TURÁN NUMBER OF DISJOINT TRIANGLES IN 4-PARTITE GRAPHS

JIE HAN AND YI ZHAO

Abstract. Let $k \geqslant 2$ and $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4$ be integers such that n_4 is sufficiently larger than k. We determine the maximum number of edges of a 4-partite graph with parts of sizes $n_1, \dots, n_4$ that does not contain k vertex-disjoint triangles. For any $r > t \geqslant 3$, we give a conjecture on the maximum number of edges of an r-partite graph that does not contain k vertex-disjoint cliques K_t.

1. Introduction

Given two graphs G and F, we say that G is *F-free* if G does not contain F as a subgraph. Let K_t denote a complete graph on t vertices, and $T_{n,t}$ denote a balanced complete t-partite graph on n vertices (now known as the *Turán graph*). In 1941, Turán [9] proved that $T_{n,t}$ has the maximum number of edges among all K_{t+1}-free graphs (the case $t = 2$ was previously solved by Mantel [7]). Turán's result initiates the study of Extremal Graph Theory, an important area of research in modern Combinatorics (see the monograph of Bollobás [3]). Let kK_t denote k disjoint copies of K_t. Simonovits [8] studied the Turán problem for kK_t and showed that when n is sufficiently large, the (unique) extremal graph on n vertices is the join of K_{k-1} and the Turán graph $T_{n-k+1,t-1}$.

In this paper we consider Turán problems in multi-partite graphs. Let $K_{n_1,n_2,\dots,n_r}$ denote the complete r-partite graph on parts of sizes $n_1, n_2, \dots, n_r$. This variant of the Turán problem was first considered by Zarankiewicz [11], who was interested in the case of forbidding $K_{s,t}$ in (subgraphs of) $K_{a,b}$. Formally, given graphs H and F, we define $\mathrm{ex}(H, F)$ as the maximum number of edges in an F-free subgraph of H. Bollobás, Erdős, and Straus [2] (see also [3, Page 544]) proved the following result. For any subset $I \subseteq [r]$, write $n_I := \sum_{i \in I} n_i$.

Theorem 1.1. [2] *The extremal number* $\mathrm{ex}(K_{n_1,\dots,n_r}, K_t)$ *is equal to*

$$\max_{\mathcal{P}} \sum_{I \neq I' \in \mathcal{P}} n_I \cdot n_{I'},$$

where the maximum is taken over all partitions $\mathcal{P}$ of $[r]$ into $t-1$ parts.

The problem of forbidding disjoint copies of cliques in multi-partite graphs has been studied recently. Chen, Li and Tu [4] determined $\mathrm{ex}(K_{n_1,n_2}, kK_2)$ and De Silva, Heysse and Young [5] showed that $\mathrm{ex}(K_{n_1,\dots,n_r}, kK_2) = (k-1)(n_1 + \cdots + n_{r-1})$ for $n_1 \geqslant \cdots \geqslant n_r$. De Silva, Heysse, Kapilow, Schenfisch and Young [6] determined $\mathrm{ex}(K_{n_1,\dots,n_r}, kK_r)$ and raised the question of determining $\mathrm{ex}(K_{n_1,\dots,n_r}, kK_t)$ when $r > t$. After giving another proof of Theorem 1.1, Bennett, English and Talanda-Fisher [1] reiterated this question.

Problem 1.2. [6] *Determine* $\mathrm{ex}(K_{n_1,\dots,n_r}, kK_t)$ *when* $r > t$.

Jie Han is partially supported by a Simons Collaboration Grant 630884. Yi Zhao is partially supported by an NSF grant DMS 1700622 and Simons Collaboration Grant 710094.

In this paper we solve Problem 1.2 for $r = 4$ and $t = 3$ when all n_i's are sufficiently large. To state our result, for $k \geqslant 1$, we define a function of positive integers $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4$:

$$g_k(n_1, n_2, n_3, n_4) := \max\{(n_1+n_4)(n_2+n_3) + (k-1)n_1, n_1(n_2+n_3+n_4) + (k-1)(n_2+n_3)\}$$
$$= \begin{cases} (n_1+n_4)(n_2+n_3) + (k-1)n_1 & \text{if } n_1 \leqslant n_2+n_3, \\ n_1(n_2+n_3+n_4) + (k-1)(n_2+n_3), & \text{if } n_1 > n_2+n_3. \end{cases}$$

When G is a 4-partite graph with parts of sizes $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4$, we define $g_k(G) := g_k(n_1, n_2, n_3, n_4)$. For arbitrary positive integers a, b, c, d, we define $g_k(a, b, c, d) = g_k(a_1, a_2, a_3, a_4)$, where $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4$ is a reordering of a, b, c, d.

Theorem 1.3. *Given $k \geqslant 1$, there exists $N_0(k)$ such that if G is a kK_3-free 4-partite graph with parts of sizes $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4 \geqslant 6k^2$ and $n_1+n_2+n_3+n_4 \geqslant N_0(k)$, then $e(G) \leqslant g_k(n_1, n_2, n_3, n_4)$. In other words, $\text{ex}(K_{n_1,n_2,n_3,n_4}, kK_3) \leqslant g_k(n_1, n_2, n_3, n_4)$.*

Theorem 1.3 is tight due to two constructions G_1 and G_2 below. In fact, a subgraph of G_2 was given by De Silva et al. [6] as a potential extremal construction; later Wagner [10] realized that G_1 was a better construction for the $n_1 = n_2 = n_3 = n_4$ case. Let $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4 \geqslant k$. We define two 4-partite graphs with parts $V_1, \dots, V_4$ such that $|V_i| = n_i$. Fix a set Z of $k-1$ vertices in V_4. Let

$$G_1 := K_{V_1 \cup V_4, V_2 \cup V_3} \cup K_{Z, V_1} \text{ and } G_2 := K_{V_1, V_2 \cup V_3 \cup V_4} \cup K_{Z, V_2 \cup V_3},$$

where $K_{V_1,\dots,V_r}$ denotes the complete r-partite graph with parts $V_1, \dots, V_r$. Note that each triangle must intersect Z and thus both G_1 and G_2 are kK_3-free. Moreover, $e(G_1) = (n_1+n_4)(n_2+n_3) + (k-1)n_1$ and $e(G_2) = n_1(n_2+n_3+n_4) + (k-1)(n_2+n_3)$. Thus $e(G_2) \leqslant e(G_1)$ if and only if $n_1 \leqslant n_2+n_3$ and equality holds when $n_1 = n_2+n_3$.

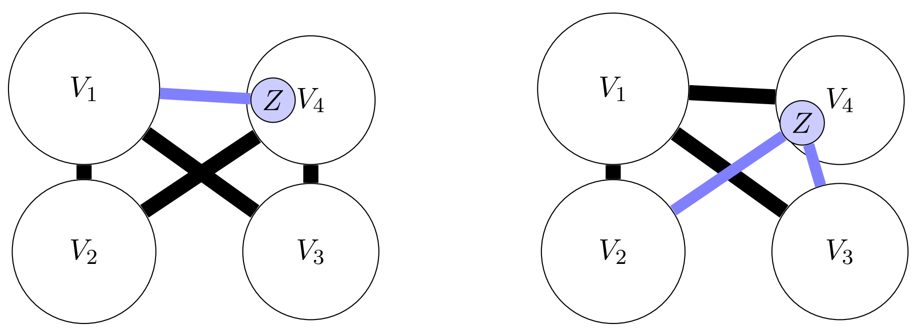

FIGURE 1. The extremal graphs G_1 and G_2

Our proof uses a *progressive induction* (an induction without a base case) on the total number of vertices and a standard induction on k that uses Theorem 1.1 as the base case.

