^2 n / b^2$ for all $i \in [t]$;
- (ii) $|\mathcal{F}'| \leq 2q|\mathcal{T}| \leq \beta n$;
- (iii) there are at most $8b^2 q^2 n^{2b-1} p^{2a} = 2\beta^2 n / b^2$ pairs of overlapping members of \mathcal{F}' .

By deleting one ordered b -set from each overlapping pair and all ordered b -sets not in $\bigcup_{i \in [t]} \mathcal{F}_i$, we obtain a collection \mathcal{F} of disjoint ordered b -sets such that $|\mathcal{F}| \leq \beta n$, and for every $i \in [t]$, $|\mathcal{F}_i \cap \mathcal{F}| \geq 3\beta^2 n/b^2 - 2\beta^2 n/b^2 = \beta^2 n/b^2$. Moreover, since $\mathcal{F} \subseteq \mathcal{T}$, each member of \mathcal{F} spans a labeled copy of $P_{a,x}$ in G . \square

We now prove a connecting lemma that provides connectors for any two ℓ -sets with large degree. Throughout the rest of the paper, let

$$t_1 := \lceil \ell/(k-\ell) \rceil, \quad t_2 := t_1(k-\ell) - \ell, \quad \text{and} \quad t_3 := 3t_1(k-\ell) - \ell.$$

Given a k -graph H , we say that an ordered t_3 -set C connects two ordered ℓ -sets A and B if $C \cap A = C \cap B = \emptyset$ and the concatenation ACB spans an ℓ -path. Note that in this case, C spans a copy of P_{t_1, t_2} in H .

Lemma 3.3. *Suppose $1 \leq \ell < k$ and $1/n \ll 1/C \ll \beta \ll \eta \ll 1/k$ and $0 < \epsilon \leq \ell/(3t_1)$. Let H' and G be two n -vertex k -graphs on the same vertex set V such that for any ℓ -set $S \subseteq V$, either $\deg_{H'}(S) = 0$ or $\deg_{H'}(S) \geq \eta n^{k-\ell}$ and $G := \mathbb{G}^{(k)}(n, p)$ satisfies (1.1). Then for any set $W \subseteq V$ of size at most $\eta n/3$, a.a.s. $H' \cup G$ contains a set \mathcal{C} of disjoint t_3 -sets such that $V(\mathcal{C}) \cap W = \emptyset$, $|\mathcal{C}| \leq \beta n$, and for every two disjoint ordered ℓ -sets S, S' in V with $\deg_{H'}(S), \deg_{H'}(S') \geq \eta n^{k-\ell}$, there are at least $3\beta^3 n$ members of \mathcal{C} that connect them.*

Proof. Fix two disjoint ordered ℓ -sets $S := (v_1, \dots, v_\ell)$ and $S' := (w_\ell, \dots, w_1)$ such that $\deg_{H'}(S), \deg_{H'}(S') \geq \eta n^{k-\ell}$. We first claim that we can greedily extend S to an ℓ -path $v_1, \dots, v_{\ell+t_1(k-\ell)}$ of length t_1 in H' such that the new vertices are disjoint from $S' \cup W$ and there are at least $(\eta/2)^{t_1} n^{t_1(k-\ell)}$ choices for them. Indeed, we iteratively extend the path from the current ℓ -end T by adding $k-\ell$ new vertices. By the degree assumption, we know that $\deg_{H'}(T) \geq \eta n^{k-\ell}$ (in the first step $T = S$). Since the number of $(k-\ell)$ -sets that intersect the existing vertices or W is $\eta n^{k-\ell}/3 + O(n^{k-\ell-1})$, there are at least $\eta n^{k-\ell}/2$ choices for the new $k-\ell$ vertices.

