THE NUMBER OF CLIQUES IN HYPERGRAPHS WITH FORBIDDEN SUBGRAPHS

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Abstract. We study the maximum number of r-vertex cliques in $(r-1)$-uniform hypergraphs not containing complete r-partite hypergraphs $K_r^{(r-1)}(a_1, \ldots, a_r)$. By using the hypergraph removal lemma, we show that this maximum is $o(n^{r-1/(a_1 \cdots a_{r-1})})$. This immediately implies the corresponding results of Mubayi and Mukherjee and of Balogh, Jiang, and Luo for graphs. We also provide a lower bound by using hypergraph Turán numbers.

1. Introduction

Given integers $r \geq 2$ and $n > 0$ and two r-uniform hypergraphs T and F, let $\mathrm{ex}(n, T, F)$ denote the maximum number of copies of T in any F-free (i.e., not containing F as a subgraph) r-uniform hypergraph on n vertices. The case when T is an edge (i.e., $T = K_2$) is the Turán problem $\mathrm{ex}(n, F)$. The parameter $\mathrm{ex}(n, T, F)$ has been studied for different choices of graphs T and F by many authors (for example, see [1, 2, 4, 7, 9, 10, 14, 18]).

Given an r-uniform hypergraph F with vertex set $V(F) = \{v_1, \ldots, v_\ell\}$, let $F(a_1, \ldots, a_\ell)$ denote a *blowup of* F, i.e., the hypergraph obtained from F by replacing each vertex v_i by a set V_i of size a_i, and every edge $\{v_{i_1}, \ldots, v_{i_r}\}$ by a complete r-partite r-uniform hypergraph on the vertex sets $V_{i_1}, \ldots, V_{i_r}$. If $a_1 = \cdots = a_\ell = a$, then we denote $F(a_1, \ldots, a_\ell)$ by $F(a)$. For $r \leq \ell$, we denote by $K_\ell^{(r)}(a_1, \ldots, a_\ell)$ the complete ℓ-partite r-uniform hypergraph with $a_1, \ldots, a_\ell$ vertices in its parts. We often omit the superscript when $r = 2$, for example, K_3 is a graph triangle and $K_3(1, a, b)$ is a complete tripartite graph with parts of size 1, a and b.

In this note, we consider the parameter $\mathrm{ex}(n, T, F)$, when T is an r-uniform hypergraph, and F is a blowup of T. This problem is related to the following classical result of Erdős [5] on the Turán number $\mathrm{ex}(n, K_r^{(r)}(a_1, \ldots, a_r))$. It states that given integers $r \geq 2$ and $1 \leq a_1 \leq \cdots \leq a_r$,

$$\mathrm{ex}(n, K_r^{(r)}(a_1, \ldots, a_r)) = O(n^{r-1/(a_1 \cdots a_{r-1})}). \tag{1}$$

Note that the $r = 2$ case of (2) follows from a result of Kövari, Sós, and Turán [15].

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Given $2 \le s \le r$ and an r-uniform hypergraph F, the *s-uniform shadow* $\partial^{(s)}F$ of F is an s-uniform hypergraph on $V(F)$ whose edge set consists all s-subsets $A \subseteq V(F)$ such that $A \subseteq B$ for some edge $B \in F$. We observe the following simple fact (and will prove it in Section 2).

Fact 1. *Given $r \ge 3$ and an r-uniform hypergraph F, we have*

$$\mathrm{ex}(n, K_r, \partial^{(2)}F) \le \cdots \le \mathrm{ex}(n, K_r^{(r-1)}, \partial^{(r-1)}F) \le \mathrm{ex}(n, F).$$

Fact 1 and (1) together imply that, given positive integers $r \ge 3$ and $a_1 \le \cdots \le a_r$,

$$\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1, \dots, a_r)) \le \mathrm{ex}(n, K_r^{(r)}(a_1, \dots, a_r)) = O(n^{r-1/(a_1 \cdots a_{r-1})}). \quad (2)$$

Using the triangle removal lemma, Mubayi and Mukherjee [18] showed that for any $1 \le a \le b$, $\mathrm{ex}(n, K_3, K_3(1, a, b)) = o(n^{3-1/a})$.[1] Our main result extends this to hypergraphs.

