TURÁN NUMBER OF COMPLETE MULTIPARTITE GRAPHS IN MULTIPARTITE GRAPHS

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ABSTRACT. In this paper we study a multi-partite version of the Erdős–Stone theorem. Given integers $r < k$ and $t \geqslant 1$, let $\mathrm{ex}_k(n, K_{r+1}(t))$ be the maximum number of edges of $K_{r+1}(t)$-free k-partite graphs with n vertices in each part, where $K_{r+1}(t)$ is the t-blowup of K_{r+1}. An easy consequence of the supersaturaion result gives that $\mathrm{ex}_k(n, K_{r+1}(t)) = \mathrm{ex}_k(n, K_{r+1}) + o(n^2)$. Similar to a result of Erdős and Simonovits for the non-partite case, we find that the error term is closely related to the (multi-partite) Zarankiewicz problem. Using such Zarankiewicz numbers, for $t = 2, 3$, we determine the error term up to an additive linear term; using some natural assumptions on such Zarankiewicz numbers, we determine the error term up to an additive constant depending on k, r and t. We actually obtain exact results in many cases, for example, when $k \equiv 0, 1 \pmod r$. Our proof uses the stability method and starts by proving a stability result for K_{r+1}-free multi-partite graphs.

1. INTRODUCTION

Generalizing Mantel's theorem from 1907 [14], Turán's theorem from 1941 [17] started the systemetic study of Extremal Graph Theory. Given a graph F, let $\mathrm{ex}(n, F)$ denote the largest number of edges in a graph not containing F as a subgraph (called F-free). Let K_r denote the complete graph on r vertices and $T_r(n)$ denote the complete r-partite graph on n vertices with $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ in each part (known as the Turán graph); and $t_r(n)$ be the size of $T_r(n)$. Turán's theorem [17] states that $\mathrm{ex}(n, K_{r+1}) = t_r(n)$ for all $n \geqslant r \geqslant 1$ and in addition, $T_r(n)$ is the unique extremal graph.

Let $K_{t_1,\dots,t_r}$ denote the complete r-partite graph with parts of size $t_1, \dots, t_r$ and write $K_r(t) = K_{t,\dots,t}$ with r parts. A celebrated result of Erdős and Stone [8] determines $\mathrm{ex}(n, K_{r+1}(t))$ asymptotically:

$$\mathrm{ex}(n, K_{r+1}(t)) = t_r(n) + o(n^2) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} + o(n^2).$$

Erdős [4] and Simonovits [15] independently improved the error term above to $O(n^{2-1/t})$. Simonovits [15] also showed that any extremal graph for $K_{r+1}(t)$ can be obtained from $T_r(n)$ by adding or removing $O(n^{2-1/t})$ edges. Later Erdős and Simonovits [7] determined the structure of extremal graphs for $K_{r+1}(t)$ for $t \leqslant 3$ as follows.

Theorem 1. [7] *For $t \leqslant 3$, every extremal graph G for $K_{r+1}(t)$ has a vertex partition $U_1, \dots, U_r$ such that*

- *$G[U_i, U_j]$ is complete for all $i \neq j$,*
- *$G[U_i] = n/r + o(n)$,*
- *$G[U_1]$ is extremal for $K_{t,t}$, and*
- *$G[U_2], \dots, G[U_r]$ are extremal for $K_{1,t}$.*

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The restriction $t \leqslant 3$ in Theorem 1 comes from our knowledge on $\mathrm{ex}(n, K_{t,t})$. A well-known open problem in Extremal Graph Theory is proving $\mathrm{ex}(n, K_{t,t}) = \Omega(n^{2-1/t})$ and this is only known for $t \leqslant 3$.

Extremal problems whose host graphs are multipartite graphs have been studied since 1951, when Zarankiewicz proposed the study of the largest number of edges in a bipartite graph not containing a copy of $K_{s,t}$. Let $\mathcal{G}(n_1, \dots, n_k)$ denote the family of k-partite graphs with $n_1, \dots, n_k$ vertices in its parts and write $\mathcal{G}_k(n) = \mathcal{G}(n, \dots, n)$ with k parts. Given a graph F, define $\mathrm{ex}(n_1, \dots, n_k; F)$ as the largest number of edges in F-free graphs from $\mathcal{G}(n_1, \dots, n_k)$, and let $\mathrm{ex}_k(n, F) = \mathrm{ex}(n, \dots, n; F)$ (with k parts). (Trivially $\mathrm{ex}_k(n, F) = \binom{k}{2} n^2$ if the chromatic number $\chi(F) > k$.) In 1975 Bollobás, Erdős, and Szemerédi [1] investigated several Turán-type problem for multipartite graphs. Applying a simple counting argument, they showed that

$$\mathrm{ex}_k(n, K_{r+1}) = t_r(k) n^2 \tag{1.1}$$

for any $n, k, r \in \mathbb{N}$ with $k \geqslant r$. The main results of [1] concern the minimum degree version of this problem, which has been intensively studied, see [9–11, 13, 16].

In this paper we study $\mathrm{ex}_k(n, K_{r+1}(t))$, the multi-partite versions of the Erdős–Stone theorem and Theorem 1.

We first prove a stability theorem for $\mathrm{ex}_k(n, K_{r+1})$. This result was independently obtained by Chen, Lu, and Yuan [3]. Given $r, k \in \mathbb{N}$ with $k \geqslant r$, write $k = ar + b$ for $0 \leqslant b \leqslant r - 1$. By Turán's theorem, the Turán graph $T_r(k) = K_{a,\dots,a,a+1,\dots,a+1}$ (with b parts of size $a + 1$ and $r - b$ parts of size a) is the unique largest K_{r+1}-free graph on k vertices. We now describe extremal graphs for $\mathrm{ex}_k(n, K_{r+1})$. Let $\mathcal{T}_{r,k}(n)$ be the collection of k-partite graphs with parts $V_1, \dots, V_k$ of size n defined as follows. If $b > 0$, we arbitrarily divide $V_{ar+1}, \dots, V_k$ into r sets $W_1, \dots, W_r$ (some of them may be empty) such that each W_i is a subset of V_j for some j; if $b = 0$, then let $W_1, \dots, W_r$ be empty sets. Now let T be the r-partite graph with parts $U_1, \dots, U_r$ such that

$$U_i = W_i \cup Z_i, \text{ where } Z_i := V_{(i-1)a+1} \cup \dots \cup V_{ia},$$

obtained from the complete r-partite graph $K(U_1, \dots, U_r)$ by removing edges between W_i and $W_{i'}$, $i \neq i'$, whenever $W_i, W_{i'} \subseteq V_j$ for some j (in other words, $T = K(U_1, \dots, U_r) \cap K(V_1, \dots, V_k)$). Since T is r-partite, it is K_{r+1}-free. Let $U = \bigcup_{i \in [r]} U_i$ and $W = \bigcup_{i \in [r]} W_i$. Note that $e_T(U) = \binom{r}{2} a^2 n^2$ and $e_T(W, T) = |W|(k - a - 1)n = b(k - a - 1)n^2$. Since $t_r(k) = \binom{r}{2} a^2 + b(k - a - 1)$, it follows that $e(T) = t_r(k) n^2$. By (1.1), T is an extremal graph for K_{r+1}. Let $\mathcal{T}_{r,k}(n)$ be the collection of all such T.

Given two graphs $G, H \in \mathcal{G}_k(n)$ on the same parts $V_1, \dots, V_k$, we say that G and H are γ-close if $|E(G) \triangle E(H)| \leqslant \gamma n^2$.

Theorem 2. *For any positive integers $r \leqslant k$ and any $\gamma > 0$, there exist $\varepsilon > 0$ and n_0 such that the following holds for every integer $n \geqslant n_0$. Suppose $G \in \mathcal{G}_k(n)$ is K_{r+1}-free and $e(G) \geqslant (t_r(k) - \varepsilon) n^2$. Then G is γ-close to a member of $\mathcal{T}_{r,k}(n)$.*

We now consider $\mathrm{ex}_k(n, K_{r+1}(t))$, We assume that $k \geqslant r + 1$ because otherwise $\mathrm{ex}_k(n, K_{r+1}(t)) = \binom{k}{2} n^2$ trivially. Applying (1.1) and either the Regularity Lemma or the Graph Removal Lemma, one can easily derive the following Erdős–Stone theorem for multipartite graphs:

$$\mathrm{ex}_k(n, K_{r+1}(t)) = t_r(k) n^2 + o(n^2).$$

Applying Theorem 2 and the Graph Removal Lemma, we derive the following stability result for $\mathrm{ex}_k(n, K_{r+1}(t))$, which handles the *non-extremal case* for $\mathrm{ex}_k(n, K_{r+1}(t))$.

Theorem 3. *For any $k, r, t \in \mathbb{N}$ and any $\gamma > 0$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for every integer $n \geqslant n_0$. Suppose $G \in \mathcal{G}_k(n)$ is $K_{r+1}(t)$-free and $e(G) \geqslant (t_r(k) - \varepsilon)n^2$. Then G is γ-close to a member of $\mathcal{T}_{r,k}(n)$. In particular, we have $e(G) \leqslant (t_r(k) + \gamma)n^2$.*

To give the precise value of $\mathrm{ex}_k(n, K_{r+1}(t))$, we need the following definitions. Given $a, t, n_1, \ldots, n_a \in \mathbb{N}$, let $z_t^{(a)}(n_1, \ldots, n_a)$ be the a-partite Zarankiewicz number for $K_{t,t}$, namely, the maximum number of edges in a $K_{t,t}$-free a-partite graph with part sizes $n_1, \ldots, n_a$. We also write $z_t^{(a)}(n)$ for $z_t^{(a)}(n, \ldots, n)$. Given $r, k, t, n \in \mathbb{N}$ with $k \geqslant r + 1$, assume $k = ar + b$ for some $0 \leqslant b \leqslant r - 1$, in particular, $b \geqslant 1$ if $a = 1$. Let

$$t_1 := \left\lfloor \frac{t-1}{4a^2 + 6a} \right\rfloor, \quad \text{and} \quad b' := \min\left\{b - 1, \left\lfloor \frac{r-b}{2} \right\rfloor\right\}.$$

Note that $b' = b$ if and only if $b \leqslant (r + 2)/3$. The following two functions will appear in our lower and upper bounds for $\mathrm{ex}_k(n, K_{r+1}(t))$, respectively.

$$h_1(n, r, k, t) := \begin{cases} \lfloor \frac{t-1}{2}(k - a - 1)n \rfloor & \text{if } a \geqslant 2,\ b \geqslant 2 \text{ and } t_1 = 0, \\ \lfloor \frac{t-1}{2}(k - a - 1)n \rfloor + \min\{b - 1, r - b\} \frac{(2a^2+3a)t_1^2}{2} - \frac{r-1}{2} & \text{if } a \geqslant 2,\ b \geqslant 2 \text{ and } t_1 > 0, \\ (r - 1)\lfloor \frac{t-1}{2} an \rfloor & \text{if } a \geqslant 2 \text{ and } b = 0, 1, \\ (t - 1)(b - 1)n + b' \lfloor \frac{(t-1)^2}{4} \rfloor & \text{if } a = 1 \text{ and } b \geqslant 1, \end{cases}$$

and

$$h_2(n, r, k, t) := \begin{cases} \lfloor \frac{t-1}{2}(k - a - 1)n \rfloor + (r - b) \frac{(t-1)^2}{16(a-1)} & \text{if } a \geqslant 2, t \geqslant a, \\ \lfloor \frac{t-1}{2}(k - a - 1)n \rfloor & \text{if } a \geqslant 2, t < a, \\ (r - 1)\lfloor \frac{t-1}{2} an \rfloor & \text{if } a \geqslant 2 \text{ and } b = 0, 1, \\ (t - 1)(b - 1)n + (b - 1)\lfloor \frac{(t-1)^2}{4} \rfloor & \text{if } a = 1 \text{ and } b \geqslant 1. \end{cases}$$

Note that $h_1(n, r, k, t) = h_2(n, r, k, t)$ in the following cases:

$$(1)\ a \geqslant 2 \text{ and } t < a, \quad (2)\ a \geqslant 2 \text{ and } b = 0, 1, \quad (3)\ a = 1 \text{ and } 1 \leqslant b \leqslant (r + 2)/3. \tag{1.2}$$

For $i = 1, 2$, we accordingly define

$$g_i(n, r, k, t) := t_r(k)n^2 + z_t^{(\lceil k/r \rceil)}(n) + h_i(n, r, k, t),$$

and note that $\lceil k/r \rceil$ is equal to $a + 1$ if $b > 0$ and a otherwise.

We have the following lower bound for $\mathrm{ex}_k(n, K_{r+1}(t))$.

Theorem 4. *Given $r, k, t, n \in \mathbb{N}$, we have $\mathrm{ex}_k(n, K_{r+1}(t)) \geqslant g_1(n, r, k, t)$.*

The second terms in the second and the fourth cases of the definition of $h_1(n, r, k, t)$ show that $\mathrm{ex}_k(n, K_{r+1}(t)) > t_r(k)n^2 + z_t^{(\lceil k/r \rceil)}(n) + \lfloor \frac{t-1}{2}(k - a - 1)n \rfloor$ when $a, b \geqslant 2$ and $t_1 > 0$, and $\mathrm{ex}_k(n, K_{r+1}(t)) > t_r(k)n^2 + z_t^{(\lceil k/r \rceil)}(n) + (t - 1)(b - 1)n$ when $b \geqslant a = 1$, and we shall elaborate on this at the end of this section.

Although we can use $z_t^{(a)}(n)$ without knowing its precise value in the lower bound, our proofs of the upper bounds need several estimates on it. Kővári, Sós, Turán [12] showed that $z_t^{(2)}(n) = O(n^{2-1/t})$ for $t \geqslant 2$ and proving a matching lower bound is a well-known open problem:

(*Z*) $z_t^{(2)}(n) = \Omega(n^{2-1/t})$ for $t \geqslant 2$.

Note that this is known for $t = 2, 3$ [2,6]. In addition, we will need the following properties.

($E1$) There exists $\delta > 0$ such that for large n, $z_t^{(a+1)}(n) - z_t^{(a)}(n) \geqslant \delta n^{2-1/t}$.
($E2$) for any $\varepsilon \in (0,1]$ and integer $a \geqslant 2$, there exists $\delta > 0$ such that for large n,

$$z_t^{(a)}(n) - z_t^{(a)}((1-\varepsilon)n, n, \dots, n) \geqslant \delta n^{2-1/t}.$$

($E3$) $z_t^{(a)}(n_1, \dots, n_a) - z_t^{(a)}(n_1 - 1, \dots, n_a) \gg 1$. That is, for any constant C^*, there exists $n_0 \in \mathbb{N}$ such that the $z_t^{(a)}(n_1, \dots, n_a) - z_t^{(a)}(n_1 - 1, \dots, n_a) \geqslant C^*$ whenever $n_1, \dots, n_a \geqslant n_0$.

($E1$) is a special case of ($E2$) with $\varepsilon = 1$. However, we distinguish them because we can indeed derive ($E1$) from (Z).

Theorem 5. *Suppose (Z) holds. For any $r, k, t \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that if $n \geqslant n_0$ is an integer and $G \in \mathcal{G}_k(n)$ is $K_{r+1}(t)$-free, then $e(G) = t_r(k)n^2 + z_t^{(\lceil k/r \rceil)}(n) + O(n)$.*

Our loose estimate on $\mathrm{ex}_k(n, K_{r+1}(t))$ from Theorem 5 only assumes (Z) and thus holds for $t = 2, 3$.

Theorem 6. *Suppose (Z), ($E2$) and ($E3$) hold. For any $r, k, t \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that if $n \geqslant n_0$ is an integer and $G \in \mathcal{G}_k(n)$ is $K_{r+1}(t)$-free, then $e(G) \leqslant g_2(n, r, k, t)$.*

Since $h_1(n, r, k, t) = h_2(n, r, k, t)$ and $g_1(n, r, k, t) = g_2(n, r, k, t)$ hold under (1.2), Theorems 3, 4 and 6 together determine $\mathrm{ex}_k(n, K_{r+1}(t))$ in the cases of (1.2) for large n and under the hypotheses (Z), ($E2$) and ($E3$). In all other cases, the difference between our upper and lower bounds, $h_2(n, r, k, t) - h_1(n, r, k, t)$, is $O_{r,k}(t^2)$.

Remark. Theorem 1 says that any extremal graph for $K_{r+1}(t)$ is the join of an extremal graph for $K_{t,t}$ and $r-1$ extremal graphs for $K_{1,t}$. Inspired by by this result, a natural guess of an extremal graph for our r-partite problem is starting from an n-blowup of $T_r(k)$, add a $K_{t,t}$-free $(a+1)$-partite graph to one of its parts with $a+1$ partition sets (assuming $b > 0$), and $K_{1,t}$-free graphs to other $r-1$ parts. This graph has

$$t_r(k)n^2 + z_t^{a+1}(n) + \left\lfloor \frac{(k-a-1)(t-1)}{2} \right\rfloor$$

edges. However, Theorem 4 says that this is *not* always extremal. Indeed, by moving vertices around, we obtain graphs with more edges and such graphs do not admit a partition similar to the ones in Theorem 1 (see the proof of Theorem 4, Section 3). This shows that our problem is different and more challenging than the non-partite version considered by Erdős and Simonovits [7].

Nevertheless, when $b = 0$ this complexity does not exist, and we give an analog of Theorem 1.

Theorem 7. *For $r, k \in \mathbb{N}$ with $r \mid k$ and $t = 2, 3$, there exist $C_0, n_0 \in \mathbb{N}$ such that the following holds for $n \geqslant n_0$. Let G be a $K_{r+1}(t)$-free k-partite graph with n vertices in each part and $\mathrm{ex}_k(n, K_{r+1}(t))$ edges. Then there is a partition of the k vertex clusters of G into r groups $U_1, \dots, U_r$, each with k/r clusters, and a vertex set $Z \subseteq V(G)$ with $|Z| \leqslant C_0$ such that*

- *$G[U_i \backslash Z, U_j \backslash Z]$ is almost complete for all $i \neq j$,*
- *$G[U_1 \backslash Z]$ is $K_{t,t}$-free, and*
- *for $i \in [2, r]$, $G[U_i \backslash Z]$ is $K_{1,t}$-free.*

Assuming ($E2$), we can show that $Z = \varnothing$ in the theorem above. However, we choose to present a result that resembles Theorem 1 and requires no additional condition.

Organization. In Section 2 we prove the stability theorems (Theorems 2 and 3), that is, on K_r-free and $K_r(t)$-free multi-partite graphs. Then we prove the lower bound, Theorem 4, by giving the corresponding constructions in Section 3. Finally we prove the upper bounds, Theorems 5, 6 and 7, together in Section 4.

Notation. We omit floors and ceilings unless they are crucial, e.g., we may choose a set of εn vertices even if our assumption does not guarantee that εn is an integer.

When $X, Y \subseteq V(G)$ intersect, $E_G(X,Y)$ is defined as the collection of ordered pairs in $(x,y) \in X \times Y$ such that $\{x,y\} \in E(G)$. Write $e_G(X,Y) = |E_G(X,Y)|$. For a vertex v in G, let $N(v,X) = N(v) \cap X$ and $d(v,X) = |N(v,X)|$. Moreover, given $X \subseteq V(G)$, let $e(X,G)$ be the number of edges of G incident to the vertices of X. Given two graphs G and H on a common vertex set V, $G \cap H$ denotes a graph on V with $E(G \cap H) = E(G) \cap E(H)$. Given a k-partition $\{V_1, V_2, \ldots, V_k\}$, a set S is called *crossing* if $|S \cap V_i| \leqslant 1$, $i \in [k]$.

When we choose constants $x, y > 0$, $x \ll y$ means that for any $y > 0$ there exists $x_0 > 0$ such that for any $x < x_0$ the subsequent statement holds. Hierarchies of other lengths are defined similarly. Furthermore, all constants in the hierarchy are positive and for a constant appearing in the form $1/s$, we always mean to choose s as an integer.

2. Proof of the Stability Theorems (Theorems 2 and 3)

The extremal problem $\mathrm{ex}_k(n, K_{r+1})$ (instead of its stability version) has a simple probabilistic proof. Indeed, let $G \in \mathcal{G}_k(n)$ with $e(G) > t_r(k)n^2$. Denote the parts of $V(G)$ by $V_1, \ldots, V_k$ and the densities of the bipartite graphs $G[V_i, V_j]$ as d_{ij}, $i, j \in [k]$. For each $i \in [r]$, uniformly choose a random vertex $v_i \in V_i$ independent of other choices. Thus, the probability for $v_iv_j \in E(G)$ is precisely d_{ij}. Let X be the number of edges spanned on $\{v_1, \ldots, v_k\}$, and note that $\mathbb{E}(X) = \sum d_{ij} = e(G)/n^2 > t_r(k)$. This implies that there exists a choice of the k-set $\{v_1, \ldots, v_k\}$ which spans more than $t_r(k)$ edges. By the definition of $t_r(k)$, this k-set contains a copy of K_{r+1}.

Our proof starts with this simple argument – we obtain $\mathbb{E}(X) \geqslant t_r(k) - \varepsilon$ from $e(G) \geqslant (t_r(k)-\varepsilon)n^2$. Since G is K_{r+1}-free, we know deterministically that $X \leqslant t_r(k)$, and thus by Markov's inequality we obtain that almost all crossing k-sets span exactly $t_r(k)$ edges and thus are isomorphic to the Turán graph $T_r(k)$.

