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Exact minimum degree thresholds for perfect matchings in uniform hypergraphs

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ABSTRACT

Given positive integers k and ℓ where 4 divides k and $k/2 \leqslant \ell \leqslant k-1$, we give a minimum ℓ-degree condition that ensures a perfect matching in a k-uniform hypergraph. This condition is best possible and improves on work of Pikhurko who gave an asymptotically exact result. Our approach makes use of the absorbing method, as well as the hypergraph removal lemma and a structural result of Keevash and Sudakov relating to the Turán number of the expanded triangle.

1. Introduction

A *perfect matching* in a hypergraph H is a collection of vertex-disjoint edges of H which cover the vertex set $V(H)$ of H. It is unlikely that there exists a characterisation of all those k-uniform hypergraphs that contain a perfect matching for $k \geqslant 3$. Indeed, Garey and Johnson [6] showed that the decision problem whether a k-uniform hypergraph contains a perfect matching is NP-complete for $k \geqslant 3$. (In contrast, a theorem of Tutte [24] gives a characterisation of all those graphs which contain a perfect matching.) It is natural therefore to seek simple sufficient conditions that ensure a perfect matching in a k-uniform hypergraph.

Given a k-uniform hypergraph H with an ℓ-element vertex set S (where $0 \leqslant \ell \leqslant k-1$) we define $d_H(S)$ to be the number of edges containing S. The *minimum ℓ-degree* $\delta_\ell(H)$ of H is the minimum of $d_H(S)$ over all ℓ-element sets of vertices in H. Clearly $\delta_0(H)$ is the number of edges in H. We also refer to $\delta_1(H)$ as the *minimum vertex degree* of H and $\delta_{k-1}(H)$ the *minimum codegree* of H.

One of the earliest results on perfect matchings was given by Daykin and Häggkvist [4], who showed that a k-uniform hypergraph H on n vertices contains a perfect matching provided that

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$\delta_1(H) \geqslant (1-1/k)\binom{n-1}{k-1}$. Recently there has been much interest in establishing minimum ℓ-degree thresholds that force a perfect matching in a k-uniform hypergraph. See [19] for a survey on matchings (and Hamilton cycles) in hypergraphs. In particular, Rödl, Ruciński and Szemerédi [22] determined the minimum codegree threshold that ensures a perfect matching in a k-uniform hypergraph for all $k \geqslant 3$. The threshold is $n/2 - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of n and k. This improved bounds given in [13,21]. A k-partite version was proved by Aharoni, Georgakopoulos and Sprüssel [1].

Kühn, Osthus and Treglown [14] and independently Khan [11] determined the precise minimum vertex degree threshold that forces a perfect matching in a 3-uniform hypergraph. (This improved on an "asymptotically exact" result of Hàn, Person and Schacht [8].) Recently a 3-partite version was proved by Lo and Markström [15]. Khan [12] has also determined the exact minimum vertex degree threshold for 4-uniform hypergraphs. (Lo and Markström [16] have a proof of an approximate version of this result.) For $k \geqslant 5$, the precise minimum vertex degree threshold which ensures a perfect matching in a k-uniform hypergraph is not known.

The situation for ℓ-degrees where $1 < \ell < k-1$ is also still open. Hàn, Person and Schacht [8] provided conditions on $\delta_\ell(H)$ that ensure a perfect matching in the case when $1 \leqslant \ell < k/2$. These bounds were subsequently lowered by Markström and Ruciński [17]. Recently, Alon et al. [2] gave a connection between the minimum ℓ-degree that forces a perfect matching in a k-uniform hypergraph and the minimum ℓ-degree that forces a *perfect fractional matching*. As a consequence of this result they determined, asymptotically, the minimum ℓ-degree which forces a perfect matching in a k-uniform hypergraph for the following values of (k, ℓ): $(4,1)$, $(5,1)$, $(5,2)$, $(6,2)$, and $(7,3)$.

Pikhurko [18] showed that if $\ell \geqslant k/2$ and H is a k-uniform hypergraph whose order n is divisible by k then H has a perfect matching provided that $\delta_\ell(H) \geqslant (1/2 + o(1))\binom{n}{k-\ell}$. This result is best possible up to the $o(1)$-term (see the constructions in $\mathcal{H}_{\mathrm{ext}}(n,k)$ below).

In this paper we strengthen Pikhurko's result for k-uniform hypergraphs when 4 divides k. In order to state our results, we need more definitions. Fix a set V of n vertices. Given a partition V into non-empty sets A, B, let $E_{\mathrm{odd}}(A,B)$ ($E_{\mathrm{even}}(A,B)$) denote the family of all k-element subsets of V that intersect A in an odd (even) number of vertices. (Notice that the ordering of the vertex classes A, B is important.) Define $\mathcal{B}_{n,k}(A,B)$ to be the k-uniform hypergraph with vertex set $V = A \cup B$ and edge set $E_{\mathrm{odd}}(A,B)$. Note that the complement $\overline{\mathcal{B}}_{n,k}(A,B)$ of $\mathcal{B}_{n,k}(A,B)$ has edge set $E_{\mathrm{even}}(A,B)$.

Suppose $n, k \in \mathbb{N}$ such that k divides n and $k \geqslant 2$. Define $\mathcal{H}_{\mathrm{ext}}(n,k)$ to be the collection of the following hypergraphs. First, $\mathcal{H}_{\mathrm{ext}}(n,k)$ contains all hypergraphs $\overline{\mathcal{B}}_{n,k}(A,B)$ where $|A|$ is odd. Second, if n/k is odd then $\mathcal{H}_{\mathrm{ext}}(n,k)$ also contains all hypergraphs $\mathcal{B}_{n,k}(A,B)$ where $|A|$ is even; if n/k is even then $\mathcal{H}_{\mathrm{ext}}(n,k)$ also contains all hypergraphs $\mathcal{B}_{n,k}(A,B)$ where $|A|$ is odd.

It is easy to see that no hypergraph in $\mathcal{H}_{\mathrm{ext}}(n,k)$ contains a perfect matching. Indeed, first assume that $|A|$ is even and n/k is odd. Since every edge of $\mathcal{B}_{n,k}(A,B)$ intersects A in an odd number of vertices, one cannot cover A with an odd number of disjoint odd sets. Similarly $\mathcal{B}_{n,k}(A,B)$ does not contain a perfect matching if $|A|$ is odd and n/k is even. Finally, if $|A|$ is odd then since every edge of $\overline{\mathcal{B}}_{n,k}(A,B)$ intersects A in an even number of vertices, $\overline{\mathcal{B}}_{n,k}(A,B)$ does not contain a perfect matching.

Given $\ell \in \mathbb{N}$ such that $k/2 \leqslant \ell \leqslant k-1$ define $\delta(n,k,\ell)$ to be the maximum of the minimum ℓ-degrees among all the hypergraphs in $\mathcal{H}_{\mathrm{ext}}(n,k)$. For example, it is not hard to see that

$$\delta(n,k,k-1) = \begin{cases} n/2 - k + 2 & \text{if } k/2 \text{ is even and } n/k \text{ is odd,} \\ n/2 - k + 3/2 & \text{if } k \text{ is odd and } (n-1)/2 \text{ is odd,} \\ n/2 - k + 1/2 & \text{if } k \text{ is odd and } (n-1)/2 \text{ is even,} \\ n/2 - k + 1 & \text{otherwise.} \end{cases} \tag{1}$$

The following is our main result.

Theorem 1.1. *Suppose $r, \ell \in \mathbb{N}$ such that $2r \leqslant \ell \leqslant 4r-1$. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose H is a $4r$-uniform hypergraph on $n \geqslant n_0$ vertices where $4r$ divides n. If*

$$\delta_\ell(H) > \delta(n, 4r, \ell)$$

then H contains a perfect matching.

As explained before, the minimum ℓ-degree condition in Theorem 1.1 is best possible. When k is divisible by 4, Theorem 1.1 and (1) together give the aforementioned result of Rödl, Ruciński and Szemerédi [22].

In general, the precise value of $\delta(n,k,\ell)$ is unknown because it is not known what value of $|A|$ maximizes the minimum ℓ-degree of $\mathcal{B}_{n,k}(A,B)$ (or $\overline{\mathcal{B}}_{n,k}(A,B)$). Clearly one needs to know the degree of every ℓ-tuple of vertices from $\mathcal{B}_{n,k}(A,B)$ to establish the minimum ℓ-degree of $\mathcal{B}_{n,k}(A,B)$. Further, if one knows this then one can compute the total number of edges in $\mathcal{B}_{n,k}(A,B)$. However, for even k, it is shown in [10, Section 3.1] that finding the value of $|A|$ that maximizes the number of edges in $\mathcal{B}_{n,k}(A,B)$ is equivalent to finding the minima of binary Krawtchouk polynomials, which is an open problem. Thus, this would suggest that calculating $\delta(n,k,\ell)$ is likely a challenging task.

In Appendix A we give a tight upper bound on $\delta(n,4,2)$, which together with Theorem 1.1 gives the minimum 2-degree threshold that forces a perfect matching in a 4-uniform hypergraph. This result was recently independently proven by Czygrinow and Kamat [3].

Theorem 1.2. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 4-uniform hypergraph on $n \geqslant n_0$ vertices where n is divisible by 4. If*

$$\delta_2(H) > \frac{n^2}{4} - \frac{5n}{4} - \frac{\sqrt{n-3}}{2} + \frac{3}{2}$$

then H contains a perfect matching. Furthermore, this minimum degree condition is best possible.

Note that Theorem 1.2, together with the results of Rödl, Ruciński and Szemerédi [22] and Khan [12], characterize the minimum ℓ-degree threshold that forces a perfect matching in a 4-uniform hypergraph for all $1 \leqslant \ell \leqslant 3$.

The overall strategy for the proof of Theorem 1.1 is similar to that of Rödl, Ruciński and Szemerédi in [22], which in turn is typical for proving sharp results. Indeed, we split the argument into 'extremal' and 'non-extremal' cases, and use the absorbing method developed by Rödl, Ruciński and Szemerédi [20] in the non-extremal case. However, our non-extremal case is somewhat different from [22]. We concentrate on the $\ell = 2r$ case and study the structure of an auxiliary graph $G(H)$, whose vertices are all $2r$-subsets of $V(H)$, and two $2r$-sets U, W are joined by an edge if and only if $U \cup W \in E(H)$. Furthermore, we use the hypergraph removal lemma (see e.g. [7,23]) and a structural result of Keevash and Sudakov [10].

In fact, the proof of Theorem 1.1 is such that most of the argument extends to a more general setting. For example, we deal with the extremal case for k-uniform hypergraphs for all integers $k \geqslant 2$. Several parts of the non-extremal case also generalize to $2r$-uniform hypergraphs (where $r \in \mathbb{N}$). Thus, it seems likely that our methods may be useful in making Pikhurko's result exact for k-uniform hypergraphs for all $k \geqslant 2$.

Conjecture 1.3. *Suppose $k, \ell \in \mathbb{N}$ such that $k/2 \leqslant \ell \leqslant k-1$. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose H is a k-uniform hypergraph on $n \geqslant n_0$ vertices where k divides n. If*

$$\delta_\ell(H) > \delta(n,k,\ell)$$

then H contains a perfect matching.

2. Notation and preliminaries

2.1. Definitions and notation

Given a set X and an integer $r \geqslant 2$, we write $\binom{X}{r}$ for the set of all r-element subsets (r-subsets, for short) of X. Let $k, \ell \in \mathbb{N}$. Suppose $H = (V, E)$ is a k-uniform hypergraph. Let $\{v_1, \dots, v_l\}$ be an ℓ-subset of $V(H)$. Often we will use the notation $\underline{v}$, for example, to abbreviate $\{v_1 \cdots v_\ell\}$. When it is clear from the context we may also write $v_1 \cdots v_\ell$ (i.e. we drop the brackets). Given $\underline{v} \in \binom{V(H)}{\ell}$, we write $N_H(\underline{v})$ or $N(\underline{v})$ to denote the *neighborhood of* $\underline{v}$, that is, the family of those $(k-\ell)$-subsets of $V(H)$ which, together with $\underline{v}$, form an edge in H. Then $|N_H(\underline{v})| = d_H(\underline{v})$. When considering ℓ-degree together with ℓ'-degree for some $\ell' \neq \ell$, the following proposition is very useful (the proof is a standard counting argument, which we omit).

Proposition 2.1. *Let $0 \leqslant \ell \leqslant \ell' < k$ and H be a k-uniform hypergraph. If $\delta_{\ell'}(H) \geqslant x\binom{n-\ell'}{k-\ell'}$ for some $0 \leqslant x \leqslant 1$, then $\delta_\ell(H) \geqslant x\binom{n-\ell}{k-\ell}$.*

We denote the *complement of* H by $\overline{H}$. That is, $\overline{H} := (V(H), \binom{V(H)}{k} \setminus E(H))$. Given a set $A \subseteq V(H)$, $H[A]$ denotes the k-uniform subhypergraph of H *induced by* A, namely, $H[A] := (A, E(H) \cap \binom{A}{k})$. We define $H \setminus A := H[V(H) \setminus A]$. Given $B \subseteq E(H)$, we define $H[B] := (V(H), B)$.

Let $\varepsilon > 0$. Suppose that H and H' are k-uniform hypergraphs on n vertices. We say that H is *ε-close to* H', and write $H = H' \pm \varepsilon n^k$, if H becomes a copy of H' after adding and deleting at most εn^k edges. More precisely, let $A \triangle B := (A \setminus B) \cup (B \setminus A)$ denote the *symmetric difference* of two sets A and B. Then H is ε-close to H' if there is an isomorphic copy $\tilde{H}$ of H such that $V(\tilde{H}) = V(H')$ and $|E(\tilde{H}) \triangle E(H')| \leqslant \varepsilon n^k$.

Given a graph G, $x \in V(G)$ and $Y \subseteq V(G)$, we denote by $d_G(x, Y)$ the number of vertices $y \in Y$ such that $xy \in E(G)$. A bipartite graph is called *balanced* if its vertex classes have equal size.

We will often write $0 < a_1 \ll a_2 \ll a_3$ to mean that we can choose the constants a_1, a_2, a_3 from right to left. More precisely, there are increasing functions f and g such that, given a_3, whenever we choose some $a_2 \leqslant f(a_3)$ and $a_1 \leqslant g(a_2)$, all calculations needed in our proof are valid. Hierarchies with more constants are defined in the obvious way. Throughout the paper we omit floors and ceilings whenever this does not affect the argument.

2.2. The extremal graphs $\mathcal{B}_{n,k}$ and $\mathcal{B}_{n,k}(t)$

Given a k-uniform hypergraph H and a partition A, B of $V(H)$, an edge e of H is called an $A^r B^{k-r}$ *edge* if $|e \cap A| = r$ and $|e \cap B| = k - r$. An $A^r B^{k-r}$ edge is called an *(A, B)-even edge* if r is even; otherwise we call such an edge *(A, B)-odd*. We refer to such edges as *even* and *odd* respectively when it is clear from the context what our partition of $V(H)$ is. Two edges of H have the same *parity* if both are even or both are odd. As defined earlier, $E_{\text{odd}}(A, B)$ ($E_{\text{even}}(A, B)$) is the family of all (A, B)-odd (-even) edges.