Similarly, we can greedily extend (w_1, \dots, w_ℓ) to an ℓ -path $w_1, \dots, w_{\ell+t_1(k-\ell)}$ of length t_1 in H' such that new vertices are disjoint with $\{v_1, \dots, v_{\ell+t_1(k-\ell)}\} \cup W$ and there are at least $(\eta/2)^{t_1} n^{t_1(k-\ell)}$ choices for them. At last, if $t_2 > 0$, then we pick t_2 arbitrary vertices $\{u_1, \dots, u_{t_2}\}$ that are disjoint from the existing vertices and W , and there are at least $n^{t_2}/2$ choices for them. Note that $t_3 = 2t_1(k-\ell) + t_2$. So there are at least $(\eta/2)^{2t_1} n^{t_3}/2 \geq 24\beta n^{t_3}$ choices for the ordered t_3 -sets

$$(v_{\ell+1}, \dots, v_{\ell+t_1(k-\ell)}, u_1, \dots, u_{t_2}, w_{\ell+t_1(k-\ell)}, \dots, w_{\ell+1}).$$

Let $\mathcal{C}_{S, S'}$ be a collection of exactly $24\beta n^{t_3}$ such ordered t_3 -sets. By this definition, if some $C \in \mathcal{C}_{S, S'}$ spans a labeled copy of P_{t_1, t_2} , then C connects S and S' . We apply Lemma 3.2 to $\mathcal{C}_{S, S'}$ for all pairs of S, S' such that $\deg_{H'}(S), \deg_{H'}(S') \geq \eta n^{k-\ell}$ and $G = \mathbb{G}^{(k)}(n, p)$, and conclude that a.a.s. there exists a family \mathcal{C} of disjoint t_3 -sets such that $|\mathcal{C}| \leq \beta n$, and for ordered ℓ -sets S, S' with $\deg_{H'}(S), \deg_{H'}(S') \geq \eta n^{k-\ell}$, there are at least $\beta^2 n/t_3^2 \geq 3\beta^3 n$ t_3 -sets that connect them. In particular, $V(\mathcal{C}) \cap W = \emptyset$ by our construction. \square

Given a k -graph H , let $W = \{w_1, \dots, w_{k-\ell}\} \subseteq V(H)$. The W -absorber is defined as follows. Let

$$t_4 := \lceil (3k-\ell-2)/(k-\ell) \rceil \quad \text{and} \quad t_5 := t_4(k-\ell) \quad (\text{thus } 3k-\ell-2 \leq t_5 \leq 4k).$$

Suppose $X_i, Y_i, Z_i, i \in [k-\ell]$, and T are pairwise disjoint ordered sets from $V(H) \setminus W$ satisfying the following properties:

- (i) $|X_i| = k-1$, $|Y_i| = t_5 - k - i + 1$, and $|Z_i| = i-1$ for every $i \in [k-\ell]$ and $|T| = \ell$;
- (ii) $Q := X_1 Z_2 Y_1 X_2 Z_3 Y_2 \cdots X_{k-\ell-1} Z_{k-\ell} Y_{k-\ell-1} X_{k-\ell} Z_1 Y_{k-\ell} T$ spans a copy of P_{t_5-1} ;
- (iii) $Q' := X_1 w_1 Z_1 Y_1 X_2 w_2 Z_2 Y_2 \cdots X_{k-\ell} w_{k-\ell} Z_{k-\ell} Y_{k-\ell} T$ spans a copy of P_{t_5} .

By definition, Q is a W -absorber. Note that $|Y_i| \geq k-1$ for $i \in [k-\ell]$ by the definition of t_5 . Let B_i be the ordered set $X_i w_i Z_i Y_i$. Since $|X_i|, |Y_i| \geq k-1$, all the edges of Q' that intersect $\{w_i\} \cup Z_i$ are completely in B_i . Furthermore, when counting from the left end, all X_i and Y_i are placed at the same location in Q as in Q' , except that $Y_{k-\ell}$ is shifted $k-\ell$ vertices to the right in Q' (thus $Z_2, \dots, Z_{k-\ell}$ are simply place-holders). Consequently, if $H[B_i] \supseteq Q'[B_i]$ for $i \in [k-\ell]$ and Q is a path, then Q' is a path.