Theorem 2. *Given positive integers $r \ge 3$ and $a_1 \le \cdots \le a_r$, we have,*

$$\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1, \dots, a_r)) = o(n^{r-1/(a_1 \cdots a_{r-1})}). \quad (3)$$

Further, given integers $a \ge 1$, $\ell \ge r$, and any $(r-1)$-uniform hypergraph F on ℓ vertices,

$$\mathrm{ex}(n, F, F(a)) = o\left(n^{\ell - \frac{1}{a^{\ell-1}}}\right). \quad (4)$$

Very recently and independently, Balogh, Jiang, and Luo [2] proved that for any integers $r \ge 3$ and $1 \le a_1 \le \cdots \le a_r$,

$$\mathrm{ex}(n, K_r, K_r(a_1, \dots, a_r)) = o(n^{r-1/(a_1 \cdots a_{r-1})}). \quad (5)$$

Note that Fact 1 and Theorem 2 together imply (5).

Both (3) and (5) generalize the result of [18] and all three proofs are similar in the sense that they all reduce the problem to an application of the removal lemma. While in [2], the authors remarked that the proof idea in [18] seemed not to work when proving (5) for $r > 3$ and $a_1 > 1$, here we present a proof of Theorem 2 (which also implies (5)) by taking an approach along the lines of [18].

The following lower bound complements Theorem 2 and will be proved in Section 3. Note that similar constructions (for graphs) appeared in [2, 18].

Proposition 3. *Given integers $1 \le a_1 \le \cdots \le a_r$,*

$$\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1, \dots, a_r)) = \Omega\left(n \cdot \mathrm{ex}\left(n,\ K_{r-1}^{(r-1)}(a_1, \dots, a_{r-1})\right)\right). \quad (6)$$

Observe that if $a_1 = \cdots = a_{r-1} = 1$ and $a_r \ge 2$, then the right hand side of (6) is zero and hence the lower bound is trivial. In this case, a construction in [8] implies the following lower bound. Let $r_r(n)$ denotes the size of the largest subset of $[n]$ that does not contain an arithmetic progression of length r.

Proposition 4. *For every $r \ge 3$,*

$$\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(1, \dots, 1, 2)) \ge n^{r-2} r_r(n).$$

For the proof of Proposition 4, see Section 3.

[1] It was mentioned in [2, 18] that this result was also proved by several other researchers.

2. Proof of Fact 1 and Theorem 2

In this section, we will prove Fact 1 and Theorem 2.

Proof of Fact 1. It suffices to show that, for every $2 \le s \le r-1$,

$$\mathrm{ex}(n, K_r^{(s)}, \partial^{(s)} F) \le \mathrm{ex}(n, K_r^{(s+1)}, \partial^{(s+1)} F),$$

(trivially $\partial^{(r)} F = F$). Indeed, let G be an $\partial^{(s)} F$-free s-graph on $[n]$ with $\mathrm{ex}(n, K_r^{(s)}, \partial^{(s)} F)$ copies of $K_r^{(s)}$. Let H be the $(s+1)$-graph on $[n]$ whose edges are $(s+1)$-sets that span a copy of $K_{s+1}^{(s)}$ in G. We claim that H is $\partial^{(s+1)} F$-free. Suppose instead, that H contains a copy of $\partial^{(s+1)} F$ on some set $S \subset [n]$ under a bijection $\phi : V(F) \to S$. Consider an s-set $A \in \partial^{(s)} F$. We know $A \subset B$ for some $B \in \partial^{(s+1)} F$. Thus, $\phi(B) \in H$ by the definition of ϕ and consequently, $\phi(A) \in G$ by the definition of H. This implies that S spans a copy of $\partial^{(s)} F$ in G, contradicting that G is $\partial^{(s)} F$-free.