Suppose that $n \in \mathbb{N}$ such that $n \geqslant k \geqslant 2$. Let A, B be a partition of a set of n vertices. Recall that $\mathcal{B}_{n,k}(A, B)$ is the k-uniform hypergraph with vertex set $A \cup B$ and edge set $E_{\text{odd}}(A, B)$, and its complement $\overline{\mathcal{B}}_{n,k}(A, B)$ has edge set $E_{\text{even}}(A, B)$. When $|A| = \lfloor n/2 \rfloor$ and $|B| = \lceil n/2 \rceil$, we simply denote $\mathcal{B}_{n,k}(A, B)$ by $\mathcal{B}_{n,k}$, and $\overline{\mathcal{B}}_{n,k}(A, B)$ by $\overline{\mathcal{B}}_{n,k}$. When $|A| = \lfloor n/2 \rfloor + t$ and $|B| = \lceil n/2 \rceil - t$ for some integer t such that $-\lfloor n/2 \rfloor < t < \lceil n/2 \rceil$, we may denote $\mathcal{B}_{n,k}(A, B)$ by $\mathcal{B}_{n,k}(t)$. We refer to A and B as the *vertex classes of* $\mathcal{B}_{n,k}$ and $\mathcal{B}_{n,k}(t)$.

2.3. Absorbing sets

Following the ideas of Rödl, Ruciński and Szemerédi [20,22], we define *absorbing sets* as follows: Given a k-uniform hypergraph H, a set $S \subseteq V(H)$ is called an *absorbing set for* $Q \subseteq V(H)$, if both $H[S]$ and $H[S \cup Q]$ contain perfect matchings. In this case, if the matching covering S is M, we also say M *absorbs* Q.

When constructing our absorbing sets in Section 5 we will use the following Chernoff bound for binomial distributions (see e.g. [9, Corollary 2.3]). Recall that the binomial random variable with parameters (n, p) is the sum of n independent Bernoulli variables, each taking value 1 with probability p or 0 with probability $1 - p$.

Proposition 2.2. *Suppose X has binomial distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geqslant a\mathbb{E}X) \leqslant 2e^{-\frac{a^2}{3}\mathbb{E}X}$.*

2.4. Two structural results for hypergraphs

In Section 5.3 we will show that if our hypergraph H does not contain a certain type of absorbing set then H is in the extremal case. To deduce this, we will obtain structural information about two auxiliary (hyper)graphs. This will in turn provide structural information about H. The following two powerful results will be required for this.

Theorem 2.3 *(Hypergraph Removal Lemma). (See [7,23].) Let $\gamma > 0$ and $k, t \in \mathbb{N}$ such that $2 \leqslant k \leqslant t$. Given any k-uniform hypergraph F on t vertices, there exists $\alpha = \alpha(F, \gamma) > 0$ and $n_0 = n_0(F, \gamma) \in \mathbb{N}$ such that the following holds. Suppose H is a k-uniform hypergraph on $n \geqslant n_0$ vertices such that H contains at most αn^t copies of F. Then H can be made F-free by deleting at most γn^k edges.*

Given $r \in \mathbb{N}$, let $\mathcal{C}_3^{2r}$ denote the *expanded* $2r$*-uniform triangle*. That is, $\mathcal{C}_3^{2r}$ consists of three disjoint sets P_1, P_2, P_3 of vertices of size r, and the edges $P_1 \cup P_2$, $P_2 \cup P_3$, $P_3 \cup P_1$. Keevash and Sudakov [10] used the following theorem to prove a conjecture of Frankl [5] concerning the Turán number of $\mathcal{C}_3^{2r}$.

Theorem 2.4. *(See [10].) For every $\gamma > 0$ and $r \in \mathbb{N}$, there exists $\beta = \beta(\gamma, r) > 0$ such that if H is a $\mathcal{C}_3^{2r}$-free $2r$-uniform hypergraph on n vertices with*

$$e(H) > \left(\frac{1}{2} - \beta\right)\binom{n}{2r},$$

then $H = \mathcal{B}_{n,2r} \pm \gamma n^{2r}$.

3. Proof of Theorem 1.1

Most of the paper is devoted to the proof of the following two results: we will prove Theorem 3.1 in Section 5 and Theorem 3.2 in Section 4.

Theorem 3.1. *Let $\varepsilon > 0$ and $r, \ell \in \mathbb{N}$ such that $2r \leqslant \ell \leqslant 4r - 1$. Then there exist $\alpha, \xi > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a $4r$-uniform hypergraph on $n \geqslant n_0$ vertices where $4r$ divides n. If*

$$\delta_\ell(H) \geqslant \left(\frac{1}{2} - \alpha\right)\binom{n-\ell}{4r-\ell}$$

then H is ε-close to $\mathcal{B}_{n,4r}$ or $\overline{\mathcal{B}}_{n,4r}$, or H contains a matching M of size $|M| \leqslant \xi n/(4r)$ that absorbs any set $W \subseteq V(H) \setminus V(M)$ such that $|W| \in 4r\mathbb{N}$ with $|W| \leqslant \xi^2 n$.

Notice that the minimum ℓ-degree condition in Theorem 3.1 is weaker than that in Theorem 1.1. Theorem 3.1 says that either H contains a reasonably small absorbing set which can absorb any small set of vertices or H is ‘close’ to $\mathcal{B}_{n,4r}$ or $\overline{\mathcal{B}}_{n,4r}$. The next result shows that in the latter, ‘extremal case’, H contains a perfect matching.

Theorem 3.2. *Given $1 \leqslant \ell \leqslant k-1$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a k-uniform hypergraph on $n \geqslant n_0$ vertices such that n is divisible by k. If $\delta_\ell(H) > \delta(n, k, \ell)$ and H is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$, then H contains a perfect matching.*

The following result of Markström and Ruciński [17] is needed in the ‘non-extremal’ case.

Theorem 3.3. *(See [17, Lemma 2].) For each integer $k \geqslant 3$, every $1 \leqslant \ell \leqslant k-2$ and every $\gamma > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a k-uniform hypergraph on $n \geqslant n_0$ vertices such that*

$$\delta_\ell(H) \geqslant \left(\frac{k-\ell}{k} - \frac{1}{k^{(k-\ell)}} + \gamma\right)\binom{n-\ell}{k-\ell}.$$

Then H contains a matching covering all but at most $\sqrt{n}$ vertices.

In [17], Markström and Ruciński only stated Theorem 3.3 for $1 \leqslant \ell < k/2$. In fact, their proof works for all values of ℓ such that $1 \leqslant \ell \leqslant k-2$. In the case when $\ell = k-1$, we need a result of Rödl, Ruciński and Szemerédi [22, Fact 2.1]: if $\delta_{k-1}(H) \geqslant n/k$, then H contains a matching covering all but at most k^2 vertices in H.

We now show that, to prove Theorem 1.1, it suffices to prove Theorems 3.1 and 3.2.

Proof of Theorem 1.1. Let ε be as in Theorem 3.2 and α, ξ be as in Theorem 3.1. That is,

$$0 < \alpha, \xi \ll \varepsilon \ll 1/r.$$

Assume that $2r \leqslant \ell \leqslant 4r-1$. Consider any sufficiently large $4r$-uniform hypergraph H on n vertices such that $4r$ divides n and

$$\delta_\ell(H) > \delta(n, 4r, \ell).$$

For any $k \geqslant 2$, it is clear that $\delta_{k-1}(\mathcal{B}_{n,k}) \geqslant n/2 - (k-1)$. Thus, by Proposition 2.1, $\delta_\ell(\mathcal{B}_{n,4r}) \geqslant (1/2-\alpha)\binom{n-\ell}{4r-\ell}$. Consequently $\delta_\ell(H) \geqslant (1/2-\alpha)\binom{n-\ell}{4r-\ell}$. Theorem 3.1 implies that either H is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$ or H contains a matching M of size $|M| \leqslant \xi n/(4r)$ that absorbs any set $W \subseteq V(H) \setminus V(M)$ such that $|W| \in 4r\mathbb{N}$ with $|W| \leqslant \xi^2 n$. In the former case Theorem 3.2 implies that H contains a perfect matching. In the latter case set $H' := H \backslash V(M)$ and $n' := |V(H')|$. Since $\ell \geqslant 2r$, $\alpha, \xi \ll 1/r$ and n is sufficiently large,

$$\delta_\ell\bigl(H'\bigr) \geqslant \delta_\ell(H) - \bigl|V(M)\bigr|\binom{n}{4r-\ell-1} \geqslant \left(\frac{4r-\ell}{4r} - \frac{1}{(4r)^{(4r-\ell)}} + \alpha\right)\binom{n'-\ell}{4r-\ell}.$$

Hence, if $\ell \leqslant 4r-2$, Theorem 3.3 implies that H' contains a matching M' covering all but at most $\sqrt{n'}$ vertices in H'. If $\ell = 4r-1$, then since $\delta_\ell(H') \geqslant n'/(4r)$, Fact 2.1 from [22] implies that H' contains a matching M' covering all but at most $(4r)^2$ vertices in H'. In both cases set $W := V(H') \backslash V(M')$. Then $|W| \leqslant \sqrt{n'} \leqslant \xi^2 n$. By definition of M, there is a matching M'' in H which covers $V(M) \cup W$. Thus, $M' \cup M''$ is a perfect matching of H, as desired. □

4. The extremal case

In this section we prove Theorem 3.2: for sufficiently small $\varepsilon > 0$ and sufficiently large $n \in k\mathbb{N}$, any k-uniform n-vertex hypergraph H with $\delta_\ell(H) > \delta(n, k, \ell)$ and which is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$ contains a perfect matching. Recall that $\delta(n, k, \ell)$ is the maximum of the minimum ℓ-degrees among all the hypergraphs in $\mathcal{H}_{\text{ext}}(n, k)$, and $\mathcal{H}_{\text{ext}}(n, k)$ contains all hypergraphs $\overline{\mathcal{B}}_{n,k}(A, B)$ with $|A|$ odd, and all hypergraphs $\mathcal{B}_{n,k}(A, B)$ where n/k is odd and $|A|$ is even, and where n/k is even and $|A|$ is odd.

Given two k-uniform hypergraphs H and H' on n vertices, we say H *ε-contains* H' if, after adding at most εn^k edges to H, the resulting hypergraph contains a copy of H'. More precisely, H ε-contains H' if there is an isomorphic copy $\tilde{H}$ of H such that $V(\tilde{H}) = V(H')$ and $|E(H') \setminus E(\tilde{H})| \leqslant \varepsilon n^k$. Trivially if H is ε-close to H', then H ε-contains H'.

The following theorem thus implies Theorem 3.2.

Theorem 4.1. *Given $1 \leqslant \ell \leqslant k-1$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a k-uniform hypergraph on $n \geqslant n_0$ vertices such that n is divisible by k. Then H contains a perfect matching if the following holds*:

- $\delta_\ell(H) > \delta(n, k, \ell)$;
- *H ε-contains $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$.*

Furthermore, by modifying the proof of Theorem 4.1 slightly one can obtain another structural extremal case result (we omit its proof).

Theorem 4.2. *Given an integer $k \geqslant 2$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a k-uniform hypergraph on $n \geqslant n_0$ vertices such that n is divisible by k. Then H contains a perfect matching if the following holds.*

(i) $\delta_1(H) \geqslant (\frac{1}{2} - \varepsilon)\binom{n-1}{k-1}$;
(ii) *Under any partition A, B of $V(H)$, there always exist at least one (A, B)-even edge and at least one (A, B)-odd edge*;
(iii) *H ε-contains $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$.*

The rest of this section is devoted to the proof of Theorem 4.1.

4.1. Preliminaries and proof outline

Given a set A, we denote by $K^k(A)$ the complete k-uniform hypergraph on A (the superscript k is often omitted). Given integers $0 \leqslant r \leqslant k$ and two disjoint sets A and B, let $K_r^k(A, B)$ or simply $K_r(A, B)$ denote the k-uniform hypergraph on $A \cup B$ whose edges are all k-sets intersecting A with precisely r vertices.

Let H, H' be two k-uniform hypergraphs on the same vertex set V. Let $H' \setminus H := (V, E(H') \setminus E(H))$. Suppose that $0 \leqslant \alpha \leqslant 1$ and $|V| = n$. A vertex $v \in V$ is called *α-good in H* (otherwise *α-bad*) with respect to H' if $d_{H' \setminus H}(v) \leqslant \alpha n^{k-1}$. Sometimes we also say that v is *α-good* (*in H*) *with respect to* $E(H')$.

We use the following result [22, Fact 4.1] and include a proof for completeness.

Lemma 4.3. *Let $k, r \in \mathbb{N}$ such that $k \geqslant 2$ and $r \leqslant k$. Let $0 < \alpha < \frac{1}{k(2k(k-1))^{k-1}}$. Suppose that H is a k-uniform hypergraph on $V = A \cup B$ such that $|A| = tr$, $|B| = t(k-r)$ for some integer $t \geqslant 2(k-1)$, and every vertex of H is α-good with respect to $K_r^k(A, B)$. Then H contains a perfect matching.*

Proof. Let M be a largest matching of H consisting of only $A^r B^{k-r}$ edges. Set $m := |M|$ and $n := |V| = tk$. We claim that $m = t$, namely, M is a perfect matching of H. Suppose $m < t$ instead. Let $A_0 := A \setminus V(M)$ and $B_0 := B \setminus V(M)$. Then $|A_0| = (t-m)r \geqslant r$ and $|B_0| = (t-m)(k-r) \geqslant k-r$. The maximality of M implies that there are no $A_0^r B_0^{k-r}$ edges. Fix $v \in A_0$. Since v is α-good with respect to $K_r^k(A, B)$, it follows that $\binom{|A_0|-1}{r-1}\binom{|B_0|}{k-r} \leqslant \alpha n^{k-1}$, which implies that

$$\left(\frac{|A_0|}{r}\right)^{r-1}\left(\frac{|B_0|}{k-r}\right)^{k-r} \leqslant \alpha n^{k-1}$$

and thus, $(t-m)^{k-1} \leqslant \alpha (tk)^{k-1}$. Since $\alpha < 1/(2k)^{k-1}$, this implies that $t - m \leqslant t/2$ or $m \geqslant t/2$.

Fix a k-set $S = \{v_1, v_2, \dots, v_k\}$ with $v_1, \dots, v_r \in A_0$ and $v_{r+1}, \dots, v_k \in B_0$. Given a vertex $v \in V$, we call a collection $e_1, \dots, e_{k-1}$ of $k-1$ distinct edges *feasible for* v if every k-set T with $v \in T$, $|T \cap e_i| = 1$ for all $1 \leqslant i \leqslant k-1$ and $|T \cap A| = r$ is an edge of H. We claim that there are $k-1$ (distinct) edges $e_1, \dots, e_{k-1}$ of M that are feasible for all the vertices of S. This contradicts the maximality of M since it is easy to see that $\bigcup_{i=1}^{k-1} e_i \cup S$ contains k disjoint $A^r B^{k-r}$ edges of H.