The following is our absorbing lemma.

Lemma 3.4. *Let $1 \leq \ell < k$ be integers and suppose $0 < \epsilon \leq \ell/(3t_5)$ and $1/n \ll \beta \ll \eta, \alpha, 1/k, 1/t_5$. Let V be a set of n vertices and let V', U be two (not necessarily disjoint) subsets of V such that $|U| \leq \eta n/3$. Let H be a k -graph on V such that $\deg_H(v) \geq \alpha n^{k-1}$ for all $v \in V'$, and for all ℓ -sets $S \subseteq V$, either $\deg_H(S) = 0$ or $\deg_H(S) \geq \eta n^{k-\ell}$. Suppose $G := \mathbb{G}^{(k)}(n, p)$ has vertex set V and satisfies (1.1). Then $H \cup G$ a.a.s. contains a family \mathcal{A} of at most βn vertex-disjoint copies of P_{t_5-1} with ends in $\partial_\ell H$ such that $V(\mathcal{A}) \subseteq V \setminus U$, and every $(k-\ell)$ -set $W \subseteq V'$ has at least $\beta^3 n$ W -absorbers in \mathcal{A} .*

Proof. For each $W = \{w_1, \dots, w_{k-\ell}\} \subseteq V'$, we will find W -absorbers from $V \setminus U$ satisfying (i) – (iii). We achieve this in two steps. In the first step, for each $i \in [k-\ell]$, we will find a path Q_i of length t_4 with $V(Q_i) = \{v_1, \dots, v_{t_5+\ell}\} \subseteq V \setminus U$ such that $v_k = w_i$, and there are at least $\frac{\alpha}{2} (\frac{\eta}{2})^{t_4-1} n^{t_5+\ell-1}$ choices for $V(Q_i)$. Indeed, we first choose an unordered set $\{v_1, \dots, v_{k-1}\} \in N_H(w_i)$. Since $\deg_H(w_i) \geq \alpha n^{k-1}$ and at most $|U| + \ell + t_5(k-\ell) \leq \eta n/2$ vertices are either in U or used in this step, there are at least $\frac{\alpha}{2} n^{k-1}$ choices for $\{v_1, \dots, v_{k-1}\}$. Next, let $S = \{v_{k-\ell+1}, \dots, v_k\}$. Since $\deg_H(S) > 0$, we have $\deg_H(S) \geq \eta n^{k-\ell}$. Hence we can choose an unordered set $\{v_{k+1}, \dots, v_{2k-\ell}\} \in N_H(S)$ while avoiding U and the vertices already used in this step. There are at least $\frac{\eta}{2} n^{k-\ell}$ choices. We repeat this to obtain the desired path Q_i and there are at least $\frac{\alpha}{2} (\frac{\eta}{2})^{t_4-1} n^{t_5+\ell-1}$ choices for $V(Q_i)$ as an ordered set. Let B_i be the ordered set $\{v_1, \dots, v_{t_5}\}$. It follows that there are at least $\frac{\alpha}{2} (\frac{\eta}{2})^{t_4-1} n^{t_5-1}$ choices for B_i . Now let $A = B_1 \cdots B_{k-\ell-1} V(Q_{k-\ell})$. We have at least $((\frac{\alpha}{2})(\frac{\eta}{2})^{t_4-1})^{k-\ell} n^{\ell+(t_5-1)(k-\ell)} \geq 24\beta n^{\ell+(t_5-1)(k-\ell)}$ choices for A .