Proposition 2.1. *For any $r, k \in \mathbb{N}$ and $\varepsilon > 0$, there exist $\gamma > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for every $n \geqslant n_0$. Suppose $G \in \mathcal{G}_k(n)$ is K_{r+1}-free and $e(G) \geqslant (t_r(k) - \varepsilon)n^2$. Then for every $k' \in [1, k]$, all but at most $\gamma n^{k'}$ crossing k'-sets K in G satisfy that all but at most $\gamma n^{k-k'}$ crossing k-sets containing K are isomorphic to $T_r(k)$.*

Proof. Denote the parts of $V(G)$ by $V_1, \ldots, V_k$ and the densities of the bipartite graphs $G[V_i, V_j]$ as d_{ij}, $i, j \in [k]$. For $i \in [k]$, let v_i be a uniformly random vertex in V_i. Then let X be the number of edges spanned on $\{v_1, ..., v_k\}$ and note that $\mathbb{E}(X) = \sum d_{ij} = e(G)/n^2 \geqslant t_r(k) - \varepsilon$. Since $G[\{v_1, ..., v_k\}]$ is K_{r+1}-free, we have $X \leqslant t_r(k)$ deterministically. Let $Y = t_r(k) - X$. Then Y is a nonnegative random integer with $\mathbb{E}(Y) = t_r(k) - \mathbb{E}(X) \leqslant \varepsilon$. By Markov's inequality, $\mathbb{P}(Y > 0) \leqslant \mathbb{E}(Y) \leqslant \varepsilon$, which implies that $\mathbb{P}[X = t_r(k)] = \mathbb{P}(Y = 0) \geqslant 1 - \varepsilon$. Therefore, all but at most εn^k crossing k-sets in G span a copy of $T_r(k)$. The proposition follows by counting and choosing $\gamma^2 \geqslant \binom{k}{k'}\varepsilon$. □

We also need the celebrated Graph Removal Lemma for cliques due to Erdős, Frankl and Rödl [5].

Lemma 2.2 (Graph Removal Lemma)**.** *For any $r \in \mathbb{N}$ and any $\varepsilon > 0$, there exist $\beta > 0$ and integer $n_0 \in \mathbb{N}$ such that the following holds for every integer $n \geqslant n_0$. If G is a graph with at most βn^r copies of K_r, then it can be made K_r-free by removing at most εn^2 edges.*

A crossing copy of $T_r(k)$ in the k-partite graph G gives us a partition of $[k]$ into r parts of size a or $a+1$, where each part corresponds to a color class of $T_r(k)$. Throughout the rest of this section, a *partition of* $[k]$ refers to such a partition.

Now we give our proof.

Proof of Theorem 2. Write $k = ar + b$ with $0 \leqslant b \leqslant r-1$. Without loss of generality, assume $\gamma \ll 1/r$. Choose constants

$$1/n \ll \varepsilon := \varepsilon_k \ll \cdots \ll \varepsilon_{ar} \ll \gamma' \ll \gamma_{ar} \ll \cdots \ll \gamma_k =: \gamma \ll 1/r, 1/k.$$

We use induction on both $r \geqslant 1$ and $b \geqslant 0$. Let G be a k-partite K_{r+1}-free graph with $e(G) \geqslant (t_r(k) - \varepsilon_k)n^2$. The base case $r = 1$ is trivial, and thus we assume $r \geqslant 2$.

Here is an outline of the proof. The base case $b = 0$ (i.e., $k = ar$) is simple. By Proposition 2.1 and averaging, there is a partition of $[k]$ such that many crossing k-sets of $V(G)$ induce a copy of $K_r(a)$ under this partition. We call the partition sets under this partition *rows*. For every $i \in [r]$, we find a crossing $a(r-1)$-set S_i intersecting all but the i-th row such that if we extend S_i to crossing ar-sets, then almost all these ar-sets are isomorphic to $K_r(a)$. This implies that the i-th row (and thus every row) has $o(n^2)$ edges. Together with $e(G) \geqslant (t_r(k) - \varepsilon_k)n^2$, we conclude that G is close to a blow-up of $K_r(a)$.

For the $b > 0$ case, we first show that we may assume that every V_i is incident to many edges (Claim 2.3). Indeed, if say, V_k is incident to too few edges, then we obtain the structure of $G - V_k$ by induction hypothesis. Next we add the vertices of V_k to the partition of $G - V_k$ joining W_i's. A key observation is that if we tend to add a set $V_k^i \subseteq V_k$ of vertices to the i-th row, then (current) $|W_i| = o(n)$ and thus there are only $o(n^2)$ edges in $G[W_i, V_k^i]$ (as a result, adding the vertices of V_k^i will not create too many edges inconsistent with the new partition). Finally we need to move things around to make sure that after the operation all parts have at most n vertices.

Now we may assume that all parts are incident to a good amount of edges as in (2.1). Then for $a \geqslant 2$, by (2.1) and Proposition 2.1, we show that every part V_i is in a group of a parts which only induces $o(n^2)$ edges as in (†). Repeated applications of (†) give a contradiction with (2.1). The case $a = 1$ is nevertheless more complicated. We first find a pair of parts, say (V_1, V_2), which is not too dense (because G is K_{r+1}-free). Together with Proposition 2.1, this implies that almost all crossing $(k-2)$-sets in $\prod_{3 \leqslant i \leqslant k} V_i$ form a copy of $T' := K_{1,\dots,1,2,\dots,2}$, which has $b-1$ clusters of size 2 and $r-b$ clusters of size 1, see (‡). Then we can show that $G' := G - (V_1 \cup V_2)$ is close to a member of $\mathcal{T}_{r-1,k-2}(n)$, due to the fact that G' has few copies of K_r and $(t_{r-1}(k-2) - o(1))n^2$ edges. Indeed, by the removal lemma, G' has a subgraph G'' which is K_r-free and still has $(t_{r-1}(k-2) - o(1))n^2$ edges, which is close to a member of $\mathcal{T}_{r-1,k-2}(n)$ by induction, and so is G'. The rest of the proof is to find an edge $uv \in V_1 \times V_2$ such that u and v have large common neighborhood in each part V_i, $i \in [3,k]$, which leads to a copy of K_{r-1} in its common neighborhood, and thus $K_{r+1} \subseteq G$, a contradiction. We find such uv by dedicate counting and the fact that $G[V_1, V_2] = pn^2$ where p is bounded away from 1.

The base case. We first prove the base case $b = 0$, that is, $k = ar$. In this case $t_r(k) = \binom{r}{2}a^2$ and $\mathcal{T}_{r,k}(n)$ contains a unique member $T_{r,k}(n)$ which is the complete r-partite graph with exactly a parts in each color classes. If $a = 1$, then $G \in \mathcal{G}_r(n)$ is a subgraph of $T_{r,k}(n)$ and G is ε_k-close to $T_{r,k}(n)$ because $e(G) \geqslant (t_r(k) - \varepsilon_k)n^2$.

Now assume $a \geqslant 2$. Since $e(G) \geqslant (t_r(k) - \varepsilon_k)n^2$, by Proposition 2.1, almost all crossing k-sets induce a copy of $T^* := K_r(a)$. Since there are $\frac{k!}{(a!)^r r!}$ partitions of $[k]$, by averaging, there is one partition such that at least $n^k/(2k!)$ crossing k-sets of G induces T^* under this partition. Without loss of generality, denote this partition as $[1,a] \cup [a+1, 2a] \cup \cdots \cup [(r-1)a+1, ar]$. Now, for each

$i \in [r]$, we shall choose a crossing $(a(r-1))$-set S_i from

$$Y_i := \prod_{j \neq i} \prod_{j' \in [a]} V_{(j-1)a+j'}$$

such that all but at most $\gamma' n^a$ crossing k-sets containing S_i induce T^* (in particular, the a new vertices would form an independent set). Indeed, this is possible as there are at least $(n^k/(2k!))/n^a = n^{a(r-1)}/(2k!)$ choices for crossing sets S_i in Y_i so that $G[S_i]$ is isomorphic to $K_{r-1}(a)$, and among them, by Proposition 2.1, at most $\gamma' n^{a(r-1)}$ crossing sets violate the extension property.

For $i \in [r]$, let $U_i := \bigcup_{j' \in [a]} V_{(i-1)a+j'}$ and we shall show that G is γ_k-close to $K(U_1, \ldots, U_r)$. We claim that $G[U_i]$ has at most $\binom{a}{2}\gamma' n^2$ edges. Indeed, every edge of $G[U_i]$ together with S_i can be extended to n^{a-2} crossing k-sets not isomorphic to T^*. If $G[U_i]$ contains more than $\binom{a}{2}\gamma' n^2$ edges, then there are more than $\binom{a}{2}\gamma' n^2 \cdot n^{a-2}/\binom{a}{2} = \gamma' n^a$ crossing k-sets containing S_i not isomorphic to T^* (a crossing k-set contains at most $\binom{a}{2}$ edges in $G[U_i]$ and is thus counted at most $\binom{a}{2}$ times), contradicting the definition of S_i. Therefore, $\bigcup_{i \in [r]} G[U_i]$ contains at most $r\binom{a}{2}\gamma' n^2$ edges. Furthermore, since $e(G) \geqslant (t_r(k) - \varepsilon_k)n^2$, $G[U_1, \ldots, U_r]$ (as an r-partite graph) has at most

$$\varepsilon_k n^2 + r\binom{a}{2}\gamma' n^2$$

crossing non-edges. Overall, this yields that one can obtain $T_{r,k}(n)$ by altering at most $\varepsilon_k n^2 + 2r\binom{a}{2}\gamma' n^2 \leqslant \gamma_{ar} n^2$ edges from G and we are done.

The general case. Now suppose $b > 0$ and the result holds for $k-1$ with $(\varepsilon_{k-1}, \gamma_{k-1})$ in place of (ε, γ) and we will verify it for k. We first prove the following claim. Let $\varepsilon' := \varepsilon_{k-1}/2$.

Claim 2.3. *If $e(V_i, G) \leqslant (k-1-a+\varepsilon')n^2$ for some $i \in [k]$, then G is γ-close to some member of $\mathcal{T}_{r,k}(n)$.*

Proof. Without loss of generality, assume $i = k$. By the assumption on $e(G)$, we have

$$e(G - V_k) = e(G) - e(V_k, G) \geqslant (t_r(k) - \varepsilon)n^2 - (k-1-a+\varepsilon')n^2 \geqslant (t_r(k-1) - \varepsilon_{k-1})n^2.$$

Then by the induction hypothesis, $G - V_k$ is γ_{k-1}-close to a member $T \in \mathcal{T}_{r,k-1}(n)$. Since $G - V_k$ is K_{r+1}-free, $e(G - V_k) \leqslant t_r(k-1)n^2$, and thus $e(V_k, G) \geqslant (k-1-a-\varepsilon)n^2$. Recalling the definition of $\mathcal{T}_{r,k-1}(n)$, let us denote the color classes of T by $U_1, \ldots, U_r$ and for $i \in [r]$ let $U_i = W_i \cup Z_i$, such that each Z_i is a union of a clusters of $V_1, \ldots, V_{k-1}$.

We first show the following claim.

(∗) Given any vertex $v \in V_k$, if there exists $i_0 \in [r]$ such that $|N(v) \cap Z_i| \geqslant \gamma^4 n$ for every $i \in [r]\backslash\{i_0\}$, then $|N(v) \cap U_{i_0}| < \gamma^4 n$.

Indeed, otherwise let $X_{i_0} \subseteq N(v) \cap U_{i_0}$ of size $\gamma^4 n$ and $X_i \subseteq N(v) \cap Z_i$ of size $\gamma^4 n$ for every $i \in [r]\backslash\{i_0\}$. Note that $T[X_i, X_j]$ for distinct $i, j \in [r]$ is a complete bipartite graph. Since $G - V_k$ is γ_{k-1}-close to T, it follows that $e(G[X_i, X_j]) \geqslant (\gamma^4 n)^2 - \gamma_{k-1}n^2$ for any distinct $i, j \in [r]$. As $\gamma_{k-1} \ll \gamma$, by the Turán result, we find a copy of K_r in $\bigcup_{i \in [r]} X_i$, which forms a copy of K_{r+1} together with v, a contradiction. In particular, (∗) implies that $|N(v) \cap Z_{i_0}| \leqslant \gamma^4 n$ for some $i_0 \in [r]$. Since v misses at least $(a - \gamma^4)n$ vertices in Z_{i_0} and $n-1$ vertices in V_k, we have $d_G(v) \leqslant (k-1-a+\gamma^4)n$ for any $v \in V_k$.

Let V_k' be the set of vertices $v \in V_k$ such that $d_G(v) < (k-1-a-\gamma^2)n$, then we have

$$(k-1-a-\varepsilon)n^2 \leqslant e(V_k, G) < |V_k'|(k-1-a-\gamma^2)n + (n - |V_k'|)(k-1-a+\gamma^4)n$$

and thus $(\gamma^2+\gamma^4)n(n-|V_k'|) \geqslant (\gamma^2-\varepsilon)n^2$, giving that

$$|V_k'| \leqslant \left(1-\frac{\gamma^2-\varepsilon}{\gamma^2+\gamma^4}\right)n = \frac{\gamma^4+\varepsilon}{\gamma^2+\gamma^4}n \leqslant \gamma^2 n.$$

For each $v \in V_k \backslash V_k'$, since $d_G(v) \geqslant (k-1-a-\gamma^2)n$, we have that $|N(v)\cap Z_i| \geqslant \gamma^4 n$ for all but possibly one $i \in [r]$. However, by (∗), there must be one such $i \in [r]$, that is, for v there exists $i_0 \in [r]$ such that $|N(v)\cap Z_i| \geqslant \gamma^4 n$ for all $i \in [r]\backslash\{i_0\}$ and $|N(v)\cap U_{i_0}| \leqslant \gamma^4 n$. This gives a partition of $V_k\backslash V_k'$ to r (possibly empty) sets $V_k^1, \ldots, V_k^r$. Furthermore, observe that if $|U_i| \geqslant (a+2\gamma^2)n$ (that is, $|W_i| \geqslant 2\gamma^2 n$), then $V_k^i = \varnothing$, because v can not have $|U_i| - \gamma^4 n > (a+\gamma^2)n$ non-neighbors in U_i.

Now we define a new partition $U_1', \ldots, U_r'$ as follows. If $|W_i| < 2\gamma^2 n$, then we find a $j \in [r]$ such that W_j and W_i are from the same cluster of G, add W_i to W_j and redefine $W_i := V_k^i$. Denote the resulting partition as $U_1', \ldots, U_r'$, where $U_i' = Z_i \cup W_i$ and we add the vertices of V_k' to any U_i' so that it does not exceed $(a+1)n$ vertices. Let $T' = K(U_1', \ldots, U_r') \cap K(V_1, \ldots, V_k)$ and we shall show that T' and G are γ-close. Indeed, recall that $G - V_k$ is γ_{k-1}-close to T, and for every $v \in V_k^i$, v is incident to at most $\gamma^4 n$ edges within Z_i and thus at least $(k-1-a-\gamma^2-\gamma^4)n$ edges outside Z_i. Moreover, for the set V_k' and all W_i's that have been moved, the union of them contains at most $r\cdot 2\gamma^2 n$ vertices, and thus for these vertices we may need to edit at most $2r\gamma^2 n \cdot kn = 2kr\gamma^2 n^2$ edges. Therefore, the distance of T' and G is at most

$$\gamma_{k-1}n^2 + n\cdot(\gamma^2+2\gamma^4)n + 2kr\gamma^2 n^2 < \gamma n^2,$$

where the second term is on (the changes brought by) the vertices of $V_k\backslash V_k'$. We are done. □

Choose a new constant $\varepsilon \ll \gamma'' \ll \varepsilon'$. By Claim 2.3, we may assume that for every $i \in [k]$,

$$e(V_i, G) \geqslant (k-1-a+\varepsilon')n^2. \tag{2.1}$$

Next we first deal with the easier case $a \geqslant 2$.

Case 1. $a \geqslant 2$**.** By (2.1), for every $i_0 \in [k]$, take $v \in V_{i_0}$ such that $d(v) \geqslant (k-1-a+\varepsilon')n$. Note that there exist $k-a$ sets V_i such that $|N(v)\cap V_i| \geqslant (\varepsilon'/a)n$ – otherwise there exist a sets V_i such that $|N(v)\cap V_i| < (\varepsilon'/a)n$, yielding $d(v) < a(\varepsilon'/a)n + (k-a-1)n$, a contradiction. Therefore there exists $I \subseteq [k]$ of size $k-a$ such that for any $j \in I$, $|N(v)\cap V_j| \geqslant (\varepsilon'/a)n$. As $\varepsilon \ll \gamma'' \ll \varepsilon'$, by Proposition 2.1, there exists a crossing $(k-a)$-set S in $\prod_{j\in I}(N(v)\cap V_j)$ such that all but at most $\gamma'' n^a$ crossing k-sets containing S are isomorphic to $T^* := K_{a,\ldots,a,a+1,\ldots,a+1}$. Since $S \subseteq N(v)$ and G is K_{r+1}-free, S must be K_r-free. Moreover, S is an induced subgraph of T^* with $k-a$ vertices, and we infer that $G[S]$ must be isomorphic to the graph obtained from T^* with a color class of size a removed. This yields that all but at most $\gamma'' n^a$ crossing a-sets from $\prod_{j\notin I} V_j$ form independent sets. Thus, we conclude that for any $j, j' \notin I$, $e_G(V_j, V_j') \leqslant \gamma'' n^2$. Note that $i_0 \notin I$, and thus we obtain

(†) for any $i \in [k]$, there exists $I' \subseteq [k]$ of size a such that $i \in I'$ and for any $j, j' \in I'$, $e_G(V_j, V_j') \leqslant \gamma'' n^2$.

By Proposition 2.1, there is a partition of $[k]$ that induces at least $n^k/(2k!)$ labelled copies of T^*. For two clusters V_i and V_j that are in different color classes, we have $e_G(V_i, V_j) \geqslant n^2/(2k!)$ because each edge in $V_i \times V_j$ is contained in at most n^{k-2} labelled copies of T^*. Now fix a color class of size $a+1$. Without loss of generality, denote this color class by $V_1, \ldots, V_{a+1}$ and consider V_1. By (†), there exists $I' \subseteq [k]$ of size a such that $1 \in I'$ and for any $j, j' \in I'$, $e_G(V_j, V_j') \leqslant \gamma'' n^2$. Since clusters from different color classes have densities at least $1/(2k!) > \gamma''$, we conclude that $I' \subseteq [a+1]$. Without loss of generality, suppose $[a+1]\backslash I' = \{2\}$. Applying (†) again, we obtain a set $I'' \subseteq [a+1]$ such that $2 \in I''$ and for any $j, j' \in I''$, $e_G(V_j, V_j') \leqslant \gamma'' n^2$. Note that any element $i \in I' \cap I''$ satisfies that $e_G(V_i, V_j) \leqslant \gamma'' n^2$ for every $j \in [a+1]$. However, this implies $e(V_i, G) \leqslant (k-1-a+a\gamma'')n^2$, contradicting (2.1) as $\gamma'' \ll \varepsilon'$.

Case 2. $a = 1$. Choose new constants $\gamma'' \ll \varepsilon'' \ll \beta \ll \varepsilon' \leqslant 1 - t_r(k)/\binom{k}{2}$. Since G is K_{r+1}-free, we have $\sum_{i \neq j} d(V_i, V_j)/\binom{k}{2} \leqslant t_r(k)/\binom{k}{2} \leqslant 1 - \varepsilon'$. Fix a pair of clusters, say, (V_1, V_2) with density $p \leqslant 1 - \varepsilon'$. Since there are more than $\gamma'' n^2$ non-edges in $V_1 \times V_2$, by Proposition 2.1, there exists a non-edge in $V_1 \times V_2$ such that all but at most $\gamma'' n^{k-2}$ crossing k-sets containing this pair form a copy of $T^* := K_{1,\dots,1,2,\dots,2}$. That is,

(‡) all but at most $\gamma'' n^{k-2}$ crossing $(k-2)$-sets in $\prod_{3 \leqslant i \leqslant k} V_i$ form a copy of $T' := K_{1,\dots,1,2,\dots,2}$, which has $b-1$ clusters of size 2 and $r-b$ clusters of size 1.

We now show that $G' := G[\bigcup_{3 \leqslant i \leqslant k} V_i]$ is close to a member of $\mathcal{T}_{r-1,k-2}(n)$. We may assume $k \geqslant 4$. Note the $e(G')n^{k-4} = \sum_X e(X)$, where the sum is over all crossing $(k-2)$-sets X in $\prod_{3 \leqslant i \leqslant k} V_i$. By (‡), this sum is at least $(n^{k-2} - \gamma'' n^{k-2})e(T') = (1-\gamma'')n^{k-2} t_{r-1}(k-2)$. Hence,

$$e(G') \geqslant t_{r-1}(k-2)(1-\gamma'')n^2. \tag{2.2}$$

Moreover, since T' is K_r-free, (‡) implies that the number of copies of K_r in G' is at most $\binom{k-2}{r}\gamma'' n^r$ because every copy of K_r in G' is contained in n^{k-2-r} crossing $(k-2)$-sets not isomorphic to T' and every crossing $(k-2)$-set not isomorphic to T' contains at most $\binom{k-2}{r}$ copies of K_r. The graph removal lemma (Lemma 2.2) implies that there exists $G'' \subseteq G'$ on $V(G')$ such that G'' is K_r-free and $e(G') - e(G'') \leqslant \varepsilon'' n^2$. By (2.2) and the assumption $\gamma'' \ll \varepsilon''$, we have

$$e(G'') \geqslant e(G') - \varepsilon'' n^2 \geqslant t_{r-1}(k-2)(1-\gamma'')n^2 - \varepsilon'' n^2 \geqslant (1-\varepsilon'')t_{r-1}(k-2)n^2.$$

By the induction hypothesis, we derive that G'' is β-close to some graph $T'_0 \in \mathcal{T}_{r-1,k-2}(n)$. This implies that G' is (2β)-close to this T'_0.