To find $k-1$ feasible edges for all the vertices of S, we consider all $(k-1)$-tuples of M. There are $\binom{|M|}{k-1} \geqslant \binom{t/2}{k-1}$ $(k-1)$-tuples of M. Since each v_i is α-good, at most αn^{k-1} $(k-1)$-sets that are neighbors of v in $K_r^k(A, B)$ are not neighbors of v_i in H. Thus at most αn^{k-1} $(k-1)$-tuples of M are not feasible for v_i. In total, at most $k\alpha n^{k-1}$ $(k-1)$-tuples of M are not feasible for at least one vertex of S. Since $t/2 \geqslant k-1$ and $\alpha < \frac{1}{k(2k(k-1))^{k-1}}$, we have $\binom{t/2}{k-1} \geqslant (\frac{t}{2(k-1)})^{k-1} > k\alpha n^{k-1}$. Hence there always exists a $(k-1)$-tuple of M feasible for all the vertices of S. $\square$

To derive Corollary 4.5, we also need a simple claim.

Claim 4.4. *Let H and H' be two k-uniform hypergraphs on an n-vertex set V. Suppose that $\alpha > 0$ and v is α-good in H with respect to H'. Let H'' be a subgraph of H' on $U \subset V$ such that $v \in U$ and $|U| \geqslant cn$ for some $c > 0$. Then v is α'-good in $H[U]$ with respect to H'', where $\alpha' := \alpha/c^{k-1}$.*

Proof. This follows from

$$d_{H''\setminus H[U]}(v) \leqslant d_{H'\setminus H}(v) \leqslant \alpha n^{k-1} = \alpha'(cn)^{k-1} \leqslant \alpha'|U|^{k-1}. \qquad \square$$

Corollary 4.5. *Given an even integer $k \geqslant 2$, there exist $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geqslant n_0$ with $n \in 2k\mathbb{N}$. Suppose that H is an n-vertex k-uniform hypergraph with a partition A, B of $V(H)$ such that $|A| = |B| = n/2$. If every vertex of H is α-good with respect to $\mathcal{B}_{n,k}(A, B)$, then H contains a perfect matching.*

Furthermore, if $k/2$ is odd, then $n \in 2k\mathbb{N}$ can be weakened to $n \in k\mathbb{N}$.

Proof. First assume that $n \in 2k\mathbb{N}$. Then $|A| = |B|$ is divisible by k. We arbitrarily partition A into two subsets A_1 of size $|A|/k$ and A_2 of size $|A|(k-1)/k$, and partition B into two subsets B_1 of size $|B|(k-1)/k$ and B_2 of size $|B|/k$. Let $H_i = H[A_i \cup B_i]$ for $i = 1, 2$. Since all the vertices of H are α-good with respect to $\mathcal{B}_{n,k}(A, B)$, by Claim 4.4, all the vertices in $A_1 \cup B_1$ are α'-good in H_1 with respect to $K_1(A_1, B_1)$, where $\alpha' := 2^{k-1}\alpha$. Similarly, every vertex in $A_2 \cup B_2$ is α'-good in H_2 with respect to $K_1(A_2, B_2)$. As $\alpha' \ll 1/k$, we can apply Lemma 4.3 to H_1 and H_2 obtaining a perfect matching M_1 of H_1 and a perfect matching M_2 of H_2. Thus $M_1 \cup M_2$ is a perfect matching of H.

Second assume that $k/2$ is odd and $n \in k\mathbb{N}$. Then $|A| = |B|$ is divisible by $k/2$. Since every vertex of H is α-good with respect to $K_{k/2}(A, B)$, we can apply Lemma 4.3 with $r = k/2$ obtaining a perfect matching of H. $\square$

Now we give an outline of our proof of Theorem 4.1.

Step 1: Since H ε-contains $\mathcal{B}_{n,k}$ (or $\overline{\mathcal{B}}_{n,k}$), all but at most $\varepsilon_1 n$ vertices in H are ε_2-good with respect to $\mathcal{B}_{n,k}$ (or $\overline{\mathcal{B}}_{n,k}$) for some $\varepsilon \ll \varepsilon_1 \ll \varepsilon_2$. Denote the set of ε_2-bad vertices by V_0. Let A and B denote the vertex classes of $\mathcal{B}_{n,k}$ (or $\overline{\mathcal{B}}_{n,k}$). We move the vertices of V_0 to the other side (from A to B or from B to A) and denote the resulting sets by A_1 and B_1.

Step 2: In some cases, we will obtain a special edge e_0, which is an (A_1, B_1)-even edge when H ε-contains $\mathcal{B}_{n,k}$ or an (A_1, B_1)-odd edge when H ε-contains $\overline{\mathcal{B}}_{n,k}$. Note that e_0 may contain vertices of V_0.

Step 3: We remove a matching M_1 of size $|M_1| \leqslant \varepsilon_1 n$ containing all the vertices in $V_0 \setminus e_0$. Denote the resulting sets by A_2 and B_2.

Step 4: We remove a small matching from $H[A_2 \cup B_2]$ such that the resulting sets A_3, B_3 satisfy:
- If k is even and H ε-contains $\overline{\mathcal{B}}_{n,k}$, then $|A_3| \equiv 0 \pmod{k}$.
- If k is even and H ε-contains $\mathcal{B}_{n,k}$, then $|A_3| = |B_3|$. Furthermore, if k is divisible by 4, we also need $|A_3| \equiv 0 \pmod{k}$.
- If k is odd, then $|A_3| \equiv 0 \pmod{k-1}$.

In many cases the special edge e_0 is needed in this step.

Step 5: If e_0 was introduced in Step 2 but not used in Step 4 and $e_0 \cap V_0 \neq \emptyset$, we remove a small matching containing all the vertices in $e_0 \cap V_0$ while preserving the property mentioned in Step 4.

Step 6: We apply Lemma 4.3 or Corollary 4.5 to $H[A_3 \cup B_3]$ and find a perfect matching of $H[A_3 \cup B_3]$.

In the next three subsections, we give details of these steps based on the three cases listed in Step 4. Full details for each step are only given when the step is needed at the first time. Note that Steps 1 and 3 are essentially the same for all the three cases but Steps 2 and 5 are not necessary in some cases.

Indeed, we may only apply Step 2 in the case when, after applying Step 1, (i) H ε-contains $\mathcal{B}_{n,k}$ and $\mathcal{B}_{n,k}(A_1, B_1) \in \mathcal{H}_{\text{ext}}(n,k)$ or; (ii) H ε-contains $\overline{\mathcal{B}}_{n,k}$ and $\overline{\mathcal{B}}_{n,k}(A_1, B_1) \in \mathcal{H}_{\text{ext}}(n,k)$. In these cases, we will need to use the condition that $\delta_\ell(H) > \delta(n,k,\ell)$ to ensure H contains our desired edge e_0. This is the only place in the proof of Theorem 4.1 (and in fact, the only part of the proof of Theorem 1.1) where we use the full force of our minimum ℓ-degree condition.

The edge e_0 acts as a "parity-breaker", helping us to construct our desired perfect matching. However, if H does not satisfy (i) or (ii) then no parity-breaking edge is required, and so we do not need Step 2.

4.2. k is even and H ε-contains $\overline{\mathcal{B}}_{n,k}$

In this subsection, we prove Theorem 4.1 under the assumption that k is even and H ε-contains $\overline{\mathcal{B}}_{n,k}$, where $0 < \varepsilon \ll 1/k$. Define $\varepsilon_1 := k^{\frac{1}{2}}\varepsilon^{\frac{2}{3}}$ and $\varepsilon_2 := k^{\frac{1}{2}}\varepsilon^{\frac{1}{3}}$. Let H be a k-uniform hypergraph on an n-vertex set V for sufficiently large $n \in k\mathbb{N}$. Note that n is even because k is even. Suppose that H ε-contains $\overline{\mathcal{B}}_{n,k}$, namely, there exists a partition A, B of V such that $|A| = |B| = n/2$, $\overline{\mathcal{B}}_{n,k} = (V, E_{\text{even}}(A,B))$ and $|E_{\text{even}}(A,B) \setminus E(H)| \leqslant \varepsilon n^k$.

Step 1: Recall that a vertex $v \in V(H)$ is ε_2-*bad* with respect to $\overline{\mathcal{B}}_{n,k}$ if $d_{\overline{\mathcal{B}}_{n,k} \setminus H}(v) > \varepsilon_2 n^{k-1}$. In other words, if v is ε_2-good then all but at most $\varepsilon_2 n^{k-1}$ of the (A,B)-even edges that contain v belong to H. We observe that at most $\varepsilon_1 n$ vertices in H are ε_2-bad. Otherwise

$$k\left|E(\overline{\mathcal{B}}_{n,k}) \setminus E(H)\right| = \sum_{v \in V}\left|N_{\overline{\mathcal{B}}_{n,k}}(v) \setminus N_H(v)\right| > \varepsilon_2 n^{k-1}\varepsilon_1 n = k\varepsilon n^k,$$

contradicting the assumption that $|E_{\text{even}}(A,B) \setminus E(H)| \leqslant \varepsilon n^k$.

Let A_0 and B_0 denote the sets of ε_2-bad vertices in A and in B, respectively, and set $V_0 := A_0 \cup B_0$. Then $|A_0| + |B_0| = |V_0| \leqslant \varepsilon_1 n$. Notice that $\delta_1(H) \geqslant (\frac{1}{2} - \varepsilon)\binom{n-1}{k-1}$ by Proposition 2.1. Consider $v \in V_0$. We know that $d_{\overline{\mathcal{B}}_{n,k}}(v) \leqslant (\frac{1}{2} + \varepsilon)\binom{n-1}{k-1}$. Since $d_{\overline{\mathcal{B}}_{n,k} \setminus H}(v) > \varepsilon_2 n^{k-1}$, it follows that

$$d_{H \setminus \overline{\mathcal{B}}_{n,k}}(v) \geqslant \left(\frac{1}{2} - \varepsilon\right)\binom{n-1}{k-1} - \left(d_{\overline{\mathcal{B}}_{n,k}}(v) - \varepsilon_2 n^{k-1}\right) \geqslant \varepsilon_2 n^{k-1} - 2\varepsilon\binom{n-1}{k-1} \geqslant \frac{\varepsilon_2}{2} n^{k-1}. \quad (2)$$

In other words, v lies in at least $\varepsilon_2 n^{k-1}/2$ (A,B)-odd edges in H.

Define $A_1 := (A \setminus A_0) \cup B_0$ and $B_1 := (B \setminus B_0) \cup A_0$. Then A_1, B_1 is a partition of $V(H)$ with $|A_1|, |B_1| \geqslant (1/2 - \varepsilon_1)n$.

We now separate cases based on the parity of $|A_1|$.

First assume that $|A_1|$ is even. Then $\overline{\mathcal{B}}_{n,k}(A_1, B_1) \notin \mathcal{H}_{\text{ext}}(n,k)$. Thus, we do not need Step 2 (and therefore Step 5) in this case.

Step 3: We remove a matching M_1 from H such that

- $|M_1| = |V_0| \leqslant \varepsilon_1 n$;
- each edge of M_1 contains exactly one vertex of V_0;
- all the edges of M_1 are (A_1, B_1)-even.

To find M_1, we consider the vertices of V_0 in an arbitrary order and apply the following simple claim repeatedly.

Claim 4.6. *Let $k \geqslant 2$ be an integer and α_1, α_2 be constants such that $\alpha_2 > \alpha_1/(k-2)! \geqslant 0$ (here $0! := 1$). Let H be a k-uniform hypergraph on n vertices such that $d_H(v) \geqslant \alpha_2 n^{k-1}$ and $|U| \leqslant \alpha_1 n$ for some $U \subset V(H)$ with $v \notin U$. Then v lies in an edge disjoint from U.*

Proof. There are at most $\alpha_1 n\binom{n-2}{k-2} \leqslant \frac{\alpha_1}{(k-2)!}n^{k-1}$ edges of H containing v and at least one vertex from U. Since $\alpha_2 > \alpha_1/(k-2)!$, there exists an edge containing v and no vertex of U. □

Suppose that we have found i edges in M_1 and consider the next vertex $v \in V_0$. Then $|V_0 \cup V(M_1)| \leqslant k\varepsilon_1 n$. Because of (2) and $\varepsilon_1 \ll \varepsilon_2$, we can apply Claim 4.6 with $U = (V_0 \setminus \{v\}) \cup V(M_1)$ to find an (A, B)-odd edge containing v but no other vertex of V_0 and which is disjoint from the existing edges of M_1. By the definition of A_1, B_1, any (A, B)-odd edge containing v and no other vertex of V_0 is an (A_1, B_1)-even edge. We thus add this edge to M_1. At the end of this process, let $A_2 := A_1 \setminus V(M_1)$ and $B_2 := B_1 \setminus V(M_1)$.

Step 4: Since $|A_1|$ is even, the third property of M_1 implies that $s := |A_2| \pmod k$ is also even. If $s \neq 0$, we remove an $A_2^s B_2^{k-s}$ edge e_2. Such an edge exists because all the vertices in $A_2 \cup B_2$ are ε_2-good with respect to $\overline{\mathcal{B}}_{n,k}$. More precisely, since $A_2 \subseteq A$, $B_2 \subseteq B$, and $|A_2|, |B_2| \geqslant (\frac{1}{2} - (k+1)\varepsilon_1)n$, Claim 4.4 implies that all the vertices in $A_2 \cup B_2$ are $2\varepsilon_2$-good with respect to $K_s(A_2, B_2)$. As $\varepsilon_2 \ll 1/k$ and consequently

$$2\varepsilon_2 n^{k-1} < \binom{(\frac{1}{2} - (k+1)\varepsilon_1)n - 1}{s-1}\binom{(\frac{1}{2} - (k+1)\varepsilon_1)n}{k-s},$$

there exists an $A_2^s B_2^{k-s}$ edge containing *any* vertex in A_2.

Let $A_3 := A_2 \setminus e_2$ and $B_3 := B_2 \setminus e_2$. Then $|A_3| \equiv 0 \pmod k$. Since $|A_3| + |B_3| \equiv |A| + |B| \equiv 0 \pmod k$, we have $|B_3| \equiv 0 \pmod k$.

Step 6: Since $|A_3| \geqslant (1/2 - 2k\varepsilon_1)n \geqslant n/3$, by Claim 4.4, all the vertices of A_3 are $(3^{k-1}\varepsilon_2)$-good in $H[A_3]$ with respect to $\overline{\mathcal{B}}_{n,k}[A_3] = K^k(A_3)$, the complete k-uniform hypergraph on A_3. As $\varepsilon_2 \ll 1/k$, by Lemma 4.3 (with $r = k$), there is a perfect matching M_3 of $H[A_3]$. Similarly we can find a perfect matching M_3' of $H[B_3]$ (note that $\overline{\mathcal{B}}_{n,k}[B_3] = K^k(B_3)$ because k is even). The union $M_1 \cup \{e_2\} \cup M_3 \cup M_3'$ is the desired perfect matching of H.

Now assume that $|A_1|$ is odd. In this case we need Step 2 (but not Step 5). Note that $\overline{\mathcal{B}}_{n,k}(A_1, B_1) \in \mathcal{H}_{\text{ext}}(n,k)$ since $|A_1|$ is odd. As $\delta_\ell(H) > \delta(n,k,\ell) \geqslant \delta_\ell(\overline{\mathcal{B}}_{n,k}(A_1, B_1))$, we can find an (A_1, B_1)-odd edge e_0. We apply Step 3 as before though now we require that M_1 is chosen to be disjoint from e_0. In particular, this means M_1 is chosen to cover $V_0 \setminus e_0$. After Step 3, we let $A_2' := A_2 \setminus e_0$ and $B_2' := B_2 \setminus e_0$. Then $s := |A_2'| \pmod k$ is even. The rest of the argument is the same as in the case when $|A_1|$ is even.