We conjecture an answer to Problem 1.2 in general, which includes all aforementioned results [1, 4, 5] and Theorem 1.3.

Conjecture 1.4. Given $r > t \geqslant 3$ and $k \geqslant 2$, let $n_1, \dots, n_r$ be sufficiently large. For $I \subseteq [r]$, write $m_I := \min_{i \in I} n_i$. Given a partition $\mathcal{P}$ of $[r]$, let $n_{\mathcal{P}} := \max_{I \in \mathcal{P}}\{n_I - m_I\}$. The Turán number $\text{ex}(K_{n_1,\dots,n_r}, kK_t)$ is equal to

$$\max_{\mathcal{P}} \left\{ (k-1)n_{\mathcal{P}} + \sum_{I \neq I' \in \mathcal{P}} n_I \cdot n_{I'} \right\}, \tag{1.1}$$

where the maximum is taken over all partitions $\mathcal{P}$ of $[r]$ into $t-1$ parts.

The bound (1.1) is achieved by the following graph. Given integers k, t and $n_1, \dots, n_r$ with $r > t$ and $n_i \geqslant k$ for all i, let $\mathcal{P}$ be a partition of $[r]$ into $t-1$ parts that maximizes (1.1). Let G be an r-partite graph whose parts have sizes $n_1, \dots, n_r$. Partition G into $t-1$ parts according to $\mathcal{P}$, namely, let $V_I = \bigcup_{i \in I} V_i$ for every $I \in \mathcal{P}$ and include all edges between V_I and $V_{I'}$ for all $I \neq I' \in \mathcal{P}$. In addition, let $I_0 \in \mathcal{P}$ such that $n_{\mathcal{P}} = n_{I_0} - m_{I_0}$ and let V_{i_0} be the smallest part in V_{I_0}. We choose a set $Z \subseteq V_{i_0}$ of $k-1$ vertices and add all edges between Z and $V_{I_0} \backslash V_{i_0}$.

Verifying Conjecture 1.4 seems hard due to the complexity of (1.1) – we shall discuss this in the last section.

Notation. Given a graph $G = (V, E)$, let $|G|$ denote the order of G. Suppose A, B are two disjoint subsets of V. Let $e(A) := e(G[A])$ be the number of edges of G in A and $e(A, B)$ be the number of edges of G with one end in A and the other in B. Moreover, let $G\backslash A := G[V\backslash A]$. Denote by

$$e(A; G) := e(G) - e(G\backslash A),$$

the number of edges of G incident to A. Given a vertex x, let $N(x)$ denote the set of neighbors of x. For vertices x, y and z, we often write xyz for $\{x, y, z\}$. We sometimes abuse this notation by using $xy \in A \times B$ to indicate that $x \in A$ and $y \in B$. Given an r-partite graph G, a *crossing set* is a set that contains at most one vertex from each part of G.

2. Proof of Theorem 1.3

In this section we prove Theorem 1.3. Define two sequences $N_0(k)$ and $M_0(k)$ recursively by letting $N_0(1) = 1$,

$$M_0(k) = \max\{72(k-1)^3, 96k^2, N_0(k-1) + 3\}, \quad \text{and} \quad N_0(k) = M_0(k)^2 \tag{2.1}$$

for $k \geqslant 2$. Given a 4-partite graph G, let $v_4(G)$ denote the size of the smallest part of G. Define $\varphi(G) := e(G) - g_k(G)$. The following theorem is the main step in the proof of Theorem 1.3.

Theorem 2.1. *Suppose $k \geqslant 2$ and Theorem 1.3 holds for $k-1$. Let G be a 4-partite graph of order $|G| > M_0(k)$ and with $v_4(G) \geqslant 6k^2$. If G is kK_3-free and $\varphi(G) > 0$, then we can find a subgraph G' of G such that $|G| - 2 \leqslant |G'| \leqslant |G| - 1$, $v_4(G') \geqslant 6k^2$, and $\varphi(G') > \varphi(G)$.*

Theorem 1.3 nows follows from Theorem 2.1 by an induction on k and a progressive induction on $|G|$ (e.g., used in [8]).

Proof of Theorem 1.3. The base case $k = 1$ follows from Theorem 1.1 with $N_0(1) = 1$. Let $k \geqslant 2$ and G be a 4-partite graph of order $|G| \geqslant N_0(k)$ and with $v_4(G) \geqslant 6k^2$. Suppose G is kK_3-free and $\varphi(G) > 0$. By Theorem 2.1, we find a subgraph $G_1 \subset G$ such that $|G| - 2 \leqslant |G_1| \leqslant |G| - 1$, $v_4(G_1) \geqslant 6k^2$, and $\varphi(G_1) > \varphi(G) \geqslant 1$. Repeating this process, we obtain subgraphs $G_1 \supset G_2 \supset G_3 \supset \cdots \supset G_t$ such that $|G| - 2i \leqslant |G_i| \leqslant |G| - i$ and $\varphi(G_i) > i$ for $i = 1, \dots, t$. We stop at G_t because $|G_t| \leqslant M_0(k)$. Hence,

$$t \geqslant \frac{|G| - |G_t|}{2} \geqslant \frac{N_0(k) - M_0(k)}{2} = \frac{M_0(k)^2 - M_0(k)}{2} = \binom{M_0(k)}{2}.$$

Consequently, $\varphi(G_t) > \binom{M_0(k)}{2}$. However, this is impossible because $\varphi(G_t) \leqslant e(G_t) \leqslant \binom{M_0(k)}{2}$. ☐

The rest of this section is devoted to the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $k \geqslant 2$ and suppose that

(∗) for any $(k-1)K_3$-free 4-partite graph $\tilde{G}$ with part sizes $n'_1 \geqslant n'_2 \geqslant n'_3 \geqslant n'_4 \geqslant 6(k-1)^2$ and $\sum_{i\in[4]} n'_i \geqslant N_0(k-1)$, we have $e(\tilde{G}) \leqslant g_{k-1}(n'_1, n'_2, n'_3, n'_4)$.

Let G be a 4-partite graph of order $|G| > M_0(k)$ and with parts of size $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4 \geqslant 6k^2$. Assume that G is kK_3-free and $\varphi(G) > 0$. Without loss of generality, we assume that G contains $k-1$ disjoint triangles – otherwise we keep adding edges to G until it contains $k-1$ disjoint triangles (as a result, $\varphi(G)$ increases). Our goal is to show that there exists a crossing set $T \subset V(G)$ of size at most 2 such that $\varphi(G) < \varphi(G\backslash T)$ and $v_4(G\backslash T) \geqslant 6k^2$.