Now we proceed to the second step. For each $i \in [k-\ell]$, recall that $V(Q_i) = \{v_1, \dots, v_{t_5+\ell}\}$. Define (ordered) sets

$$X_i = \{v_1, \dots, v_{k-1}\}, \quad Z_i = \{v_{k+1}, \dots, v_{k+i-1}\}, \quad \text{and} \quad Y_i = \{v_{k+i}, \dots, v_{t_5}\}.$$

In addition, let $T = \{v_{t_5+1}, \dots, v_{t_5+\ell}\}$ from $Q_{k-\ell}$. It is clear that X_i, Y_i, Z_i and T satisfy (i). Recall that $B_i = X_i w_i Z_i Y_i$. For Q' defined in (iii), our first step already provides the edges of $Q'[B_i]$ for $i \leq k-\ell-1$ and the edges of $Q'[B_{k-\ell} \cup T]$. Following the discussion right after (iii), we achieve both (ii) and (iii) if Q is a path. To this end, we use the edges of G . Let \mathcal{F}_W be the family of $24\beta n^{\ell+(t_5-1)(k-\ell)}$ copies of A , each re-ordered as in Q . We apply Lemma 3.2 to G with $x = 0$, families \mathcal{F}_W for all ordered $(k-\ell)$ -sets $W \subseteq V'$, and conclude that a.a.s. there exists a collection \mathcal{A} of at most βn vertex-disjoint copies of P_{t_5-1} such that for every $(k-\ell)$ -set $W \subseteq V'$, at least $\beta^3 n$ members of \mathcal{A} are from \mathcal{F}_W , and thus are W -absorbers. At last, because of the first step, both ℓ -ends of these paths are in $\partial_\ell H$. \square

In the proof of Theorem 1.2 we need a lemma to cover most of the vertices with constantly many paths. This is done in the following lemma. In the proofs of the following lemma and Theorem 1.2, we use the trick of *multi-round exposure*, namely, in each of the steps later, we expose one or several independent copies of the binomial random hypergraph, each of them with edge probability a constant fraction of the original edge probability.

Lemma 3.5. *Let $1 \leq \ell < k$, and suppose $1/n \ll 1/C \ll \zeta \ll \alpha \ll 1/k$ and $0 < \epsilon \leq \zeta^3 \ell/6$. Suppose V is a set of n vertices and $V_0 \subseteq V$ with $|V_0| \leq \alpha n$, and furthermore, when $\ell = 1$, suppose that $V_0 = \emptyset$. Suppose $G := \mathbb{G}^{(k)}(n, p)$ on V satisfying (1.1). Let L be an ℓ -graph on $V \setminus V_0$ with $|E(L)| \geq \alpha n^\ell$. Then a.a.s. G contains a set \mathcal{P} of at most $2\zeta^3 n$ vertex-disjoint ℓ -paths such that their ends are in L , $V_0 \subseteq V(\mathcal{P})$ and $|V \setminus V(\mathcal{P})| \leq 2\zeta n$.*

Proof. Since $|L| \geq \alpha n^\ell$, by averaging, there exists a set $R \subseteq V \setminus V_0$ of size ζn such that $|L \cap \binom{R}{\ell}| \geq \alpha |R|^\ell/2$.

We find our path cover in two phases. In the first phase, we use relatively long paths with ends from R to cover most of the vertices of V . In the second phase, we greedily cover the remaining vertices of $V_0 \setminus R$ with short paths. We therefore expose G in two rounds such that $G = G_1 \cup G_2$, where each G_i is $\mathbb{G}^{(k)}(n, p')$ with $(1-p')^2 = 1-p$. Thus $p' > p/2 > n^{-(k-\ell)-2\epsilon}$ when $\ell \geq 2$.