4.3. *k is even and H ε-contains $\mathcal{B}_{n,k}$*

Assume that k is even, and n is sufficiently large and divisible by k (thus n is also even). Recall that $\mathcal{B}_{n,k}$ is the k-uniform hypergraph whose vertex set is partitioned into $A \cup B$ such that $|A| = |B| = n/2$ and whose edge set $E_{\text{odd}}(A, B)$ consists of all k-sets that intersect A in an odd number of vertices. Suppose that H is a k-uniform hypergraph on n vertices such that H ε-contains $\mathcal{B}_{n,k}$, namely, $|E_{\text{odd}}(A, B) \setminus E(H)| \leqslant \varepsilon n^k$.

Step 1 is the same as in Section 4.2, except for replacing $\overline{\mathcal{B}}_{n,k}$ by $\mathcal{B}_{n,k}$. Therefore again A_0 and B_0 denote the sets of ε_2-bad vertices in A and B respectively and $V_0 := A_0 \cup B_0$, $A_1 := (A \setminus A_0) \cup B_0$ and $B_1 := (B \setminus B_0) \cup A_0$.

If $\mathcal{B}_{n,k}(A_1, B_1) \in \mathcal{H}_{\text{ext}}(n,k)$ then as $\delta_\ell(H) > \delta_\ell(\mathcal{B}_{n,k}(A_1, B_1))$, we can apply Step 2. That is, H contains an (A_1, B_1)-even edge e_0. Then $r_0 := |e_0 \cap A_1|$ is even. If $\mathcal{B}_{n,k}(A_1, B_1) \notin \mathcal{H}_{\text{ext}}(n,k)$ then we do not apply Step 2. (So in what follows, we take $e_0 = \emptyset$ in this case.)

In Step 3, we remove a matching M_1 such that

- $|M_1| = |V_0 \setminus e_0| \leqslant \varepsilon_1 n$;
- each edge of M_1 contains exactly one vertex of $V_0 \setminus e_0$;
- all the edges of M_1 are (A_1, B_1)-*odd* and are disjoint from e_0.

Further, in the case when $\mathcal{B}_{n,k}(A_1, B_1) \notin \mathcal{H}_{\text{ext}}(n,k)$ we add at most 3 extra (A_1, B_1)-odd edges to M_1 to ensure that M_1 is a matching with $|M_1|$ divisible by 4. Set $A_2 := A_1 \setminus V(M_1)$ and $B_2 := B_1 \setminus V(M_1)$. Without loss of generality, assume that $|A_2| \geqslant |B_2|$. Let $d := |A_2| - |B_2|$. Then d is even because $|A_2| + |B_2|$ is even. We also know that $d \leqslant k|M_1| + 2|V_0| \leqslant (k+2)\varepsilon_1 n + 3k$. We now separate cases based on the parity of $k/2$.

4.3.1. $k/2$ is even

Step 4: We remove a matching M_2 that consists of $d/2$ $A_2^{k/2+1}B_2^{k/2-1}$ edges that are disjoint from M_1 and e_0 (note that $k/2+1$ and $k/2-1$ are odd). In a similar way to Step 4 of Section 4.2, these edges exist because all the vertices in $(A_2 \cup B_2) \setminus e_0$ are ε_2-good with respect to $\mathcal{B}_{n,k}$. The resulting sets $A_3 := A_2 \setminus V(M_2)$ and $B_3 := B_2 \setminus V(M_2)$ thus have the same size

$$|A_2| - \frac{d}{2}\left(\frac{k}{2}+1\right) = |B_2| - \frac{d}{2}\left(\frac{k}{2}-1\right).$$

Let $s := |A_3| = |B_3| \pmod k$. Since $|A_3| + |B_3| \equiv 0 \pmod k$, it follows that either $s = 0$ or $s = k/2$.

Notice that if $\mathcal{B}_{n,k}(A_1, B_1) \notin \mathcal{H}_{\text{ext}}(n,k)$, then $s = 0$. Indeed, suppose not. Then $s = k/2$ and so $|A_3| = |B_3| = km + k/2$ for some $m \in \mathbb{N}$. Thus, $|A_3| + |B_3| = 2km + k$. Hence, $(|A_3| + |B_3|)/k$ is odd but $|A_3|$ is even. Since the edges in $M_1 \cup M_2$ are (A_1, B_1)-odd, this implies that either $(|A_1| + |B_1|)/k = n/k$ is odd and $|A_1|$ is even or n/k is even and $|A_1|$ is odd. In both cases this implies that $\mathcal{B}_{n,k}(A_1, B_1) \in \mathcal{H}_{\text{ext}}(n,k)$, a contradiction.

Case 1a: $s = 0$. If $e_0 \cap V_0 = \emptyset$, then we proceed to Step 6 directly. Since $|A_3| = |B_3| \equiv 0 \pmod k$ and $|A_3|, |B_3| \geqslant (\frac{1}{2} - 2k^2\varepsilon_1)n$, we can apply Corollary 4.5 obtaining a perfect matching M_3 of $H[A_3 \cup B_3]$. Consequently $M_1 \cup M_2 \cup M_3$ is the desired perfect matching of H. (Note that this covers the case when $\mathcal{B}_{n,k}(A_1, B_1) \notin \mathcal{H}_{\text{ext}}(n,k)$, since $s = 0$ and $e_0 = \emptyset$ in this case.)

If $e_0 \cap V_0 \neq \emptyset$, then we need Step 5, in which we remove a small matching containing all the vertices of $e_0 \cap V_0$. Let $v \in e_0 \cap V_0$. By a similar calculation as in (2), v is contained in at least $\varepsilon_2 n^{k-1}/2$ (A, B)-even edges. Applying Claim 4.6 with $U = V(M_1 \cup M_2) \cup (e_0 \setminus v)$, we find an (A, B)-even edge of $H[A_3 \cup B_3]$ containing v. Since v changes 'side' (from A to B_1 or from B to A_1), and by the choice of U, this edge is an $A_3^r B_3^{k-r}$ edge for some *odd* r. To keep the numbers of the remaining vertices in A_3 and B_3 the same and divisible by k, when we remove an $A_3^r B_3^{k-r}$ edge e containing v we immediately remove an $A_3^{k-r} B_3^r$ edge disjoint from e (such an edge exists because all the vertices in $(A_3 \cup B_3) \setminus e_0$ are ε_2-good with respect to $\mathcal{B}_{n,k}$). Repeat this process for all the vertices in $e_0 \cap V_0$. Denote by M_3 the set of all removed edges in this step. Then $|M_3| \leqslant 2k$. Let $A_4 := A_3 \setminus V(M_3)$ and $B_4 := B_3 \setminus V(M_3)$. Then $|A_4| = |B_4| \equiv 0 \pmod k$. Finally in Step 6 we find a perfect matching M_4 of $H[A_4 \cup B_4]$ by Corollary 4.5. Thus $M_1 \cup M_2 \cup M_3 \cup M_4$ is the desired perfect matching of H.

Case 1b: $s = k/2$. Recall that $|e_0 \cap A_1| = r_0$ for some even r_0. Thus, $|e_0 \cap A_3| = r_0$. We continue on Step 4 as follows. If $r_0 \leqslant k/2$, then we remove e_0 together with $\frac{k}{2} - r_0$ disjoint $A_3^{k/2+1} B_3^{k/2-1}$ edges; otherwise we remove e_0 together with $r_0 - \frac{k}{2}$ disjoint $A_3^{k/2-1} B_3^{k/2+1}$ edges. Denote by M_3 the set of these removed edges. Let $A_4 := A_3 \setminus V(M_3)$ and $B_4 := B_3 \setminus V(M_3)$. It is easy to see that $|A_4| = |B_4| = |A_3| - (|\frac{k}{2} - r_0| + 1)\frac{k}{2}$. Since $s = k/2$ and $k/2, r_0$ are even, we have $|A_4| \equiv 0 \pmod k$. Since e_0 has been used, we now skip Step 5 and proceed to Step 6. As in Case 1a, we find a perfect matching M_4 of $H[A_4 \cup B_4]$ by Corollary 4.5. Consequently $M_1 \cup M_2 \cup M_3 \cup M_4$ is the desired perfect matching of H.

4.3.2. $k/2$ is odd

Recall that $d := |A_2| - |B_2| \geqslant 0$ is even. We will separate cases based on the parity of $d/2$. Firstly though, notice that if $\mathcal{B}_{n,k}(A_1, B_1) \notin \mathcal{H}_{\text{ext}}(n,k)$ then d is divisible by 4. Indeed, suppose instead that $d \equiv 2 \pmod 4$. First consider the case when $|A_2| + |B_2|$ is divisible by 4. Since $|M_1|$ is divisible by 4, this implies that $|A_1| + |B_1| = n$ is divisible by 4. But since k is not divisible by 4, this implies that n/k is even. Further, since $d \equiv 2 \pmod 4$, we derive that $|A_2|$ is odd. Since $|A_1 \setminus A_2|$ is even, this implies that $|A_1|$ is odd. Therefore $\mathcal{B}_{n,k}(A_1, B_1) \in \mathcal{H}_{\text{ext}}(n,k)$, a contradiction. Second assume that $|A_2| + |B_2| \equiv 2 \pmod 4$ (recall that $|A_2| + |B_2|$ is even). Since $|M_1|$ is divisible by 4, this implies that $n \equiv 2 \pmod 4$. As k is even, this implies that n/k is odd. So as $d \equiv 2 \pmod 4$, we derive that $|A_2|$ is even, and consequently $|A_1|$ is even. Therefore $\mathcal{B}_{n,k}(A_1, B_1) \in \mathcal{H}_{\text{ext}}(n,k)$, a contradiction.

Case 2a: 4 divides d. In Step 4, we remove $d/4$ disjoint $A_2^{k/2+2} B_2^{k/2-2}$ edges (these edges exists because $k/2+2$ is odd and all the vertices $(A_2 \cup B_2) \setminus e_0$ are ε_2-good with respect to $\mathcal{B}_{n,k}$). Denote by M_2 the set of these edges. Let $A_3 := A_2 \setminus V(M_2)$ and $B_3 := B_2 \setminus V(M_2)$. Then

$$|A_3| = |A_2| - \frac{d}{4}\left(\frac{k}{2}+2\right) = |B_2| - \frac{d}{4}\left(\frac{k}{2}-2\right) = |B_3|.$$

If $e_0 \cap V_0 = \emptyset$, then we proceed to Step 6. Claim 4.4 implies that all the vertices in $H[A_3 \cup B_3]$ are $2\varepsilon_2$-good with respect to $E_{\text{odd}}(A_3, B_3)$. Since $k/2$ is odd, we can apply the second assertion in Corollary 4.5 and find a perfect matching M_3 in $H[A_3 \cup B_3]$ (here we do not require $|A_3| = |B_3| \equiv 0 \pmod k$). Thus, $M_1 \cup M_2 \cup M_3$ is our desired perfect matching in H. (Note that this covers the case when $\mathcal{B}_{n,k}(A_1, B_1) \notin \mathcal{H}_{\text{ext}}(n,k)$, since $e_0 = \emptyset$ in this case.)

If $e_0 \cap V_0 \neq \emptyset$, we need to apply Step 5. As in Case 1a, we remove a matching M_3 of size at most $2k$ containing all the vertices of $e_0 \cap V_0$ such that $A_4 := A_3 \setminus V(M_3)$ and $B_4 := B_3 \setminus V(M_3)$ have the same size. Finally in Step 6 we find a perfect matching M_4 of $H[A_4 \cup B_4]$ by the second assertion in Corollary 4.5. Thus, $M_1 \cup M_2 \cup M_3 \cup M_4$ is a perfect matching in H.

Case 2b: $d \equiv 2 \pmod 4$. We remove e_0 immediately. Let $A_2' := A_2 \setminus e_0$ and $B_2' := B_2 \setminus e_0$. Since $k \equiv 2 \pmod 4$ and r_0 is even, we have $k - 2r_0 \equiv 2 \pmod 4$. Consequently $|A_2'| - |B_2'| = (|A_2| - r_0) - (|B_2| - k + r_0) = d + (k - 2r_0) \equiv 0 \pmod 4$. We then follow the procedure of Case 2a (since e_0 has been removed, we can skip Step 5).

4.4. *k is odd*

Let H be a k-uniform hypergraph such that it ε-contains $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$.

Recall that $\overline{\mathcal{B}}_{n,k}$ is the n-vertex k-uniform hypergraph on $V = A \cup B$ such that $|A| = \lfloor n/2 \rfloor$, $|B| = \lceil n/2 \rceil$, with edge set $E_{\text{even}}(A, B)$. Since k is odd, $\mathcal{B}_{n,k}$ can be viewed as the n-vertex k-uniform hypergraph on $V = A \cup B$ such that $|A| = \lceil n/2 \rceil$, $|B| = \lfloor n/2 \rfloor$, with edge set $E_{\text{even}}(A, B)$. We thus assume that $V(H) = A \cup B$ such that either $|A| = \lfloor n/2 \rfloor$ or $|A| = \lceil n/2 \rceil$ and $|E_{\text{even}}(A, B) \setminus E(H)| \leqslant \varepsilon n^k$.

Our Step 1 is the same as in Section 4.2. After applying Step 1 we have a partition A_1, B_1 of $V(H)$. If $\overline{\mathcal{B}}_{n,k}(A_1, B_1) \notin \mathcal{H}_{\text{ext}}(n,k)$ then, by definition of $\mathcal{H}_{\text{ext}}(n,k)$, $|A_1|$ is even. Thus, in this case $|A_1| \bmod k-1$ is even.

If $|A_1| \bmod k-1$ is odd, then we need Step 2: find an (A_1, B_1)-odd edge e_0. Note that in this case $\overline{\mathcal{B}}_{n,k}(A_1, B_1) \in \mathcal{H}_{\text{ext}}(n,k)$, and thus our minimum ℓ-degree condition ensures we can find such an edge e_0.

Our Step 3 is again the same as in Section 4.2. (Note though, if $|A_1| \bmod k-1$ is odd, then we introduced e_0. Thus in this case we select M_1 to cover $V_0 \setminus e_0$ so that M_1 is disjoint from e_0.) Since each edge in the matching M_1 is an $A_1^r B_1^{k-r}$ edge for some even $r \leqslant k-1$, it follows that $|A_1| \bmod k-1$ and $|A_2| \bmod k-1$ have the same parity.

Assume that $|A_2| \equiv s \pmod{k-1}$. In Step 4, if s is even, then we simply remove an arbitrary $A_2^s B_2^{k-s}$ edge e_2 and let $M_2 = \{e_2\}$. If s is odd, then we remove e_0, which is an $A_2^{r_0} B_2^{k-r_0}$ edge for some odd r_0. Set $A_2' := A_2 \setminus e_0$. Thus, $|A_2'| \equiv s - r_0 \bmod k-1$ and since s, r_0 are odd, $s' := |A_2'| \bmod k-1$ is even. Select an arbitrary $A_2^{s'} B_2^{k-s'}$ edge e_2 that is disjoint from e_0 and set $M_2 = \{e_0, e_2\}$.