We proceed in the following cases. It is easy to see that these cases cover all possibilities. In each case we verify $v_4(G\backslash T) \geqslant 6k^2$ immediately.

Case 0. $n_1 > n_2 + n_3$**.** We will select a one-element set $T \subset V_1$. Since $n_1 > 2n_4$, we have $n_1 - 1 > n_4$ and thus $v_4(G\backslash T) = n_4 \geqslant 6k^2$.

We assume $n_1 \leqslant n_2 + n_3$ in the remaining cases.

Case 1. $n_1 > n_3$ **and** $n_2 > n_4$**.** We will select a crossing set $T \subset V_1 \cup V_2$. Since $n_1 - 1 \geqslant n_2 - 1 \geqslant n_4$, we have $v_4(G\backslash T) = n_4 \geqslant 6k^2$.

Case 2. $n_1 = n_2 = n_3 \geqslant n_4 > 6k^2$**.** We select a one-element set $T \subset V(G)$. Then $v_4(G\backslash T) \geqslant n_4 - 1 \geqslant 6k^2$.

Case 3. $n_1 = n_2 = n_3 > n_4 = 6k^2$**.** We will select a one-element set $T \subset V_1 \cup V_2 \cup V_3$. Since $n_3 - 1 \geqslant n_4$, we have $v_4(G\backslash T) = n_4 = 6k^2$.

Case 4. $n_1 > n_2 = n_3 = n_4$**.** We will select a one-element set $T \subset V_1$. Since $n_1 > n_4$, $v_4(G\backslash T) = n_4 \geqslant 6k^2$.

It remains to show $\varphi(G) < \varphi(G\backslash T)$ in **Cases 0–4**. This is actually easy in **Case 0**.

Case 0. Recall that $\varphi(G) = e(G) - g_k(n_1, n_2, n_3, n_4) > 0$. Since $n_1 > n_2 + n_3$,

$$g_k(n_1, n_2, n_3, n_4) = n_1(n_2 + n_3 + n_4) + (k-1)(n_2 + n_3).$$

First assume that some vertex $v \in V_1$ satisfies $d(v) < n_2 + n_3 + n_4$. Let $T = \{v\}$. Since $n_1 - 1 \geqslant n_2 + n_3$,

$$\begin{aligned} g_k(n_1 - 1, n_2, n_3, n_4) &= (n_1 - 1)(n_2 + n_3 + n_4) + (k-1)(n_2 + n_3) \\ &= g_k(n_1, n_2, n_3, n_4) - (n_2 + n_3 + n_4). \end{aligned}$$

It follows that

$$\varphi(G\backslash\{v\}) = e(G) - d(v) - g_k(n_1 - 1, n_2, n_3, n_4) > e(G) - g_k(n_1, n_2, n_3, n_4) = \varphi(G),$$

as desired. Otherwise, $G[V_1, V_2 \cup V_3 \cup V_4]$ must be complete. Since G is kK_3-free, it follows that $G[V_2 \cup V_3 \cup V_4]$ contains no matching of size k. The result of [5] or a simple induction on k[1] yields that $e(G[V_2 \cup V_3 \cup V_4]) \leqslant (k-1)(n_2 + n_3)$. This shows that $e(G) \leqslant n_1(n_2 + n_3 + n_4) + (k-1)(n_2 + n_3)$, namely, $\varphi(G) = 0$, a contradiction.

In the rest of the proof we assume $n_1 \leqslant n_2 + n_3$ and will resolve **Cases 1–4**.

One difficulty in these cases is that, after we delete a set $T \subseteq V(G)$, the sizes of the four parts of $G\backslash T$ may not follow the order in G. For instance, suppose $n_1 \leqslant n_2 + n_3$ and $T = \{v\} \subseteq V_1$. If $n_1 > n_2$, then the order of the part sizes of $G\backslash T$ is $n_1 - 1 \geqslant n_2 \geqslant n_3 \geqslant n_4$, the same as in G. However, when $n_1 = n_2 > n_3 \geqslant n_4$, the order of the part sizes of $G\backslash T$ is $n_2 \geqslant n_1 - 1 \geqslant n_3 \geqslant n_4$, and the degree estimates we obtain are quite different. Another complication comes from the fact that there are two possible extremal graphs. Even under the assumption that $n_1 \leqslant n_2 + n_3$, we still have to consider the possibility of $n'_1 > n'_2 + n'_3$ in $G\backslash T$, where n'_1, n'_2, n'_3, n'_4 are the part sizes of $G\backslash T$.

---

[1]If there is a vertex of degree at least $2k-1$, then we can delete it and apply induction; otherwise, as the size of the maximum matching is $k-1$, there are at most $2(k-1)(2k-1) \leqslant (k-1)(n_2 + n_3)$ edges (using $k \ll n_3 \leqslant n_2$).

Although a case analysis is inevitable, we study the structure of G in Section 2.1 and use it to simplify the presentation of the proofs of **Cases 1–4** in Section 2.2.

2.1. **Preparation.** We first give several preliminary results. An edge of G is called *rich* if it is contained in at least k triangles whose third vertices are located in the same part of $V(G)$. We show that every triangle in G must contain a rich edge and G contains at most $6(k-1)^2$ rich edges. Let Z be the set of vertices incident to at least one rich edge. Thus, not only is $G\backslash Z$ triangle-free, but also *every edge in $G\backslash Z$ is not contained in any triangle of G* because such a triangle would not contain any rich edge.

We shall use the following simple fact.

Fact 2.2. *Let G be a 4-partite graph with parts $V_1, \dots, V_4$ and suppose $x \in V_1$ and $y \in V_2$. Let $n_i := |V_i|$ for $i \in [4]$. Then x and y have at least $d(x) + d(y) - \sum_{i\in[4]} n_i$ common neighbors in G. In particular, if x and y have no common neighbor, then $d(x)+d(y) = \sum_{i\in[4]} n_i$ implies that $xy \in E(G)$, $V_2 \subseteq N(x)$ and $V_1 \subseteq N(y)$. Moreover, if $d(x) + d(y) \geqslant \sum_{i\in[4]} n_i + 2k - 1$, then xy is rich.*

Proof. Note that $|N(x) \cap (V_3 \cup V_4)| = d(x) - |N(x) \cap V_2| \geqslant d(x) - n_2$ and $|N(y) \cap (V_3 \cup V_4)| = d(y) - |N(y) \cap V_1| \geqslant d(y) - n_1$. Let m denote the number of common neighbors of x and y. Then $m \geqslant |N(x) \cap (V_3 \cup V_4)| + |N(y) \cap (V_3 \cup V_4)| - n_3 - n_4 \geqslant d(x) + d(y) - \sum_{i\in[4]} n_i$. So the first part of the fact follows. In particular, if $m = 0$, then $d(x) + d(y) \leqslant \sum_{i\in[4]} n_i$. Moreover, if the equality holds, then the inequalities in previous calculations must be equalities. In particular, $V_2 \subseteq N(x)$ and $V_1 \subseteq N(y)$, which also imply that $xy \in E(G)$.