Note that $e(V_1 \cup V_2, G) = e(G) - e(G')$ and $e(G') \leqslant e(G'') + \varepsilon'' n^2 \leqslant (t_{r-1}(k-2) + \varepsilon'')n^2$. Thus, we have

$$e(V_1 \cup V_2, G) \geqslant (t_r(k) - \varepsilon)n^2 - (t_{r-1}(k-2) + \varepsilon'')n^2 \geqslant (2(k-2) - 2\varepsilon'')n^2.$$

For $i \in [k]$, let $c_i := e(V_i, G)/n^2$ and recall that $p = e(V_1, V_2)/n^2$. Then we have $2(t_r(k) - \varepsilon) \leqslant \sum_{i \in [k]} c_i \leqslant 2t_r(k)$ and $c_1 + c_2 - p \geqslant 2(k-2) - 2\varepsilon''$. Our ultimate goal is *finding an edge $uv \in V_1 \times V_2$ such that $|N(u) \cap N(v) \cap V_i| \geqslant 2\sqrt{\beta}n$ for every $i \in [3,k]$*, which allows us to find a copy of K_{r-1} in $N(u) \cap N(v)$, contradicting that G is K_{r+1}-free. Indeed, if there is an edge $uv \in V_1 \times V_2$ such that

$$d(u, V_3 \cup \cdots \cup V_k) + d(v, V_3 \cup \cdots \cup V_k) \geqslant (c_1 + c_2 - p - 1 + 3\sqrt{\beta})n \geqslant (2k - 5 + 2\sqrt{\beta})n, \tag{2.3}$$

then for every $i \in [3,k]$, there exists $B_i \subseteq N(u) \cap N(v) \cap V_i$ of size $2\sqrt{\beta}n$. Recall that G' is (2β)-close to a member T'_0 of $\mathcal{T}_{r-1,k-2}(n)$. In particular, there exist $r-1$ B_i's such that T'_0 induced on the union of these sets forms a copy of $K_{r-1}(2\sqrt{\beta}n)$. Thus, to destroy all these copies of K_{r-1}, one needs to remove at least $(2\sqrt{\beta}n)^2 = 4\beta n^2$ edges from T'_0, while $e(T'_0) - e(G') \leqslant 2\beta n^2$. Therefore $G[B_3 \cup \cdots \cup B_k]$ contains a copy of K_{r-1}, which together with uv forms a copy of K_{r+1}, a contradiction.

To find such uv, we first show that most vertices in V_i, $i \in [k]$ have large degrees. Fix $i \in [k]$ and let V'_i be the number of vertices $v \in V_i$ with $d(v) < c_i n - 2k\gamma'' n$. We claim that $|V'_i| \leqslant \gamma'' n/k$. Indeed, for any $j \in [k]$, since $e(V_j, G) = c_j n^2$, there exists a set V^*_j of size $\gamma'' n/k$ such that $d(v) \leqslant (1 + 2\gamma''/k)c_j n$ for every $v \in V^*_j$. Now consider the subgraph $G^* := G - (V''_i \cup \bigcup_{j \neq i} V^*_j)$, where V''_i is a subset of V'_i of size $\gamma'' n/k$. Since G^* is K_{r+1}-free, we have $e(G^*) \leqslant t_r(k)(1 - \gamma''/k)^2 n^2$. By $c_i \leqslant k-1$, $j \in [k]$

and $\sum_{i\in[k]} c_i \leqslant 2t_r(k)$, we get

$$\begin{aligned} e(G) &\leqslant t_r(k)(1-\gamma''/k)^2n^2 + (\gamma''n/k)\left(c_in - 2k\gamma''n + \sum_{j\neq i}(1+2\gamma''/k)c_jn\right) \\ &\leqslant t_r(k)(1-\gamma''/k)^2n^2 + (\gamma''/k)2t_r(k)n^2 - 2\gamma''n^2 \leqslant t_r(k)n^2 - \gamma''n^2, \end{aligned}$$

contradicting that $e(G) > t_r(k)n^2 - \varepsilon n^2$.

Pick a vertex $u \in V_1\backslash V_1'$ such that $d(u, V_2) \leqslant (1+\gamma'')pn$. Such a vertex exists – otherwise by $|V_1'| \leqslant \gamma''n/k$ we have $e_G(V_1, V_2) \geqslant (1-\gamma''/k)n \cdot (1+\gamma'')pn > pn^2$ contradicting our assumption. Consider $R := N(u, V_2\backslash V_2')$ and an arbitrary vertex $v \in R$. By the definition of V_i', we have that $d(u) + d(v) \geqslant c_1n + c_2n - 4k\gamma''n$. We may assume that (2.3) fails (otherwise we are done). Together with $d(u) + d(v) \geqslant c_1n + c_2n - 4k\gamma''n$, this implies that

$$d(u, V_2) + d(v, V_1) \geqslant (1 + p - 4k\gamma'' - 3\sqrt{\beta})n.$$

In particular, we obtain that $|R| = d(u, V_2\backslash V_2') \geqslant (p - 4k\gamma'' - 3\sqrt{\beta} - \gamma''/k)n \geqslant (p - 4\sqrt{\beta})n$ and $d(v, V_1) \geqslant (1 - p\gamma'' - 4k\gamma'' - 3\sqrt{\beta})n > (1 - 4\sqrt{\beta})n$ (for every $v \in R$). This gives at least

$$(p - 4\sqrt{\beta})(1 - 4\sqrt{\beta})n^2 \geqslant (p - 8\sqrt{\beta})n^2$$

edges in $G[V_1, R]$. On the other hand, by (2.1), we have $c_2 \geqslant k - 2 + \varepsilon'$, and thus for every vertex $w \in V_2\backslash V_2'$, we have

$$d(w, V_1) \geqslant c_2n - 2k\gamma''n - (k-2)n \geqslant \varepsilon'n/2.$$

By $|R| \leqslant (1+\gamma'')pn$ and $p < 1 - \varepsilon'$, we get $|V_2\backslash(V_2' \cup R)| \geqslant n - (1+\gamma'')pn - \gamma''n/k \geqslant \varepsilon'n/2$. Therefore, we have $e(G[V_1, V_2\backslash(V_2' \cup R)]) \geqslant (\varepsilon'n/2)^2$. However, combining these two estimates and recalling $\beta \ll \varepsilon'$ we see that

$$e(G[V_1, V_2]) \geqslant (p - 8\sqrt{\beta})n^2 + (\varepsilon'/2)^2n^2 > pn^2,$$

a contradiction. □

We now prove Theorem 3 by using the Graph Removal Lemma and a well-known result on supersaturation.

Lemma 2.4 (Supersaturation)**.** *For any $r, t \in \mathbb{N}$ and any $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds for every integer $n \geqslant n_0$. If G is a graph with βn^r copies of K_r, then it contains a copy of $K_r(t)$.*

Proof of Theorem 3. Choose constants $1/n \ll \beta \ll \varepsilon \ll \gamma, 1/k, 1/r, 1/t$. Suppose $G \in \mathcal{G}_k(n)$ is $K_{r+1}(t)$-free and $e(G) \geqslant (t_r(k) - \varepsilon)n^2$. We further assume that G contains fewer than βn^{r+1} copies of K_{r+1} – otherwise Lemma 2.4 implies that G contains a copy of $K_{r+1}(t)$, a contradiction. Thus, by Lemma 2.2, G contains a K_{r+1}-free spanning subgraph G' with at least $e(G) - \varepsilon n^2 \geqslant (t_r(k) - 2\varepsilon)n^2$ edges. By Theorem 2, G' is $\gamma/2$-close to some $T \in \mathcal{T}_{r,k}(n)$. Consequently, G is γ-close to T, as desired. □

3. Lower bound – proof of Theorem 4

In this section we prove Theorem 4. We start with the following proposition on $K_{2,2}$-free bipartite graphs.

Proposition 3.1. *Let $t \geqslant 1$ and $n > 5t^4$. Suppose $G = (A, B; E)$ is a $K_{2,2}$-free bipartite graph with n vertices in each part and with $\Delta(G) \leqslant t-1$. Then there is a perfect matching M in the complement of G such that $G \cup M$ is still $K_{2,2}$-free. In particular, there exists a t-regular $K_{2,2}$-free bipartite graph with n vertices in each part.*

Proof. When we construct M, there are two types of $K_{2,2}$ that we need to avoid in $G \cup M$ – in the first type M has an edge uv where G contains a path of length three with ends u and v, and the second type consists of two edges of M and two edges of G on the same four vertices. The first one is easy to avoid as for any vertex v in G, the number of vertices that can be reached from v via a walk of length three (equivalently, a path of length one or three) is at most $(\Delta(G))^3 \leqslant (t-1)^3$. Therefore, when matching a vertex $v \in A \cup B$ in M, we need to avoid at most $(t-1)^3$ vertices that do not depend on M. However, this is not the case for the second type of $K_{2,2}$ because the matching of a vertex in M affects the matching (more precisely, the vertices that need to be avoided) of another vertex. To overcome this, we proceed in three phases.

Choose a set S of arbitrary $2t^3$ vertices from A, and note that $N = \bigcup_{v \in S} N_G(v)$ has size at most $|S|t \leqslant 2t^4$. For the first phase of our process, we greedily find a matching M_0 that covers the vertices of N and avoids the vertices of S. To achieve this, each time we match a vertex $v \in N$, we need to avoid the vertices of S, the vertices of A that are already matched in M_0, and those vertices that have distance 1 or 3 from v in the graph $G \cup M_0$. Since $\Delta(G \cup M_0) \leqslant t$, we need to avoid at most $|S| + (|N|-1) + t^3 < n$ vertices and thus this is possible. For the second phase, we greedily match all unmatched vertices $v \in A \backslash S$ and denote the resulting matching by M_1 (containing M_0). Similarly, each time we need to avoid the vertices that are already matched and the vertices that have distance 1 or 3 from v in the graph $G \cup M_1$. This is possible as there are at most $(n - |S| - 1) + t^3 < n$ such vertices.

For the third phase, we are left with sets of unmatched vertices $S \subseteq A$ and $T \subseteq B$ such that $T \cap N(S) = \varnothing$. The key point is that $G[S, T]$ is an empty graph so no second type of $K_{2,2}$ will be created on $S \cup T$. For each vertex $v \in S \cup T$, we need to avoid the vertices that have distance 1 or 3 from v in the graph $G \cup M_1$, and there are at most $(t-1)^2 t < t^3 = |S|/2$ such vertices. Therefore, we can choose a perfect matching on $S \cup T$ by Hall's Marriage Theorem. This gives the desired perfect matching M.

The “in particular” part follows by starting with an empty bipartite graph and iteratively adding perfect matchings. □

Now we prove our lower bound on $\mathrm{ex}_k(n, K_{r+1}(t))$.

Proof of Theorem 4. We first deal with the case $a = 1$. Let $V_{i,j}$, $(i,j) \in [r] \times [2]$ be vertex sets, where each of $V_{b+1,2}, \ldots, V_{r,2}$ are empty sets and all other sets have size n. Let $G := K(V_{1,1} \cup V_{1,2}, \ldots, V_{r,1} \cup V_{r,2})$ be the blowup of the Turán graph $T_r(k)$. Thus $e(G) = t_r(k)n^2$.

We first revise the partition as follows. Let $t' := \lceil (t-1)/2 \rceil$ and $b' := \min\{b-1, \lfloor (r-b)/2 \rfloor\}$. Let $\{V'_{i,j}, (i,j) \in [r] \times [2]\}$ be obtained from $\bigcup V_{i,j}$ by moving t' vertices from $V_{i,1}$ to $V_{i+b-1,1}$, and moving t' vertices from $V_{i,2}$ to $V_{i+b+b'-1,2}$, for every $i \in [2, b'+1]$. For $i \in [r]$, let $U_i := V'_{i,1} \cup V'_{i,2}$ and $H := K(U_1, \ldots, U_r) \cap K(V_1, \ldots, V_k)$. Let H' be obtained from H by adding

- a $K_{t,t}$-free bipartite graph on U_1 of size $Z_t^{(2)}(n)$, and
- a maximum $\{K_{1,t}, K_{2,2}\}$-free bipartite graph on U_i for $2 \leqslant i \leqslant b + 2b'$.

Note that, for $i \in [2, b]$, Proposition 3.1 implies that each $H'[U_i]$ is $(t-1)$-regular and thus for $i \in [2, b'+1]$, $H'[U_i]$ has $(n-t')(t-1)$ edges, and for $i \in [b'+2, b]$ it has $n(t-1)$ edges; for $i \in [b+1, b+2b']$, each $H'[U_i]$ consists of t' vertex-disjoint stars $K_{1,t-1}$ centered at the t' new vertices.

We first observe that H contains $t_r(k)n^2 - b'(t')^2$ edges. Indeed, for every $i \in [2, b'+1]$, the t' vertices moved from $V_{i,1}$ to $V_{i+b-1,1}$ lose nt' edges to $V_{i+b-1,1}$ and gain $(n-t')t'$ edges to $V_{i,2}$, thus having a net loss $(t')^2$ edges between U_i and U_{i+b-1}. The same holds for the t' vertices moved from $V_{i,2}$ to $V_{i+b+b'-1,2}$. On the other hand, the t' new vertices in $V'_{i+b-1,1}$ and t' new vertices in $V'_{i+b+b'-1,2}$ are all joined, giving additional $(t')^2$ edges. Thus, the net loss of changing G to H is $b'(t')^2$ edges.

Furthermore, the vertices in $V_{2,2} \cup \cdots \cup V_{b,2}$ and the $b't'$ vertices moved from $V_{2,1} \cup \cdots \cup V_{b'+1,1}$ each have degree $t-1$ in $\bigcup_{2\leqslant i\leqslant r} H'[U_i]$ and thus the number of edges in $H'\backslash H$ is

$$z_t^{(2)}(n) + (t-1)(b-1)n + (t-1)b't'.$$

Thus, we have $e(H') = t_r(k)n^2 + z_t^{(2)}(n) + (t-1)(b-1)n + b'\lfloor\frac{(t-1)^2}{4}\rfloor = g_1(n, r, k, t)$.

At last, we observe that H' is $K_{r+1}(t)$-free. Indeed, by construction, every U_i is triangle-free, U_1 is $K_{t,t}$-free and other U_i's are $\{K_{1,t}, K_{2,2}\}$-free. Therefore, a copy of $K_{r+1}(t)$ can contain at most $2t-1$ vertices from U_1 and at most t vertices from other U_i's, which is impossible.

Now we assume $a \geqslant 2$. Let $V_{i,j}$, $(i,j) \in [r] \times [a+1]$ be disjoint vertex sets, where $V_{i,a+1} = \varnothing$ for $b < i \leqslant r$, and all other sets have size n. Let H be a graph defined as follows. Let

$$t_1 := \left\lfloor \frac{t-1}{4a^2+6a} \right\rfloor \quad \text{and} \quad t' \in [t - at_1 - 1, t - at_1] \text{ be even.}$$

Let $\{V'_{i,j}, (i,j) \in [r] \times [a+1]\}$ be a partition of $\bigcup_{(i,j)\in[r]\times[a+1]} V_{i,j}$ obtained from moving t_1 vertices from each of $V_{i,j}$, $2 \leqslant i \leqslant \min\{b, r+1-b\}$, $1 \leqslant j \leqslant a+1$ to $V_{i+b-1,j}$. Denote by $Z_{i+b-1,j}$ the set of vertices moved from $V_{i,j}$ and let $Z_{i+b-1} := \bigcup_{j=1}^{a+1} Z_{i+b-1,j}$. For $i \in [r]$, let $U_i := \bigcup_{j\in[a+1]} V'_{i,j}$. Let H be obtained from $K(U_1, \ldots, U_r) \cap K(V_1, \ldots, V_k)$ by adding graphs in each U_i as follows.

- (S1) Add a maximum $K_{t,t}$-free $(a+1)$-partite graph in U_1.
- (S2) for $i \in [2, b] \cup [2b, r]$,[1] add on U_i a $\{K_{1,t}, K_{2,2}\}$-free bipartite graph with $\lfloor |U_i| \frac{t-1}{2} \rfloor$ edges.
- (S3) for $i \in [b+1, \min\{2b-1, r\}]$, we add a triangle-free graph H_i on U_i such that $e(H_i) \geqslant \lfloor |U_i| \frac{t-1}{2} \rfloor + \frac{t_1 t'}{2}$, and for any $j \in [a+1]$, $H_i - Z_{i,j}$ contains no biclique of order larger than t.

We first show that (S2) is indeed possible. For $i \in [2, b] \cup [2b, r]$, each U_i consists of a or $a+1$ parts of equal size n', where $n' = n$ or $n' = n - t_1$. We first show that there exist disjoint sets $X_1, \ldots, X_s, Y_1, \ldots, Y_s \subseteq U_i$, such that for each $j \in [s]$, X_j, Y_j belong to distinct parts of U_i, $|X_j| = |Y_j| \geqslant (n - t_1)/2$ and $\bigcup_{i=1}^{s}(X_i \cup Y_i)$ is either U_i (if $|U_i|$ is even) or $U_i\backslash\{v\}$ for some $v \in V'_{i,1}$ (if $|U_i|$ is odd). Indeed, this is trivial if U_i has an even number of parts. If U_i has an odd number of parts, since $a \geqslant 2$, there are at least three parts. We arbitrarily pair up all but the first three parts. Now if n' is even, then $|U_i|$ is even, and let $V'_{i,1} = X_1 \cup X_2$, $V'_{i,2} = Y_1 \cup X_3$ and $V'_{i,3} = Y_2 \cup Y_3$, where each small part has size $n'/2$ and we are done. Otherwise both n' and $|U_i|$ are odd. In this case we take any $v \in V'_{i,1}$, and let $V'_{i,1}\backslash\{v\} = X_1 \cup X_2$, $V'_{i,2} = Y_1 \cup X_3$ and $V'_{i,3} = Y_2 \cup Y_3$, where $|X_1| = |Y_1| = |X_2| = |Y_2| = (n'-1)/2$ and $|X_3| = |Y_3| = (n'+1)/2$.

Now we define the desired bipartite graph on U_i. If $|U_i|$ is even, then apply Proposition 3.1 to each pair (X_j, Y_j). Since the resulting graph is $(t-1)$-regular and $K_{2,2}$-free, it satisfies the desired properties. Otherwise $|U_i|$ is odd, and we apply Proposition 3.1 and add $K_{2,2}$-free $(t-1)$-regular bipartite graphs on pairs of sets except (X_3, Y_3) (as defined above). For (X_3, Y_3) and $v \in V'_{i,1}$, we connect $\lfloor (t-1)/2 \rfloor$ edges from v to each of X_3 and Y_3 and denote the set of these neighbors by X^*. Then by Proposition 3.1, we add a $(t-2)$-regular $K_{2,2}$-free bipartite graph on (X_3, Y_3), and then add a matching of size $|X_3| - \lfloor (t-1)/2 \rfloor$ such that the resulting graph is still $K_{2,2}$-free, all the

[1]When $2b > r$, the interval $[2b, r]$ is an empty set.

vertices in $X_3 \cup Y_3$ have degree $t-1$, and X^* remains an independent set. Indeed, this is possible by applying Proposition 3.1 to the current graph with a complete bipartite graph added to X^*. By construction, the resulting graph is $K_{2,2}$-free, and all vertices except v have degree $t-1$, while v has degree $t-1$ or $t-2$. So it follows that the resulting graph has $\lfloor |U_i|\frac{t-1}{2}\rfloor$ edges as desired.

We now verify (S3). Recall that t' is even. Choose $A_{i,1} \subseteq V_{i,1}$ and $A_{i,2} \subseteq V_{i,2}$, each of size $t'/2$. Add in H_i a complete bipartite graph with parts Z_i and $A_{i,1} \cup A_{i,2}$ (which is a copy of $K_{t',(a+1)t_1}$). Next, if a is even, then we add a $(t-1)$-regular $K_{2,2}$-free bipartite graph on each of $(V_{i,3}, V_{i,4}), (V_{i,5}, V_{i,6}), \ldots, (V_{i,a-1}, V_{i,a})$; if a is odd, then we add a $(t-1)$-regular $K_{2,2}$-free bipartite graph on each of $(V_{i,4}, V_{i,5}), (V_{i,6}, V_{i,7}), \ldots, (V_{i,a-1}, V_{i,a})$. Thus, we are left with $V_{i,1}, V_{i,2}$ and possibly $V_{i,3}$ when a is odd. If a is odd, then for $j = 1, 2$, take arbitrary $B_{i,j} \subseteq V_{i,j}\backslash A_{i,j}$ of size $\lceil n/3\rceil - t'/2$; if a is even, then let $B_{i,j} = V_{i,j}\backslash A_{i,j}$. We first connect vertices of Z_i to $B_{i,1} \cup B_{i,2}$ such that every vertex of Z_i has $t-1-t'$ neighbors in $B_{i,1} \cup B_{i,2}$ and all these neighbors are distinct and distributed as evenly as possible in the two sets $B_{i,1}, B_{i,2}$. In particular, if one of $B_{i,1}, B_{i,2}$ receives one more edge than the other, then we remove one arbitrary such edge and denote the vertex of this edge in Z_i by w. Denote by $B^*_{i,1}$ and $B^*_{i,2}$ the neighbors we have just obtained. So as for now for any vertex $v \in Z_i$, $N_{H_i}(v) \subseteq A_{i,1} \cup A_{i,2} \cup B^*_{i,1} \cup B^*_{i,2}$ and $d_{H_i}(v) \leqslant t-1$. Using Proposition 3.1, we add a $(t-1-at_1)$-regular $K_{2,2}$-free bipartite graph G' on $(A_{i,1} \cup B_{i,1}, A_{i,2} \cup B_{i,2})$ such that

(A1) $G'[A_{i,1} \cup B^*_{i,1}, A_{i,2} \cup B^*_{i,2}]$ contains no edge and
(A2) no two vertices in $A_{i,1} \cup A_{i,2} \cup B^*_{i,1} \cup B^*_{i,2}$ have any common neighbors in $U_i\backslash Z_i$.