Let $A_3 := A_2 \setminus V(M_2)$ and $B_3 := B_2 \setminus V(M_2)$. The choice of M_2 is such that $|A_3| \equiv 0 \pmod{k-1}$. We skip Step 5 and proceed to Step 6. Arbitrarily partition B_3 into B_3^1 and B_3^2 such that $|B_3^1| = |A_3|/(k-1)$ (this is possible because $|A_3| \approx |B_3| \approx n/2$). Note that $|A_3| + |B_3| \equiv 0 \bmod k$. Hence, as $|A_3| + |B_3^1| = k|A_3|/(k-1) \equiv 0 \bmod k$, we have that $|B_3^2| \equiv 0 \bmod k$. Let $H_1 := H[A_3 \cup B_3^1]$ and $H_2 := H[B_3^2]$. Since $|A_3| + |B_3^1| \geqslant (1/2 - 2k\varepsilon_1)nk/(k-1) \geqslant n/2$, by Claim 4.4, all the vertices of H_1 are $(2^{k-1}\varepsilon_2)$-good with respect to $K_{k-1}(A_3, B_3^1)$. Since $k \geqslant 3$ (because $k \geqslant 2$ is odd), we have $|B_3^2| \approx \frac{n}{2}\frac{k-2}{k-1} \geqslant \frac{n}{2k}$. By Claim 4.4, all the vertices of H_2 are $((2k)^{k-1}\varepsilon_2)$-good with respect to $K^k[B_3^2]$. We therefore apply Lemma 4.3 to H_1 (with $r = k-1$) and to H_2 (with $r = k$) to obtain a perfect matching M_3 of H_1 and a perfect matching M_3' of H_2. Thus $M_1 \cup M_2 \cup M_3 \cup M_3'$ is a perfect matching of H. □

5. The non-extremal case

In this section we prove Theorem 3.1. Let $\alpha > 0$ and $r, \ell \in \mathbb{N}$ such that $2r \leqslant \ell \leqslant 4r-1$. Given a $4r$-uniform hypergraph H on n vertices such that $\delta_\ell(H) \geqslant (\frac{1}{2} - \alpha)\binom{n-\ell}{4r-\ell}$, by Proposition 2.1, we have $\delta_{2r}(H) \geqslant (\frac{1}{2} - \alpha)\binom{n-2r}{2r}$. Thus, in order to prove Theorem 3.1 it suffices to prove the following result.

Theorem 5.1. *Given any $\varepsilon > 0$ and $r \in \mathbb{N}$, there exist $\alpha, \xi > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a $4r$-uniform hypergraph on $n \geqslant n_0$ vertices where $4r$ divides n. If*

$$\delta_{2r}(H) \geqslant \left(\frac{1}{2} - \alpha\right)\binom{n-2r}{2r}$$

then H is ε-close to $\mathcal{B}_{n,4r}$ or $\overline{\mathcal{B}}_{n,4r}$, or H contains a matching M of size $|M| \leqslant \xi n/(4r)$ that absorbs any set $W \subseteq V(H) \setminus V(M)$ such that $|W| \in 4r\mathbb{N}$ with $|W| \leqslant \xi^2 n$.

Theorem 5.1 immediately follows from Lemmas 5.2–5.4. Following the ideas in [20,22], we first show in Lemma 5.2 that in order to find the absorbing set described in Theorem 5.1, it suffices to prove that there are at least ξn^{8r} absorbing $8r$-sets for every fixed $4r$-set from $V(H)$.

Lemma 5.2 *(Absorbing Lemma). Given $0 < \xi \ll 1$ and an integer $k \geqslant 2$, there exists an $n_0 \in \mathbb{N}$ such that the following holds. Consider a k-uniform hypergraph H on $n \geqslant n_0$ vertices. Suppose that any k-set of vertices $Q \subseteq V(H)$ can be absorbed by at least ξn^{2k} $2k$-sets of vertices from $V(H)$. Then H contains a matching M of size $|M| \leqslant \xi n/k$ that absorbs any set $W \subseteq V(H) \setminus V(M)$ such that $|W| \in k\mathbb{N}$ and $|W| \leqslant \xi^2 n$.*

Given a $2r$-uniform hypergraph H (for some $r \geqslant 2$), we define the graph $G(H)$ with vertex set $\binom{V(H)}{r}$ in which two vertices $x_1 \cdots x_r, y_1 \cdots y_r \in V(G(H))$ are adjacent if and only if $x_1 \cdots x_r y_1 \cdots y_r \in E(H)$. When it is clear from the context, we will often refer to $G(H)$ as G.

Lemma 5.3 *(Lemma on G). Given any $\beta > 0$ and an integer $r \geqslant 2$, there exist $\alpha, \xi > 0$, and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a $2r$-uniform hypergraph on $n \geqslant n_0$ vertices so that $2r$ divides n and*

$$\delta_r(H) \geqslant \left(\frac{1}{2} - \alpha\right)\binom{n-r}{r}.$$

Set $G := G(H)$ and $N := \binom{n}{r}$ (then N is even because $2r$ divides n). Then at least one of the following assertions holds.

- *$G = K_{\frac{N}{2},\frac{N}{2}} \pm \beta N^2$ or $\overline{G} = K_{\frac{N}{2},\frac{N}{2}} \pm \beta N^2$; in other words, either G or $\overline{G}$ becomes a copy of $K_{\frac{N}{2},\frac{N}{2}}$ after adding or deleting at most βN^2 edges.*
- *There are at least ξn^{4r} absorbing $4r$-sets in $\binom{V(H)}{4r}$ for every $2r$-subset of $V(H)$.*

Lemma 5.4. *Given any $\varepsilon > 0$ and $r \in \mathbb{N}$, there exist $\beta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a $4r$-uniform hypergraph on $n \geqslant n_0$ vertices where $4r$ divides n. Suppose further that $G := G(H)$ satisfies $G = K_{\frac{N}{2},\frac{N}{2}} \pm \beta N^2$ or $\overline{G} = K_{\frac{N}{2},\frac{N}{2}} \pm \beta N^2$, where $N := \binom{n}{2r}$. Then H is ε-close to $\mathcal{B}_{n,4r}$ or $\overline{\mathcal{B}}_{n,4r}$.*

Notice we have stated Lemmas 5.2 and 5.3 in a more general setting than we require. (That is, we consider k-uniform hypergraphs in Lemma 5.2 for all $k \geqslant 2$ and $2r$-uniform hypergraphs in Lemma 5.3 for $r \geqslant 2$.) However, for Lemma 5.4, our proof is such that we can only consider $4r$-uniform hypergraphs for $r \in \mathbb{N}$. (This is the main obstacle in extending our proof to work for all $2r$-uniform hypergraphs.) The rest of the section is devoted to the proof of Lemmas 5.2–5.4.

5.1. Proof of Lemma 5.2

For a k-set $Q \subseteq V(H)$, let L_Q denote the family of all absorbing $2k$-sets for Q. By assumption, $|L_Q| \geqslant \xi n^{2k}$. Let F be the family of $2k$-sets obtained by selecting each of the $\binom{n}{2k}$ elements of $\binom{V(H)}{2k}$ independently with probability $p := \xi/n^{2k-1}$. Then

$$\mathbb{E}\big(|F|\big) = p\binom{n}{2k} < \frac{\xi}{(2k)!}n \quad \text{and} \quad \mathbb{E}\big(|L_Q \cap F|\big) \geqslant p\xi n^{2k} = \xi^2 n$$

for every set $Q \subseteq \binom{V(H)}{k}$.

Since n is sufficiently large, Proposition 2.2 implies that with high probability we have

$$|F| \leqslant 2\mathbb{E}\big(|F|\big) < \frac{2\xi}{(2k)!}n, \tag{3}$$

$$|L_Q \cap F| \geqslant \frac{1}{2}\mathbb{E}\big(|L_Q \cap F|\big) \geqslant \frac{\xi^2}{2}n \quad \text{for all } Q \in \binom{V(H)}{k}. \tag{4}$$

Let Y be the number of intersecting pairs of members of F. Then

$$\mathbb{E}(Y) \leqslant p^2\binom{n}{2k}2k\binom{n}{2k-1} \leqslant \frac{\xi^2 n}{(2k-1)!(2k-1)!}.$$

By Markov's bound, the probability that $Y \leqslant \frac{2\xi^2}{(2k-1)!(2k-1)!}n$ is at least $\frac{1}{2}$. Therefore we can find a family F of $2k$-sets satisfying (3) and (4) and having at most $\frac{2\xi^2}{(2k-1)!(2k-1)!}n$ intersecting pairs. Removing all non-absorbing $2k$-sets and one set from each of the intersecting pairs in F, we obtain a family F' of disjoint absorbing $2k$-sets such that $|F'| \leqslant |F| \leqslant \frac{2\xi}{(2k)!}n \leqslant \xi n/2k$ and for all $Q \in \binom{V(H)}{k}$,

$$\big|L_Q \cap F'\big| \geqslant \frac{\xi^2}{2}n - \frac{2\xi^2}{(2k-1)!(2k-1)!}n > \frac{\xi^2}{k}n. \tag{5}$$

Since F' consists of disjoint absorbing sets and each absorbing set is covered by a matching, $V(F')$ is covered by a matching M. Now let $W \subseteq V(H)\backslash V(F')$ be a set of at most $\xi^2 n$ vertices such that $|W| = k\ell$ for some $\ell \in \mathbb{N}$. We arbitrarily partition W into k-sets $Q_1, \ldots, Q_\ell$. Because of (5), we are able to absorb each Q_i with a different $2k$-set from $L_{Q_i} \cap F'$. Therefore $V(F') \cup W$ is covered by a matching, as desired.

5.2. Proof of Lemma 5.3

Given $\beta > 0$, we choose additional constants γ, α, ξ such that

$$0 < \xi \ll \alpha \ll \gamma \ll \beta. \tag{6}$$

Without loss of generality we may assume that $\beta \ll 1/r$. We also assume that n is sufficiently large.

Let $Q \subseteq V(H)$ be a $2r$-set. It is easy to see that if Q has at least $\gamma^3 n^{2r}$ absorbing $2r$-sets then Q has at least ξn^{4r} absorbing $4r$-sets. Indeed, let P be an absorbing $2r$-set for Q. Then $P \cup e$ is an absorbing $4r$-set for Q for any edge $e \in E(H - (P \cup Q))$. Since n is sufficiently large,

$$\big|E(H)\big| \geqslant \left(\frac{1}{2} - \alpha\right)\binom{n-r}{r} \times \frac{\binom{n}{r}}{\binom{2r}{r}} = \left(\frac{1}{2} - \alpha\right)\binom{n}{2r}.$$

Hence, as n is sufficiently large, there are at least

$$\left(\frac{1}{2} - \alpha\right)\binom{n}{2r} - 4r\binom{n}{2r-1} \geqslant \frac{n^{2r}}{4(2r)!}$$

edges in $H - (P \cup Q)$. Since an absorbing $4r$-set may be counted at most $\binom{4r}{2r}$ times when counting the number of P, e, there are at least

$$\gamma^3 n^{2r} \times \frac{n^{2r}}{4(2r)!} \times \frac{1}{\binom{4r}{2r}} \overset{(6)}{\geqslant} \xi n^{4r}$$

absorbing $4r$-sets for Q.

Therefore, in order to prove Lemma 5.3, it suffices to prove the following two claims.

Claim 5.5. *If either of the following cases holds, then we can find $\gamma^3 n^{2r}$ absorbing $2r$-sets or $\gamma^3 n^{4r}$ absorbing $4r$-sets for every $2r$-set $Q \in \binom{V(H)}{2r}$.*

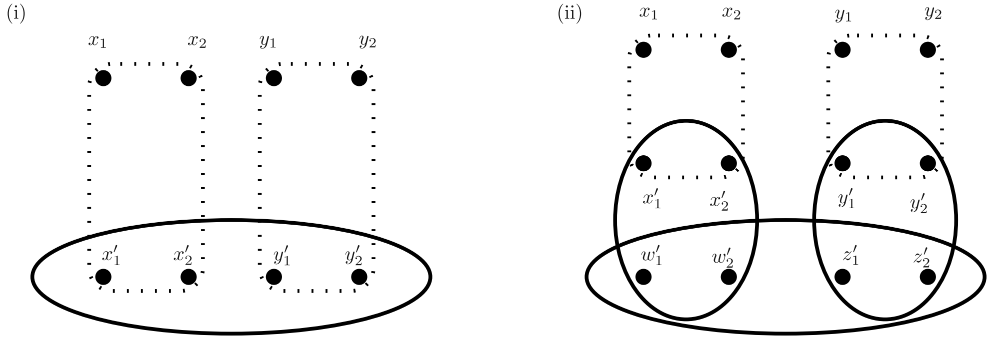

Fig. 1. The (i) absorbing $2r$-set and (ii) absorbing $4r$-set in the case when $r = 2$.

Case (a): *For any r-tuple* $\underline{a} \in \binom{V(H)}{r}$, *there are at least* $(\frac{1}{2} + \gamma)\binom{n}{r}$ *r-tuples* $\underline{b} \in \binom{V(H)}{r}$ *such that* $|N_H(\underline{a}) \cap N_H(\underline{b})| \geqslant \gamma\binom{n}{r}$.

Case (b): $|\{\underline{a} \in \binom{V(H)}{r}: d_H(\underline{a}) \geqslant (\frac{1}{2} + \gamma)\binom{n}{r}\}| \geqslant 2\gamma\binom{n}{r}$.

Claim 5.6. *If neither Case* (a) *or Case* (b) *holds, then* $G = K_{\frac{N}{2},\frac{N}{2}} \pm \beta N^2$ *or* $\overline{G} = K_{\frac{N}{2},\frac{N}{2}} \pm \beta N^2$.

Proof of Claim 5.5. Given a $2r$-set $Q = \{x_1, \dots, x_r, y_1, \dots, y_r\} \subseteq V(H)$, we will consider two types of absorbing sets for Q:

Absorbing $2r$-sets: These consist of a single edge $x'_1 \cdots x'_r y'_1 \cdots y'_r \in E(H)$ with the property that both $x_1 \cdots x_r x'_1 \cdots x'_r$ and $y_1 \cdots y_r y'_1 \cdots y'_r$ are edges of H.

Absorbing $4r$-sets: These consist of distinct vertices $x'_1, \dots, x'_r$, $y'_1, \dots, y'_r$, $w'_1, \dots, w'_r$, $z'_1, \dots, z'_r \in V(H)$ such that $x'_1 \cdots x'_r w'_1 \cdots w'_r$, $y'_1 \cdots y'_r z'_1 \cdots z'_r$ and $w'_1 \cdots w'_r z'_1 \cdots z'_r$ are edges in H. Furthermore, $x_1 \cdots x_r x'_1 \cdots x'_r$ and $y_1 \cdots y_r y'_1 \cdots y'_r$ are also edges of H (see Fig. 1).