Furthermore, it is easy to see that for any r-subset $S \subset [n]$, S spans a copy of $K_r^{(s)}$ in G if and only if S spans a copy of $K_r^{(s+1)}$ in H. Thus, the number of $K_r^{(s+1)}$ in H equals to $\mathrm{ex}(n, K_r^{(s)}, \partial^{(s)} F)$, the number of $K_r^{(s)}$ in G. Since H is $\partial^{(s+1)} F$-free, we conclude that $\mathrm{ex}(n, K_r^{(s)}, \partial^{(s)} F) \le \mathrm{ex}(n, K_r^{(s+1)}, \partial^{(s+1)} F)$. □

Before proving Theorem 2, we will fix some notation that we use for the rest of the section. We call r-uniform hypergraphs *r-graphs*. Given an $(r-1)$-graph G and a vertex $v \in V(G)$, let $G(v)$ be the $(r-1)$-graph with vertex set $V(G) \setminus \{v\}$, and

$$\{v_1, \dots, v_{r-1}\} \in G(v) \text{ if } \{v, v_1, \dots, v_{r-1}\} \text{ induces } K_r^{(r-1)} \text{ in } G.$$

For a positive integer a, let $G(v_1) \cap \cdots \cap G(v_a)$ be the $(r-1)$-graph with vertex set $V(G) \setminus \{v_1, \dots, v_a\}$ and edge set consisting of all $\{w_1, \dots, w_{r-1}\}$ such that $\{v_i, w_1, \dots, w_{r-1}\}$ induces a $K_r^{(r-1)}$ for every $i \in \{1, \dots, a\}$.

In the following proofs we will use the hypergraph removal lemma, which we state below.

Lemma 5 (Hypergraph Removal Lemma [19, 11])**.** *For every $r \ge 3$, $\varepsilon > 0$ there exists $\delta > 0$ such that for every $(r-1)$-uniform hypergraph G on n vertices the following holds. If G contains at least εn^{r-1} edge disjoint copies of $K_r^{(r-1)}$, then it must contain at least δn^r copies of $K_r^{(r-1)}$.*

We also need the following simple claim.

Claim 6. *For every positive integer b, $r \ge 3$ and $(r-1)$-graph G on n vertices, the following holds. If $\mathscr{I}$ is a collection of cliques $K_r^{(r-1)}$ of G such that every edge $e \in G$ is contained in less than b cliques of $\mathscr{I}$, then G contains at least $\frac{|\mathscr{I}|}{r(b-1)}$ edge disjoint copies of $K_r^{(r-1)}$.*

Proof. For G and $\mathscr{I}$ satisfying the above assumptions, let $\mathscr{I}_1 \subseteq \mathscr{I}$ be a maximum collection of pairwise edge disjoint cliques $K_r^{(r-1)}$ in $\mathscr{I}$ and let $\mathscr{E}$ be the union of edge sets of the cliques in $\mathscr{I}_1$. Clearly $|\mathscr{E}| = r \cdot |\mathscr{I}_1|$. Since by assumption, each edge $e \in \mathscr{E}$ is contained in at most $b-1$ cliques $K_r^{(r-1)}$ in $\mathscr{I}$, there are at most $(b-1)r|\mathscr{I}_1|$ cliques in $\mathscr{I}$ containing some edge of $\mathscr{E}$. Due to the maximality of $\mathscr{I}_1$, it follows that $|\mathscr{I}| \leq (b-1)r|\mathscr{I}_1|$ and thus $|\mathscr{I}_1| \geq \frac{|\mathscr{I}|}{(b-1)r}$. □

Now we prove Theorem 2.