Indeed, this can be done by applying Proposition 3.1 (repeatedly) on $(A_{i,1} \cup B_{i,1}, A_{i,2} \cup B_{i,2})$ with the initial graph $K(A_{i,1} \cup B^*_{i,1}, A_{i,2} \cup B^*_{i,2})$. Note that vertices in $A_{i,1} \cup A_{i,2}$ have degree exactly $t-1-at_1+(a+1)t_1 = t-1+t_1$. Furthermore, we add a perfect matching on $(B_{i,1}\backslash B^*_{i,1}, B_{i,2}\backslash B^*_{i,2})$ and at_1-1 edge-disjoint perfect matchings on $(B_{i,1}, B_{i,2})$ such that i) they are edge-disjoint from G', ii) the resulting graph on $A_{i,1} \cup B_{i,1} \cup A_{i,2} \cup B_{i,2}$ is $K_{2,2}$-free and iii) (A1) and (A2) hold. This is possible by Proposition 3.1. By now the construction is completed if a is even. If a is odd, then for $j = 1, 2$, let $X_{i,j} \subseteq V_{i,j}\backslash(A_{i,j} \cup B_{i,j})$ of size $\lfloor n/2\rfloor$, and $X'_{i,j} = V_{i,j}\backslash(A_{i,j} \cup B_{i,j} \cup X_{i,j})$. So $|X'_{i,j}| \geqslant n/7$ is large. We split $V_{i,3}$ to $Y_{i,1}, Y_{i,2}$, each of size $\lfloor n/2\rfloor$, and possibly one vertex u (only when n is odd). By Proposition 3.1, we add $(t-1)$-regular $K_{2,2}$-free bipartite graphs on each of $(X_{i,1}, Y_{i,1})$ and $(X_{i,2}, Y_{i,2})$. If n is even, then by Proposition 3.1, we add a $(t-1)$-regular $K_{2,2}$-free bipartite graph on $(X'_{i,1}, X'_{i,2})$ and the construction is finished. If n is odd, then we connect $\lfloor (t-1)/2\rfloor$ edges from u to each of $X'_{i,1}$ and $X'_{i,2}$ and connect u and w (if w exists). Denote by X^{**} the set of these neighbors of u in $X'_{i,1}$ and $X'_{i,2}$. Next, similarly to above, by Proposition 3.1 we can add a $(t-2)$-regular $K_{2,2}$-free graph on $(X'_{i,1}, X'_{i,2})$, and then add a matching of size $|X'_{i,1}| - \lfloor (t-1)/2\rfloor$ to $(X'_{i,1}, X'_{i,2})$ so that all vertices in $X'_{i,1} \cup X'_{i,2}$ have degree $t-1$, the resulting graph is $K_{2,2}$-free, and X^{**} remains an independent set. The construction is completed. By construction,

(1) vertices in $A_{i,1} \cup A_{i,2}$ have degree $t-1+t_1$ and all other vertices have degree $t-1$ with the exception of at most one vertex (u or w above), which has degree $t-2$.;
(2) for any $j \in [a+1]$, $H_i - Z_{i,j}$ has maximum degree $t-1$ and thus is $K_{1,t}$-free;
(3) H_i is triangle-free;
(4) $H_i - Z_i$ is $K_{2,2}$-free.

Now we verify the last assertion of (S3). Let K be a biclique of order larger than t in $H_i - Z_{i,j}$ for some $j \in [a+1]$. By (2) and (4), $V(K)$ must intersect Z_i. Then as Z_i forms an independent set, $V(K) \cap Z_i$ is in one part of K, denoted by P_1, and by construction, the other part, denoted by P_2, is a subset of $A_{i,1} \cup A_{i,2} \cup B^*_{i,1} \cup B^*_{i,2} \cup \{u\}$. Since by (2) $H_i - Z_{i,j}$ is $K_{1,t}$-free, we have $|P_1| \geqslant 2$ and $|P_2| \geqslant 2$. By construction, for any $x \in A_{i,1} \cup A_{i,2} \cup B^*_{i,1} \cup B^*_{i,2}$, $N(x) \cap N(u) \cap U_i = \varnothing$. Therefore, together with (A2), the vertices of P_2 have no common neighbor outside Z_i, yielding $P_1 \subseteq Z_i \backslash Z_{i,j}$ and $|P_1| \leqslant at_1$. Now, $P_1 \subseteq Z_i$ and $|P_1| \geqslant 2$ imply that P_2 are the common neighbors of at least two vertices from Z_i, which gives that $P_2 \subseteq A_{i,1} \cup A_{i,2}$ and thus $|P_2| \leqslant t'$. Together we obtain $|V(K)| = |P_1| + |P_2| \leqslant at_1 + t' \leqslant t$, a contradiction. Therefore, $H_i - Z_{i,j}$ does not contain any bicliques on more than t vertices.

At last, we compute $e(H_i)$. Note that only vertices in $A_{i,1} \cup A_{i,2}$ have degree more than $t-1$ – they have degree $t-1+t_1$. All other vertices have degree $t-1$ with the exception of at most one vertex (u or w above), which has degree $t-2$. Since t' is even, $t_1 t'/2 \in \mathbb{N}$. Thus, we have

$$e(H_i) = \left\lfloor \frac{|U_i|(t-1) + t_1 t'}{2} \right\rfloor = \left\lfloor |U_i| \frac{t-1}{2} \right\rfloor + \frac{t_1 t'}{2}.$$

So (S3) is verified.

Claim 3.2. *The graph H is $K_{r+1}(t)$-free and $e(H) \geqslant g_1(n, r, k, t)$.*

Unlike the case $a = 1$, we need some arguments to prove that H is $K_{r+1}(t)$-free. The difference is that $H[U_1]$ might not be triangle-free (which is trivial for $a = 1$, as $H[U_1]$ is bipartite), and thus it may contribute e.g. a copy of $K_3(t-1)$ towards a copy of $K_{r+1}(t)$.

Proof. We first show that H is $K_{r+1}(t)$-free. Suppose instead, that H contains a copy of $K_{r+1}(t)$ on a vertex set K. We first claim the following.

(♡) For every $i \in [2, r]$, $V(K) \cap U_i$ is a biclique or an independent set and $|V(K) \cap U_i| \leqslant 2t$. If $|V(K) \cap U_i| > t$, then $i \in [b+1, \min\{2b-1, r\}]$ and $V(K) \cap U_{i-(b-1)} = \varnothing$.

Indeed, for $i \in [2, r]$, since $H[U_i]$ is triangle-free, $V(K) \cap U_i$ is a biclique or an independent set. By (S3), if $|V(K) \cap U_i| > t$, then $i \in [b+1, \min\{2b-1, r\}]$ and $V(K)$ intersects all $Z_{i,j}$ for $j \in [a+1]$, which imply that $V(K) \cap U_{i-(b-1)} = \varnothing$. Furthermore, we know $|V(K) \cap (U_i \backslash Z_{i,1})| \leqslant t$ and thus $|V(K) \cap U_i| \leqslant t + |Z_{i,1}| < 2t$. So (♡) is proved.

Let K_1 be the subgraph of K induced on its vertices in U_1. By (♡), for each $i \in [2, r]$, we have either $|V(K) \cap U_i| \leqslant t$ or $|V(K) \cap (U_i \cup U_{i-b+1})| \leqslant 2t$. So we have $|V(K) \backslash V(K_1)| \leqslant (r-1)t$ and thus $|V(K_1)| \geqslant 2t$ (which is now possible as commented above).

Let r' be the number of members of $\mathcal{U} := \{U_i, i \in [2, r]\}$ that intersect $V(K)$. Since the number of members of $\mathcal{U}$ not intersecting $V(K)$ is $r - 1 - r'$, by (♡), there are at most $r - 1 - r'$ parts U_i, $i > 1$ such that $|V(K) \cap U_i| > t$. Now we define an auxiliary graph G^* on $C_1, \dots, C_{r+1}$, the color classes of K. Indeed, for distinct $i, j \in [r+1]$, $C_i C_j \in E(G^*)$ if and only if there exists $l \in [2, r]$ such that $C_i \cap U_l \neq \varnothing$ and $C_j \cap U_l \neq \varnothing$. Note that by (♡), each U_l defines at most one edge of G^*. Let $\mathcal{C}_1, \dots, \mathcal{C}_s$ be the connected components of G^*. Since K_1 is $K_{t,t}$-free, there is at most one $i \in [s]$ such that $V(\mathcal{C}_i) := \bigcup_{C_j \in \mathcal{C}_i} C_j$ contains at least t vertices of $V(K_1)$: two such components each containing t vertices of K_1 will induce a copy of $K_{t,t}$ in K_1. Without loss of generality, assume that $|V(\mathcal{C}_i) \cap V(K_1)| < t$ for all $2 \leqslant i \leqslant s$. For each $i \in [s]$, let R_i be the set of indices $l \in [2, r]$ such that $|U_l \cap V(\mathcal{C}_i)| > 0$ and $R_i' \subseteq R_i$ be the set of l such that $|U_l \cap V(\mathcal{C}_i)| > t$. So we have

$$\sum_{i=1}^{s} |R_i| = r' \quad \text{and} \quad \sum_{i=1}^{s} |R_i'| \leqslant r - 1 - r'.$$

Moreover, by the definition of $\mathcal{C}_1, \dots, \mathcal{C}_s$, the R_i's are pairwise disjoint. We claim that for $i \in [2, s]$, $|R_i| \geqslant |\mathcal{C}_i| - |R_i'|$. Indeed, if $|R_i| \leqslant |\mathcal{C}_i| - |R_i'| - 1$, then by (♡), we have $|V(\mathcal{C}_i) \cap \bigcup_{l \in R_i} U_l| \leqslant |R_i'| \cdot 2t + (|\mathcal{C}_i| - 2|R_i'| - 1)t = (|\mathcal{C}_i| - 1)t$. Since $|V(\mathcal{C}_i) \cap V(K_1)| < t$, we obtain $|\mathcal{C}_i| t = |V(\mathcal{C}_i)| <$

$t+|V(\mathcal{C}_i)\cap\bigcup_{l\in R_i}U_l|\leqslant|\mathcal{C}_i|t$, a contradiction. So the claim follows. Overall, we get

$$\sum_{i=2}^{s}|R_i|\geqslant\sum_{i=2}^{s}(|\mathcal{C}_i|-|R_i'|)\geqslant r+1-|\mathcal{C}_1|-((r-1-r')-|R_1'|)\geqslant r'+2-|\mathcal{C}_1|.$$

Therefore, $|R_1|\leqslant r'-(r'+2-|\mathcal{C}_1|)=|\mathcal{C}_1|-2$ and thus G^* has at most $|\mathcal{C}_1|-2$ edges on $\mathcal{C}_1$. This contradicts with that $\mathcal{C}_1$ is connected. So H is $K_{r+1}(t)$-free.

It remains to compute $e(H)$. First suppose $b=0,1$, then for $2\leqslant i\leqslant r-1$, $|U_i|=an$ and thus $H[U_i]$ has at least $\lfloor\frac{t-1}{2}an\rfloor$ edges. It follows that $e(H)=g_1(n,r,k,t)$. Secondly suppose $b\geqslant 2$ and let $b':=\min\{b-1,r-b\}$. Note that $e_H(U_1)=z_t^{(\lceil k/r\rceil)}(n)$ and $t'\geqslant t-at_1-1\geqslant(4a^2+5a)t_1$. So

$$\sum_{i=2}^{r}e_H(U_i)\geqslant\sum_{i=2}^{r}\left\lfloor|U_i|\frac{t-1}{2}\right\rfloor+b'\frac{t_1t'}{2}\geqslant\left\lfloor(k-a-1)n\frac{t-1}{2}\right\rfloor+b'\frac{(4a^2+5a)t_1^2}{2}-\frac{r-1}{2}.$$

Let $H'=K(U_1,\dots,U_r)\cap K(V_1,\dots,V_k)$. It remains to bound $e(H')$. Note that for each $i\in[b+1,b+b']$, we have moved a set Z_i of $(a+1)t_1$ vertices from $\bigcup_j V_{i-b+1,j}$ to U_i. Thus, H' is obtained from $H^*:=K(U_1^*,\dots,U_r^*)\cap K(V_1,\dots,V_k)$ where $U_i^*=\bigcup_{j\in[a+1]}V_{i,j}$ by disconnecting the edges between Z_i and U_i^* and connecting the vertices of Z_i to U_{i-b+1} except for pairs of vertices from the same V_j's (Z_i remains to be an independent set). That is, for each vertex in Z_i the net gain is $-an+(an-at_1)=-at_1$, and thus the total net gain is $-a(a+1)t_1^2$. Since $e(H^*)=t_r(k)n^2$, we obtain $e(H')=t_r(k)n^2-b'a(a+1)t_1^2$. Together with the bounds on $e_H(U_i)$ we get

$$e(H)\geqslant t_r(k)n^2+z_t^{(\lceil k/r\rceil)}(n)+\left\lfloor(k-a-1)n\frac{t-1}{2}\right\rfloor+b'\frac{(2a^2+3a)t_1^2}{2}-\frac{r-1}{2}.$$

So we obtain $e(H)\geqslant g_1(n,r,k,t)$ for the case $t_1>0$.

Finally for the case $t_1=0$ or $b=0,1$ we give a slight improvement to avoid the loss $-(r-1)/2$. Let $W':=\bigcup_{2\leqslant i\leqslant r}V_{i,a+1}$. If $b\geqslant 2$, starting from the partition $\{V_{i,j}\}$, we move at most r vertices in W' to other rows, (e.g. move a vertex from $V_{2,a+1}$ to $V_{r,a+1}$, etc.), so that in the resulting partition $i\in[r]$, $U_i:=\bigcup_{j\in[a+1]}V_{i,j}$, all but at most one of the U_i's, $2\leqslant i\leqslant r$ have even order. If $b\leqslant 1$, then $W'=\varnothing$ and we do nothing in this step. Then as in the previous proof, we i) add a maximum $K_{t,t}$-free $(a+1)$-partite graph in U_1, and ii) for $i\in[2,r]$, add on U_i a $\{K_{1,t},K_{2,2}\}$-free bipartite graph with $\lfloor|U_i|\frac{t-1}{2}\rfloor$ edges. Let H be obtained by adding all edges between each two rows, i.e., adding $K(U_1,\dots,U_r)\cap K(V_1,\dots,V_k)$. For $b\geqslant 2$, due to our control on the parities of the sizes, we have $\sum_{i\in[r]}e_H(U_i)=z_t^{(a+1)}(n)+\lfloor(k-a-1)n\frac{t-1}{2}\rfloor$. For $b\leqslant 1$, we have $\sum_{i\in[r]}e_H(U_i)=z_t^{(\lceil k/r\rceil)}(n)+(r-1)\lfloor\frac{t-1}{2}an\rfloor$. That is, we have $\sum_{i\in[r]}e_H(U_i)=z_t^{(\lceil k/r\rceil)}(n)+h_1(n,r,k,t)$ in both cases. Moreover, note that the movement of the vertices does not change the number of edges in $J:=K(U_1,\dots,U_r)\cap K(V_1,\dots,V_k)$, because i) the induced subgraph on $\bigcup_{i\in[r],j\in[a]}V_{i,j}$ does not change, ii) the degree of every vertex of $\bigcup_{i\in[r]}V_{i,a+1}$ in the graph J does not change. Thus, the graph $K(U_1,\dots,U_r)\cap K(V_1,\dots,V_k)$ contains exactly $t_r(k)n^2$ edges and we obtain $e(H)=t_r(k)n^2+z_t^{(\lceil k/r\rceil)}(n)+h_1(n,r,k,t)$. ☐

Now the proof of Theorem 4 is completed. ☐

4. Proof of the upper bounds

Below we prove (E1).

Proof of (Z) ⇒ (E1). Given $a+1$ sets $V_1, \dots, V_{a+1}$ of size n, we define an $(a+1)$-partite graph G on $V_1, \dots, V_{a+1}$ as follows. Let V_2' be a set of n vertices consisting of $\lfloor n/2 \rfloor$ vertices from V_1 and $\lceil n/2 \rceil$ vertices from V_2. We place an extremal graph G' for $z_t^{(a)}(n)$ on $V_2', V_3, \dots, V_{a+1}$, in other words, G' is an a-partite $K_{t,t}$-free graph with $z_t^{(a)}(n)$ edges. Next we add a maximum bipartite $K_{t,t}$-free graph G'' on the remaining vertices of V_1 and V_2. By (Z), $e(G'') \geqslant z_t^{(2)}(\lfloor n/2 \rfloor) \geqslant \delta n^{2-1/t}$ for some $\delta > 0$. Thus $G = G' \cup G''$ is $K_{t,t}$-free and $e(G) = e(G') + e(G'') \geqslant z_t^{(a)}(n) + \delta n^{2-1/t}$. This gives (E1). □

Now we prove the upper bounds on $\mathrm{ex}_k(n, K_r(t))$.

The proof of Theorem 6 is done by a series of claims and estimates, and actually contains the proofs of Theorems 5 and 7. More precisely, the first half of the proof of Theorem 6 only requires (Z) and (consequently) (E1), but not (E2) or (E3). The (weaker) estimates obtained in this part of the proof suffice for Theorems 5 and 7. The second part of the proof contains refined estimates obtained from (E2) and (E3) and completes the proof of Theorem 6.

Outline of the proofs. Now we give an outline of our proofs. Let $G \in \mathcal{G}_k(n)$ be $K_{r+1}(t)$-free and has the maximum number of edges. Since $e(G) > t_r(k)n^2$, we can assume that G is γ-close to some $T \in \mathcal{T}_{r,k}(n)$. Using that $e(G)$ is maximum, we can easily derive a minimum degree condition by some symmetrization arguments.

Next we define *atypical* vertices. Roughly speaking, there are two types of *atypical* vertices: the first type of vertices, denoted by $Z'' \cup W''$, are the "wrong" ones that do not exist in $\mathcal{T}_{r,k}(n)$; the second type of vertices, denoted by $(W_i' \backslash W_i) \cup \bigcup_{j \neq i} Z_j^i$ for $i \in [r]$, are the vertices that are not in U_i but *behave* like the vertices of U_i, in other words, they are in the wrong place. In the first half of the proof (i.e., the proofs of Theorem 5 and Theorem 7), we completely ignore the first type of atypical vertices because there are only a constant number of them (see (P1) and (P3)) and they contribute only $O(n)$ to $e(G)$. For the second type of atypical vertices, there are only $o(n)$ of them (see (P1) and (P3)) and we move them to appropriate rows and redefine our partition as $\tilde{U}_1, \dots, \tilde{U}_r$ (see (4.2)). A key observation is that $Z_j^i \neq \varnothing$ (namely, there is a vertex in Z_j but behaves as a vertex in Z_i) is possible only if $|W_j| \geqslant (1 - o(1))n$.

Now we estimate $e(G)$. We split $E(G)$ into $E_G(\tilde{U}_1), \dots, E_G(\tilde{U}_r)$, and $E(G')$, where $G' := G \cap K(\tilde{U}_1, \dots, \tilde{U}_r)$. We have a relatively good estimate of $e(G')$ (see Claim 4.4) taking into account that the partition is no longer balanced. In contrast, due to the second type of atypical vertices, we can only show that each $G[\tilde{U}_i]$ is "almost" $K_{t,t}$-free (see Claim 4.5). Similarly, we show that all but at most one rows are "almost" $K_{1,t}$-free (see Claim 4.7). Assuming that $e_G(\tilde{U}_1)$ is the largest among all $e_G(\tilde{U}_i)$, $i \in [r]$, we can collect the estimates and give an upper bound of $e(G)$. Now we show that $\tilde{U}_1$ has no atypical vertices (Claim 4.8), and thus $e_G(\tilde{U}_1) \leqslant z_t^{(a+1)}(n)$. We further refine our estimate on $\tilde{U}_i$, $i > 1$, and show that each second type atypical vertex contributes at most a constant number of edges to $E(G) \backslash E(G')$ (Claims 4.9 and 4.10). In summary, $\tilde{U}_1$ is indeed $K_{t,t}$-free, $e_G(\tilde{U}_i) = O(n)$ for $i > 1$, and $|Z'' \cup W''| = O(1)$, from which we conclude the proofs of Theorem 5 and Theorem 7.