For the "moreover" part, note that $d(x) + d(y) \geqslant \sum_{i\in[4]} n_i + 2k - 1$ implies that x and y have at least $2k - 1$ common neighbors and thus at least k common neighbors in one part. Therefore xy is rich. ☐

Recall that we have assumed that $\varphi(G) > 0$ and $n_1 \leqslant n_2 + n_3$. Thus,

$$e(G) > g_k(n_1, n_2, n_3, n_4) = (n_1 + n_4)(n_2 + n_3) + (k - 1)n_1. \tag{2.2}$$

Let R be the subgraph of G induced by the rich edges of G, and let $Z = V(R)$ be the set of the vertices of G that are incident to at least one rich edge.

Claim 2.3. *Suppose (∗), (2.2), and G is kK_3-free. Then the following assertions hold:*

- (*i*) *every vertex is contained in at most $k-1$ edges of R whose other ends are located in the same part of G; in particular, the maximum degree of R is at most $3k - 3$;*
- (*ii*) *$e(R) \leqslant 6(k-1)^2$ and $|Z| \leqslant 6(k-1)^2$;*
- (*iii*) *every triangle in G contains an edge in R.*

Proof. We first show (*i*) ⇒ (*ii*). Note that if R has a matching of size k, then we can greedily build k vertex-disjoint triangles by extending each rich edge in the matching. This contradicts the assumption that G is kK_3-free. Therefore, the largest matching in R is of size at most $k-1$ and consequently, R has a vertex cover of size at most $2(k-1)$. If the maximum degree of R is at most $3k-3$, then $e(R) \leqslant 2(k-1)(3k-4)+k-1 < 6(k-1)^2$ and $|Z| \leqslant 2(k-1)(3k-4)+2(k-1) = 6(k-1)^2$, confirming (*ii*).

To see (*i*), we assume that some vertex v is incident to k rich edges whose other ends are in the same part of G. If there is a copy S of $(k-1)K_3$ in $G\backslash\{v\}$, then we can pick a rich edge in $G\backslash S$ that contains v and then extend this rich edge to a triangle that does not intersect S. This gives a kK_3 in G, a contradiction. Thus, we infer that $G\backslash\{v\}$ is $(k-1)K_3$-free.

Let $n'_1 \geqslant n'_2 \geqslant n'_3 \geqslant n'_4$ be the sizes of four parts of $G\backslash\{v\}$. By (∗), we have $e(G\backslash\{v\}) \leqslant g_{k-1}(n'_1, n'_2, n'_3, n'_4)$. To estimate $g_{k-1}(n'_1, n'_2, n'_3, n'_4)$, we first observe that there exists $i_0 \in [4]$ such that $n'_i = n_i$ for all $i \neq i_0$ and $n_{i_0} = n_{i_0} - 1$; and furthermore, $n'_i = |V_i \backslash \{v\}|$ for $i \in [4]$ after relabeling V_1, V_2, V_3, V_4 if necessary (but maintaining $n_i = |V_i|$). This is obvious when $v \in V_{i_0}$ and $n_{i_0} > n_{i_0+1}$. Otherwise, for example, assume that $v \in V_1$ and $n_1 = n_2 > n_3$ (other cases are similar). Then $n'_1 = n_2 = n_1$ and $n'_2 = n_1 - 1 = n_2 - 1$. After relabeling V_1 and V_2, we have $v \in V_2$, and $n'_i = |V_i \backslash \{v\}|$ for $i \in [4]$.

By the definition of g, we consider two cases. When $n'_1 \leqslant n'_2 + n'_3$, we have

$$
\begin{aligned}
g_{k-1}(n'_1, n'_2, n'_3, n'_4) &= (n'_1 + n'_4)(n'_2 + n'_3) + (k-2)n'_1 \\
&\leqslant \begin{cases} (n_1 + n_4 - 1)(n_2 + n_3) + (k-2)n_1 & \text{if } v \in V_1 \cup V_4, \\ (n_1 + n_4)(n_2 + n_3 - 1) + (k-2)n_1, & \text{if } v \in V_2 \cup V_3. \end{cases}
\end{aligned} \tag{2.3}
$$

Together with (2.2) and (∗), this implies that

$$
\begin{aligned}
d_G(v) = e(G) - e(G \backslash \{v\}) &> g_k(n_1, n_2, n_3, n_4) - g_{k-1}(n'_1, n'_2, n'_3, n'_4) \\
&\geqslant \begin{cases} n_1 + n_2 + n_3 & \text{if } v \in V_1 \cup V_4, \\ 2n_1 + n_4, & \text{if } v \in V_2 \cup V_3, \end{cases}
\end{aligned}
$$

which is impossible. When $n'_1 > n'_2 + n'_3$, it must be the case when $n_1 = n_2 + n_3$ and $n'_{i_0} = n_{i_0} - 1$ for $i_0 \in \{2, 3\}$. Thus

$$
\begin{aligned}
g_{k-1}(n'_1, n'_2, n'_3, n'_4) &= n'_1(n'_2 + n'_3 + n'_4) + (k-2)(n'_2 + n'_3) \\
&= (n_2 + n_3)(n_1 + n_4 - 1) + (k-2)(n_1 - 1).
\end{aligned}
$$

Together with (2.2) and (∗), this implies that $d_G(v) > n_1 + n_2 + n_3$, which is impossible for any $v \in V(G)$.

To see *(iii)*, let S be a triangle in G and consider $G \backslash S$. Since G is kK_3-free, $G \backslash S$ is $(k-1)K_3$-free. By (∗), we have $e(G \backslash S) \leqslant g_{k-1}(n'_1, n'_2, n'_3, n'_4)$ where $n'_1 \geqslant n'_2 \geqslant n'_3 \geqslant n'_4$ are the sizes of parts of $G \backslash S$. We observe that there exists $i_0 \in [4]$ such that $n'_i = n_i - 1$ for $i \neq i_0$ and $n'_{i_0} = n_{i_0}$; furthermore, $n'_i = |V_i \backslash S|$ after relabeling V_1, V_2, V_3, V_4 if necessary (while maintaining $n_i = |V_i|$). This is obvious when $S \subset \bigcup_{i \neq i_0} V_i$ and either $i_0 = 1$ or $n_{i_0 - 1} > n_{i_0}$. Otherwise, for example, assume that $S \subset V_1 \cup V_2 \cup V_3$ and $n_2 > n_3 = n_4$ (other cases are similar). We have $n'_1 = n_1 - 1$, $n'_2 = n_2 - 1$, $n'_3 = n_4 = n_3$ and $n'_4 = n_3 - 1 = n_4 - 1$. After swapping V_3 and V_4, we have $S \subset V_1 \cup V_2 \cup V_4$.