We start with Phase 1. Let s be the smallest integer such that $s \geq 1/\zeta^3$ and $s \equiv \ell \pmod{k-\ell}$, and let $s_1 = (s-\ell)/(k-\ell)$. Since $\epsilon \leq \zeta^3 \ell/3$, we have $2\epsilon \leq \ell/(3s_1)$. By Lemma 2.2 (1), a.a.s. for all $V^* \subseteq V \setminus R$, $R' \subseteq R$ satisfying $|V^*| \geq \zeta^3 n$, $|R'| \geq |R|/2$ and $|L \cap \binom{R'}{\ell}| \geq (\alpha/3)|R'|^\ell$, $G_1 = \mathbb{G}^{(k)}(n, p')$ contains a copy of P_{s_1} whose ℓ -ends are in $L \cap \binom{R'}{\ell}$ and other vertices are from V^* . Owing to this property, we repeatedly construct copies of P_{s_1} by letting V^* be the set of uncovered vertices of V' and letting R' be the set of uncovered vertices of

R , as long as $|V^*| \geq \zeta^3 n$. This is possible because we construct at most $\zeta^3 n$ vertex-disjoint copies of P_{s_1} , which consume at most $2\ell\zeta^3 n$ vertices from R . During the process, at least $|R| - 2\ell\zeta^3 n \geq |R|/2$ vertices of R are available and by our assumption, they span at least $\alpha|R|^\ell/2 - 2\ell\zeta^3 n \cdot |R|^{\ell-1} \geq \alpha|R|^\ell/3$ edges of L . Let \mathcal{P}_1 denote the set of the paths obtained in this phase.

Note that when $\ell = 1$, since $V_0 = \emptyset$ and $|V \setminus V(\mathcal{P}_1)| \leq |R| + \zeta^3 n \leq 2\zeta n$, we are done by letting $\mathcal{P} = \mathcal{P}_1$.

Now we proceed to Phase 2 and assume that $\ell \geq 2$. Let V'' be the set of uncovered vertices in $V \setminus R$ and $R' = R \setminus V(\mathcal{P}_1)$. Note that $|V''| \leq \zeta^3 n$ and $|R'| \geq |R| - 2\ell\zeta^3 n \geq |R|/2$, and $|L \cap \binom{R'}{\ell}| \geq (\alpha/3)|R|^\ell$. Using the edges of $G_2 = \mathbb{G}^{(k)}(n, p')$, we will greedily put vertices $v \in V''$ into vertex-disjoint ℓ -paths $w_1 \cdots w_{k-1} v w_k \cdots w_{t_6(k-\ell)+\ell-1}$ of length $t_6 := \lfloor (k-1)/(k-\ell) \rfloor + 1$ such that all the vertices other than v are from R' and both ℓ -ends are in L . Note that v is in every edge of the path but in neither of the ℓ -ends.

For any $v \in V''$, let G_v be the edges of G_2 that contain v and have their other $k-1$ vertices from R' . For distinct vertices $u, v \in V''$, the possible edges appear in G_u independently of the possible edges that can appear in G_v . Suppose we consider $v \in V''$ after covering some vertices of V'' by ℓ -paths. To this end, we expose G_v . Let R'' be the set of unused vertices in R' . We have $|R''| \geq |R'| - |V''|(t_6(k-\ell) + \ell) \geq |R'| - 2k\zeta^3 n \geq |R|/3$ and $|L \cap \binom{R''}{\ell}| \geq |L \cap \binom{R'}{\ell}| - 2k\zeta^3 n |R'|^{\ell-1} \geq (\alpha/4)|R|^\ell$. We choose two disjoint ℓ -sets from $L \cap \binom{R''}{\ell}$ and $t_6(k-\ell) - \ell - 1$ vertices from R'' forming an ordered $(t_6(k-\ell) + \ell - 1)$ -set $(w_1, \dots, w_{t_6(k-\ell)+\ell-1})$ – there are

$$\frac{\alpha|R|^\ell}{4} \cdot \left(\frac{\alpha|R|^\ell}{4} - \ell|R''|^{\ell-1} \right) \cdot \left(\frac{\zeta n}{4} \right)^{t_6(k-\ell)-\ell-1} \geq \alpha^3 (\zeta n)^{t_6(k-\ell)+\ell-1}$$