Proof of Theorem 2. Fix $r \geq 3$ and integers $a_1 \leq \cdots \leq a_r$. We first consider the case when $a_{r-1} = 1$. Let $\varepsilon > 0$, and let G be a $K_r^{(r-1)}(1, \ldots, 1, a_r)$-free $(r-1)$-graph on n vertices, i.e., every edge of G is in at most $(a_r - 1)$ copies of $K_r^{(r-1)}$. Assume by contradiction, that the collection $\mathscr{I}$ of all $K_r^{(r-1)}$ in G has size at least εn^{r-1}. In view of Claim 6, G must contain at least $\varepsilon' n^{r-1}$ edge disjoint copies of $K_r^{(r-1)}$ where $\varepsilon' = ((a_r - 1)r)^{-1}\varepsilon$. By the hypergraph removal lemma, this implies that there exists some $\delta > 0$ (depending on ε') such that G contains δn^r copies of $K_r^{(r-1)}$. However, this contradicts (2).

Next, we consider the case where $a_{r-1} \geq 2$. Let G be a $K_r^{(r-1)}(a_1, \ldots, a_r)$-free $(r-1)$-graph on n vertices. First we will show that for every $\{v_1, \ldots, v_{a_{r-1}}\} \subseteq V(G)$, the $(r-1)$-graph $G(v_1) \cap \cdots \cap G(v_{a_{r-1}})$ is $K_{r-1}^{(r-1)}(a_1, \ldots, a_{r-2}, a_r)$-free. Indeed, given $\{v_1, \ldots, v_{a_{r-1}}\} \subseteq V(G)$, assume by contradiction, that the $(r-1)$-graph $G(v_1) \cap \cdots \cap G(v_{a_{r-1}})$ contains a copy of $K_{r-1}^{(r-1)}(a_1, \ldots, a_{r-2}, a_r)$ with the vertex set $V_1 \sqcup V_2 \cdots \sqcup V_{r-2} \sqcup V_r$, where $|V_i| = a_i$. Let $V_{r-1} := \{v_1, \ldots, v_{a_{r-1}}\}$. Then the r-partite graph on $V_1 \sqcup V_2 \cdots \sqcup V_r$ in G forms a copy of $K_r^{(r-1)}(a_1, \ldots, a_r)$, a contradiction.

Consequently, for every $\{v_1, \ldots, v_{a_{r-1}}\} \subseteq V(G)$,

$$|G(v_1) \cap \cdots \cap G(v_{a_{r-1}})| \leq \mathrm{ex}(n, K_{r-1}^{(r-1)}(a_1, \ldots, a_{r-2}, a_r)) = O\left(n^{r-1-\frac{1}{a_1 a_2 \cdots a_{r-2}}}\right). \tag{7}$$

Our goal is using the above fact to obtain a large collection of edge disjoint $K_r^{(r-1)}$ in G. To this end we consider the family $\mathscr{A}$, elements of which are collections of a_{r-1} copies of $K_r^{(r-1)}$ that share an edge of G. More formally,

$$\mathscr{A} := \{\{T_1, \ldots, T_{a_{r-1}}\} : T_i \cong K_r^{(r-1)} \text{ and } T_1, T_2, \ldots, T_{a_{r-1}} \text{ share an edge of } G\}.$$

Next we give an upper bound on the size of $\mathscr{A}$. Given *any* element in $\mathscr{A}$, there exists vertices $\{v_1, \ldots, v_{a_{r-1}}\} \subseteq V(G)$, and an edge $e \in G$ (in particular, $e \in G(v_1) \cap \cdots \cap G(v_{a_{r-1}})$), such that $e \cup \{v_i\}$ form a $K_r^{(r-1)}$ for every $1 \leq i \leq a_{r-1}$. Consequently, the cardinality of $\mathscr{A}$ can be bounded by the number of pairs $(\{v_1, \ldots, v_{a_{r-1}}\}, e)$ with $e \in G(v_1) \cap \cdots \cap G(v_{a_{r-1}})$. Thus in view of (7),

$$|\mathscr{A}| \leq \binom{n}{a_{r-1}} |G(v_1) \cap \cdots \cap G(v_{a_{r-1}})| \leq n^{a_{r-1}} O\left(n^{r-1-\frac{1}{a_1 a_2 \cdots a_{r-2}}}\right). \tag{8}$$