If $n'_1 \leqslant n'_2 + n'_3$. then $g_{k-1}(n'_1, n'_2, n'_3, n'_4) = (n'_1 + n'_4)(n'_2 + n'_3) + (k-2)n'_1$. By our observation on the values of n'_1, n'_2, n'_3, n'_4, it follows that

$$
g_{k-1}(n'_1, n'_2, n'_3, n'_4) \leqslant \max_{j=1,2} \{(n_1 + n_4 - j)(n_2 + n_3 - (3 - j))\} + (k-2)n_1.
$$

If $n'_1 > n'_2 + n'_3$, then $g_{k-1}(n'_1, n'_2, n'_3, n'_4) = n'_1(n'_2 + n'_3 + n'_4) + (k-2)(n'_2 + n'_3)$. In this case, we must have $n_1 = n_2 + n_3 - t$ for $t = 0, 1$, $n'_2 = n_2 - 1$, and $n'_3 = n_3 - 1$. Thus $n'_i = n_i - 1$ either for $i \in [3]$ or for $i \in \{2, 3, 4\}$, and consequently

$$
\begin{aligned}
g_{k-1}(n'_1, n'_2, n'_3, n'_4) \leqslant \max\{&(n_1 - 1)(n_2 + n_3 + n_4 - 2) + (k-2)(n_2 + n_3 - 2), \\
&n_1(n_2 + n_3 + n_4 - 3) + (k-2)(n_2 + n_3 - 2).
\end{aligned}
$$

Since $n_1 = n_2 + n_3 - t$ for $t = 0, 1$, it follows that

$$
g_{k-1}(n'_1, n'_2, n'_3, n'_4) \leqslant \max_{j=1,2,3} \{(n_2 + n_3 - (3 - j))(n_1 + n_4 - j)\} + (k-2)(n_1 - 1).
$$

Putting all cases together with $e(G \backslash S) \leqslant g_{k-1}(n'_1, n'_2, n'_3, n'_4)$, we conclude that

$$
e(G \backslash S) \leqslant \max_{j=1,2,3} \{(n_1 + n_4 - j)(n_2 + n_3 - (3 - j))\} + (k-2)n_1. \tag{2.4}
$$

Recall that $e(S;G) := e(G) - e(G\backslash S)$. We next claim that $e(S;G) \geqslant \frac{3}{2}\sum_{i\in[4]} n_i + 3k$. Indeed, if the maximum in (2.4) is achieved by $j = 1, 2$, then, together with (2.2), it gives

$$e(S;G) > \sum_{i\in[4]} n_i + \min\{n_1 + n_4, n_2 + n_3\} + n_1 - 2 \geqslant \frac{3}{2}\sum_{i\in[4]} n_i + n_4 - 2 \geqslant \frac{3}{2}\sum_{i\in[4]} n_i + 3k,$$

where we used $n_4 \geqslant 6k^2$ in the last inequality. Otherwise, the maximum in (2.4) is achieved by $j = 3$, that is, $e(G\backslash S) \leqslant (n_1 + n_4 - 3)(n_2 + n_3) + (k-2)n_1$. By (2.2), we get

$$\begin{aligned} e(S;G) &> (n_1 + n_4)(n_2 + n_3) + (k-1)n_1 - (n_1 + n_4 - 3)(n_2 + n_3) - (k-2)n_1 \\ &= n_1 + 3n_2 + 3n_3 \geqslant \frac{3}{2}\sum_{i\in[4]} n_i + \frac{n_4}{2} \geqslant \frac{3}{2}\sum_{i\in[4]} n_i + 3k, \end{aligned}$$

where we used the assumption $n_2 + n_3 \geqslant n_1$ and $n_2, n_3 \geqslant n_4$.

Let $S = xyz$ and note that $d(x) + d(y) + d(z) = e(S;G) + 3$. By averaging, without loss of generality, we may assume that

$$d(x) + d(y) \geqslant \frac{2}{3}\left(\frac{3}{2}\sum_{i\in[4]} n_i + 3k\right) = \sum_{i\in[4]} n_i + 2k.$$

By the moreover part of Fact 2.2, xy is rich and we are done. □

For two disjoint sets $A, B \subseteq V(G)$, let $d(A,B) = e(A,B)/(|A||B|)$ be the density of the bipartite graph with parts A and B. A pair (V_i, V_j) is called *full* if $d(V_i\backslash Z, V_j) = d(V_j\backslash Z, V_i) = 1$; (V_i, V_j) is called *empty* if $e(V_i\backslash Z, V_j) = e(V_i, V_j\backslash Z) = 0$. We have the following observation.

Observation 2.4. *For distinct $i, j, t \in [4]$, if $d(V_i\backslash Z, V_j) = d(V_i\backslash Z, V_t) = 1$, then (V_j, V_t) must be empty because any edge in (V_j, V_t) but not in $(V_j \cap Z, V_t \cap Z)$ will create a triangle with at most one vertex in Z, contradicting (iii). In particular, if both (V_i, V_j) and (V_i, V_t) are full, then (V_j, V_t) is empty.*

Claim 2.5. *Fix $i \neq j \in [4]$. If $d(x) + d(y) \geqslant \sum_{i\in[4]} n_i$ for every edge $xy \in V_i \times V_j$, then either*

- *$e(V_i\backslash Z, V_j\backslash Z) = 0$ (this is weaker than (V_i, V_j) being empty) or*
- *$d(V_i\backslash Z, V_j) = d(V_j\backslash Z, V_i) = 1$, and $d(x) + d(y) = \sum_{i\in[4]} n_i$.*

Moreover, if $d(x) + d(y) > \sum_{i\in[4]} n_i$ for every edge $xy \in V_i \times V_j$, then (V_i, V_j) is empty.

Proof. Assume that $\{i, j, t, \ell\} = [4]$. Suppose there is an edge $xy \in (V_i\backslash Z) \times (V_j\backslash Z)$. Note that if x and y have a common neighbor z, then as $x, y \notin Z$, none of the edges of xyz is rich, contradicting (iii). Thus, x and y have no common neighbor. By Fact 2.2, $d(x) + d(y) \leqslant \sum_{i\in[4]} n_i$. If $d(x) + d(y) \geqslant \sum_{i\in[4]} n_i$, then Fact 2.2 implies that $V_j \subseteq N(x)$ and $V_i \subseteq N(y)$. In particular, $xy' \in E(G)$ for every $y' \in V_j\backslash Z$. Applying the same argument to the edge xy', we obtain that $V_i \subseteq N(y')$. Similarly, we can derive that $V_j \subseteq N(x')$ for every $x' \in V_i\backslash Z$. Thus, $d(V_i\backslash Z, V_j) = d(V_j\backslash Z, V_i) = 1$.