such sets. We observe that $w_1 \cdots w_{k-1} v w_k \cdots w_{t_6(k-\ell)+\ell-1}$ spanning a copy of P_{t_6} in G_v is equivalent to $w_1 \cdots w_{t_6(k-\ell)+\ell-1}$ spanning a $(k-1)$ -uniform $(\ell-1)$ -path in $N_{G_v}(v)$. Since $p' \geq n^{(k-1)-(\ell-1)-2\epsilon}$ and $2\epsilon \leq (\ell-1)/(3t_6)$, we can apply Lemma 2.3 to $\alpha^3 (\zeta n)^{t_6(k-\ell)+\ell-1}$ ordered $(t_6(k-\ell) + \ell - 1)$ -sets, and conclude that G_v contains a desired ℓ -path with probability at least $1 - \exp(-n^{1/3})$. By the union bound, with probability at least $1 - |V''| \exp(-n^{1/3}) = 1 - o(1)$, we can put all the vertices of V'' into vertex-disjoint ℓ -paths of length t_6 by using the vertices of R such that all the ℓ -ends are in L . This finishes Phase 2. Let \mathcal{P}_2 denote the family of the ℓ -paths found in this phase. Let $\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2$ and note that $|\mathcal{P}| \leq 2\zeta^3 n$. By construction, all the ℓ -ends of the paths in \mathcal{P} are in L . Since $V \setminus V(\mathcal{P}) \subseteq R$, we have $V_0 \subseteq V(\mathcal{P})$ and $|V \setminus V(\mathcal{P})| \leq |R| \leq 2\zeta n$. \square

4. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. We essentially follow the procedure mentioned in Section 1.3 but need additional work. We first apply Lemma 3.1 and obtain a spanning subgraph H' of H . Let V_* be the set of vertices of H' with high degree. Following the procedure outlined in Section 1.3, we obtain an absorbing path P_{abs} , a set \mathcal{C}_1 of connectors and a set \mathcal{P} of paths that cover almost all the vertices. A natural attempt is to use the connectors in \mathcal{C}_1 to connect the paths in \mathcal{P} and P_{abs} to obtain an almost spanning cycle and then absorb the remaining vertices of \mathcal{C}_1 by P_{abs} . On the other hand, when applying Lemma 3.4 to H' , we can only absorb vertices in V_* . Therefore we need to have $V(\mathcal{C}_1) \subseteq V_*$. However, we cannot strengthen Lemma 3.3 by asking $V(\mathcal{C}_1) \subseteq V_*$ because for a given ℓ -set in V_* , it is possible that all its neighbors intersect $V \setminus V_*$ (recall that $\deg_{H'}(S) \geq \eta^2 n^{k-\ell}$ and $|V \setminus V_*| \leq \eta n$). Therefore, this naive attempt fails.

To fix it, we “shave” H' again, namely, applying Lemma 3.1 to $H'[V_*]$, and obtain a spanning k -graph H_* on V_* . We thus apply Lemma 3.3 to H_* and obtain \mathcal{C}_1 such that $V(\mathcal{C}_1) \subseteq V_*$ and \mathcal{C}_1 can connect any two ℓ -sets in $L := \partial_\ell H_*$. In order to obtain P_{abs} , we apply Lemma 3.4 to H' obtaining a family \mathcal{A} of absorbers and apply Lemma 3.3 to H' obtaining another set \mathcal{C}_2 of connectors. After connecting \mathcal{A} into P_{abs} , unused members of \mathcal{C}_2 will be discarded (and the vertices in these members will be covered in a later step).

Below are the details of our proof. Let $1/n \ll 1/C \ll \zeta \ll \beta \ll \eta \ll \alpha, 1/k$ and $0 < \epsilon \leq \zeta^3 \ell/12$. Write $V = V(H)$. Let $\bigcup_{i \in [4]} G_i = \mathbb{G}^{(k)}(n, p)$ such that each G_i is $\mathbb{G}^{(k)}(n, p')$ and $(1-p')^4 = 1-p$. In particular, $p' > p/4 > n^{-(k-\ell)-2\epsilon}$ if $\ell \geq 2$, and $p' > p/4 \geq (C/4)n^{-(k-1)}$ if $\ell = 1$. When we apply Lemmas 3.3, 3.4 and 3.5, we apply them with p' in place of p and 2ϵ in place of ϵ .