In order to prove (3) of Theorem 2, assume by contradiction, that G contains $N = \Omega(n^{r-1/(a_1 \cdots a_{r-1})})$ copies of $K_r^{(r-1)}$. We will find a collection $\mathscr{I}$ of cliques $K_r^{(r-1)}$ in G satisfying

$$|\mathscr{I}| = \Omega(n^{r-1}) \quad \text{and} \quad \mathscr{I}^{(a_{r-1})} \cap \mathscr{A} = \emptyset, \tag{9}$$

i.e., for every $S \subseteq \mathscr{I}$ with $|S| = a_{r-1}$, S is not an element of $\mathscr{A}$. Note that if $|\mathscr{A}| \leq N/2$, then one can obtain $\mathscr{I}$ from the collection of $K_r^{(r-1)}$ in G, by deleting a copy of $K_r^{(r-1)}$ for each element of $\mathscr{A}$. Thus, $\mathscr{I}^{(a_{r-1})} \cap \mathscr{A} = \emptyset$ and $|\mathscr{I}|$ is at least $N/2 = \Omega(n^{r-1/(a_1 \cdots a_{r-1})})$ which is bigger than $\Omega(n^{r-1})$.

Now we consider the case where $|\mathscr{A}| \geq N/2$. Let $\mathbf{I}$ be a random subset of copies of $K_r^{(r-1)}$ where each copy of $K_r^{(r-1)}$ in G is chosen with probability $p > 0$ independently. Let $\mathbf{I}^{(a_{r-1})}$ denote the collection of a_{r-1}-subsets of $\mathbf{I}$. We have that,

$$\mathbb{E}[|\mathbf{I}| - |\mathscr{A} \cap \mathbf{I}^{(a_{r-1})}|] = pN - p^{a_{r-1}}|\mathscr{A}|.$$

Let p be chosen such that, $p^{a_{r-1}}|\mathscr{A}| = pN/2$, which implies

$$p = \left(\frac{N}{2|\mathscr{A}|}\right)^{\frac{1}{a_{r-1}-1}} \leq 1, \text{ (since } N \leq 2|\mathscr{A}|) \quad \text{and} \quad pN = \frac{N^{\frac{a_{r-1}}{a_{r-1}-1}}}{(2|\mathscr{A}|)^{\frac{1}{a_{r-1}-1}}}.$$

Consequently, there exists a choice of $\mathscr{I}'$ such that,

$$|\mathscr{I}'| - |\mathscr{A} \cap \mathscr{I}'^{(a_{r-1})}| \geq \frac{pN}{2}.$$

Let $\mathscr{I} \subseteq \mathscr{I}'$ be the collection of $K_r^{(r-1)}$ of G formed by deleting one $K_r^{(r-1)}$ in $\mathscr{I}'$ from every a_{r-1} subset in $\mathscr{A} \cap \mathscr{I}'^{(a_{r-1})}$. Consequently, $\mathscr{A} \cap \mathscr{I}^{(a_{r-1})}$ is empty. Further,

$$|\mathscr{I}| \geq \frac{pN}{2} = \frac{1}{2}\frac{N^{\frac{a_{r-1}}{a_{r-1}-1}}}{(2|\mathscr{A}|)^{\frac{1}{a_{r-1}-1}}}. \tag{10}$$

Using the value of N (by assumption) and $|\mathscr{A}|$ in (8), the exponent of n in the RHS of (10) is equal to

$$\left(r - \frac{1}{a_1 a_2 a_3 \cdots a_{r-1}}\right)\frac{a_{r-1}}{a_{r-1}-1} - \left(a_{r-1} + r - 1 - \frac{1}{a_1 \cdots a_{r-2}}\right)\frac{1}{a_{r-1}-1}$$
$$= \frac{a_{r-1}r - a_{r-1} - (r-1)}{a_{r-1}-1} = r - 1,$$

which implies $|\mathscr{I}| = \Omega(n^{r-1})$. Hence $\mathscr{I}$ satisfies (9).