Now assume $d(x) + d(y) > \sum_{i\in[4]} n_i$ for every edge $xy \in V_i \times V_j$. If $e(V_i\backslash Z, V_j\backslash Z) \neq 0$, then the arguments in the previous paragraph provide a contradiction. Suppose there is an edge $xy \in (V_i \cap Z) \times (V_j\backslash Z)$. As $d(x) + d(y) > \sum_{i\in[4]} n_i$, x and y have some common neighbors in $V_t \cup V_\ell$. But since $y \notin Z$, by (iii), their common neighbors must be in $(V_t \cup V_\ell) \cap Z$. Since $e(V_i\backslash Z, V_j\backslash Z) = 0$, we know that $N(y) \cap V_i \subseteq V_i \cap Z$. Altogether, we obtain that $d(x) + d(y) \leqslant n_j + n_t + n_\ell + |Z| < \sum_{i\in[4]} n_i$, a contradiction. Analogous arguments show that there is no edge in $(V_i\backslash Z) \times (V_j \cap Z)$. Thus, $e(V_i\backslash Z, V_j) = e(V_i, V_j\backslash Z) = 0$, that is, (V_i, V_j) is empty. □

Consider a set $T \subseteq V(G)$ defined in **Cases 1–4** and let n_1', n_2', n_3', n_4' denote the sizes of the parts of $G\backslash T$. Then $\varphi(G) < \varphi(G\backslash T)$ is equivalent to

$$e(G) - g_k(n_1, n_2, n_3, n_4) < e(G\backslash T) - g_k(n_1', n_2', n_3', n_4'),$$

or $e(T; G) < g_k(n_1, n_2, n_3, n_4) - g_k(n_1', n_2', n_3', n_4')$. We will prove by contradiction, assuming that $\varphi(G) \geqslant \varphi(G\backslash T)$, equivalently,

$$e(T; G) \geqslant (n_1 + n_4)(n_2 + n_3) + (k-1)n_1 - g_k(n_1', n_2', n_3', n_4') \tag{2.5}$$

for every $T \subseteq V(G)$ defined in **Cases 1–4**.

The case when $T = \{v\} \subseteq V_1$ occurs in all four cases so we consider it before the cases. Since $n_1 \leqslant n_2 + n_3$, we have three possibilities:

- if $n_1 > n_2$, then $g_k(n_1 - 1, n_2, n_3, n_4) = (n_1 - 1 + n_4)(n_2 + n_3) + (k-1)(n_1 - 1)$;
- if $n_1 = n_2 > n_4$, then $g_k(n_1 - 1, n_2, n_3, n_4) = (n_1 + n_4)(n_2 + n_3 - 1) + (k-1)n_1$;
- if $n_1 = n_4$, then $g_k(n_1 - 1, n_2, n_3, n_4) = (n_1 + n_4 - 1)(n_2 + n_3) + (k-1)n_1$;

Thus (2.5) implies that for every $v \in V_1$,

$$d(v) \geqslant \begin{cases} n_2 + n_3 + k - 1, & \text{if } n_1 > n_2, \\ n_1 + n_4, & \text{if } n_1 = n_2. \end{cases} \tag{2.6}$$

2.2. **Proof of Cases 1–4.** After these preparations, we return to the proof of **Cases 1–4**. Recall that $n_1 \leqslant n_2 + n_3$ in all these cases. Recall also that $n_i \geqslant 6k^2$ for $i \in [4]$, so we can always assume that $V_i\backslash Z \neq \varnothing$. Moreover, by (2.1), we have $M_0(k) \geqslant N_0(k-1) + 3$, and thus we can apply the induction hypothesis $(*)$ on any $(k-1)K_3$-*free* subgraph $G\backslash S$, whenever $|S| \leqslant 3$ (and thus $v_4(G\backslash S) \geqslant 6k^2 - 3 \geqslant 6(k-1)^2$).

Case 1. $n_1 > n_3$ **and** $n_2 > n_4$**.**

In this case (2.5) holds for every crossing set $T = xy \in V_1 \times V_2$. Since the part sizes of $G\backslash\{x, y\}$ are $n_1 - 1 \geqslant \{n_2 - 1, n_3\} \geqslant n_4$. By (2.5), we have

$$\begin{aligned} e(xy; G) &\geqslant (n_1 + n_4)(n_2 + n_3) + (k-1)n_1 - ((n_1 + n_4 - 1)(n_2 + n_3 - 1) + (k-1)(n_1 - 1)) \\ &= \sum_{i \in [4]} n_i + k - 2. \end{aligned}$$

If $xy \in E(G)$, then $d(x) + d(y) = e(xy; G) + 1 \geqslant \sum_{i \in [4]} n_i + k - 1 > \sum_{i \in [4]} n_i$. By Claim 2.5, (V_1, V_2) is empty. For every $x \in V_1\backslash Z$, we thus have $d(x) \leqslant n_3 + n_4 < \min\{n_2 + n_3, n_1 + n_4\}$, contradicting (2.6).

Case 2. $n_1 = n_2 = n_3 \geqslant n_4 > 6k^2$**.**

In this case (2.5) holds for any one-element set $T \subset V(G)$. Write $n_1 = n_2 = n_3 = n$. For any $x \in V_1 \cup V_2 \cup V_3$, by (2.5), we have

$$d(x) = e(\{x\}; G) \geqslant 2n(n + n_4) + (k-1)n - g_k(n, n, n-1, n_4),$$

where $g_k(n, n, n-1, n_4) = (2n-1)(n + n_4) + (k-1)n$ if $n > n_4$ and $g_k(n, n, n-1, n_4) = 2n(n + n_4 - 1) + (k-1)n$ if $n = n_4$. Thus, we have $d(x) \geqslant \min\{2n, n + n_4\} = n + n_4$. Similarly, for $y \in V_4$, by (2.5), we have

$$d(y) = e(\{y\}; G) \geqslant 2n(n + n_4) + (k-1)n - \big(2n(n + n_4 - 1) + (k-1)n\big) = 2n. \tag{2.7}$$

These together imply $d(x) + d(y) \geqslant \sum n_i$ for every edge $xy \in (V_1 \cup V_2 \cup V_3) \times V_4$. For $i = 1, 2, 3$, Claim 2.5 implies that either (V_i, V_4) is full or $e(V_i\backslash Z, V_4\backslash Z) = 0$. If $e(V_i\backslash Z, V_4\backslash Z) = 0$ holds for at least two values of $i \in \{1, 2, 3\}$, then for every $y \in V_4\backslash Z$, we have $d(y) \leqslant n + |Z| < 2n$ (as $n \geqslant M_0(k)/4 > 6k^2$), contradicting (2.7).

This implies that at least two of (V_1, V_4), (V_2, V_4), and (V_3, V_4) must be full. Without loss of generality, assume (V_1, V_4) and (V_2, V_4) are full. By Observation 2.4, (V_1, V_2) is empty. Next, we claim that (V_3, V_4) is empty. Indeed, let $x \in V_2 \backslash Z$ and recall that $d(x) \geqslant n + n_4$. Since (V_1, V_2) is empty, we have $d(x) \leqslant n + n_4$. Thus, $d(x) = n + n_4$ and in particular $V_3 \subseteq N(x)$. Since this holds for every $x \in V_2 \backslash Z$, it follows that $d(V_2 \backslash Z, V_3) = 1$. Thus (V_3, V_4) is empty by Observation 2.4. Together with *(ii)*, we infer

$$e(G) = e(G[Z]) + e(V \backslash Z; G) < \binom{|Z|}{2} + (n_1 + n_2)(n_3 + n_4) \leqslant (n_1 + n_2)(n_3 + n_4) + (k-1)n_1,$$

contradicting (2.2), The previous inequality follows from $\binom{|Z|}{2} \leqslant 18(k-1)^4 \leqslant (k-1)n_1$, which follows from $n_1 \geqslant M_0(k)/4$ and (2.1).