Write $\underline{x} := x_1 \cdots x_r$ and $\underline{y} := y_1 \cdots y_r$. For any two (not necessarily disjoint) r-tuples $\underline{a}, \underline{b} \in \binom{V(H)}{r}$ we call $\underline{a}$ a *good r-tuple for* $\underline{b}$ if $|N_H(\underline{a}) \cap N_H(\underline{b})| \geqslant \gamma\binom{n}{r}/2$. We first observe that Q has at least $\gamma^3 n^{2r}$ absorbing $2r$-sets if there are

$$\text{at least } \frac{\gamma}{2}\binom{n}{r} \text{ good } r\text{-tuples in } N_H(\underline{x}) \text{ for } \underline{y},$$
$$\text{or at least } \frac{\gamma}{2}\binom{n}{r} \text{ good } r\text{-tuples in } N_H(\underline{y}) \text{ for } \underline{x}. \tag{7}$$

Indeed, assume that there are at least $\gamma\binom{n}{r}/2$ good r-tuples in $N_H(\underline{x})$ for $\underline{y}$. There are at most $r\binom{n}{r-1}$ r-tuples in $\binom{V(H)}{r}$ that contain at least one element from $\{y_1, \dots, y_r\}$. So there are at least $\gamma\binom{n}{r}/2 - r\binom{n}{r-1}$ r-tuples in $N_H(\underline{x})$ that are good for $\underline{y}$ and disjoint from $\underline{y}$. Let us pick such an r-tuple $\underline{x}' = (x'_1 \cdots x'_r)$. Thus, $|N_H(\underline{x}') \cap N_H(\underline{y})| \geqslant \gamma\binom{n}{r}/2$. We pick $\underline{y}' = (y'_1 \cdots y'_r) \in N_H(\underline{x}') \cap N_H(\underline{y})$ such that $\underline{y}'$ is disjoint from $\underline{x}$. Note that there are at least $\gamma\binom{n}{r}/2 - r\binom{n}{r-1}$ choices for $\underline{y}'$. Notice that the $2r$-set $\{x'_1, \dots, x'_r, y'_1, \dots, y'_r\}$ is an absorbing set for Q since $x'_1 \cdots x'_r y'_1 \cdots y'_r$, $x_1 \cdots x_r x'_1 \cdots x'_r$ and $y_1 \cdots y_r y'_1 \cdots y'_r$ are edges in H. Since an absorbing $2r$-set may be counted $\binom{2r}{r}$ times, this argument implies that there are at least

$$\left(\frac{\gamma}{2}\binom{n}{r} - r\binom{n}{r-1}\right)^2 \frac{1}{\binom{2r}{r}} \geqslant \gamma^3 n^{2r}$$

absorbing $2r$-sets for Q. We reach the same conclusion when there are at least $\gamma\binom{n}{r}/2$ good r-tuples in $N_H(\underline{y})$ for $\underline{x}$.

Now assume that Case (a) holds. This implies that there are at least $(\frac{1}{2}+\gamma)\binom{n}{r}$ good r-tuples for $\underline{x}$. By the minimum r-degree condition, $d_H(\underline{y}) \geqslant (\frac{1}{2}-\alpha)\binom{n}{r}$. So there are at least $(\gamma-\alpha)\binom{n}{r} \geqslant \gamma\binom{n}{r}/2$ r-tuples in $N_H(\underline{y})$ that are good for $\underline{x}$. Thus (7) holds and consequently Q has at least $\gamma^3 n^{2r}$ absorbing $2r$-sets.

Next assume Case (b) holds. Let $\Lambda := \{\underline{a} \in \binom{V(H)}{r}\colon\ d_H(\underline{a}) \geqslant (\frac{1}{2}+\gamma)\binom{n}{r}\}$. So by assumption, $|\Lambda| \geqslant 2\gamma\binom{n}{r}$. We also assume that (7) fails (otherwise we are done). Every r-tuple $\underline{a} \in \Lambda$ is good for arbitrary $\underline{b} \in \binom{V(H)}{r}$ because $|N_H(\underline{a}) \cap N_H(\underline{b})| \geqslant (\gamma-\alpha)\binom{n}{r} \geqslant \gamma\binom{n}{r}/2$. Hence $|\Lambda \cap N_H(\underline{y})| < \gamma\binom{n}{r}/2$. On the other hand, less than $\gamma\binom{n}{r}/2$ r-tuples in $N_H(\underline{x})$ are good for $\underline{y}$ and consequently at least $(\frac{1}{2}-\alpha)\binom{n}{r} - \frac{\gamma}{2}\binom{n}{r}$ r-tuples $\underline{x}' \in N_H(\underline{x})$ satisfy $|N_H(\underline{x}') \cap N_H(\underline{y})| < \gamma\binom{n}{r}/2$. We pick such an r-tuple $\underline{x}'$ that is disjoint from $\underline{y}$; there are at least $(\frac{1}{2}-\alpha)\binom{n}{r} - \frac{\gamma}{2}\binom{n}{r} - r\binom{n}{r-1} \geqslant (\frac{1}{2}-\gamma)\binom{n}{r}$ r-tuples with this property. Since

$$\left|N_H(\underline{x}') \cup N_H(\underline{y})\right| \geqslant 2\left(\frac{1}{2}-\alpha\right)\binom{n}{r} - \frac{\gamma}{2}\binom{n}{r} \geqslant \binom{n}{r} - \gamma\binom{n}{r},$$

it follows that

$$\begin{aligned}\left|\Lambda \cap N_H(\underline{x}')\right| &\geqslant |\Lambda| - \left|\Lambda \cap N_H(\underline{y})\right| - \left|\overline{\left(N_H(\underline{x}') \cup N_H(\underline{y})\right)}\right| \\ &\geqslant 2\gamma\binom{n}{r} - \frac{\gamma}{2}\binom{n}{r} - \gamma\binom{n}{r} = \frac{\gamma}{2}\binom{n}{r}. \end{aligned} \tag{8}$$

Now pick any $\underline{w}' \in \Lambda \cap N_H(\underline{x}')$ that is disjoint from Q. (Note there are at least $\frac{\gamma}{2}\binom{n}{r} - 2r\binom{n}{r-1} \geqslant \frac{\gamma}{3}\binom{n}{r}$ choices for $\underline{w}'$.) Next pick an r-tuple $\underline{y}' \in N_H(\underline{y})$ such that $\underline{y}'$ is disjoint from $\underline{x}$, $\underline{x}'$ and $\underline{w}'$. (There are at least $(\frac{1}{2}-\alpha)\binom{n}{r} - 6r\binom{n}{r-1} \geqslant (\frac{1}{2}-\gamma)\binom{n}{r}$ choices for $\underline{y}'$ here.) By the definition of Λ, there are at least $(\gamma-\alpha)\binom{n}{r}$ pairs in $N_H(\underline{w}') \cap N_H(\underline{y}')$. We pick $\underline{z}' \in N_H(\underline{w}') \cap N_H(\underline{y}')$ such that $\underline{z}'$ is disjoint from $\underline{x}$, $\underline{y}$ and $\underline{x}'$. (There are at least $(\gamma-\alpha)\binom{n}{r} - 6r\binom{n}{r-1} \geqslant \gamma\binom{n}{r}/2$ choices for $\underline{z}'$ here.)

Let S denote the $4r$-set consisting of the vertices contained in $\underline{x}'$, $\underline{y}'$, $\underline{w}'$ and $\underline{z}'$. By the choice of $\underline{x}'$, $\underline{y}'$, $\underline{w}'$ and $\underline{z}'$, S is an absorbing $4r$-set for Q.

In summary, there are at least $(\frac{1}{2}-\gamma)\binom{n}{r}$ choices for $\underline{x}'$, at least $\frac{\gamma}{3}\binom{n}{r}$ choices for $\underline{w}'$, at least $(\frac{1}{2}-\gamma)\binom{n}{r}$ choices for $\underline{y}'$ and at least $\frac{\gamma}{2}\binom{n}{r}$ choices for $\underline{z}'$. Since each absorbing $4r$-set may be counted $\binom{4r}{r}\binom{3r}{r}\binom{2r}{r}$ times, there are at least

$$\left[\left(\frac{1}{2}-\gamma\right)\binom{n}{r}\right]^2 \frac{\gamma}{3}\binom{n}{r}\frac{\gamma}{2}\binom{n}{r} \times \frac{1}{\binom{4r}{r}\binom{3r}{r}\binom{2r}{r}} \overset{(6)}{\geqslant} \gamma^3 n^{4r}$$

absorbing $4r$-sets for Q, as desired. □

Claim 5.6 follows from the following lemma (by letting $G = G(H)$) immediately.

Lemma 5.7. *For any $\beta > 0$, there exist $\gamma > 0$ and $n_0 \in \mathbb{N}$ such that following holds. Let $G = (V, E)$ be a graph on an even $N \geqslant n_0$ number of vertices such that $\delta(G) \geqslant (1/2-\gamma)N$. In addition, G satisfies*

(a) *There exists $a \in V$ such that at most $(\frac{1}{2}+\gamma)N$ vertices $b \in V$ satisfy $|N(a) \cap N(b)| \geqslant \gamma N$.*
(b) *$|\{v \in V\colon\ d(v) \geqslant (\frac{1}{2}+\gamma)N\}| < 2\gamma N$.*

Then either $G = K_{N/2,N/2} \pm \beta N^2$ or $\overline{G} = K_{N/2,N/2} \pm \beta N^2$.

Proof. Let $A := N(a)$ and $B := \{b \in V\colon\ |A \cap N(b)| < \gamma N\}$. Then $|A| \geqslant (\frac{1}{2}-\gamma)N$ and $|B| \geqslant (\frac{1}{2}-\gamma)N$.

We also need an upper bound on $|A|$. Fix $b \in B$. Since $|N(b)| \geqslant (\frac{1}{2} - \gamma)N$, we have

$$|A| + \left(\frac{1}{2} - \gamma\right)N \leqslant |A| + \left|N(b)\right| = \left|A \cup N(b)\right| + \left|A \cap N(b)\right| \leqslant N + \gamma N,$$

which gives $|A| \leqslant (\frac{1}{2} + 2\gamma)N$.

Let $e(A, B)$ denote the number of *ordered* pairs a, b such that $a \in A$, $b \in B$, and $ab \in E$. (Therefore, if $a, b \in A \cap B$ and $ab \in E$ then ab counts twice to the value of $e(A, B)$.) By the definition of B, we have $e(A, B) \leqslant \gamma N|B|$. Since $\delta(G) \geqslant (\frac{1}{2} - \gamma)N$ we have that

$$e(\bar{A}, B) \geqslant (1/2 - 2\gamma)N|B|, \tag{9}$$

where, as usual $\bar{A} := V \setminus A$. Next we show that $e(\bar{A}, \bar{B})$ is very small.

Claim 5.8. $e(\bar{A}, \bar{B}) \leqslant 8\sqrt{\gamma}|\bar{A}||\bar{B}|$.

Proof. Assume for a contradiction that the claim is false. Set $A_1 := \{x \in \bar{A}\colon\ d(x, \bar{B}) \geqslant 4\sqrt{\gamma}|\bar{B}|\}$. By assumption

$$8\sqrt{\gamma}|\bar{A}||\bar{B}| \leqslant e(\bar{A}, \bar{B}) \leqslant |A_1||\bar{B}| + 4\sqrt{\gamma}|\bar{B}||\bar{A}|,$$

which gives that $|A_1| \geqslant 4\sqrt{\gamma}|\bar{A}|$. By (9), as $|\bar{A}| \leqslant (\frac{1}{2} + \gamma)N$, we derive that

$$e(\bar{A}, B) \geqslant \left(\frac{1}{2} - 2\gamma\right)N|B| \geqslant (1 - 6\gamma)\left(\frac{1}{2} + \gamma\right)N|B| \geqslant (1 - 6\gamma)|\bar{A}||B|. \tag{10}$$

Let $A_2 := \{x \in \bar{A}\colon\ d(x, B) \geqslant (1 - 3\sqrt{\gamma})|B|\}$. We claim that $|A_2| \geqslant (1 - 3\sqrt{\gamma})|\bar{A}|$. Indeed, for convenience, consider $\bar{e}(\bar{A}, B)$, the number of ordered pairs a, b such that $a \in \bar{A}$, $b \in B$, and $ab \notin E$. If $|A_2| < (1 - 3\sqrt{\gamma})|\bar{A}|$, then $\bar{e}(\bar{A}, B) \geqslant 3\sqrt{\gamma}|\bar{A}|3\sqrt{\gamma}|B| = 9\gamma|\bar{A}||B|$, contradicting (10).

Let $A_0 := A_1 \cap A_2$. We have $|A_0| \geqslant (4\sqrt{\gamma} - 3\sqrt{\gamma})|\bar{A}|$. Since $|\bar{A}| \geqslant N/3$ and $\gamma \leqslant 1/36$, we derive that $|A_0| \geqslant \sqrt{\gamma}N/3 \geqslant 2\gamma N$. For every $x \in A_0$, we have

$$\begin{aligned} d(x) &= d(x, B) + d(x, \bar{B}) \\ &\geqslant (1 - 3\sqrt{\gamma})|B| + 4\sqrt{\gamma}|\bar{B}| = (1 - 7\sqrt{\gamma})|B| + 4\sqrt{\gamma}N \\ &\geqslant \left(\frac{1}{2} - \frac{7}{2}\sqrt{\gamma} + 4\sqrt{\gamma} - \gamma\right)N \geqslant \left(\frac{1}{2} + \gamma\right)N. \end{aligned}$$

(The penultimate inequality follows since $|B| \geqslant (\frac{1}{2} - \gamma)N$.) This is a contradiction to the assumption (b). □

Now we go back to the proof of Lemma 5.7. We separate the cases by whether $|\overline{A \cup B}| \leqslant \gamma^{1/4}N$ or not.

First assume that $|\overline{A \cup B}| \leqslant \gamma^{1/4}N$. Since $(\frac{1}{2} - \gamma)N \leqslant |A| \leqslant (\frac{1}{2} + 2\gamma)N$, we can find a set $V_1 \subseteq V(G)$ of size $N/2$ such that $|V_1 \triangle A| \leqslant 2\gamma N$. Let $V_2 := V(G) \setminus V_1$. Thus,

$$\begin{aligned} e(V_1, V_2) &\leqslant e(A \cap V_1, B \cap V_2) + e(V_1 \setminus A, V_2) + e(V_1, V_2 \cap A) + e\big(V_1, V_2 \setminus (A \cup B)\big) \\ &\leqslant e(A, B) + \left(|V_1 \setminus A|\frac{N}{2} + |V_2 \cap A|\frac{N}{2}\right) + |\overline{A \cup B}|\frac{N}{2} \\ &\leqslant \gamma N|B| + 2\gamma N\frac{N}{2} + \gamma^{1/4}N\frac{N}{2} \leqslant \gamma^{1/4}N^2. \end{aligned}$$

Since $\delta(G) \geqslant (\frac{1}{2} - \gamma)N$, we derive that for $i = 1, 2$,

$$2e(V_i)=e(V_i,V_i)\geqslant \frac{N}{2}\left(\frac{1}{2}-\gamma\right)N-\gamma^{1/4}N^2\geqslant \frac{N^2}{4}-2\gamma^{1/4}N^2.$$

Thus, we can delete at most $\gamma^{1/4}N^2$ edges between V_1 and V_2 and add at most $\gamma^{1/4}N^2$ edges in each of V_1 and V_2 to turn G into a graph consisting of two vertex-disjoint cliques; one on V_1, the other on V_2. In other words, $\overline{G}=K_{N/2,N/2}\pm 3\gamma^{1/4}N^2$.