Next we obtain a family of edge disjoint $K_r^{(r-1)}$ in G from $\mathscr{I}$. By construction, $\mathscr{I}$ is a collection of cliques $K_r^{(r-1)}$ in G such that every edge $e \in G$ is contained in less than a_{r-1} cliques of $\mathscr{I}$. In view of Claim 6, this implies that G contains at least $|\mathscr{I}|/r(a_{r-1}-1)$ edge disjoint copies of $K_r^{(r-1)}$. Since $|\mathscr{I}| = \Omega(n^{r-1})$, this implies that G contains $\Omega(n^{r-1})$ copies of edge disjoint $K_r^{(r-1)}$.

To summarise, this implies that given any $\varepsilon > 0$, and $(r-1)$-graph G on n vertices that is $K_r^{(r-1)}(a_1, \ldots, a_r)$-free the following holds. Assuming by contradiction that G contains $N = \varepsilon n^{r-1/(a_1 \cdots a_{r-1})}$ copies of $K_r^{(r-1)}$, there exists some $\varepsilon' > 0$ (depending only on ε, r, a_i) such that G contains $\varepsilon' n^{r-1}$ edge disjoint copies of $K_r^{(r-1)}$. By the hypergraph removal lemma, this implies that there exists some $\delta > 0$ (depending only on ε, r, a_i) such that G contains δn^r copies of $K_r^{(r-1)}$. In view of (2), however, this implies that G contains $K_r^{(r-1)}(a_1, \ldots, a_r)$. Thus (3) holds.

Now we prove the upper bound in (4) on $\mathrm{ex}(n, F, F(a))$ for any given $(r-1)$-graph F. Label the vertices of F $v_1, \dots, v_\ell$. Let G be an $F(a)$-free $(r-1)$-graph on n vertices, and assume by contradiction, that G contains $N = \Omega(n^{\ell - \frac{1}{a^{\ell-1}}})$ copies of F. Given an ℓ-partition of $V(G) = W_1 \sqcup \cdots \sqcup W_\ell$, we call a set $X \subseteq V(G)$ *crossing* if $|X \cap W_i| \leq 1$ for $1 \leq i \leq \ell$. We call a copy of F in G on a vertex set $\{x_1, \dots, x_\ell\}$ *aligned* with respect to $W_1 \sqcup W_2 \sqcup \cdots \sqcup W_\ell$ if

(1) $x_i \in W_i$ for $i = 1, 2, \dots, \ell$, and
(2) $x_i \mapsto v_i$ is an isomorphism.

We will denote such a copy by $\overrightarrow{F}$. A simple averaging argument yields that there exists a partition of $V(G) = W_1 \sqcup \cdots \sqcup W_\ell$ with at least $\ell^{-\ell} N$ copies of $\overrightarrow{F}$.

Let $\mathscr{H}$ be an *auxiliary* ℓ-partite $(\ell-1)$-graph with vertex set $W_1 \sqcup \cdots \sqcup W_\ell$. Let the edges of $\mathscr{H}$ be those crossing $(\ell-1)$-tuples that extend to a copy of $\overrightarrow{F}$. Formally,

$$\mathscr{H} = \bigsqcup_{i=1}^{\ell} \left\{ (x_j)_{j \in [\ell] \setminus \{i\}} : \text{ there exists } x_i \in W_i \text{ such that } (x_1, \dots, x_\ell) \text{ is a copy of } \overrightarrow{F} \right\}.$$

Note that each *aligned copy* $\overrightarrow{F}$ in G forms a $K_\ell^{(\ell-1)}$ in $\mathscr{H}$. Consequently, the number of copies of $K_\ell^{(\ell-1)}$ in $\mathscr{H}$ is at least $(\ell^{-\ell})N = \Omega(n^{\ell - \frac{1}{a^{\ell-1}}})$.