Case 3. $n_1 = n_2 = n_3 > n_4 = 6k^2$**.**

Write $n_1 = n_2 = n_3 = n$. We assume that

$$n_1 \geqslant 30k^2, \tag{2.8}$$

as otherwise $\sum n_i \leqslant 3 \cdot 30k^2 + 6k^2 \leqslant M_0(k)$ by (2.1), contradicting the assumption $|G| > M_0(k)$. By (2.6) and the similarity of V_1, V_2, and V_3, we have $d(x) \geqslant n + n_4$ for every $x \in V_1 \cup V_2 \cup V_3$. We claim that for $y \in V_4$,

$$d(y) \leqslant 2n + 2k - 1. \tag{2.9}$$

Otherwise, pick k neighbors $x_1, \ldots, x_k$ of y from the same part of G. For each i, since $d(x_i) \geqslant n + n_4$, we have $d(x_i) + d(y) \geqslant \sum n_i + 2k - 1$, yielding that $x_i y$ is rich by Fact 2.2. However, this contradicts *(i)*.

Claim. The graph $G[V_1 \cup V_2 \cup V_3]$ is K_3-free.

Proof. Suppose instead, there exists a triangle $xyz \in V_1 \times V_2 \times V_3$. Without loss of generality, assume that $d(x) \geqslant d(y) \geqslant d(z)$. We first claim that

$$d(x) + d(y) + d(z) \geqslant 5n + 2n_4 + k. \tag{2.10}$$

Otherwise $d(x)+d(y)+d(z) \leqslant 5n+2n_4+k-1$ and $e(xyz; G) = d(x)+d(y)+d(z)-3 \leqslant 5n+2n_4+k-4$. Then, by (2.2),

$$\begin{aligned} e(G \backslash \{x, y, z\}) &= e(G) - e(xyz; G) > g_k(n, n, n, n_4) - (5n + 2n_4 + k - 4) \\ &= 2n(n + n_4) + (k-1)n - (5n + 2n_4 + k - 4) \\ &= (2n-2)(n - 1 + n_4) + (k-2)(n-1) \\ &= g_{k-1}(n-1, n-1, n-1, n_4). \end{aligned}$$

By induction hypothesis $(*)$, we obtain a copy of $(k-1)K_3$ in $G \backslash \{x, y, z\}$. Together with the triangle xyz, this contradicts the assumption G is kK_3-free.

We next claim that at least two of xy, yz, xz are rich and thus all $x, y, z \in Z$. Indeed, if $d(x) < 2n + n_4 - k$, then by (2.10),

$$d(y) + d(z) > 5n + 2n_4 + k - (2n + n_4 - k) = 3n + n_4 + 2k > \sum n_i + 2k - 1.$$

By Fact 2.2, yz is rich. Since $d(x)$ is the largest, this argument implies that all three edges of xyz are rich, as desired. Otherwise, $d(x) \geqslant 2n + n_4 - k$ and recall that $d(y) \geqslant d(z) \geqslant n + n_4$. Thus

$$d(x) + d(y) \geqslant d(x) + d(z) \geqslant 3n + 2n_4 - k \geqslant \sum n_i + 2k - 1$$

because $n_4 = 6k^2 \geqslant 3k - 1$. By Fact 2.2, both xy and xz are rich, as desired.

The claim in the previous paragraph applies to all triangles in $V_1 \cup V_2 \cup V_3$. Therefore, all the common neighbors of x and y in $V_1 \cup V_2 \cup V_3$ are in Z and consequently, $|N(x) \cap N(y)| \leqslant |Z| + |V_4| \leqslant 6k^2 + n_4$, and consequently, $d(x) + d(y) \leqslant \sum n_i + 6k^2 + n_4 = 3n + 2n_4 + 6k^2$. On the other hand, (2.10) and the assumption $d(x) \geqslant d(y) \geqslant d(z)$ imply that

$$d(x) + d(y) \geqslant \frac{2}{3}(5n + 2n_4 + k) = \frac{10}{3}n + \frac{4}{3}n_4 + \frac{2}{3}k > 3n + 2n_4 + 6k^2 \tag{2.11}$$

because $n \geqslant 30k^2 = 2n_4 + 18k^2$ by (2.8). This gives a contradiction. □

By the claim, $G[V_1 \cup V_2 \cup V_3]$ is K_3-free, and thus has at most $2n^2$ edges by Theorem 1.1. Together with (2.9) and (2.8), we obtain that

$$e(G) \leqslant 2n^2 + n_4 \cdot (2n + 2k - 1) = 2n(n + n_4) + (2k - 1)n_4 < 2n(n + n_4) + (k - 1)n,$$

contradicting (2.2).

Case 4. $n_1 > n_2 = n_3 = n_4$**.**

Assume $n_2 = n_3 = n_4 = n$ and recall that $n_1 \leqslant 2n$. We first claim that

$$d(x) \leqslant 3n \text{ for all } x \in V_1, \text{ and } d(y) \leqslant n_1 + n + k - 1 \text{ for all } y \in V_2 \cup V_3 \cup V_4. \tag{2.12}$$

Indeed, the bound $d(x) \leqslant 3n$ for $x \in V_1$ is trivial. Suppose to the contrary, that there is a vertex $y \in V_2 \cup V_3 \cup V_4$ with $d(y) \geqslant n_1 + n + k$. It follows that $|N(y) \cap V_1| \geqslant d(y) - 2n \geqslant k$. Assume that $x_1, \ldots, x_k \in N(y) \cap V_1$. By (2.6), we have $d(x_j) \geqslant 2n + k - 1$. Thus, we infer that $d(x_j) + d(y) \geqslant n_1 + 3n + 2k - 1$. By Fact 2.2, we have $x_1 y, \ldots, x_k y \in E(R)$. However, this contradicts *(i)*.

We next claim that there is no rich edge in $V_1 \times (V_2 \cup V_3 \cup V_4)$. Suppose to the contrary, that $xy \in V_1 \times (V_2 \cup V_3 \cup V_4)$ is a rich edge. By (2.12), we have $e(xy; G) = d(x) + d(y) - 1 \leqslant n_1 + 4n + k - 2$. By (2.2), it follows that

$$\begin{aligned} e(G \backslash \{x, y\}) &= e(G) - e(xy; G) > 2n(n_1 + n) + (k - 1)n_1 - (n_1 + 4n + k - 2) \\ &= 2n(n_1 + n - 2) + (k - 2)(n_1 - 1) \\ &= g_{k-1}(n_1 - 1, n, n, n - 1). \end{aligned}$$

By induction hypothesis (∗), $G \backslash \{x, y\}$ contains a copy S of $(k - 1)K_3$. Since xy is rich, we can find a triangle in $G \backslash S$ containing xy, contradicting the assumption that G is kK_3-free.