Now assume that $|\overline{A\cup B}|\geqslant \gamma^{1/4}N$. Then $|B\setminus A|\leqslant |\bar{A}|-\gamma^{1/4}N\leqslant (\frac{1}{2}+\gamma-\gamma^{1/4})N$. By Claim 5.8, $e(B\setminus A,\overline{A\cup B})\leqslant e(\bar{A},\bar{B})\leqslant 8\sqrt{\gamma}|\bar{A}||\bar{B}|\leqslant 8\sqrt{\gamma}N^2$. Together with $e(B\setminus A,A)\leqslant e(B,A)\leqslant |B|\gamma N\leqslant \gamma N^2$, it gives that

$$\begin{aligned}|B\setminus A|\left(\frac{1}{2}-\gamma\right)N&\leqslant e(B\setminus A,V)\leqslant e(B\setminus A,B\setminus A)+e(B\setminus A,A)+e(B\setminus A,\overline{A\cup B})\\&\leqslant |B\setminus A|\left(\frac{1}{2}+\gamma-\gamma^{1/4}\right)N+\gamma N^2+8\sqrt{\gamma}N^2.\end{aligned}$$

This implies that $(\gamma^{1/4}-2\gamma)N|B\setminus A|\leqslant 9\sqrt{\gamma}N^2$ and so $|B\setminus A|\leqslant 10\gamma^{1/4}N$. Similarly we can show that $|A\setminus B|\leqslant 10\gamma^{1/4}N$. Now pick a set $V_1\subseteq V(G)$ of size $N/2$ such that $|V_1\cap A|$ is maximized. Thus, $|V_1\setminus A|\leqslant \gamma N$. Then, as $e(A\cap B,A)\leqslant e(B,A)\leqslant \gamma N^2$, we have

$$\begin{aligned}e(V_1)&\leqslant e(V_1\setminus A,V_1)+e(A)\leqslant e(V_1\setminus A,V_1)+e(A\cap B,A)+e(A\setminus B)\\&\leqslant \gamma\frac{N^2}{2}+\gamma N^2+\frac{1}{2}\left(10\gamma^{1/4}N\right)^2\leqslant 52\sqrt{\gamma}N^2.\end{aligned}$$

Let $V_2:=V(G)\setminus V_1$. Since $\delta(G)\geqslant (\frac{1}{2}-\gamma)N$, we have

$$e(V_1,V_2)\geqslant \frac{N}{2}\left(\frac{1}{2}-\gamma\right)N-52\sqrt{\gamma}N^2\geqslant \frac{N^2}{4}-53\sqrt{\gamma}N^2.$$

Further, by Claim 5.8 and since $|A\cap V_2|=|A\setminus V_1|\leqslant 2\gamma N$,

$$\begin{aligned}e(V_2)&\leqslant e(A\cap V_2,V_2)+e(\bar{A})\leqslant e(A\cap V_2,V_2)+e(\bar{A}\cap\bar{B},\bar{A})+e(\bar{A}\cap B)\\&\leqslant |A\cap V_2|\frac{N}{2}+e(\bar{B},\bar{A})+e(B\setminus A)\\&\leqslant 2\gamma\frac{N^2}{2}+8\sqrt{\gamma}N^2+\frac{1}{2}\left(10\gamma^{1/4}N\right)^2\leqslant 59\sqrt{\gamma}N^2.\end{aligned}$$

Hence, we can add at most $53\sqrt{\gamma}N^2$ edges between V_1 and V_2 and delete at most $52\sqrt{\gamma}N^2+59\sqrt{\gamma}N^2$ edges inside V_1 and V_2 to turn G into a complete balanced bipartite graph. In other words, $G=K_{N/2,N/2}\pm 164\sqrt{\gamma}N^2$.

Since $\gamma\ll\beta$ we conclude that either $G=K_{\frac{N}{2},\frac{N}{2}}\pm\beta N^2$ or $\overline{G}=K_{\frac{N}{2},\frac{N}{2}}\pm\beta N^2$, as desired. □

This completes the proof of Lemma 5.3.

5.3. Proof of Lemma 5.4

We need the following structural result and prove it by applying Theorems 2.3 and 2.4.

Lemma 5.9 *(Structure Lemma). For any $\eta>0$ and $r\in\mathbb{N}$, there exist $\delta>0$ and $n_0\in\mathbb{N}$ such that the following holds. Suppose that K is a complete $2r$-uniform hypergraph on $n\geqslant n_0$ vertices whose edge set is partitioned into two sets R (red) and B (blue). Let Ω denote the collection of all $4r$-subsets $S\subseteq V(K)$ such that there exists a partition of $S=P_1\cup P_2\cup P_3\cup P_4$ where $|P_i|=r$ for all $1\leqslant i\leqslant 4$ and such that exactly one of the four $2r$-sets $P_1\cup P_2$, $P_2\cup P_3$, $P_3\cup P_4$, $P_4\cup P_1$ is in R or B (the other three are in the other color class). Suppose that*

(i) $|R|, |B| \geqslant (\frac{1}{2} - \delta)\binom{n}{2r}$ *and*;
(ii) $|\Omega| \leqslant \delta n^{4r}$.

Then either $K[R] = \mathcal{B}_{n,2r} \pm \eta n^{2r}$ *or* $K[B] = \mathcal{B}_{n,2r} \pm \eta n^{2r}$.

Proof. Given $\eta > 0$ define additional constants δ, δ_1, ε such that

$$0 < \delta \ll \delta_1 \ll \varepsilon \ll \eta, 1/r. \tag{11}$$

Let C_4^{2r} denote the expanded $2r$-uniform 4-cycle. That is, C_4^{2r} consists of four disjoint sets P_1, P_2, P_3, P_4 of vertices of size r, and the edges $P_1 \cup P_2$, $P_2 \cup P_3$, $P_3 \cup P_4$, $P_4 \cup P_1$. We call a 2-colored copy of C_4^{2r} *bad* if exactly one of its four edges is in R or B (and the other three are in the other color class). A $4r$-set $S \in \binom{V(K)}{4r}$ is *bad* if $K[S]$ contains a bad C_4^{2r}. Thus (ii) says that the number of bad $4r$-sets is at most δn^{4r}.

Observe that if T_1 is red copy of $\mathcal{C}_3^{2r}$ and T_2 is a blue copy of $\mathcal{C}_3^{2r}$ such that T_1 and T_2 are vertex-disjoint, then there exists at least one bad copy of C_4^{2r} whose vertex set is contained in $V(T_1) \cup V(T_2)$: Let $\underline{a}$, $\underline{b}$, $\underline{c}$ denote the r-tuples in $V(T_1)$ such that $\underline{a} \cup \underline{b}$, $\underline{b} \cup \underline{c}$, $\underline{c} \cup \underline{a} \in E(T_1)$. Define $\underline{x}, \underline{y}, \underline{z} \subseteq V(T_2)$ analogously. If there is a $\underline{v} \in \{\underline{a}, \underline{b}, \underline{c}\}$ such that $\underline{v} \cup \underline{w}_1 \in R$ and $\underline{v} \cup \underline{w}_2 \in B$ for some $\underline{w}_1, \underline{w}_2 \in \{\underline{x}, \underline{y}, \underline{z}\}$, then we obtained our desired bad copy of C_4^{2r}. For example, if $\underline{a} \cup \underline{x} \in B$ and $\underline{a} \cup \underline{z} \in R$, then the edges $\underline{a} \cup \underline{x}, \underline{x} \cup \underline{y}, \underline{y} \cup \underline{z} \in B$ and $\underline{a} \cup \underline{z} \in R$ induce a bad copy of C_4^{2r}. Similarly, if there exists $\underline{v} \in \{\underline{x}, \underline{y}, \underline{z}\}$ such that $\underline{v} \cup \underline{w}_1 \in R$ and $\underline{v} \cup \underline{w}_2 \in B$ for some $\underline{w}_1, \underline{w}_2 \in \{\underline{a}, \underline{b}, \underline{c}\}$, then we obtain a bad copy of C_4^{2r}. If neither of these two cases holds, then all the edges of the form $\underline{v} \cup \underline{w}$ receive the same color, say red (where $\underline{v} \in \{\underline{a}, \underline{b}, \underline{c}\}$ and $\underline{w} \in \{\underline{x}, \underline{y}, \underline{z}\}$). But then $\underline{a} \cup \underline{b}, \underline{a} \cup \underline{x}, \underline{b} \cup \underline{y} \in R$ and $\underline{x} \cup \underline{y} \in B$ induce a bad copy of C_4^{2r}.

Assume for a contradiction that K contains at least $\delta_1 n^{3r}$ red copies of $\mathcal{C}_3^{2r}$ and at least $\delta_1 n^{3r}$ blue copies of $\mathcal{C}_3^{2r}$. For each red copy T of $\mathcal{C}_3^{2r}$ in K, there are at most $3r\binom{n}{3r-1}\binom{3r}{2r}\binom{2r}{r}$ blue copies of $\mathcal{C}_3^{2r}$ in K which contain at least one vertex from $V(T)$. (The $\binom{3r}{2r}\binom{2r}{r}$ term comes from the fact that, given any $3r$-set $V \subseteq V(K)$, there are $\binom{3r}{2r}\binom{2r}{r}$ copies of $\mathcal{C}_3^{2r}$ in $K[V]$.) So there are at least $\delta_1 n^{3r} - 3r\binom{n}{3r-1}\binom{3r}{2r}\binom{2r}{r} \geqslant \delta_1 n^{3r}/2$ blue $\mathcal{C}_3^{2r}$ in K that are disjoint from T. Hence, there are at least $\delta_1^2 n^{6r}/2$ pairs T_1, T_2 of vertex-disjoint copies of $\mathcal{C}_3^{2r}$ such that T_1 is red and T_2 is blue.

Now consider any bad copy C of C_4^{2r}. There are $\binom{n-4r}{2r}$ $6r$-subsets of $V(K)$ which contain $V(C)$. For each such $6r$-set S, there are $\binom{6r}{3r}\binom{3r}{2r}^2\binom{2r}{r}^2$ pairs T', T'' of vertex-disjoint copies of $\mathcal{C}_3^{2r}$ in K such that $V(T') \cup V(T'') = S$. Together, this all implies that the number of bad $4r$-sets is at least

$$\frac{\delta_1^2}{2} n^{6r} \times \frac{1}{\binom{n-4r}{2r}\binom{6r}{3r}\binom{3r}{2r}^2\binom{2r}{r}^2} \overset{(11)}{>} \delta n^{4r},$$

a contradiction to (ii) as desired.

Thus, there are less than $\delta_1 n^{3r}$ blue $\mathcal{C}_3^{2r}$ in K or less than $\delta_1 n^{3r}$ red $\mathcal{C}_3^{2r}$ in K. Without loss of generality we assume there are less than $\delta_1 n^{3r}$ blue $\mathcal{C}_3^{2r}$ in K. So (i) implies that $K[B]$ is an n-vertex $2r$-uniform hypergraph with at least $(1/2 - \delta)\binom{n}{2r}$ edges and less than $\delta_1 n^{3r}$ copies of $\mathcal{C}_3^{2r}$. To show $K[B] = \mathcal{B}_{n,2r} \pm \eta n^{2r}$, we will use Theorems 2.3 and 2.4.

Since $\delta_1 \ll \varepsilon$, Theorem 2.3 implies that we may remove at most εn^{2r} edges from $K[B]$ to obtain a $\mathcal{C}_3^{2r}$-free hypergraph $K'[B]$. As $\varepsilon \ll \eta$ and

$$e\big(K'[B]\big) \geqslant \left(\frac{1}{2} - \delta\right)\binom{n}{2r} - \varepsilon n^{2r} \geqslant \left(\frac{1}{2} - \sqrt{\varepsilon}\right)\binom{n}{2r}$$

we may apply Theorem 2.4 to obtain that $K'[B] = \mathcal{B}_{n,2r} \pm \eta n^{2r}/2$. Consequently, $K[B] = \mathcal{B}_{n,2r} \pm \eta n^{2r}$, as desired. □

Given two disjoint vertex sets R and B we define $K_{R,B}$ to be the complete bipartite graph with vertex classes R and B.

Proof of Lemma 5.4. Given $\varepsilon > 0$ we define additional constants β, η such that

$$0 < \beta \ll \eta \ll \varepsilon, 1/r. \tag{12}$$

Further assume that n is sufficiently large.

By assumption, either $G = K_{\frac{N}{2},\frac{N}{2}} \pm \beta N^2$ or $\overline{G} = K_{\frac{N}{2},\frac{N}{2}} \pm \beta N^2$, where $N := \binom{n}{2r}$. It suffices to show that if $G = K_{\frac{N}{2},\frac{N}{2}} \pm 2\beta N^2$ then $H = \mathcal{B}_{n,4r} \pm \varepsilon n^{4r}$. Indeed, the edge set of $\overline{G}$ contain the edge set of $G(\overline{H})$ and all the pairs of intersecting $2r$-subsets of $V(H)$. Since there are $O(n^{4r-1})$ pairs of intersecting $2r$-subsets of $V(H)$, if $\overline{G} = K_{\frac{N}{2},\frac{N}{2}} \pm \beta N^2$, then $G(\overline{H}) = K_{\frac{N}{2},\frac{N}{2}} \pm 2\beta N^2$, which implies that $\overline{H} = \mathcal{B}_{n,4r} \pm \varepsilon n^{4r}$, equivalently, $H = \overline{\mathcal{B}}_{n,4r} \pm \varepsilon n^{4r}$, as desired.

Assume that $G = K_{\frac{N}{2},\frac{N}{2}} \pm 2\beta N^2$, namely, there is partition R, B of $V(G) = \binom{V(H)}{2r}$ such that $|R| = |B| = N/2$ and $|E(G) \triangle E(K_{R,B})| \leqslant 2\beta N^2$. Let $K(H)$ denote the complete $2r$-uniform hypergraph whose vertex set is $V(H)$. Since R, B is a partition of $\binom{V(H)}{2r}$ we may view R and B as the color classes of a 2-coloring of the edge set of $K(H)$. Let $K[R]$ denote the spanning subhypergraph of $K(H)$ induced by the edges of R. Define $K[B]$ analogously.

Given a $4r$-set Q of vertices from $V(H)$ we say that Q is *bad* if there exists a partition of $Q = P_1 \cup P_2 \cup P_3 \cup P_4$ where $|P_i| = r$ for all $1 \leqslant i \leqslant 4$ and such that exactly one of the four $2r$-sets $P_1 \cup P_2$, $P_2 \cup P_3$, $P_3 \cup P_4$, $P_4 \cup P_1$ receives one of the colors. First assume that this color is B. Without loss of generality, assume that $P_1 \cup P_2, P_2 \cup P_3, P_3 \cup P_4 \in R$ and $P_4 \cup P_1 \in B$. If $Q \in E(H)$, then $\{P_1 \cup P_2, P_3 \cup P_4\} \in E(G) \cap \binom{R}{2}$. On the other hand, if $Q \notin E(H)$, then $\{P_4 \cup P_1, P_2 \cup P_3\} \in E(\overline{G}) \cap E(K_{R,B})$. Therefore, one of $\{P_1 \cup P_2, P_3 \cup P_4\}$ and $\{P_4 \cup P_1, P_2 \cup P_3\}$ is in $E(G) \triangle E(K_{R,B})$. The same holds when exactly one of $P_1 \cup P_2$, $P_2 \cup P_3$, $P_3 \cup P_4$, $P_4 \cup P_1$ is colored R. Clearly two distinct bad $4r$-sets lead to two different members of $E(G) \triangle E(K_{R,B})$. Since $|E(G) \triangle E(K_{R,B})| \leqslant 2\beta N^2$, the number of bad $4r$-sets is at most $2\beta N^2$.