To prove Theorem 6, we refine earlier estimates as follows. We first show that $|W_1'| = (1 - o(1))n$ and $Z'' \cup W'' = \varnothing$, where we use (E2). The rest of the proofs are further refinements of our estimates for various cases (see, e.g. the definition of h_2). In particular, at the very last step, we use (E3) to show that all of Z_1^i, $i \in [2, r]$ should be empty and $|W_1'|$ should be equal to n.

We start with the following simple proposition.

Proposition 4.1. *Given $r, t \in \mathbb{N}$ and reals $\gamma, \varepsilon > 0$ such that $\varepsilon^2 > 3r^2t^2\gamma$, and let n be sufficiently large. Suppose G is a $K_{r+1}(t)$-free graph with vertex partition $V = U_1 \cup \cdots \cup U_r$ such that $|U_i| \geqslant n$*

for $i \in [r]$ and $d(U_i, U_j) \geqslant 1-\gamma$, $i, j \in [r]$, $i \neq j$. Let $X \subseteq V$ be the set of vertices v satisfies that $d(v, U_i) \geqslant \varepsilon|U_i|$ for all $i \in [r]$. Then $|X| \leqslant 2(t-1)\varepsilon^{-rt}$.

Proof. We call a copy of $K_r(t)$ in G *useful* if it consists of exactly t vertices from each of $U_1, \dots, U_r$. We first show that for every $v \in X$, $N(v)$ contains many useful copies of $K_r(t)$. Indeed, $d(U_i, U_j) \geqslant 1-\gamma$ for every $i, j \in [r]$, $i \neq j$ implies that $G[U_i, U_j]$ has at most $\gamma|U_i||U_j|$ non-edges. Since $d(v, U_i) \geqslant \varepsilon|U_i|$ for all $i \in [r]$, take $W_i \subseteq N(v) \cap U_i$ of size exactly $\varepsilon|U_i|$. We can find $\prod_{i \in [r]} \binom{\varepsilon|U_i|}{t}$ rt-sets which consists of t vertices from each U_i, amongst which, at most

$$\sum_{i,j \in [r], i \neq j} \gamma|U_i||U_j| \cdot \left(\prod_{i' \in [r]} \binom{\varepsilon|U_{i'}|}{t}\right) \cdot \frac{t}{\varepsilon|U_i|} \frac{t}{\varepsilon|U_j|} \leqslant \frac{r^2t^2\gamma}{\varepsilon^2} \prod_{i' \in [r]} \binom{\varepsilon|U_{i'}|}{t}$$

of them contain crossing non-edges. Therefore, $N(v)$ contains at least

$$\left(1 - \frac{r^2t^2\gamma}{\varepsilon^2}\right) \prod_{i' \in [r]} \binom{\varepsilon|U_{i'}|}{t} \geqslant \frac{\varepsilon^{rt}}{2} \prod_{i' \in [r]} \binom{|U_{i'}|}{t}$$

useful copies of $K_r(t)$, where we used that $r^2t^2\gamma\varepsilon^{-2} < 1/3$. Since G is $K_{r+1}(t)$-free, each useful copy K of $K_r(t)$ is in $N(v)$ for at most $t-1$ choices of $v \in X$. Double counting on the number of pairs (v, K) such that $K \subseteq N(v)$ is useful, we obtain that

$$|X| \frac{\varepsilon^{rt}}{2} \prod_{i' \in [r]} \binom{|U_{i'}|}{t} \leqslant (t-1) \prod_{i' \in [r]} \binom{|U_{i'}|}{t},$$

which gives $|X| \leqslant 2(t-1)\varepsilon^{-rt}$. □

Given integers $1 \leqslant s \leqslant t$ and sufficiently large m, n, Kővári, Sós, Turán [12] showed that $z(m, n, s, t) \leqslant Cmn^{1-1/s}$ for some $C = C(t) > 0$, that is, a bipartite graph G with parts of size m and n has at most $Cmn^{1-1/s}$ edges if G has no copy of $K_{s,t}$ where the part of size s is in the part of G of size m. This easily implies that $\mathrm{ex}(n, K_{s,t}) \leqslant Cn^{2-1/t}$ for sufficiently large n.

Proofs of Theorems 5, 6, and 7. Suppose (Z) holds, that is, $z_t^{(2)}(n) \geqslant cn^{2-1/t}$ for some $c > 0$. Take $C = C(t)$ as in the Kővári–Sós–Turán result in the previous paragraph. We choose constants

$$1/n \ll \gamma \ll \varepsilon \ll \varepsilon' \ll 1/k, 1/t, c, C.$$

Suppose G is $K_{r+1}(t)$-free and has the maximum number of edges, that is, $e(G) = \mathrm{ex}_k(n, K_{r+1}(t))$. Suppose further that $e(G) > g_2(n, r, k, t) > t_r(k)n^2$. By Theorem 3, G is γ-close to some $T \in \mathcal{T}_{r,k}(n)$, where the color classes of T are denoted by $U_1, \dots, U_r$ with $U_i = W_i \cup Z_i$ such that $Z_i = V_{(i-1)a+1} \cup \dots \cup V_{ia}$, $W_i = \varnothing$ if $b = 0$ and W_i is a subset of V_j for some $j \geqslant ar+1$ otherwise, and $T = K(U_1, \dots, U_r) \cap K(V_1, \dots, V_k)$. For $i \in [r]$, if $W_i \neq \varnothing$, then let q_i be the index such that $W_i \subseteq V_{q_i}$ and we know $ar < q_i \leqslant k$. For simplicity, we write $z_t(n) = z_t^{(\lceil k/r \rceil)}(n)$.

The fact that G is γ-close to T gives the following observation.

(D0) for any $i \in [r]$, there exists $B_i \subseteq U_i$ of size at most $2\sqrt{\gamma}n$ such that for any $v \in U_i \backslash B_i$ and $A \subseteq \bigcup_{j \in [r] \backslash \{i\}} U_j$ satisfying that none of the vertices of A is in the same cluster as v is, we have $\overline{d}(v, A) \leqslant \sqrt{\gamma}n$.

To see it, fix $i \in [r]$ and write $U^* := \bigcup_{j \neq i} U_j$. Since G is γ-close to T, we have

$$e_G(Z_i, U^*) \geqslant |Z_i||U^*| - \gamma n^2, \quad \text{and} \quad e_G(W_i, U^* \backslash V_{q_i}) \geqslant |W_i||U^* \backslash V_{q_i}| - \gamma n^2.$$

Let $B_i' \subseteq Z_i$ be the set of vertices v such that $\overline{d}(v, U^*) > \sqrt{\gamma}n$, and $B_i'' \subseteq W_i$ be the set of vertices w such that $\overline{d}(w, U^* \backslash V_{q_i}) > \sqrt{\gamma}n$. The displayed line above implies that $|B_i'| \leqslant \sqrt{\gamma}n$ and $|B_i''| \leqslant \sqrt{\gamma}n$. Now (D0) holds by setting $B_i = B_i' \cup B_i''$.

Minimum degree. For $i \in [k]$, let $N_i := N_T(u_i)$ for some $u_i \in V_i$. Note that this is well-defined as the vertices of V_i share the same neighborhood in T. Using the maximality of $e(G)$, we derive that for every $u \in V_i$, $i \in [k]$

$$d_G(u) \geqslant d_T(u) - 2t\gamma n.$$

Indeed, since G is γ-close to T, that is, $|E(G) \triangle E(T)| \leqslant \gamma n^2$, for each $i \in [k]$, we can greedily pick distinct $u_1, \dots, u_t \in V_i$, such that $|N_G(u_j) \triangle N_i| \leqslant \gamma n^2/(n-j) \leqslant 2\gamma n$, for $j \in [t]$. Let $N_i' := \bigcap_{j \in [t]} N_G(u_j)$ and note that $|N_i' \triangle N_i| \leqslant 2t\gamma n$. In particular, $|N_i'| \geqslant |N_i| - 2t\gamma n$. Now for a contradiction suppose there is $u \in V_i$ such that $d_G(u) < d_T(u) - 2t\gamma n = |N_i| - 2t\gamma n$. Then we replace $N_G(u)$ by N_i', that is, we disconnect all the edges of u in G and connect u to the vertices of N_i'. Thus, we obtain a k-partite graph on the same vertex set as G and has more edges than G. Therefore, by the maximality of G, this new graph contains a copy of $K_{r+1}(t)$, denoted by K. Clearly, K must contain the vertex u, as G is $K_{r+1}(t)$-free. Moreover, K must miss at least one vertex from $u_1, \dots, u_t$, say u_j, because the set $\{u, u_1, \dots, u_t\}$ is independent in G and K has independence number t. However, as the neighborhood of u N_i' is a subset of $N_G(u_j)$, we can replace u by u_j and still get a copy of $K_{r+1}(t)$, which is in G, a contradiction.

Therefore, comparing with the degrees in T, we derive that for any vertex u,

$$d_G(u) \geqslant \begin{cases} (k-a)n - |W_i| - 2t\gamma n, & \text{if } u \in Z_i \text{ for } i \in [r], \\ (k-1-a)n - 2t\gamma n, & \text{if } u \in W. \end{cases} \tag{4.1}$$

Atypical vertices. In this step we identify a set of atypical vertices, that is, those behave differently from the majority of the vertices. Let $W := \bigcup_{i \in [r]} W_i = V_{ar+1} \cup \dots \cup V_k$. We define $W'' := \{v \in W : d(v, Z_j) \geqslant \varepsilon n, \text{ for all } j \in [r]\}$ and $W_i' := \{v \in W : d(v, Z_i) < \varepsilon n\}$. Then we have $W = W'' \cup W_1' \cup \dots \cup W_r'$. Next, for $i \in [r]$, let $Z'' := \bigcup_{i \in [r]} Z_i''$, where

$$Z_i'' := \{v \in Z_i : d(v, Z_j) \geqslant \varepsilon n, \text{ for all } j \in [r] \backslash \{i\} \quad \text{and} \quad d(v, U_i) \geqslant \varepsilon n\}.$$

Let $Z_i' := Z_i \backslash Z_i''$ for all i. We write Z_i' as $\bigcup_{j \in [r]} Z_i^j$, where Z_i^j, $j \neq i$, consists of the vertices v such that $d(v, Z_j) < \varepsilon n$, and Z_i^i consists of the vertices v such that $d(v, U_i) < \varepsilon n$. Below are some useful properties of these sets.

Claim 4.2. *The following properties hold for all $i \in [r]$.*

- $(P1)$ $|W_i' \backslash W_i| \leqslant 2\gamma n$ *and* $|W''| \leqslant C_0 := 2t\varepsilon^{-rt}$.
- $(P2)$ $W = W'' \cup W_1' \cup \dots \cup W_r'$ *is a partition of* W.
- $(P3)$ $|Z_i''| \leqslant C_0$, $|Z_i^j| \leqslant \sqrt{\gamma}n$ *for* $j \neq i$, *and* $|Z_i^i| \geqslant (1-\sqrt{\gamma})an$.
- $(P4)$ $\bigcup_{j \in [r]} Z_i^j$ *is a partition of* Z_i'.

Proof. Recall the definition of W'' and that $d(Z_i, Z_j) \geqslant 1 - \gamma$ for distinct $i, j \in [r]$. Applying Proposition 4.1 to the graph $G[W'' \cup Z]$ with vertex partition $(U_1, \dots, U_r)$, we obtain that $|W''| \leqslant C_0 := 2t\varepsilon^{-rt}$. We next show that $|W_i' \backslash W_i| \leqslant 2\gamma n$ for each $i \in [r]$. Indeed, because G is γ-close to T, we have $e_G(Z_i, W_i' \backslash W_i) \geqslant an|W_i' \backslash W_i| - \gamma n^2$. On the other hand, by definition, $e_G(Z_i, W_i' \backslash W_i) < |W_i' \backslash W_i| \cdot \varepsilon n$. Thus, we get $|W_i' \backslash W_i| < \gamma n/(a - \varepsilon) < 2\gamma n$, verifying $(P1)$.

To see $(P2)$, suppose there is a vertex $v \in W_i' \cap W_j'$. By definition, $d(v) \leqslant (k-1)n - 2(a-\varepsilon)n < (k-1-a)n - \sqrt{\gamma}n$, contradicting (4.1).

Next we show (*P*3). Fix $i \in [r]$. Since G is γ-close to T, we have $d(Z_j, Z_{j'}) \geqslant 1-\gamma$ and $d(U_i, Z_j) \geqslant 1-\gamma$ for distinct $j, j' \in [r]\backslash\{i\}$. Thus, we can apply Proposition 4.1 on $G[U_i \cup \bigcup_{j\neq i} Z_j]$ (with the obvious r-partition) and obtain $|Z_i''| \leqslant C_0$. Moreover, for $i \neq j$, from $d(Z_i, Z_j) \geqslant 1-\gamma$ we infer $|Z_i^j| \leqslant (\gamma/\varepsilon)n \leqslant \sqrt{\gamma}n$, as $\gamma \ll \varepsilon$. Therefore, we also get $|Z_i^i| \geqslant |Z_i| - |Z_i''| - \sum_{j\neq i}|Z_i^j| \geqslant an - C_0 - (r-1)\gamma n/\varepsilon \geqslant (1-\sqrt{\gamma})an$.

Now we show (*P*4). By definition, if $v \in Z_i^i$, then $d(v, U_i) < \varepsilon n$; if $v \in Z_i^j$ for $j \neq i$, then $d(v, Z_j) < \varepsilon n$. Thus, we have $Z_i' \subseteq \bigcup_{j\in[r]} Z_i^j$ by definition. A vertex $v \in Z_i^i \cap Z_i^j$, $j \neq i$, satisfies that $d(v) < kn - (|U_i| - \varepsilon n) - (a-\varepsilon)n \leqslant (k-a)n - |W_i| - (1-2\varepsilon)n$, contradicting (4.1). A vertex $v \in Z_i^j \cap Z_i^{j'}$ for distinct $j, j' \in [r]\backslash\{i\}$ satisfies that $d(v) < (k-1)n - 2(a-\varepsilon)n \leqslant (k-a-2)n + 2\varepsilon n$, contradicting (4.1) as well. Thus, $\bigcup_{j\in[r]} Z_i^j$ is a partition of Z_i'. □

For $i \in [r]$, our refined partition is defined by

$$\tilde{U}_i := \tilde{Z}_i \cup W_i', \text{ where } \tilde{Z}_i := \bigcup_{j\in[r]} Z_j^i. \tag{4.2}$$

Then $V(G) = Z'' \cup W'' \cup \bigcup_{i\in[r]} \tilde{U}_i$. Note that for any $v \in \tilde{U}_i$, we have $d(v, Z_i^i) \leqslant d(v, Z_i) \leqslant \varepsilon n$, and thus $d(v, \tilde{Z}_i) \leqslant \varepsilon n + (r-1)\sqrt{\gamma}n$ by (*P*3).

For every $i \in [r]$, note that (*P*1) implies that $|W_i\backslash W_i'| \leqslant C_0 + (r-1)2\gamma n \leqslant 2r\gamma n$, and similarly (*P*3) implies that $|Z_i\backslash\tilde{Z}_i| \leqslant C_0 + (r-1)\sqrt{\gamma}n \leqslant r\sqrt{\gamma}n$.

We now derive a more handy minimum degree condition. For convenience, define $\overline{d}(v, A) = |A| - d(v, A)$. For $v \in Z_i^i$, we have $\overline{d}(v, \tilde{U}_i) \geqslant \overline{d}(v, U_i) - |U_i\backslash\tilde{U}_i|$. Since $\overline{d}(v, U_i) > an + |W_i| - \varepsilon n$ and $|U_i\backslash\tilde{U}_i| \leqslant |Z_i\backslash\tilde{Z}_i| + |W_i\backslash W_i'| \leqslant \varepsilon n/2$, we have $\overline{d}(v, \tilde{U}_i) \geqslant an + |W_i| - \varepsilon n - \varepsilon n/2$. By (4.1), $\overline{d}(v) \leqslant an + |W_i| + \sqrt{\gamma}n$. It follows that $\overline{d}(v, V\backslash\tilde{U}_i) \leqslant 2\varepsilon n$. Now consider $v \in \tilde{U}_i\backslash Z_i^i$. The definition of $\tilde{U}_i$ implies that $d(v, Z_i) < \varepsilon n$ and $\overline{d}(v, Z_i) > an - \varepsilon n$. Assume $v \in V_j$. Then $V_j \cap Z_i = \varnothing$ and trivially $\overline{d}(v, V_j) = n$. It follows that $\overline{d}(v, Z_i \cup V_j) > (a+1)n - \varepsilon n$. Hence $\overline{d}(v, \tilde{Z}_i \cup V_j) \geqslant \overline{d}(v, Z_i \cup V_j) - |Z_i\backslash\tilde{Z}_i| > (a+1)n - \frac{3}{2}\varepsilon n$. On the other hand, either case of (4.1) implies that $\overline{d}(v) \leqslant (a+1)n + \sqrt{\gamma}n$. Consequently, $\overline{d}(v, V\backslash(\tilde{Z}_i \cup V_j)) \leqslant 2\varepsilon n$. In summary, for $i \in [r]$ and $j \in [k]$,

(Deg) If $v \in Z_i^i$, then $\overline{d}(v, V\backslash\tilde{U}_i) \leqslant 2\varepsilon n$; if $v \in (\tilde{U}_i\backslash Z_i^i) \cap V_j$, then $\overline{d}(v, V\backslash(\tilde{Z}_i \cup V_j)) \leqslant 2\varepsilon n$.

Next we prove further properties on Z_i^j and $\tilde{Z}_j$.

Claim 4.3. *If $Z_i^j \neq \varnothing$ for some $i \neq j$, then the following holds.*

- (*Q*1) *For $v \in Z_i^j$ and $A \subseteq V(G)\backslash(Z_i \cup Z_j)$, we have $d(v, A) \geqslant |A| - \varepsilon n - \sqrt{\gamma}n$.*
- (*Q*2) *$|W_i| \geqslant (1-\varepsilon-\sqrt{\gamma})n$.*
- (*Q*3) *If $|\tilde{Z}_j\backslash Z_j| \geqslant t$, then $|W_j| \leqslant 2t\varepsilon n$.*

Proof. Note that $d(v, Z_j) \leqslant \varepsilon n$ and $d(v, Z_i) \leqslant (a-1)n$, that is, v has at least $n + (an - \varepsilon n) = (a+1)n - \varepsilon n$ non-neighbors in $Z_i \cup Z_j$. On the other hand, (4.1) says that v has at most $an + |W_i| + \sqrt{\gamma}n$ non-neighbors in G. Combining these two we get that v has at most $|W_i| - n + \varepsilon n + \sqrt{\gamma}n \leqslant \varepsilon n + \sqrt{\gamma}n$ non-neighbors outside $Z_i \cup Z_j$, and thus (*Q*1) holds. The fact that $|W_i| - n + \varepsilon n + \sqrt{\gamma}n \geqslant 0$ implies (*Q*2).

For (*Q*3), suppose to the contrary, $|\tilde{Z}_j\backslash Z_j| \geqslant t$ and $|W_j| > 2t\varepsilon n$. By (*Q*1) with $A = W_j$, arbitrary t vertices in $\tilde{Z}_j\backslash Z_j$ have at least $|W_j| - t(\varepsilon + \sqrt{\gamma})n \geqslant t$ common neighbors in W_j. We thus obtain a copy of $K_{t,t}$ with one part in $\tilde{Z}_j\backslash Z_j$ and the other part in W_j – denote its vertex set by B. For any $i' \in [r]\backslash\{j\}$ such that $B \cap Z_{i'}^j \neq \varnothing$, we have $|W_{i'}| \geqslant (1-\varepsilon-\sqrt{\gamma})n$ by (*Q*2). Since $|W_j| > 2t\varepsilon n$, $W_{i'}$ and

W_j do not belong to the same cluster, and thus no vertex of B is in the same cluster that contains $W_{i'}$, which implies that the vertices of B have at least $|W_{i'}| - 2t(2\varepsilon n) \geqslant n/2$ common neighbors in $W_{i'}$ by (Deg). For any $i'' \in [r]\backslash\{j\}$ such that $B \cap Z_{i''}^j = \varnothing$ (and thus $B \cap Z_{i''} = \varnothing$), by (Deg) we have that the vertices of B have at least $n/2$ common neighbors in $Z_{i''}$. Because G is γ-close to T, these common neighborhoods, each of size at least $n/2$, have densities close to one between each pair, and thus contain a copy of $K_{r-1}(t)$. Together with B, they form a copy of $K_{r+1}(t)$ in G, a contradiction. □

In particular, when $b = 0$ (and thus $W_i = \varnothing$ for all i), (*Q*2) implies that $Z_i^j = \varnothing$ whenever $i \neq j$. Consequently,

$$\tilde{U}_i = Z_i^i = Z_i \backslash Z'' \quad \text{for all } i \in [r] \text{ when } b = 0. \tag{4.3}$$

Let $L \subseteq [r]$ be the set of indices i such that $|W_i| \geqslant (1 - \varepsilon - \sqrt{\gamma})n$. (*Q*2) and (*Q*3) imply that

- for $i \in [r]\backslash L$, we have $Z_i^j = \varnothing$ for $j \neq i$.
- for $i \in L$, $|\tilde{Z}_i \backslash Z_i| \leqslant t - 1$ and thus $|\tilde{Z}_i| \leqslant an + t - 1$.