By the first part of Theorem 2, this implies that $\mathscr{H}$ contains a copy of $K_\ell^{(\ell-1)}(a)$ with vertex sets $U_i \subseteq W_i$ for $1 \leq i \leq \ell$. Let $(x_1, \dots, x_\ell) \in U_1 \times \cdots \times U_\ell$. Since $\ell \geq r$, for every edge $\{v_{i_1}, \dots, v_{i_{r-1}}\}$ of F, there exists an $\ell - 1$ subset $S \subseteq [\ell]$ such that $\{i_1, \dots, i_{r-1}\} \subseteq S$. By definition of $\mathscr{H}$, the tuple $(x_s : s \in S)$ must extend to some copy of $\overrightarrow{F}$, which implies $\{x_{i_1}, \dots, x_{i_{r-1}}\}$ must be an edge in G.

Consequently, for every $(x_1, \dots, x_\ell) \in U_1 \times \cdots \times U_\ell$, we have that the subgraph of G induced by $\{x_1, \dots, x_\ell\}$ contains an aligned copy $\overrightarrow{F}$. This implies that G contains a copy of $F(a)$, contradicting the assumption that G is $F(a)$-free. □

3. Lower Bound Constructions

In this section, we will prove Propositions 3 and 4.

Proof of Proposition 3. We construct an $(r-1)$-graph H whose vertex set is partitioned into $A \sqcup B$ such that

- $|A| = \lfloor n/r \rfloor$ and $|B| = \lceil (r-1)n/r \rceil$;
- $H[B]$ is $K_{r-1}^{(r-1)}(a_1, \dots, a_{r-1})$-free and has $\mathrm{ex}(\lceil \frac{r-1}{r} n \rceil, K_{r-1}^{(r-1)}(a_1, \dots, a_{r-1}))$ edges;
- every vertex of A and every $(r-2)$-subset of B form an edge and there are no other edges intersecting A. In other words, the link of every vertex in A is the complete $(r-2)$-graph on the vertex set B.

The number of $K_r^{(r-1)}$ is at least $\lfloor \frac{n}{r} \rfloor \mathrm{ex}(\lceil \frac{r-1}{r} n \rceil, K_{r-1}^{(r-1)}(a_1, \dots, a_{r-1}))$ because every vertex of A together with any edge of B forms a copy of $K_r^{(r-1)}$. It remains to show that H

contains no $K_r^{(r-1)}(a_1, \dots, a_r)$. Assume by contradiction, it does. Since there is no edge containing two vertices from A, and $a_1 \le a_2 \le \cdots \le a_r$, the subgraph induced by H on B needs to contain a $K_{r-1}^{(r-1)}(a_1, \dots, a_{r-1})$, thus contradicting the construction of H. $\square$

The proof of Proposition 4 is based on a construction given in [8].

Proof of Proposition 4. In the proof of [8, Proposition 2.1], it was shown that for every $r \ge 3$, there exists an r-partite r-graph H with parts $V_1, \dots, V_r$ satisfying the following properties.

(1) For every $\{x_1, \dots, x_{r-1}\} \subseteq V(H)$, there exists at most one edge in H containing $\{x_1, \dots, x_{r-1}\}$.
(2) For every collection of subsets $\{\{x_i, y_i\} \subseteq V_i : 1 \le i \le r\}$, there exists $1 \le i \le r$ such that $\{x_1, \dots, x_r\} \setminus \{x_i\} \cup \{y_i\}$ is not an edge of H.
(3) H has $(r-1)rn$ vertices and $n^{r-2}r_r(n)$ edges.

Let G be the $(r-1)$-uniform shadow of H, i.e., $G = \partial^{(r-1)}H$. We claim that G is $K_r^{(r-1)}(1, \dots, 1, 2)$-free and contains $n^{r-2}r_r(n)$ copies of $K_r^{(r-1)}$. Since G is the shadow of H, the number of copies of $K_r^{(r-1)}$ in G is at least the number of edges in H.