Since $\beta \ll \eta$, we may apply Lemma 5.9 to $K(H)$ to obtain that either $K[R] = \mathcal{B}_{n,2r} \pm \eta n^{2r}$ or $K[B] = \mathcal{B}_{n,2r} \pm \eta n^{2r}$. Since the roles of $K[R]$ and $K[B]$ are interchangeable, we may assume that $K[R] = \mathcal{B}_{n,2r} \pm \eta n^{2r}$. Let X, Y denote a partition of $V(H)$ such that $|E(K[R]) \triangle E(\mathcal{B}_{n,2r}[X,Y])| \leqslant \eta n^{2r}$. We now use the structural information we have about G and $K[R]$ to piece together that of H.

Claim 5.10. $H = \mathcal{B}_{n,4r} \pm \varepsilon n^{4r}$.

Recall that given a $2r$-tuple $\underline{x} \in \binom{V(H)}{2r}$ we say that $\underline{x}$ is *even* if $\underline{x}$ contains an even number of elements from X (and so an even number of elements from Y). Otherwise, we say that $\underline{x}$ is *odd*. Thus, the edge set of $\mathcal{B}_{n,2r}[X,Y]$ is precisely the set of odd $2r$-tuples.

Our ultimate aim is to show that

$$\big|E(H) \triangle E\big(\mathcal{B}_{n,4r}[X,Y]\big)\big| \leqslant \varepsilon n^{4r}. \tag{13}$$

First we show that $|E(H) \backslash E(\mathcal{B}_{n,4r}[X,Y])| \leqslant \varepsilon n^{4r}/2$. Consider any $4r$-tuple Q from $E(H) \backslash E(\mathcal{B}_{n,4r}[X,Y])$. Since $Q \notin E(\mathcal{B}_{n,4r}[X,Y])$ (thus $|Q \cap X|$ is even), Q can be partitioned into $2r$-tuples $\underline{x}$, $\underline{y}$ such that both $\underline{x}$ and $\underline{y}$ are even. (For example, if $|Q \cap X| \geqslant 2r$, then let $\underline{x}$ be a $2r$-subset of $Q \cap X$; otherwise let $\underline{x}$ be a $2r$-subset of $Q \cap Y$. Since $|Q \cap X|$ is even, $\underline{y}$ is even.) As $Q \in E(H)$ we have that $\{\underline{x}, \underline{y}\} \in E(G)$. Thus,

$$\big|E(H) \backslash E\big(\mathcal{B}_{n,4r}[X,Y]\big)\big| \leqslant |\Sigma|,$$

where Σ is the set of all disjoint pairs of $2r$-tuples $\underline{w}, \underline{z} \in \binom{V(H)}{2r}$ such that $\underline{w}$ and $\underline{z}$ are even and $\{\underline{w}, \underline{z}\} \in E(G)$.

Since $K[R] = \mathcal{B}_{n,2r}[X,Y] \pm \eta n^{2r}$, there are at most $\binom{\eta n^{2r}}{2} \leqslant \eta^2 n^{4r}$ pairs $\{\underline{w}, \underline{z}\} \in \Sigma$ such that $\underline{w}, \underline{z} \in R$. Similarly, there are at most $\eta n^{2r}|B| \leqslant \eta n^{4r}$ pairs $(\underline{w}, \underline{z}) \in \Sigma$ such that $\underline{w} \in B$ and $\underline{z} \in R$.

Given any pair $(\underline{w}, \underline{z}) \in \Sigma$ such that $\underline{w}, \underline{z} \in B$, by definition of Σ, we have $\{\underline{w}, \underline{z}\} \in E(G)$. However, $G = K_{R,B} \pm 2\beta N^2$, so there are most $2\beta N^2 \leqslant 2\beta n^{4r}$ such pairs in Σ. Together, this all implies that $|\Sigma| \leqslant (\eta^2 + \eta + 2\beta)n^{4r} \leqslant \varepsilon n^{4r}/2$. So indeed, $|E(H)\backslash E(\mathcal{B}_{n,4r}[X,Y])| \leqslant \varepsilon n^{4r}/2$.

Next we show that $|E(\mathcal{B}_{n,4r}[X,Y])\backslash E(H)| \leqslant \varepsilon n^{4r}/2$. Consider any $4r$-tuple Q from $E(\mathcal{B}_{n,4r}[X,Y])\backslash E(H)$. Since $Q \in E(\mathcal{B}_{n,4r}[X,Y])$, Q can be partitioned into $2r$-tuples $\underline{x}$, $\underline{y}$ such that $\underline{x}$ is even and $\underline{y}$ is odd. (For example, if $|Q \cap X| \geqslant 2r$, then let $\underline{x}$ be a $2r$-subset of $Q \cap X$; otherwise let $\underline{x}$ be a $2r$-subset of $Q \cap Y$. Since $|Q \cap X|$ is odd, $\underline{y}$ is odd.) As $Q \notin E(H)$ we have that $\{\underline{x}, \underline{y}\} \in E(\overline{G})$. Thus,

$$\left|E\left(\mathcal{B}_{n,4r}[X,Y]\right)\backslash E(H)\right| \leqslant |\Gamma|,$$

where Γ is the set of all disjoint pairs of $2r$-tuples $\underline{w}, \underline{z} \in \binom{V(H)}{2r}$ such that $\underline{w}$ is even, $\underline{z}$ is odd and $\{\underline{w}, \underline{z}\} \in E(\overline{G})$.

Since $K[R] = \mathcal{B}_{n,2r}[X,Y] \pm \eta n^{2r}$, we have that $K[B] = \overline{\mathcal{B}}_{n,2r}[X,Y] \pm \eta n^{2r}$. Thus, there are at most $\eta n^{2r}\binom{n}{2r} \leqslant \eta n^{4r}$ pairs $\{\underline{w}, \underline{z}\} \in \Gamma$ such that $\underline{w}$ is even and $\underline{w} \in R$. Similarly, there are at most ηn^{4r} pairs $\{\underline{w}, \underline{z}\} \in \Gamma$ such that $\underline{z}$ is odd and $\underline{z} \in B$. Given any pair $\{\underline{w}, \underline{z}\} \in \Gamma$ such that $\underline{w} \in R$ is odd and $\underline{z} \in B$ is even, by definition of Γ, $\{\underline{w}, \underline{z}\} \in E(\overline{G})$. However, $\overline{G} = \overline{K}_{R,B} \pm 2\beta N^2$, so there are most $2\beta N^2 \leqslant 2\beta n^{4r}$ such pairs in Γ. Together this all implies that $|\Gamma| \leqslant (2\eta + \eta + 2\beta)n^{4r} \leqslant \varepsilon n^{4r}/2$. So indeed, $|E(\mathcal{B}_{n,4r}[X,Y])\backslash E(H)| \leqslant \varepsilon n^{4r}/2$. Therefore (13) is satisfied, as desired. □

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Appendix A

In this section we prove Theorem 1.2. Because of Theorem 1.1, it suffices to prove the following fact.

Fact A.1. *For all $n \geqslant 12$ divisible by 4,*

$$\delta(n,4,2) \leqslant \frac{n^2}{4} - \frac{5n}{4} - \frac{\sqrt{n-3}}{2} + \frac{3}{2}.$$

Furthermore, there are infinitely many values of n such that the following holds:

- $\delta(n,4,2) = \delta_2(\overline{\mathcal{B}}_{n,4}(t)) = \frac{n^2}{4} - \frac{5n}{4} - \frac{\sqrt{n-3}}{2} + \frac{3}{2}$ *for some t;*
- $\overline{\mathcal{B}}_{n,4}(t)$ *does not contain a perfect matching.*

Proof. Suppose that $n \in \mathbb{N}$ is divisible by 4 and let t be an integer such that $0 \leqslant t < n/2$. Denote the vertex classes of $\mathcal{B}_{n,4}(t)$ by A and B. Therefore $|A| = n/2 + t$ and $|B| = n/2 - t$.

Given distinct $v_1, v_2 \in A$,

$$d_{\mathcal{B}_{n,4}(t)}(v_1 v_2) = (n/2 + t - 2)(n/2 - t) = \frac{n^2}{4} - n - t^2 + 2t.$$

Given distinct $w_1, w_2 \in B$,

$$d_{\mathcal{B}_{n,4}(t)}(w_1 w_2) = (n/2 + t)(n/2 - t - 2) = \frac{n^2}{4} - n - t^2 - 2t.$$

Given any $v_1 \in A$ and $w_1 \in B$,

$$d_{\mathcal{B}_{n,4}(t)}(v_1 w_1) = \binom{n/2+t-1}{2} + \binom{n/2-t-1}{2}$$
$$= \frac{1}{2}\big[(n/2+t-1)(n/2+t-2) + (n/2-t-1)(n/2-t-2)\big]$$
$$= \frac{n^2}{4} - \frac{3n}{2} + t^2 + 2.$$

Thus, $d_{\mathcal{B}_{n,4}(t)}(v_1 v_2) \geqslant d_{\mathcal{B}_{n,4}(t)}(w_1 w_2)$ for all $v_1, v_2 \in A$ and $w_1, w_2 \in B$. Notice that $n^2/4 - n - t^2 - 2t$ decreases as t increases and that $n^4/4 - 3n/2 + t^2 + 2$ increases as t increases (for $t \geqslant 0$). For fixed n consider the equation

$$\frac{n^2}{4} - n - t_1^2 - 2t_1 = \frac{n^2}{4} - \frac{3n}{2} + t_1^2 + 2 \quad \text{where } t_1 \geqslant 0.$$

It gives that $t_1^2 + t_1 + (1 - n/4) = 0$ and so

$$t_1 = \frac{-1 + \sqrt{n-3}}{2}.$$

This analysis implies that, for all $0 \leqslant t < n/2$,

$$\delta_2\big(\mathcal{B}_{n,4}(t)\big) \leqslant \frac{n^2}{4} - \frac{3n}{2} + t_1^2 + 2 = \frac{n^2}{4} - \frac{5n}{4} - \frac{\sqrt{n-3}}{2} + \frac{3}{2}. \tag{14}$$

Further, since $\mathcal{B}_{n,4}(t)$ is isomorphic to $\mathcal{B}_{n,4}(-t)$ for all $0 \leqslant t < n/2$, (14) holds for all $-n/2 < t < n/2$.

Now consider $\overline{\mathcal{B}}_{n,4}(t)$ for any $0 \leqslant t < n/2$ and assume A and B are the vertex classes of $\overline{\mathcal{B}}_{n,4}(t)$. Given distinct $v_1, v_2 \in A$,

$$d_{\overline{\mathcal{B}}_{n,4}(t)}(v_1 v_2) = \binom{n/2+t-2}{2} + \binom{n/2-t}{2}$$
$$= \frac{1}{2}\big[(n/2+t-2)(n/2+t-3) + (n/2-t)(n/2-t-1)\big]$$
$$= \frac{n^2}{4} - \frac{3n}{2} + t^2 - 2t + 3.$$

Given distinct $w_1, w_2 \in B$,

$$d_{\overline{\mathcal{B}}_{n,4}(t)}(w_1 w_2) = \binom{n/2-t-2}{2} + \binom{n/2+t}{2} = \frac{n^2}{4} - \frac{3n}{2} + t^2 + 2t + 3.$$

Given any $v_1 \in A$ and $w_1 \in B$,

$$d_{\overline{\mathcal{B}}_{n,4}(t)}(v_1 w_1) = (n/2+t-1)(n/2-t-1) = \frac{n^2}{4} - n - t^2 + 1.$$

Notice that $d_{\overline{\mathcal{B}}_{n,4}(t)}(v_1 v_2) \leqslant d_{\overline{\mathcal{B}}_{n,4}(t)}(w_1 w_2)$ for all $v_1, v_2 \in A$ and $w_1, w_2 \in B$. Further, when $t \geqslant 1$, $n^2/4 - 3n/2 + t^2 - 2t + 3$ increases as t increases and that $n^2/4 - n - t^2 + 1$ decreases as t increases. Thus, for a fixed n the value of $t \geqslant 1$ which maximizes the minimum 2-degree of $\overline{\mathcal{B}}_{n,4}(t)$ satisfies

$$n^2/4 - 3n/2 + t^2 - 2t + 3 = n^2/4 - n - t^2 + 1,$$

which gives that $t^2 - t + (1 - n/4) = 0$. Therefore as $t \geqslant 1$ we have that

$$t = \frac{1 + \sqrt{n-3}}{2}.$$

This analysis implies that, for all $1 \leqslant t < n/2$,

$$\delta_2\bigl(\overline{\mathcal{B}}_{n,4}(t)\bigr) \leqslant \frac{n^2}{4} - n - \left(\frac{1+\sqrt{n-3}}{2}\right)^2 + 1 = \frac{n^2}{4} - \frac{5n}{4} - \frac{\sqrt{n-3}}{2} + \frac{3}{2}. \tag{15}$$

It is easy to see that $\delta_2(\overline{\mathcal{B}}_{n,4}(0)) = \frac{n^2}{4} - \frac{3n}{2} + 3 \leqslant \frac{n^2}{4} - \frac{5n}{4} - \frac{\sqrt{n-3}}{2} + \frac{3}{2}$ when $n \geqslant 12$. Thus (15) holds for all $0 \leqslant t < n/2$. Since $\overline{\mathcal{B}}_{n,4}(t)$ is isomorphic to $\overline{\mathcal{B}}_{n,4}(-t)$ for all $0 \leqslant t < n/2$, (15) actually holds for all $-n/2 < t < n/2$. Thus, (14) and (15) imply that

$$\delta(n,4,2) \leqslant \frac{n^2}{4} - \frac{5n}{4} - \frac{\sqrt{n-3}}{2} + \frac{3}{2},$$

as desired.

Notice that there are values of n such that n is divisible by 4 and where $(1+\sqrt{n-3})/2$ is an odd integer. Indeed, let $n := (4m+1)^{2s} + 3$ for some $m, s \in \mathbb{N}$. Then $n = (4m+1)^{2s} + 3 \equiv 1 + 3 \equiv 0 \bmod 4$. Since $(4m+1)^s$ is odd, clearly $(1+\sqrt{n-3})/2 = (1+(4m+1)^s)/2$ is an integer. Further if $(1+(4m+1)^s)/2 = 2x$ for some $x \in \mathbb{N}$ then $(4m+1)^s = 4x - 1 \equiv 3 \bmod 4$, a contradiction as $(4m+1)^s \equiv 1 \bmod 4$. Hence $(1+\sqrt{n-3})/2$ is odd.

For values of n where n is divisible by 4 and where $t := (1+\sqrt{n-3})/2$ is an odd integer, we have that

$$\delta_2\bigl(\overline{\mathcal{B}}_{n,4}(t)\bigr) = \frac{n^2}{4} - n - t^2 + 1 = \frac{n^2}{4} - \frac{5n}{4} - \frac{\sqrt{n-3}}{2} + \frac{3}{2}.$$

Note though that $|A| = n/2 + t$ is odd, therefore, $\overline{\mathcal{B}}_{n,4}(t) \in \mathcal{H}_{\text{ext}}(n,4)$ and so it does not contain a perfect matching. Thus, the second part of Fact A.1 is proven. □

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