Estimate $e(G)$**.** Let $G' = G \cap K(\tilde{U}_1, \dots, \tilde{U}_r)$. We have $e(G) = e(G') + \sum_{i=1}^r e_G(\tilde{U}_i) + e(Z'' \cup W'', G)$.

Since G' is r-partite, it is K_{r+1}-free. As G' is a subgraph of $G \in \mathcal{G}_k(n)$, we have $e(G') \leqslant t_r(k)n^2$ (but this is not good enough when $b > 0$). Below we give an upper bound for $e(G')$, which will be used throughout the proof. Recall that $T = K(V_1, \dots, V_k) \cap K(U_1, \dots, U_r)$ has precisely $t_r(k)n^2$ edges,

Claim 4.4. *We have* $e(G') \leqslant t_r(k)n^2 + \sum_{i \in [r]}(\beta_i - \alpha_i)$*, where*

$$\beta_i := \sum_{j \in L\backslash\{i\}} |Z_j^i| \left(|\tilde{Z}_j \backslash Z_j| + |W_j'| - n + |Z_i \backslash \tilde{Z}_i|\right) \text{ and}$$

$$\alpha_i := |\tilde{Z}_i \backslash Z_i||W_i'| + e_T(W_i') + e_T(\tilde{Z}_i \backslash Z_i).$$

Proof. We first obtain $G^{(0)} := K(Z_1 \cup W_1', \dots, Z_r \cup W_r') \cap K(V_1, \dots, V_k)$ from T. During this process, we lose the edges of T between W_i and W_j, $j \neq i$, if both ends of the edges are placed in W_i'. Thus

$$e(G^{(0)}) = t_r(k)n^2 - \sum_{i \in [r]} e_T(W_i'). \tag{4.4}$$

We imagine a dynamic process of obtaining G' from $G^{(0)}$ by recursively moving vertices. To estimate $e(G')$, we track the changes of the edges with respect to complete r-partite graphs (but also respecting the k-partition of G). More precisely, for $l > 0$, let

$$G^{(l)} := K(Z_1^{(l)} \cup W_1', \dots, Z_r^{(l)} \cup W_r') \cap K(V_1, \dots, V_k)$$

such that the r-partition of $G^{(l)}$ can be obtained by moving exactly one vertex from the partition of $G^{(l-1)}$. The process terminates after $m := \sum_{i \in [r]} |\tilde{Z}_i \backslash Z_i|$ steps and thus G' is a subgraph of $G^{(m)}$. Furthermore, throughout the process, we only move vertices from the color classes in L to other color classes. Therefore, we can give a linear ordering to the members of L, and for $i \in L$ we move vertices from Z_i only after we have moved the vertices in color classes j prior to i (denoted by $j <_L i$). Now, in the l-th step, suppose we move v from $Z_j^{(l-1)}$ to $Z_i^{(l-1)}$, namely, $v \in Z_j^i$, then the change is

$$e(G^{(l)}) - e(G^{(l-1)}) = |Z_j^{(l-1)} \backslash V_p| + |W_j'| - |\tilde{Z}_i^{(l-1)}| - |W_i'|,$$

where $V_p \ni v$ and $\tilde{Z}_i^{(l-1)} = Z_i^{(l-1)} \backslash V_p$.

Note that we have $|Z_j^{(l-1)}\backslash V_p| \leqslant (a-1)n+|\tilde{Z}_j\backslash Z_j|$. Moreover for any $j' <_L j$, we have $Z^i_{j'} \subseteq Z_i^{(l-1)}$. Therefore, we have $|\tilde{Z}_i^{(l-1)}| \geqslant an - |Z_i\backslash\tilde{Z}_i| + \sum_{j'<_L j}|Z^i_{j'}|$. Putting all these together, we get

$$e(G^{(l)}) - e(G^{(l-1)}) \leqslant |\tilde{Z}_j\backslash Z_j| + |W'_j| - n + |Z_i\backslash\tilde{Z}_i| - \sum_{j'<_L j}|Z^i_{j'}| - |W'_i|.$$

Recalling that we moved v from $Z_j^{(l-1)}$ to $Z_i^{(l-1)}$ at the l-th step, we obtain

$$e(G') - e(G^{(0)}) \leqslant \sum_{l=1}^{m}\left(|\tilde{Z}_j\backslash Z_j| + |W'_j| - n + |Z_i\backslash\tilde{Z}_i| - \sum_{j'<_L j}|Z^i_{j'}| - |W'_i|\right),$$

where i, j depends on l. Since $m = \sum_{i\in[r]}|\tilde{Z}_i\backslash Z_i|$, we have

$$\begin{aligned}
&\sum_{l=1}^{m}(|\tilde{Z}_j\backslash Z_j| + |W'_j| - n + |Z_i\backslash\tilde{Z}_i| - |W'_i|)\\
&= \sum_{i\in[r]}\sum_{j\in L\backslash\{i\}}|Z^i_j|(|\tilde{Z}_j\backslash Z_j| + |W'_j| - n + |Z_i\backslash\tilde{Z}_i| - |W'_i|)\\
&= \sum_{i\in[r]}\sum_{j\in L\backslash\{i\}}|Z^i_j|(|\tilde{Z}_j\backslash Z_j| + |W'_j| - n + |Z_i\backslash\tilde{Z}_i|) - \sum_{i\in[r]}|\tilde{Z}_i\backslash Z_i||W'_i|.
\end{aligned}$$

Moreover, it is not hard to see that

$$\sum_{l=1}^{m}\sum_{j'<_L j}|Z^i_{j'}| = \sum_{i\in[r]}\sum_{\{j_1,j_2\}\in\binom{L\backslash\{i\}}{2}}|Z^i_{j_1}||Z^i_{j_2}| = \sum_{i\in[r]}e_T(\tilde{Z}_i\backslash Z_i).$$

Now the claim follows by combining these estimates with (4.4). ☐

Our main task is bounding the number of edges in each $\tilde{U}_i$. For $i \in [r]$, we have $e(G[\tilde{U}_i]) = e(Z^i_i, G[\tilde{U}_i]) + e_G(\tilde{U}_i\backslash Z^i_i)$. To bound $e_G(\tilde{U}_i\backslash Z^i_i) = e_G((\tilde{Z}_i\backslash Z_i)\cup W'_i)$, we note that $e_G(\tilde{Z}_i\backslash Z_i, W'_i) \leqslant |\tilde{Z}_i\backslash Z_i||W'_i|$ and $e_G(W'_i) \leqslant e_T(W'_i)$. However, we may not have $e_G(\tilde{Z}_i\backslash Z_i) \leqslant e_T(\tilde{Z}_i\backslash Z_i)$ because each Z^i_j is an independent set in T, but may not be independent in G when $a \geqslant 2$. Thus, $e_G(\tilde{Z}_i\backslash Z_i) \leqslant e_T(\tilde{Z}_i\backslash Z_i) + \sum_{j\neq i}e_G(Z^i_j)$. Putting these together, for each $i \in [r]$, we have

$$e_G(\tilde{U}_i\backslash Z^i_i) = e_G(\tilde{Z}_i\backslash Z_i, W'_i) + e_G(W'_i) + e_G(\tilde{Z}_i\backslash Z_i) \leqslant \alpha_i + \sum_{j\neq i}e_G(Z^i_j). \tag{4.5}$$

Let $f_i := e(Z^i_i, G[\tilde{U}_i])$. Applying Claim 4.4 and (4.5), we derive that

$$e(G) = e(G') + e(Z''\cup W'', G) + \sum_{i\in[r]}f_i + e_G((\tilde{U}_i\backslash Z^i_i) \tag{4.6}$$

$$\leqslant t_r(k)n^2 + e(Z''\cup W'', G) + \sum_{i\in[r]}\left(f_i + \beta_i + \sum_{j\neq i}e_G(Z^i_j)\right) \tag{4.7}$$

The following claim shows that a large portion of $G[\tilde{U}_i]$ is $K_{t,t}$-free (though we cannot prove that the entire $G[\tilde{U}_i]$ is $K_{t,t}$-free).

Claim 4.5. *The following holds for all $i \in [r]$.*

$(K1)$ Both $G[\tilde{Z}_i]$ and $G[Z^i_i \cup W'_i]$ are $K_{t,t}$-free.
$(K2)$ If $|W'_i| > 2t\varepsilon n + 2\gamma n$, then $|W'_i\backslash V_{q_i}| \leqslant t-1$.

($K3$) *If* $|W_i'| > 2t\varepsilon n + 2\gamma n$, *then* $G[\tilde{Z}_i \cup (W_i' \cap V_{q_i})]$ *is* $K_{t,t}$*-free.*

Proof. For ($K1$), suppose there is a copy of $K_{t,t}$ in $\tilde{U}_i$, with vertex set denoted by B, contained in $\tilde{Z}_i$ or in $Z_i^i \cup W_i'$. Let N_B be the set of common neighbors of these $2t$ vertices of B. First assume that $B \subseteq \tilde{Z}_i$. Then for any $j \in L\backslash\{i\}$, by (Deg) we have $|N_B \cap W_j'| \geqslant |W_j'| - 4t\varepsilon n$, and thus by ($P1$) $|N_B \cap W_j \cap W_j'| \geqslant |W_j'| - 4t\varepsilon n - 2\gamma n \geqslant n/2$. For any $j \notin L \cup \{i\}$, because $B \cap Z_j = \varnothing$ by ($Q2$), we have $|N_B \cap Z_j| \geqslant an - 4t\varepsilon n \geqslant n/2$ by (Deg). Note that every set in $\{N_B \cap Z_j : j \notin L\} \cup \{N_B \cap W_j \cap W_j' : j \in L\}$ has size at least $n/2$ and every pair of them has density at least $1 - 4\gamma$. Therefore we can find a copy of $K_{r-1}(t)$ in the union of these sets, which gives rise to a copy of $K_{r+1}(t)$ together with B, a contradiction.

Second we assume that $B \subseteq Z_i^i \cup W_i'$. In this case we note that for any $j \neq i$, we have $B \cap Z_j = \varnothing$ and thus by (Deg), we have $|N_B \cap Z_j^j| \geqslant (1 - \sqrt{\gamma})an - 4t\varepsilon n \geqslant n/2$. Then as these sets have high pairwise densities, as in the previous case, we can find a copy of $K_{r-1}(t)$ in the union of these sets, yielding a copy of $K_{r+1}(t)$ together with B, a contradiction. Now ($K1$) is proved.

Now we turn to ($K2$), and suppose $|W_i'| > 2t\varepsilon n + 2\gamma n$ and thus $|W_i \cap W_i'| > 2t\varepsilon n$ by ($P1$). First, if W_i' contains at least t vertices which are not from V_{q_i} (the cluster containing W_i), then by (Deg), each of these vertices have at most $2\varepsilon n$ non-neighbors in $W_i \cap W_i'$, and thus we can find a copy of $K_{t,t}$ in W_i', contradicting ($K1$). So we have $|W_i'\backslash V_{q_i}| \leqslant t - 1$.

For ($K3$), suppose there is a copy of $K_{t,t}$ as stated in the claim, whose vertex set is denoted by B. As in the previous paragraph, we have $|W_i| > 2t\varepsilon n$ by ($P1$). Now observe crucially that if $B \cap Z_j^i \neq \varnothing$, then by ($Q2$) $|W_i| + |W_j| > n$, and thus, W_i and W_j are not from the same cluster. So by (Deg), for any $j \in [r-1]\backslash\{i\}$, if $B \cap Z_j^i = \varnothing$, then the vertices of B have large common neighborhoods in Z_j^j; if $B \cap Z_j^i \neq \varnothing$, then the vertices of B have large common neighborhoods in $W_j \cap W_j'$ (note that $|W_j| \geqslant (1 - \varepsilon - \sqrt{\gamma})n$ by ($Q2$)). Since each of these common neighborhoods have size at least $n/2$ and each pair of them has high density, we can find a copy of $K_{r-1}(t)$ in the union of these sets, yielding a copy of $K_{r+1}(t)$ together with B, a contradiction. □

We now derive an upper bound for $e(G)$ from Claims 4.4 and 4.5. For $i \in [r]$, we have $\beta_i \leqslant \sum_{j \in L\backslash\{i\}} |Z_j^i|(|\tilde{Z}_j\backslash Z_j| + |W_j'\backslash V_{q_j}| + |Z_i\backslash\tilde{Z}_i|)$ as $|W_j'| - n \leqslant |W_j'\backslash V_{q_j}|$. Fix $j \in L\backslash\{i\}$. Note that $|W_j| \geqslant (1 - 2\varepsilon)n$. We have $|\tilde{Z}_j\backslash Z_j| \leqslant t - 1$ by ($Q3$), and $|W_j'\backslash V_{q_j}| \leqslant t - 1$ by ($K2$). If $|W_i| > n/2$, then $|Z_j^i| \leqslant t - 1$ by ($Q3$). Furthermore, since $|Z_i\backslash\tilde{Z}_i| \leqslant (r-1)\sqrt{\gamma}n + C_0$ by ($P3$), it follows that

$$|Z_j^i| \left(|\tilde{Z}_j\backslash Z_j| + |W_j'\backslash V_{q_j}| + |Z_i\backslash\tilde{Z}_i|\right) \leqslant (t-1)(t-1+t-1+(r-1)\sqrt{\gamma}n + C_0) \leqslant (t-1)r\sqrt{\gamma}n.$$

Otherwise $|W_i| \leqslant n/2$, and by ($Q2$), we have $Z_i^{i'} = \varnothing$ for any $i' \neq i$. This implies $|Z_i\backslash\tilde{Z}_i| = |Z_i''| \leqslant C_0$. Using $|Z_j^i| \leqslant \sqrt{\gamma}n$, ($Q3$), and ($K2$), we derive that

$$|Z_j^i| \left(|\tilde{Z}_j\backslash Z_j| + |W_j'\backslash V_{q_j}| + |Z_i\backslash\tilde{Z}_i|\right) \leqslant \sqrt{\gamma}n(2(t-1) + C_0) \leqslant 2C_0\sqrt{\gamma}n.$$

Summarizing these two cases for all $j \in L\backslash\{i\}$, we obtain that $\beta_i \leqslant (r-1)2C_0\sqrt{\gamma}n$, and consequently,

$$\sum_{i \in [r]} \beta_i \leqslant 2(r-1)rC_0\sqrt{\gamma}n. \tag{4.8}$$

On the other hand, for all $i \neq j$, the graph $G[Z_j^i]$ is $K_{t,t}$-free by ($K1$) and thus, by ($P3$), $\sum_{i,j:i\neq j} e_G(Z_j^i) \leqslant r(r-1)C\left(\sqrt{\gamma}n\right)^{2-1/t}$. Applying this with (4.7), (4.8), and the fact that $e(Z'' \cup$

$W'', G) \leqslant (r+1)C_0kn$, we obtain that

$$e(G) \leqslant t_r(k)n^2 + (r+1)C_0kn + \sum_{i\in[r]} f_i + 2(r-1)rC_0\sqrt{\gamma}n + \sum_{i,j:i\neq j} e_G(Z_j^i) \tag{4.9}$$
$$\leqslant t_r(k)n^2 + \sum_{i\in[r]} f_i + r^2C\sqrt{\gamma}n^{2-1/t}.$$

Using the assumption $e(G) \geqslant g_2(n,r,k,t) \geqslant t_r(k)n^2 + z_t(n)$, we infer that

$$\sum_{i\in[r]} f_i \geqslant z_t(n) - r^2C\sqrt{\gamma}n^{2-1/t} \geqslant \frac{c}{2}n^{2-1/t} \tag{4.10}$$

by using (Z), $z_t(n) \geqslant z_t^{(2)}(n) \geqslant cn^{2-1/t}$, and $\gamma \ll 1$.

In the rest of the proof we will derive a contradiction to (4.10). We first study the existence of $K_{1,t}$ in each color class. To do so, we consider a copy of $K_3(t)$ in $G[\tilde{U}_i \cup \tilde{U}_j]$ for some $i \neq j$.

Claim 4.6. *For any $i \neq j$, if $G[\tilde{U}_i \cup \tilde{U}_j]$ contains a copy K of $K_3(t)$, then there exists $l \notin \{i,j\}$ such that $V(K)$ intersects V_{q_l} and every cluster in Z_l.*

Proof. We may assume that $r > 2$ as otherwise the claim is trivial. Suppose to the contrary that there is a copy K of $K_3(t)$ in, say, $\tilde{U}_1$ and $\tilde{U}_2$, such that for every $l \in [3,r]$, there is a cluster in U_l which does not intersect $B := V(K)$. Let V_{i_l} be a cluster in Z_l such that $B \cap V_{i_l} = \varnothing$, and if there is no such cluster in Z_l, then we choose $V_{i_l} = V_{q_l}$. Note that in the former case, we have $|\tilde{U}_l \cap V_{i_l}| = |Z_l^l \cap V_{i_l}| \geqslant (1-\sqrt{\gamma}a)n$. In the latter case, we have $Z_l^1 \neq \varnothing$ or $Z_l^2 \neq \varnothing$, which implies that $|W_l| \geqslant (1-2\varepsilon)n$ by (Q2), and thus $|\tilde{U}_l \cap V_{i_l}| = |W_l' \cap V_{i_l}| \geqslant (1-3\varepsilon)n$. Now, by (Deg), every vertex in B has at most $2\varepsilon n$ non-neighbors in $\tilde{U}_l \cap V_{i_l}$ for each $l \in [3,r]$. Since for every l we have $|\tilde{U}_l \cap V_{i_l}| \geqslant 0.9n$, one can find large common neighborhoods (e.g. of size $n/2$) of all vertices of B in each $\tilde{U}_l \cap V_{i_l}$, and then find a copy of $K_{r-2}(t)$ in these sets. Altogether we obtain a copy of $K_{r+1}(t)$, a contradiction.

Therefore, for such a copy K of $K_3(t)$, there exists $l \notin \{i,j\}$ such that K must intersect all clusters of U_l. Since $V(K) \cap Z_l \neq \varnothing$, we have $Z_l^i \neq \varnothing$ or $Z_l^j \neq \varnothing$. Then by (Q2), $|W_l| \geqslant (1-2\varepsilon)n$ and in particular, $V_{q_l} \neq \varnothing$. Therefore $V(K) \cap V_{q_l} \neq \varnothing$. □

Claim 4.7. *For all but exactly one $j \in [r]$, we have $d(v, Z_j^j) \leqslant t-1$ for all $v \in \tilde{U}_j$.*

Proof. First assume that there exists $j \in [r]$ such that $G[\tilde{U}_j]$ contains a copy of $K_{1,t}$, with vertex set denoted by $\{v, u_1, \ldots, u_t\}$, $v \in \tilde{U}_j$ and $u_1, \ldots, u_t \in Z_j^j$. Fix $i \in [r]\backslash\{j\}$ and let N' be the set of common neighbors of $u_1, \ldots, u_t$ in $\tilde{U}_i \cap U_i$. Suppose $v \in V_p$ and let N be the set of common neighbors of these $t+1$ vertices in $\tilde{U}_i \cap U_i$. In particular, $N \subseteq N'$ and N is almost equal to the union of a or $a+1$ clusters in $\tilde{U}_i$. Suppose there is a copy of $K_{t-1,t}$ with parts S_1 of size $t-1$ and S_2 of size t such that $S_1 \subseteq N'$ and $S_2 \subseteq N$. Then by Claim 4.6, there exists $l \in [r]\backslash\{i,j\}$ such that $B \cap Z_l \neq \varnothing$ and $B \cap V_{q_l} \neq \varnothing$, where B denotes the vertex set of the copy of $K_3(t)$. This is impossible since v is the only possible vertex in $B \cap (Z_l \cup V_{q_l})$ and can not satisfy both. Therefore, $G[N, N']$ is $K_{t-1,t}$-free, implying that $e_G(N, N') = O(n^{2-1/(t-1)})$.

By (P1), (P3) and (Deg), we have $|(\tilde{U}_i\backslash V_p)\backslash N| \leqslant 3(t+1)\varepsilon n$ and $|\tilde{U}_i\backslash N'| \leqslant 3t\varepsilon n$. Let E^i be the set of the edges incident to $(\tilde{U}_i\backslash V_p)\backslash N$ or $\tilde{U}_i\backslash N'$ and counted in f_i. We split it to $E^i \cap E_G(Z_i^i)$ and $E^i \cap E_G(\tilde{U}_i\backslash Z_i^i, Z_i^i)$. Note that by (K1), each of the terms can be split further into at most k

$K_{t,t}$-free bipartite graphs, each with one part of size at most $3(t+1)\varepsilon n$ and the other part of size at most $(1+(r-2)\sqrt{\gamma})an$. Therefore, we obtain that

$$f_i = O(\varepsilon n^{2-1/t}) + O(n^{2-1/(t-1)}) = O(\varepsilon n^{2-1/t}). \tag{4.11}$$

Now assume there exist distinct $j_1, j_2 \in [r]$ such that each $G[\tilde{U}_{j_i}]$ contains a copy of $K_{1,t}$ whose part of size t is in $Z^{j_i}_{j_i}$. The arguments above imply that (4.11) holds for all $i \in [r]$, and consequently, $\sum_{i\in[r]} f_i = O(\varepsilon n^{2-1/t})$, contradicting (4.10).