Step 1. Shave H twice. We define two subgraphs H' and H_* of H as follows. If $\ell = 1$, let $H' = H_* = H$, $V_* = V$, and $L = V$. If $\ell \geq 2$, then we apply Lemma 3.1 to H and obtain a subgraph H' with the following properties:

- there exists $V_0 \subseteq V$ such that $|V_0| \leq 3k\eta^2 n/\alpha \leq \eta n$ and $\deg_{H'}(v) \geq 2\alpha n^{k-1}/3$ for all $v \in V \setminus V_0$;
- for every ℓ -set $S \subseteq V$, either $\deg_{H'}(S) = 0$ or $\deg_{H'}(S) \geq \eta^2 n^{k-\ell}$.

Let $V_* = V \setminus V_0$ and $n_* := |V_*| \geq (1 - \eta)n$. We have $\delta_1(H'[V_*]) \geq 2\alpha n^{k-1}/3 - |V_0|n^{k-2} \geq \alpha n_*^{k-1}/2$. Apply Lemma 3.1 again to $H'[V_*]$ and obtain a subgraph H_* on V_* such that

- $e(H_*) \geq \alpha n_*^k/(4k)$,
- for every ℓ -set $S \subseteq V_*$, either $\deg_{H_*}(S) = 0$ or $\deg_{H_*}(S) \geq \eta^2 n_*^{k-\ell}$.

Let $L = \partial_\ell H_*$. We have

$$|L| \geq \frac{\binom{k}{\ell} e(H_*)}{\binom{n-\ell}{k-\ell}} \geq \frac{\binom{k}{\ell} \frac{\alpha}{4k} n_*^k}{n_*^{k-\ell}} \geq \frac{\alpha}{4} n_*^\ell. \quad (4.1)$$

Step 2. Build connectors \mathcal{C}_1 and \mathcal{C}_2 . We obtain \mathcal{C}_1 and \mathcal{C}_2 by applying Lemma 3.3 twice. First, we apply Lemma 3.3 to $H_* \cup G_1[V_*]$ with $W = \emptyset$, η^2 (in place of η) and ζ (in place of β), and conclude that $H_* \cup G_1[V_*]$ a.a.s. contains a set \mathcal{C}_1 of disjoint t_3 -sets such that $V(\mathcal{C}_1) \subseteq V_*$, $|\mathcal{C}_1| \leq \zeta n$ and for any two disjoint ordered ℓ -sets S, S' in L , there are at least $3\zeta^3 n$ members of \mathcal{C}_1 connecting them. Second, we apply Lemma 3.3 to $H' \cup G_2$ with $W = V(\mathcal{C}_1)$, η^2 (in place of η) and β , and conclude that $H' \cup G_2$ a.a.s. contains a set \mathcal{C}_2 of disjoint t_3 -sets such that $V(\mathcal{C}_2) \subseteq V \setminus V(\mathcal{C}_1)$, $|\mathcal{C}_2| \leq \beta n$, and for any two disjoint ordered ℓ -sets S, S' in $\partial_\ell H'$, there are at least $3\beta^3 n$ members of \mathcal{C}_2 connecting them.