Now we show that there is no triangle intersecting V_1. Suppose to the contrary, there is a triangle xyz with $x \in V_1$. If $d(x) + d(z) \geqslant n_1 + 3n + 2k - 1$, then, by Fact 2.2, xy is rich, contradicting our earlier claim. We thus assume that $d(x) + d(z) < n_1 + 3n + 2k - 1$. Together with (2.12), it gives that $d(x) + d(y) + d(z) < 2n_1 + 4n + 3k - 2$, and $e(xyz; G) = d(x) + d(y) + d(z) - 3 < 2n_1 + 4n + 3k - 5$. By (2.2), it follows that

$$\begin{aligned} e(G \backslash \{x, y, z\}) &= e(G) - e(xyz; G) > 2n(n_1 + n) + (k - 1)n_1 - (2n_1 + 4n + 3k - 5) \\ &= (n_1 + n - 2)(2n - 1) + (k - 2)(n_1 - 1) + n - 2k + 1 \\ &= g_{k-1}(n_1 - 1, n, n - 1, n - 1) + n - 2k + 1. \end{aligned}$$

By (∗), $G \backslash \{x, y, z\}$ contains a copy of $(k - 1)K_3$. Together with the triangle xyz, this contradicts the assumption that G is kK_3-free.

We assumed that G contains $k - 1$ disjoint triangles. Let T_1 be a triangle of G. By the claim of the previous paragraph, T_1 must be in $V_2 \cup V_3 \cup V_4$. Moreover, by *(iii)*, T_1 must contain a rich edge xy. Below we show that

$$e(G \backslash \{x, y\}) > g_{k-1}(n_1, n, n - 1, n - 1). \tag{2.13}$$

Then, by (∗), $G\backslash\{x, y\}$ contains a copy S of $(k-1)K_3$. Since xy is rich, we can find a triangle in $G\backslash S$ containing xy, contradicting the assumption that G is kK_3-free.

We first assume that $n_1 = 2n$. If $d(x) + d(y) > 6n$, then x and y have a common neighbor in V_1, contradicting the earlier claim that there is no triangle intersecting V_1. We thus assume that $d(x) + d(y) \leqslant 6n$. Thus $e(xy; G) \leqslant 6n - 1$. By (2.2), it follows that

$$\begin{aligned} e(G\backslash\{x, y\}) &> g_k(2n, n, n, n) - (6n - 1) \\ &= 3n \cdot 2n + 2n(k-1) - (6n - 1) \\ &= 2n(3n - 2) + (k - 2)(2n - 1) + k - 1 \\ &= g_{k-1}(2n, n, n - 1, n - 1) + k - 1. \end{aligned}$$

Thus (2.13) holds. Second, assume $n_1 < 2n$. By (2.12), we have $e(xy; G) = d(x) + d(y) - 1 \leqslant 2(n_1 + n + k - 1) - 1$. By (2.2), it follows that

$$\begin{aligned} e(G\backslash\{x, y\}) &> g_k(n_1, n, n, n) - (2n_1 + 2n + 2k - 3) \\ &= (n_1 + n)2n + (k - 1)n_1 - (2n_1 + 2n + 2k - 3) \\ &= (n_1 + n - 1)(2n - 1) + (k - 2)n_1 + n - 2k + 2 \\ &= g_{k-1}(n_1, n, n - 1, n - 1) + n - 2k + 2. \end{aligned}$$

Thus (2.13) holds.

The proof of Theorem 2.1 is now completed. □

3. Concluding remarks

In this paper we solved Problem 1.2 for $r = 4$ and $t = 3$ when all n_i's are large. The idea in our proof should be helpful for proving Conjecture 1.4 in general. However, to determine the maximum in (1.1), there are quite a few cases to consider even when $r = 5$ and $t = 3$. Indeed, suppose $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_5$ and $\{I, I'\}$ is the bipartition of $[5]$ that attained the maximum in (1.1). Assume $1 \in I$. Depending on the values of $n_1, \ldots, n_5$, it is possible to have

$$I = \{1\} \text{ or } \{1, 2\} \text{ or } \{1, 3\} \text{ or } \{1, 4\} \text{ or } \{1, 5\} \text{ or } \{1, 4, 5\}.$$

Another open problem is to find the smallest $N_0(k)$ such that Theorem 1.3 holds. The $N_0(k)$ provided in our proof is a double exponential function of k. Indeed, by (2.1) and $N_0(1) = 1$, we have $M_0(2) = 96 \cdot 2^2 = 384$ and $N_0(2) = 384^2$. It is easy to see that $N_0(k) = (N_0(k-1) + 3)^2$ for $k \geqslant 3$. Thus $N_0(k-1)^2 \leqslant N_0(k) \leqslant 2N_0(k-1)^2$ for $k \geqslant 3$. It follows that

$$N_0(2)^{2^{k-2}} \leqslant N_0(k) \leqslant \big(2N_0(2)\big)^{2^{k-2}}.$$

It is interesting to know whether one can reduce $N_0(k)$ to a polynomial function (or even a linear function) of k.

Acknowledgements. We would like to thank Chunqiu Fang and Longtu Yuan for valuable feedbacks on an earlier version of the manuscript and thank Ming Chen, Jie Hu and Donglei Yang for helpful discussions. We also thank two anonymous referees for their helpful comments that improved the presentation of this paper.

References

[1] P. Bennett, S. English, and M. Talanda-Fisher, *Weighted Turán problems with applications*, Discrete Math. **342** (2019), no. 8, 2165–2172. MR3952133 ↑1, 1

[2] B. Bollobás, P. Erdős, and E. G. Straus, *Complete subgraphs of chromatic graphs and hypergraphs*, Utilitas Math. **6** (1974), 343–347. MR379256 ↑1, 1.1

[3] B. Bollobás, *Extremal graph theory*, London Mathematical Society Monographs, vol. 11, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1978. MR506522 ↑1
[4] H. Chen, X. Li, and J. Tu, *Complete solution for the rainbow numbers of matchings*, Discrete Math. **309** (2009), no. 10, 3370 –3380. ↑1, 1
[5] J. De Silva, K. Heysse, and M. Young, *Rainbow number for matchings in r-partite graphs, preprint.* ↑1, 1, 2
[6] J. De Silva, K. Heysse, A. Kapilow, A. Schenfisch, and M. Young, *Turán numbers of vertex-disjoint cliques in r-partite graphs*, Discrete Math. **341** (2018), no. 2, 492–496. MR3724116 ↑1, 1.2, 1
[7] W. Mantel, *Problem 28*, Wiskundige Opgaven **10** (1907), 60–61. ↑1
[8] M. Simonovits, *A method for solving extremal problems in graph theory, stability problems*, Theory of Graphs (Proc. Colloq., Tihany, 1966), 1968, pp. 279–319. MR0233735 ↑1, 2
[9] P. Turán, *On an extremal problem in graph theory, (Hungarian)*, Mat. Fiz. Lapok **48** (1941), 436–452. ↑1
[10] A. Z. Wagner, *Refuting conjectures in extremal combinatorics via linear programming*, J. Combin. Theory Ser. A **169** (2020), 105130. ↑1
[11] K. Zarankiewicz, *Problem p 101*, Colloq. Math. **3** (1954), 19–30. ↑1

Department of Mathematics, University of Rhode Island, 5 Lippitt Road, Kingston, RI, 02881, USA
E-mail address: `jie_han@uri.edu`

Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303
E-mail address: `yzhao6@gsu.edu`