While the edges of H correspond to a collection of edge disjoint cliques ("real cliques") in G, we will now show that G contains no other cliques $K_r^{(r-1)}$. Assume by contradiction that $\{x_1, \dots, x_r\}$ induces such a "fake clique" $K_r^{(r-1)}$, i.e., $\{x_1, \dots, x_r\} \notin H$ but induces a $K_r^{(r-1)}$ in G. Since every edge of this clique belongs to some "real clique", for every $1 \le i \le r$, there must exist $y_i \ne x_i$ in V_i such that $\{x_1, \dots, x_r\} \setminus \{x_i\} \cup \{y_i\} \in H$, contradicting (2). Consequently, by (1), no two $K_r^{(r-1)}$ in G share an edge and hence G is $K_r^{(r-1)}(1, \dots, 1, 2)$-free. $\square$

4. Concluding remarks

As mentioned earlier, in the case where $a_1 = \cdots = a_{r-1} = 1$ and $a_r \ge 2$ the lower bound in (6) is trivial. We ask if there are other sequences of integers $a_1 \le \cdots \le a_r$ for which (6) can be improved.

Question 7. *Given integer $r \ge 3$, for what sequence of integers $1 \le a_1 \le \cdots \le a_r$,*

$$\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1, \dots, a_r)) \ge n^{1+\varepsilon} \cdot \mathrm{ex}\left(n,\ K_{r-1}^{(r-1)}(a_1, \dots, a_{r-1})\right)$$

for some $\varepsilon = \varepsilon(n) > 0$?

The order of magnitude for $\mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1, \dots, a_r))$ is not known in any non-trivial case. The case when $r \ge 3$ and $a_1 = \cdots = a_r = 2$ is related to a problem of Erdős, see, e.g., [3, 12, 13]. Theorem 2 and Proposition 3, together with the lower bound in [3] imply that

$$\Omega\left(n^{r-\left\lceil \frac{2^{r-1}-1}{r-1} \right\rceil^{-1}}\right) \le \mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(2, \dots, 2)) \le o\left(n^{r-\frac{1}{2^{r-1}}}\right).$$

It was conjectured in [17], that $\mathrm{ex}(n, K_{r-1}^{(r-1)}(a_1, \dots, a_{r-1})) = \Omega(n^{r-1-1/(a_1 \cdots a_{r-2})})$. This was confirmed for some cases in [16, 17, 20]. If this conjecture is true, then Theorem 2 and Proposition 3 would imply that,

$$\Omega(n^{r-1/(a_1 \cdots a_{r-2})}) \leq \mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a_1, \dots, a_r)) \leq o(n^{r-1/(a_1 \cdots a_{r-1})}).$$

When $a_1 = \cdots = a_r = a \geq 2$, one can obtain that $\mathrm{ex}(n, K_{r-1}^{(r-1)}(a)) = \Omega\left(n^{r-1-\frac{(r-1)(a-1)}{a^{r-1}-1}}\right)$ by using the probabilistic deletion method [6]. Together with Proposition 3 and Theorem 2, this gives

$$\Omega\left(n^{r-\frac{(r-1)(a-1)}{a^{r-1}-1}}\right) \leq \mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(a)) \leq o\left(n^{r-\frac{1}{a^{r-1}}}\right).$$

When $a_1 = 1$ and $a_2 = \cdots = a_r = a \geq 2$, instead of Proposition 3, one can employ the deletion method directly to an random $(r-1)$-uniform hypergraph on n vertices by removing copies of $K_r^{(r-1)}(1, a, \dots, a)$. Together with Theorem 2, this implies that

$$\Omega\left(n^{r-\frac{r(r-1)}{a^{r-2}}}\right) \leq \mathrm{ex}(n, K_r^{(r-1)}, K_r^{(r-1)}(1, a, \dots, a)) \leq o(n^{r-\frac{1}{a^{r-2}}}).$$

It would be interesting to improve the gaps in any of the above cases.

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