Step 3. Build an absorbing path. Note that $|V(\mathcal{C}_1 \cup \mathcal{C}_2)| \leq 2\beta n \cdot t_3 < 6k\beta n$ (as $t_3 < 3k$). We apply Lemma 3.4 to $H' \cup G_3$ with $V' = V_*$, $U = V(\mathcal{C}_1 \cup \mathcal{C}_2)$, $2\alpha/3$ (in place of α), η^2 (in place of η) and β^3 (in place of β). Then a.a.s. there exists a collection \mathcal{A} of at most $\beta^3 n$ vertex-disjoint copies of P_{t_5-1} such that for every $(k-\ell)$ -set $S \subseteq V_*$, there are at least $\beta^9 n$ S -absorbers in \mathcal{A} . Note that each member of \mathcal{A} contains $(t_5-1)(k-\ell) + \ell \leq 4k^2$ vertices. Moreover, all the members of \mathcal{A} have their ℓ -ends in $\partial_\ell H'$ and $V(\mathcal{A}) \subseteq V \setminus V(\mathcal{C}_1 \cup \mathcal{C}_2)$. Next, we pick two disjoint ℓ -sets $E_1, E_2 \in L$, which are also disjoint from $V(\mathcal{A}) \cup V(\mathcal{C}_1 \cup \mathcal{C}_2)$. This is possible because $|V(\mathcal{A}) \cup V(\mathcal{C}_1 \cup \mathcal{C}_2)| \leq 4k^2\beta^3 n + 6k\beta n \leq 7k\beta n$ and $|L| \geq \alpha n_*^\ell/4$. Finally, we use the members of \mathcal{C}_2 to connect the members of \mathcal{A} and E_1, E_2 to an ℓ -path P_{abs} with ends E_1 and E_2 , which is possible because all the absorbers have ends in $\partial_\ell H'$ and $|\mathcal{A}| \leq \beta^3 n$.

Step 4. Cover most of the remaining vertices. Let $V' = V \setminus (V(P_{abs}) \cup V(\mathcal{C}_1))$. Note that $|V(P_{abs}) \cup V(\mathcal{C}_1)| \leq 7k\beta n + 2\ell \leq 8k\beta n$. Let $L' := L[V_* \setminus (V(P_{abs}) \cup V(\mathcal{C}_1))]$. By (4.1), we have

$$|L'| \geq \alpha n_*^\ell/4 - |V(P_{abs}) \cup V(\mathcal{C}_1)| \cdot n_*^{\ell-1} \geq \alpha n_*^\ell/5 \geq \alpha |V'|^\ell/6.$$

So we can apply Lemma 3.5 with V' (in place of V), V_0 , L' (in place of L), $\alpha/6$ (in place of α), $G = G_4$, and a.a.s. obtain a collection \mathcal{P} of at most $2\zeta^3 n$ vertex-disjoint paths with ends in L , which leaves a set W of at most $2\zeta n$ vertices in $V \setminus V_0 \subseteq V_*$ uncovered. Next, we connect P_{abs} and the paths in \mathcal{P} by the connectors in \mathcal{C}_1 and denote the resulting ℓ -cycle by Q . This is possible because the ends of these paths are in L , and $1 + |\mathcal{P}| \leq 1 + 2\zeta^3 n \leq 3\zeta^3 n$.

Step 5. Finish the Hamiltonian ℓ -cycle. Let $X = V \setminus V(Q)$. The construction of Q implies that $|X| \in (k-\ell)\mathbb{N}$, $X \subseteq W \cup V(\mathcal{C}_1) \subseteq V_*$ and $|X| \leq 2\zeta n + t_3\zeta n \leq 2t_3\zeta n$ (because $t_3 \geq 2\ell \geq 2$). We arbitrarily partition X into disjoint sets of size $k-\ell$. By the definition of \mathcal{A} , every $(k-\ell)$ -set $S \subseteq X$ has at least $\beta^9 n$ S -absorbers in \mathcal{A} . Since each member of \mathcal{A} is a subpath of Q and $2t_3\zeta \leq \beta^9$, we can absorb all these $(k-\ell)$ -sets greedily and obtain the desired Hamiltonian ℓ -cycle.

Each of Steps 2, 3 and 4 can be done with probability $1 - o(1)$ (while Steps 1 and 5 are deterministic). Hence, by the union bound, a.a.s. we complete all the steps and obtain a Hamiltonian ℓ -cycle of H .

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