On the other hand, if $d(v, Z^j_j) \leqslant t-1$ for all $j \in [r]$ and all $v \in \tilde{U}_j$, then $\sum_{j\in[r]} f_j \leqslant (t-1)kn$, again contradicting (4.10). □

By Claim 4.7, without loss of generality, we assume that,

$$\text{for } i \geqslant 2, \quad d(v, Z^i_i) \leqslant t-1 \text{ for all } v \in \tilde{U}_i, \quad \text{and thus,} \quad f_i \leqslant (t-1)|\tilde{U}_i|. \tag{4.12}$$

If $b = 0$, then $\tilde{U}_i = Z^i_i = Z_i \backslash Z''$ for all i by (4.3). In this case $\tilde{U}_1$ is $K_{t,t}$-free by $(K1)$ and $\tilde{U}_i$ is $K_{1,t}$-free for all $i \geqslant 2$ by (4.12). Since G is γ-close to $K_r(an)$, $G[U_i \backslash Z'', U_j \backslash Z'']$ is almost complete for all $i \neq j$. This completes the proof of Theorem 7 with $Z := Z''$ (note that (Z) holds because $t = 2, 3$).

Furthermore, when $b = 0$, together with $e(G') \leqslant t_r(k)n^2$, we conclude that

$$e(G) \leqslant t_r(k)n^2 + z_t(n) + (r-1)\left\lfloor \frac{t-1}{2} an \right\rfloor + |Z''|kn. \tag{4.13}$$

By $(P3)$, $|Z''| \leqslant rC_0$, and this proves Theorem 5 for the case $b = 0$.

We thus assume $b > 0$ for the rest of the proof of Theorem 5. By (4.10) and (4.12), we get

$$f_1 \geqslant z_t^{(a+1)}(n) - \varepsilon n^{2-1/t}. \tag{4.14}$$

In particular, we claim that

$$|W_1| > 3t\varepsilon n \tag{4.15}$$

(which we will refine a moment later). Indeed, the edges counted in f_1 can be covered by $G[Z^1_1]$, $G[Z^1_1, W_1 \cap W'_1]$, and at most k $K_{t,t}$-free bipartite graphs, each with a part of size at most $\sqrt{\gamma}n$ and a part of size at most an. If $|W_1| \leqslant 3t\varepsilon n$, then $e_G(Z^1_1, W_1 \cap W'_1) = O(\varepsilon n^{2-1/t})$. Together with $e_G(Z^1_1) \leqslant z_t^{(a)}(n)$, we have

$$f_1 \leqslant z_t^{(a)}(n) + O(\varepsilon n^{2-1/t}) < z_t^{(a+1)}(n) - \varepsilon n^{2-1/t}$$

by $(E1)$, contradicting (4.14).

Now we can give a much cleaner structure, shown in a series of claims below.

Claim 4.8. *Suppose $b > 0$. Then $\tilde{U}_1 = Z^1_1 \cup W'_1$ and $W'_1 \subseteq V_{q_1}$.*

Proof. Suppose to the contrary, there is a vertex v in $\tilde{U}_1 \backslash (Z^1_1 \cup W'_1)$ or $W'_1 \backslash V_{q_1}$, namely, $v \in Z^1_i$ for some $2 \leqslant i \leqslant r$ or $v \in W'_1 \backslash V_{q_1}$. Suppose $v \in V_l$. Then $l \neq q_1$. Morepver, if i is defined, then $W'_1 \cap V_{q_1} \subseteq V \backslash (\tilde{Z}_i \cup V_l)$; otherwise, $W'_1 \cap V_{q_1} \subseteq V \backslash V_l$. By (Deg), we have $\overline{d}(v, W'_1 \cap V_{q_1}) \leqslant 2\varepsilon n$. Let $N := W'_1 \cap V_{q_1} \cap N(v)$. We have $|(W'_1 \cap V_{q_1}) \backslash N| \leqslant 2\varepsilon N$. Since $|W'_1 \backslash V_{q_1}| \leqslant |W'_1 \backslash W_1| \leqslant 2\gamma n$, it follows that $|W'_1 \backslash N| \leqslant 2\varepsilon n + 2\gamma n \leqslant 3\varepsilon n$.

Recall (4.15), $|W_1| > 3t\varepsilon n$. By $(K3)$ (if $v \in \tilde{Z}_1 \backslash Z_1$) or $(K1)$ (if $v \in W'_1 \backslash V_{q_1}$), we know that $G[Z^1_1, N]$ contains no $K_{t-1,t}$ with the part of size t in N. This implies that $e_G(Z^1_1, N) = O(n^{2-1/(t-1)})$. Furthermore, by $(P3)$ and $(K1)$, $G[\tilde{Z}_1 \backslash Z^1_1, Z^1_1]$ is a $K_{t,t}$-free graph with one part of size at most

$(r-1)\sqrt{\gamma}n$ and the other part of size at most an. Thus, $e_G(\tilde{Z}_1\backslash Z_1^1, Z_1^1) \leqslant C(r-1)\sqrt{\gamma}n(an)^{1-1/t}$. By the similar arguments, we have $e_G(W_1'\backslash N, Z_1^1) \leqslant C(3\varepsilon n)(an)^{1-1/t}$.

Putting these bounds together with $e_G(Z_1^1) \leqslant z_t^{(a)}(n)$, we get

$$\begin{aligned} f_1 &= e_G(Z_1^1) + e_G(Z_1^1, N) + e_G(\tilde{Z}_1\backslash Z_1^1, Z_1^1) + e_G(W_1'\backslash N, Z_1^1) \\ &\leqslant z_t^{(a)}(n) + O(n^{2-1/(t-1)}) + O(\sqrt{\gamma}n^{2-1/t}) + O(\varepsilon n^{2-1/t}). \end{aligned}$$

By $(E1)$, this implies that $f_1 < z_t^{(a+1)}(n) - \varepsilon n^{2-1/t}$, contradicting (4.14). ☐

Claim 4.8 shows that $\tilde{U}_1$ has no atypical vertices and is thus $K_{t,t}$-free by $(K1)$. Furthermore, since $\tilde{U}_1 = Z_1^1 \cup W_1'$ and $W_1' \subseteq V_{q_1}$, it follows that

$$\alpha_1 = \beta_1 = 0, \quad \text{and} \quad e_G(\tilde{U}_1) = f_1 \leqslant z_t^{(a+1)}(|Z_1^1 \cap V_1|, \ldots, |Z_1^1 \cap V_a|, |W_1'|) \leqslant z_t^{(a+1)}(n). \tag{4.16}$$

Next we study $G[\tilde{U}_i]$ for $i \geqslant 2$. A key observation is that copies of $K_{1,t}$ in $G[\tilde{U}_i]$ together with copies of $K_{t-1,t}$ in $\tilde{U}_1$ may form copies of $K_3(t)$, which are restricted by Claim 4.6.

Claim 4.9. *Suppose $b > 0$ and $i \in [2, r]$.*

(1) If there is a copy of $K_{1,t}$ in $\tilde{U}_i\backslash(Z_1 \cup V_{q_1})$, then there exists $l \in [r]\backslash\{i\}$ such that the vertex set of $K_{1,t}$ intersects V_{q_l} and every cluster in Z_l.

(2) $\tilde{Z}_i\backslash Z_1$ and $Z_i^i \cup (W_i'\backslash V_{q_1})$ are $K_{1,t}$-free.

(3) If $t < a$, then $\tilde{U}_i\backslash(Z_1 \cup V_{q_1})$ is $K_{1,t}$-free.

Proof. For Part (1), let B be the vertex set of a copy of $K_{1,t}$ in $\tilde{U}_i\backslash(Z_1 \cup V_{q_1})$. Since $B \cap (Z_1 \cup V_{q_1}) = \varnothing$ and $\tilde{U}_1 \subseteq Z_1 \cup V_{q_1}$, by (Deg), all vertices of B have at most $2\varepsilon n$ non-neighbors in $\tilde{U}_1$. Letting $N := \tilde{U}_1 \cap \bigcap_{w \in B} N(w)$, we have $|N| \geqslant |\tilde{U}_1| - (t+1)2\varepsilon n$.

First assume that N is $K_{t-1,t}$-free and thus $e_G(N) = O(n^{2-1/(t-1)})$. Note that, since $|\tilde{U}_1\backslash N| \leqslant (t+1)2\varepsilon n$, the edges in $\tilde{U}_1$ incident to $\tilde{U}_1\backslash N$ can be split into $a+1$ bipartite $K_{t,t}$-free graphs each with one part of size at most $(t+1)2\varepsilon n$ and the other part of size at most an. Thus, the number of such edges is $O(\varepsilon n^{2-1/t})$. This gives $f_1 = O(n^{2-1/(t-1)}) + O(\varepsilon n^{2-1/t})$, contradicting (4.14).

We thus assume N contains a copy of $K_{t-1,t}$. Together with B, they form a copy of $K_3(t)$ in $G[\tilde{U}_1 \cup \tilde{U}_i]$ and we denote its vertex set by B'. By Claim 4.6, there exists $l \notin \{1, i\}$ such that B' intersects V_{q_l} and every cluster of Z_l. By Claim 4.8, $\tilde{U}_1 \cap U_l = \varnothing$, so $B' \cap Z_l = B \cap Z_l$ and B indeed intersects every cluster of Z_l. Since $\tilde{U}_i \cap Z_l \supseteq B \cap Z_l \neq \varnothing$, we infer that $|W_l| \geqslant (1-2\varepsilon)n$ from $(Q3)$, which implies that $q_l \neq q_1$ because of (4.15). It follows that $W_1 \cap V_{q_l} = \varnothing$ and thus $B \cap V_{q_l} = B' \cap V_{q_l} \neq \varnothing$, as desired.

For Part (2), let $A_i := Z_i^i \cup (W_i'\backslash V_{q_1})$ and B be the vertex set of a copy of $K_{1,t}$ in $\tilde{Z}_i\backslash Z_1$ or in A_i. Then, by the first part of the claim, there exists $l \in [r]\backslash\{i\}$ such that B intersects V_{q_l} and every cluster in Z_l. This is impossible if $B \subseteq A_i$ because $A_i \cap Z_l = \varnothing$ for any $l \notin \{1, i\}$, and also impossible if $B \subseteq \tilde{Z}_i\backslash Z_1$ because in which case $B \cap W = \varnothing$ and thus $B \cap V_{q_l} = \varnothing$ for any $l \notin \{1, i\}$.

Part (3) follows from Part (1) immediately. ☐

The next claim bounds $e_G(Z_1^i)$ for $i > 1$.

Claim 4.10. *For every $i \in [2, r]$, $e_G(Z_1^i) \leqslant \binom{a}{2}(t-1)|Z_1^i|$.*

Proof. Suppose $|W_1| \geqslant (1 - \varepsilon - \sqrt{\gamma})n$ (otherwise $Z_1^i = \varnothing$ for $i > 1$ by $(Q2)$ and there is nothing to prove). Fix $i \in [2, r]$. Suppose $a \geqslant 2$ (if $a = 1$ then $e_G(Z_1^i) = 0$). We claim that for distinct $i_1, i_2 \in [a]$, there can not be two copies of $K_{1,t}$ in the bipartite graph $G[Z_1^i \cap V_{i_1}, Z_1^i \cap V_{i_2}]$, one centered in $Z_1^i \cap V_{i_1}$ and the other centered in $Z_1^i \cap V_{i_2}$. We show this for $i_1 = 1$ and $i_2 = 2$. Suppose $w \in Z_1^i \cap V_1$ and $u_1, \dots, u_t \in Z_1^i \cap V_2$ form a copy of $K_{1,t}$ while $w' \in Z_1^i \cap V_2$, $u_1', \dots, u_t' \in Z_1^i \cap V_1$ form the other copy of $K_{1,t}$. By (Deg), $w, u_1', \dots, u_t'$ each has at most $2\varepsilon n$ non-neighbors in $\tilde{U}_1 \backslash V_1$ and $w', u_1, \dots, u_t$ each has at most $2\varepsilon n$ non-neighbors in $\tilde{U}_1 \backslash V_2$. Let

$$N_1 = \left(\tilde{U}_1 \backslash (V_1 \cup V_2)\right) \cap N(w) \cap \bigcap_{i \in [t]} N(u_i) \quad \text{ and } \quad N_2 = \left(\tilde{U}_1 \backslash V_2\right) \cap \bigcap_{i \in [t]} N(u_i)$$

(thus $N_1 \cap N_2 \subseteq \tilde{U}_1$). By $(P3)$ and $|W_1 \cap W_1'| \geqslant (1 - \varepsilon - \sqrt{\gamma})n - 2r\gamma n \geqslant (1 - 2\varepsilon)n$, we have

$$|N_1| \geqslant (1 - \sqrt{\gamma})an - 2n + (1 - 2\varepsilon)n - (t+1)2\varepsilon n \geqslant (a-1)n - (2t+5)\varepsilon n$$

and similarly $|N_2| \geqslant an - (2t+3)\varepsilon n$. If $G[N_1, N_2]$ contains a copy of $K_{t-1,t}$ with the part of size t in N_1, then it together with $\{w, u_1, \dots, u_t\}$ forms a copy of $K_3(t)$ in $\tilde{U}_1 \cup \tilde{U}_i$. Let B be the vertex set of this $K_3(t)$ and note that $B \subseteq U_1$. Thus $B \cap Z_l = \varnothing$ for $l \neq 1$ contradicting Claim 4.6. This implies that $e_G(N_1, N_2) = O(n^{2-1/(t-1)})$. Now let $N_1' = (\tilde{U}_1 \backslash (V_1 \cup V_2)) \cap N(w') \cap \bigcap_{i \in [t]} N(u_i')$ and $N_2' = (\tilde{U}_1 \cap V_2) \cap \bigcap_{i \in [t]} N(u_i')$. By the same argument as above, $G[N_1', N_2']$ contains no copy of $K_{t-1,t}$ with the part of size t in N_1', which implies that $e_G(N_1', N_2') = O(n^{2-1/(t-1)})$. Note that all but $O(\varepsilon n^{2-1/t})$ edges of $G[\tilde{U}_1] \backslash G[V_1, V_2]$ are in $G[N_1', N_2']$ or $G[N_1, N_2]$. We thus derive that $f_1 \leqslant z_t^{(2)}(n) + O(n^{2-1/(t-1)}) + O(\varepsilon n^{2-1/t}) < z_t(n) - \varepsilon n^{2-1/t}$ by $(E1)$, contradicting (4.14).

The conclusion above implies that $e_G[Z_1^i \cap V_{i_1}, Z_1^i \cap V_{i_2}] \leqslant (t-1) \max\{|Z_1^i \cap V_{i_1}|, |Z_1^i \cap V_{i_2}|\}$. Therefore,

$$e_G(Z_1^i) = \sum_{i_1, i_2 \in [a]} e_G[Z_1^i \cap V_{i_1}, Z_1^i \cap V_{i_2}] \leqslant \sum_{i_1, i_2 \in [a]} (t-1)|Z_1^i| \leqslant \binom{a}{2}(t-1)|Z_1^i|. \qquad \square$$

Claims 4.9 (2) and 4.10 together give $e_G(Z_j^i) \leqslant \max\{(t-1)|Z_j^i|, \binom{a}{2}(t-1)|Z_j^i|\} = \binom{a}{2}(t-1)|Z_j^i|$ whenever $i \neq j$. Thus, by $(P3)$,

$$\sum_{i,j: i \neq j} e_G(Z_j^i) \leqslant r(r-1)\binom{a}{2}(t-1)\sqrt{\gamma} n \leqslant a^2 r^2 t \sqrt{\gamma} n \tag{4.17}$$

Together with $f_1 \leqslant z_t^{(a+1)}(n)$ and $f_i \leqslant (t-1)|\tilde{U}_i|$ for $i \geqslant 2$, we derive from (4.9) that

$$e(G) = t_r(k)n^2 + z_t^{(a+1)}(n) + O(n).$$

This concludes the proof of Theorem 5 (note that we have not used $(E2)$ or $(E3)$).

Proof of Theorem 6. We refine our earlier estimates and prove Theorem 6. Since $\sum_{i \neq j} e_G(Z_j^i) \leqslant a^2 r^2 t \sqrt{\gamma} n$ and $f_j \leqslant (t-1)|\tilde{U}_j|$ for $j \in [2, r]$, we have the following bound better than (4.14),

$$f_1 \geqslant z_t(n) - 3rC_0 kn. \tag{4.18}$$

Furthermore, we claim that

$$\text{if} \quad b > 0, \quad \text{then} \quad |W_1'| \geqslant (1-\gamma)n \quad (\text{and thus} \quad 1 \in L). \tag{4.19}$$

Indeed, by Claim 4.8, if $|W_1'| < (1-\gamma)n$, then we have $f_1 \leqslant z_t^{(a+1)}(n, \dots, n, |W_1'|) \leqslant z_t^{(a+1)}(n) - \delta n^{2-1/t}$ for some $\delta > 0$ by $(E2)$. This contradicts (4.18).

Next we show that $Z'' \cup W'' = \varnothing$.

Claim 4.11. *Suppose $v_0 \in V(G)$ and $i \in [r]$ satisfy that v_0 has at least εn neighbors in Z_j for every $j \neq i$. Then v_0 has less than εn neighbors in U_i. In particular, we have $Z_i'' = \varnothing$ for all $i \in [r]$ and $W'' = \varnothing$.*

Proof. The second part of the claim follows immediately from the definitions of Z_i'' and W''.

Suppose to the contrary, that there exist $v_0 \in V(G)$ and $i \in [r]$ such that v_0 has at least εn neighbors in Z_j for every $j \neq i$ and at least εn neighbors in U_i. Since $|Z_j^j| \geqslant (1-\sqrt{\gamma})an$ for all $j \in [r]$, there exist sets $N_1, \ldots, N_{r-1}$ each of size $\varepsilon n - \sqrt{\gamma}an$ such that $N_j \subseteq Z_j^j \cap N(v_0)$ for $j \neq i$ and $N_i \subseteq (Z_i^i \cup W_i) \cap N(v_0)$. Recall that $W_1' = W_1 \cap V_{q_1}$. By averaging, there exists $N_1' \subseteq N_1$ with $|N_1'| \geqslant (\varepsilon n - \sqrt{\gamma}an - 2r\gamma n)/(a+1) \geqslant \varepsilon n/(a+2)$ such that all vertices of N_1' are in $Z_1^1 \cup W_1'$ and from the same cluster, that is,

$$N_1' \subseteq Q, \text{ where } Q \in \{V_1 \cap Z_1^1, V_2 \cap Z_1^1, \ldots, V_a \cap Z_1^1, W_1'\}.$$

Note that $N_1' \subseteq W_1'$ is possible only if $i = 1$. If $i \neq 1$, then let $N_i' := N_i \backslash ((W_i \backslash W_i') \cup V_{q_1})$ and for every $j \in [r]\backslash\{1, i\}$, let $N_j' := N_j$. By (*P*1), $|W_i \backslash W_i'| \leqslant 2r\gamma n$, and by (4.19), $|W_i \cap V_{q_1}| \leqslant \gamma n$. Thus, we have $|N_j'| \geqslant \varepsilon n/(a+2)$ for all $j \in [r]$. Because the sets N_j' are small, we can not apply the degree conditions (Deg) to them and instead, we use (D0).

Recall that B_1 is given by (D0). Next we show that $G[\tilde{U}_1 \backslash B_1]$ does not contain a copy of $K_{t-1,t}$ such that the part of size t is in N_1'. Suppose instead, there is such a copy of $K_{t-1,t}$, with parts denoted by A and B, such that $|A| = t$ and $A \subseteq N_1' \backslash B_1$ and $B \subseteq \tilde{U}_1 \backslash B_1$. Recall that $N_i' \cap V_{q_1} = \varnothing$ and for each $j \in [r]\backslash\{1, i\}$, $N_j' \subseteq Z_j^j$. Observe that for every $v \in \tilde{U}_1 \backslash B_1$, we have $d(v, N_j') \geqslant |N_j'| - \sqrt{\gamma}n$. Indeed, if $j \neq i$, then $N_j' \subseteq Z_j^j$ and we have $d(v, N_j') \geqslant |N_j'| - \sqrt{\gamma}n$ by (D0); otherwise note that $N_i' \subseteq Z_i^i \cup (W_i' \cap W_i)$, and by (D0) and $N_i' \cap V_{q_1} = \varnothing$ we have $d(v, N_i') \geqslant |N_i'| - \sqrt{\gamma}n$. Therefore, we obtain that the vertices in $A \cup B$ have at least $|N_j'| - (2t-1)\sqrt{\gamma}n \geqslant (1-\gamma^{1/3})|N_j'|$ common neighbors in each N_j', $j \in [2, r]$. Because each pair $N_j', N_{j'}'$ has a high density, we can find a copy of $K_{r-1}(t)$ in the union of these common neighborhoods, which together with $A \cup B \cup \{v_0\}$ form a copy of $K_{r+1}(t)$, a contradiction.

Now given that $G[\tilde{U}_1 \backslash B_1]$ does not contain a copy of $K_{t-1,t}$ such that the part of size t is in $N_1' \backslash B_1$, we shall give a refined estimate on f_1. Indeed, for each V_j, $j \in [a]$, we know that $G[N_1' \backslash B_1, (V_j \cap Z_1^1) \backslash B_1]$ does not contain a copy of $K_{t-1,t}$ such that the part of size t is in $N_1' \backslash B_1$, implying that $e_G(N_1' \backslash B_1, (V_j \cap Z_1^1) \backslash B_1) = O(n^{2-1/(t-1)})$. Similarly we also have $e_G(N_1' \backslash B_1, W_1' \backslash B_1) = O(n^{2-1/(t-1)})$. Suppose $N_1' \subseteq V_q$ for some $q \in [a] \cup \{q_1\}$, then we have

$$E(G[\tilde{U}_1]) = E(G[\tilde{U}_1 \backslash (N_1' \backslash B_1)]) \cup E(G[N_1' \backslash B_1, \tilde{U}_1 \backslash (B_1 \cup V_q)]) \cup E(G[N_1' \backslash B_1, B_1 \cap \tilde{U}_1]).$$

Recall that $|N_1'| \geqslant \varepsilon n/(a+2)$ and $|B_1| \leqslant 2\sqrt{\gamma}n$. Therefore, we can bound $f_1 \leqslant |E(G[\tilde{U}_1])|$ by

$$f_1 \leqslant z_t\left((1 - \tfrac{\varepsilon}{a+2})n, n, \ldots, n\right) + O(n^{2-1/(t-1)}) + O(\sqrt{\gamma}n^{2-1/t}) < z_t(n) - 3rC_0kn,$$

where we used (*E*2) and $\gamma \ll \varepsilon$. This contradicts (4.18). □

When $b = 0$, since $Z_i'' = \varnothing$ for all $i \in [r]$, we can improve (4.13) to $e(G) \leqslant t_r(k)n^2 + z_t^{(a)}(n) + (r-1)\lfloor \frac{t-1}{2}an \rfloor$, proving Theorem 6 for $b = 0$.

In the rest of the proof, we assume $b > 0$. We start with the following claim.

Claim 4.12. *For $i \in [2, r]$ such that $|W_i| \geqslant 2\varepsilon n$, we have $\tilde{U}_i \subseteq U_i \cup V_{q_i}$.*

Proof. Suppose instead, for some $i_0 \in [2, r]$ with $|W_{i_0}| \geqslant 2\varepsilon n$, there exists $v \in \tilde{U}_{i_0} \backslash (U_{i_0} \cup V_{q_{i_0}})$. By (*P*4) and the fact that $v \in \tilde{U}_{i_0} \backslash U_{i_0}$, we infer that $d(v, Z_j) \geqslant \varepsilon n$ for all $j \neq i_0$. Then, by Claim 4.11, we have $d(v, U_{i_0}) < \varepsilon n$. Consequently, $d(v, W'_{i_0} \cap W_{i_0}) < \varepsilon n$, namely, v has at least $2\varepsilon n - 2\gamma n - \varepsilon n \geqslant (3/4)\varepsilon n$ non-neighbors in $W'_{i_0} \cap W_{i_0}$ (in G). Note that v is adjacent to all the vertices of $W'_{i_0} \cap W_{i_0}$ in T. Since $G[W'_{i_0}] \subseteq T[W'_{i_0}]$, we infer that

$$e_G(W'_{i_0}) + e_G(\tilde{Z}_{i_0} \backslash Z_{i_0}, W'_{i_0}) \leqslant e_T(W'_{i_0}) + |\tilde{Z}_{i_0} \backslash Z_{i_0}||W'_{i_0}| - (3/4)\varepsilon n.$$

Since $e_G(\tilde{Z}_{i_0} \backslash Z_{i_0}) \leqslant e_T(\tilde{Z}_{i_0} \backslash Z_{i_0}) + \sum_{j \neq i_0} e_G(Z_j^{i_0})$ and $\alpha_{i_0} = |\tilde{Z}_{i_0} \backslash Z_{i_0}||W'_i| + e_T(W'_{i_0}) + e_T(\tilde{Z}_{i_0} \backslash Z_{i_0})$, we have

$$\begin{aligned} e_G(\tilde{U}_{i_0} \backslash Z_{i_0}) &= e_G(\tilde{Z}_{i_0} \backslash Z_{i_0}) + e_G(W'_{i_0}) + e_G(\tilde{Z}_{i_0} \backslash Z_{i_0}, W'_{i_0}) \\ &\leqslant e_T(\tilde{Z}_{i_0} \backslash Z_{i_0}) + \sum_{j \neq i_0} e_G(Z_j^{i_0}) + e_T(W'_{i_0}) + |\tilde{Z}_{i_0} \backslash Z_{i_0}||W'_{i_0}| - (3/4)\varepsilon n \\ &\leqslant \alpha_{i_0} + \sum_{j \neq i_0} e_G(Z_j^{i_0}) - (3/4)\varepsilon n. \end{aligned} \tag{4.20}$$

Combining (4.5) and (4.20) gives $\sum_{i \in [r]} e_G(\tilde{U}_i \backslash Z_i^i) \leqslant \sum_{i \in [r]} \alpha_i + \sum_{i, j \neq i} e_G(Z_j^i) - (3/4)\varepsilon n$. Since $\sum_{i, j \neq i} e_G(Z_j^i) \leqslant a^2 r^2 t \sqrt{\gamma} n$ by (4.17) and $\gamma \ll \varepsilon$, it follows that

$$\sum_{i \in [r]} e_G(\tilde{U}_i \backslash Z_i^i) \leqslant \sum_{i \in [r]} \alpha_i - \varepsilon n/2 \tag{4.21}$$

Recall that $f_1 \leqslant z_t(n)$ by (4.16). We next bound f_i for $i \geqslant 2$, noting that

$$f_i = e\left(Z_i^i, G[Z_i^i \cup (W'_i \backslash V_{q_1})]\right) + e_G\left(Z_i^i, (\tilde{Z}_i \backslash Z_i) \cup (W'_i \cap V_{q_1})\right).$$

By Claim 4.9 (2), $G[Z_i^i \cup (W'_i \backslash V_{q_1})]$ has the maximum degree at most $t-1$, and thus,

$$e\left(Z_i^i, G[Z_i^i \cup (W'_i \backslash V_{q_1})]\right) \leqslant e_G(Z_i^i \cup (W'_i \backslash V_{q_1})) \leqslant \frac{t-1}{2}(|Z_i^i| + |W'_i \backslash V_{q_1}|).$$

By (4.12), $e\left(Z_i^i, (\tilde{Z}_i \backslash Z_i) \cup (W'_i \cap V_{q_1})\right) \leqslant (t-1)\left(|\tilde{Z}_i \backslash Z_i| + |W'_i \cap V_{q_1}|\right)$. Putting these together,

$$f_i \leqslant \frac{t-1}{2}(|Z_i^i| + |W'_i \backslash V_{q_1}|) + (t-1)\left(|\tilde{Z}_i \backslash Z_i| + |W'_i \cap V_{q_1}|\right) = \frac{t-1}{2}\left(|\tilde{U}_i| + |\tilde{Z}_i \backslash Z_i| + |W'_i \cap V_{q_1}|\right).$$

By (*P*3) and (4.19), we have $|Z_1^1| \geqslant (1-\sqrt{\gamma})an$, $|W'_1| \geqslant (1-\gamma)n$, and thus, $|\tilde{U}_1| \geqslant (a+1)n(1-\sqrt{\gamma})$. By (*P*3) and (4.19), we also have $|\tilde{Z}_i \backslash Z_i| + |W'_i \cap V_{q_1}| \leqslant (r-1)\sqrt{\gamma} n + \gamma n \leqslant r\sqrt{\gamma} n$. It follows that

$$\sum_{i=2}^{r} f_i \leqslant \frac{t-1}{2}\left(kn - (a+1)n(1-\sqrt{\gamma}) + (r-1)r\sqrt{\gamma} n\right) \leqslant (k-a-1)n\frac{t-1}{2} + \sqrt[3]{\gamma} n. \tag{4.22}$$

Since $Z'' \cup W'' = \varnothing$, we rewrite (4.6) as $e(G) = e(G') + \sum_{i \in [r]} f_i + e_G(\tilde{U}_i \backslash Z_i^i)$. By Claim 4.4, it follows that $e(G) \leqslant t_r(k)n^2 + \sum_{i \in [r]}(f_i + \beta_i - \alpha_i + e_G(\tilde{U}_i \backslash Z_i^i))$. Recall that $\sum_{i=1}^{r} \beta_i \leqslant 2(r-1)rC_0\sqrt{\gamma} n$ by (4.8). Together with (4.21) and(4.22), we derive that

$$e(G) \leqslant t_r(k)n^2 + z_t(n) + (k-a-1)n\frac{t-1}{2} + \sqrt[3]{\gamma} n + 2(r-1)rC_0\sqrt{\gamma} n - \varepsilon n/2 < g_2(n, r, k, t),$$

a contradiction. □

Next we upper bound $e(\tilde{U}_i)$ and separate the discussions depending on whether $|W_i| \geqslant 2\varepsilon n$ or not. For $i \in [2, r]$, let $U_i^* := \tilde{U}_i \backslash (Z_1 \cup V_{q_1})$.

Claim 4.13. *For $i \in [2, r]$, we have*

$$e_G(\tilde{U}_i) \leqslant \begin{cases} (t-1)\min\{|Z_i^i|, |W_i'|\} & \text{if } a = 1, \text{ and } |W_i| \geqslant 2\varepsilon n \\ |\tilde{U}_i|(t-1)/2 & \text{if } a \geqslant 2 \text{ and } |W_i| \geqslant 2\varepsilon n \\ (t-1)(|\tilde{Z}_i'\backslash Z_i| + |W_i'|) + \alpha_i & \text{if } a = 1, \text{ and } |W_i| < 2\varepsilon n \\ \left\lfloor |U_i^*|\frac{t-1}{2} \right\rfloor + t' + \alpha_i + a^2 t(|Z_1^i| + |W_i' \cap V_{q_1}|) & \text{if } a \geqslant 2 \text{ and } |W_i| < 2\varepsilon n, \end{cases}$$

where $t' := (t-1)^2/(16(a-1))$. Moreover, when $|W_i| < 2\varepsilon n$, $a \geqslant 2$ and $t < a$, we have $e(\tilde{U}_i) \leqslant \left\lfloor |U_i^*|\frac{t-1}{2} \right\rfloor + \alpha_i + a^2 t(|Z_1^i| + |W_i' \cap V_{q_1}|)$.

Proof. Assume $|W_i| \geqslant 2\varepsilon n$. Since $|W_1' \geqslant (1-\gamma)n$, we know that $q_1 \neq q_i$. It follows from Claim 4.12 that $\tilde{U}_i \subseteq Z_i^i \cup (W_i' \cap V_{q_i})$. By Claim 4.9 (2), $\tilde{U}_i$ is $K_{1,t}$-free and thus, $e_G(\tilde{U}_i) \leqslant |\tilde{U}_i|(t-1)/2$. For $a = 1$, since both Z_i^i and $W_i' \subseteq V_{q_i}$ are independent sets, it follows that $e_G(\tilde{U}_i) \leqslant (t-1)\min\{|Z_i^i|, |W_i'|\}$.

Now we consider the case when $|W_i| < 2\varepsilon n$. Recall that

$$e_G(\tilde{U}_i) = f_i + e_G((Z_i'\backslash Z_i) \cup W_i').$$

When $a = 1$, we have $e(Z_j^i) = 0$ for all $j \in [r]$. Thus $f_i \leqslant (t-1)(|\tilde{Z}_i'\backslash Z_i| + |W_i'|)$ by (4.12) and $e_G((Z_i'\backslash Z_i) \cup W_i') \leqslant \alpha_i$ by (4.5). As a result, $e_G(\tilde{U}_i) \leqslant (t-1)(|\tilde{Z}_i'\backslash Z_i| + |W_i'|) + \alpha_i$ as desired.

Now assume $a \geqslant 2$. Let $Z_i^* := (\tilde{Z}_i\backslash Z_i)\backslash Z_1^i$, and $W_i^* := W_i'\backslash V_{q_1}$. For every $j \in [r]\backslash\{1, i\}$, consider the family $\mathcal{F}_j := \{Z_j^i \cap V_{(j-1)a+1}, \ldots, Z_j^i \cap V_{ja}, W_i^* \cap V_{q_j}\}$. Let B_j be the smallest set in $\mathcal{F}_j$ and A_j be an arbitrary set in $\mathcal{F}_j\backslash\{B_j, W_i^* \cap V_{q_j}\}$. Further, define $A := \bigcup_{j\in[r]\backslash\{1,i\}} A_j$ and $B := \bigcup_{j\in[r]\backslash\{1,i\}} B_j$. By Claim 4.9, both $U_i^*\backslash A$ and $U_i^*\backslash B$ are $K_{1,t}$-free. This implies that $e_G(U_i^*\backslash B) \leqslant \left\lfloor |U_i^*\backslash B|\frac{t-1}{2} \right\rfloor$ and $e(B, G[U_i^*\backslash A]) \leqslant |B|(t-1)$. Together with $e_G(A, B) \leqslant |A||B|$, we obtain that

$$e_G(U_i^*) \leqslant \left\lfloor |U_i^*\backslash B|\frac{t-1}{2} \right\rfloor + |B|(t-1) + |A||B| = \left\lfloor |U_i^*|\frac{t-1}{2} \right\rfloor + |B|\left(\frac{t-1}{2} + |A|\right).$$

Since, by definition $|B| \leqslant (|Z_i^*| - |A|)/(a-1) =: x$, unless $\frac{t-1}{2} + |A| - |Z_i^*| < 0$, we have

$$|B|\left(\frac{t-1}{2} + |A| - |Z_i^*|\right) \leqslant x\left(\frac{t-1}{2} - (a-1)x\right) \leqslant \frac{(t-1)^2}{16(a-1)} = t'.$$

Trivially $|A| \leqslant |Z_i^*|$ and $|B| \leqslant |W_i^*|$. Altogether, we get

$$e_G(U_i^*) \leqslant \left\lfloor |U_i^*|\frac{t-1}{2} \right\rfloor + t' + |Z_i^*||W_i^*|. \tag{4.23}$$

Let $X_i := Z_1^i \cup (W_i' \cap V_{q_1})$ and we bound $e(X_i, G[\tilde{U}_i])$ as follows. Since $\tilde{U}_i = \tilde{Z}_i \cup W_i' = Z_i^i \cup X_i \cup Z_i^* \cup W_i^*$, we have

$$\begin{aligned} e(X_i, G[\tilde{U}_i]) &= e_G(X_i, Z_i^i) + e(Z_1^i, G[Z_1^i \cup Z_i^* \cup W_i']) + e(W_i' \cap V_{q_1}, G[Z^* \cup W_i']) \\ &= e_G(X_i, Z_i^i) + e_G(Z_1^i) + e_G(Z_1^i, Z_i^* \cup W_i') + e_G(W_i' \cap V_{q_1}, Z_i^* \cup W_i^*). \end{aligned}$$

Note that $e_G(Z_1^i, Z_i^*) \leqslant e_T(\tilde{Z}_i\backslash Z_i)$, $e_G(W_i' \cap V_{q_1}, W_i^*) \leqslant e_T(W_i')$, and

$$e_G(Z_1^i, W_i') + e(W_1' \cap V_{q_1}, Z_i^*) \leqslant |Z_1^i||W_i'| + |W_1' \cap V_{q_1}||Z_i^*| = |\tilde{Z}_i\backslash Z_i||W_i'| - |Z^*||W^*|.$$

Recall that $\alpha_i = |\tilde{Z}_i\backslash Z_i||W_i'| + e_T(W_i') + e_T(\tilde{Z}_i\backslash Z_i)$. It follows that

$$\begin{aligned} e_G(Z_1^i, Z_i^* \cup W_i') + e_G(W_i' \cap V_{q_1}, W_i^* \cup Z_i^*) &\leqslant e_T(\tilde{Z}_i\backslash Z_i) + e_T(W_i') + |\tilde{Z}_i\backslash Z_i||W_i'| - |Z^*||W^*| \\ &= \alpha_i - |Z^*||W^*|. \end{aligned}$$

Applying $e_G(X_i, Z_i^i) \leqslant (t-1)|X_i|$ from (4.12) and $e_G(Z_1^i) \leqslant \binom{a}{2}(t-1)|Z_1^i|$ from Claim 4.10, we derive that

$$e(X_i, G[\tilde{U}_i]) \leqslant (t-1)|X_i| + \binom{a}{2}(t-1)|Z_1^i| + \alpha_i - |Z_i^*||W_i^*|. \tag{4.24}$$

Together with (4.23), this gives the desired upper bound

$$\begin{aligned} e_G(\tilde{U}_i) &\leqslant \left\lfloor |U_i^*|\frac{t-1}{2} \right\rfloor + t' + (t-1)|X_i| + \binom{a}{2}(t-1)|Z_1^i| + \alpha_i \\ &\leqslant \left\lfloor |U_i^*|\frac{t-1}{2} \right\rfloor + t' + a^2 t|X_i| + \alpha_i. \end{aligned}$$

At last, when $a \geqslant 2$ and $t < a$, we know that U_i^* is $K_{1,t}$-free (Claim 4.9 (3)) and thus $e_G(U_i^*) \leqslant \lfloor |U_i^*|\frac{t-1}{2} \rfloor$. Together with (4.24), this gives $e_G(\tilde{U}_i) \leqslant \lfloor |U_i^*|\frac{t-1}{2} \rfloor + \alpha_i + a^2 t(|Z_1^i| + |W_i' \cap V_{q_1}|)$. □

For $i \in [r]$, if $|W_i| \geqslant 2\varepsilon n$, then $\tilde{Z}_i \backslash Z_i = \varnothing$ by Claim 4.12, and therefore, $\beta_i = 0$. If $|W_i| < 2\varepsilon n$, then $Z_i \backslash \tilde{Z}_i = \varnothing$ by (Q2); for any $j \in L\backslash\{i\}$, we have $\tilde{Z}_j \backslash Z_j = \varnothing$ and $W_j' \subseteq V_{q_i}$ again by Claim 4.12. Hence $\beta_i = \sum_{j \in L\backslash\{i\}} |Z_j^i|(|W_j'| - n) \leqslant 0$ because $|W_j'| \leqslant n$. Together with (4.16), for $i \in [r]$,

$$\beta_i = \begin{cases} 0 & \text{if } |W_i| \geqslant 2\varepsilon n \text{ (including } i = 1\text{)}, \\ \sum_{j \in L\backslash\{i\}} |Z_j^i|(|W_j'| - n) \leqslant 0 & \text{otherwise.} \end{cases} \tag{4.25}$$

We are ready to finish our proof. Write $z_i := |\tilde{Z}_i|$ and $w_i := |W_i'|$ for $i \in [r]$. Further, write $n_l := |Z_1^1 \cap V_l|$ for $\in [a]$. Then $z_1 = \sum_{l \in [a]} n_l$. By (4.16), we have $e_G(\tilde{U}_1) = f_1 \leqslant z_t(n_1, \ldots, n_a, w_1)$.

We separate the cases when $a \geqslant 2$ and when $a = 1$.

The case when $a \geqslant 2$. Note that there are at most $r - b$ indices $i \in [r]$ such that $|W_i| < 2\varepsilon n$. By Claim 4.13, and using that $\sum_{i=2}^r |U_i^*| = (k-a-1)n$, we have

$$\sum_{i \in [r]} e_G(\tilde{U}_i) \leqslant f_1 + (an - z_1 + n - w_1)a^2 t + \left\lfloor \frac{t-1}{2}(k-a-1)n \right\rfloor + (r-b)t' + \sum_{i:|W_i|<2\varepsilon n} \alpha_i$$

Since $e(G) = e(G') + \sum_{i \in [r]} e_G(\tilde{U}_i)$, it follows that

$$e(G) \leqslant f_1 + (an - z_1 + n - w_1)a^2 t + \left\lfloor \frac{t-1}{2}(k-a-1)n \right\rfloor + (r-b)t' + e(G') + \sum_{i:|W_i|<2\varepsilon n} \alpha_i.$$

Claim 4.4 and (4.25) together imply that

$$e(G') \leqslant t_r(k)n^2 + \sum_{i \in [r]} (\beta_i - \alpha_i) \leqslant t_r(k)n^2 - \sum_{i:|W_i|<2\varepsilon n} \alpha_i.$$

Together with $f_1 \leqslant z_t(n_1, \ldots, n_a, w_1)$, this gives

$$e(G) \leqslant t_r(k)n^2 + z_t(n_1, \ldots, n_a, w_1) + (an - z_1 + n - w_1)a^2 t + \left\lfloor \frac{t-1}{2}(k-a-1)n \right\rfloor + (r-b)t'.$$

If $an - z_1 + n - w_1 > 0$, then by applying (E3) repeatedly, we have

$$z_t(n_1, \ldots, n_a, w_1) + (an - z_1 + n - w_1)a^2 t + (r-b)t' \leqslant z_t(n).$$

This implies that $e(G) \leqslant t_r(k)n^2 + z_t(n) + \lfloor \frac{t-1}{2}(k-a-1)n \rfloor$. Otherwise $an + n = z_1 + w_1$, which implies that

$$e(G) \leqslant t_r(k)n^2 + z_t(n) + \left\lfloor \frac{t-1}{2}(k-a-1)n \right\rfloor + (r-b)t'.$$

This is the desired bound for $t \geqslant a$. When $t < a$, the bound for $e_G(\tilde{U}_i)$ given by Claim 4.13 contains no t' term. We thus obtain $e(G) \leqslant t_r(k)n^2 + z_t(n) + \lfloor \frac{t-1}{2}(k-a-1)n \rfloor$, regardless whether $an - z_1 + n - w_1 > 0$.

Now consider the case $a \geqslant 2$ and $b = 1$. If $an - z_1 + n - w_1 > 0$, then we choose a large constant $C^* > 0$ such that

$$z_t(n_1, \ldots, n_a, w_1) + (an - z_1 + n - w_1)a^2 t + (r-b)t' \leqslant z_t(n) - C^*$$

and thus $e(G) \leqslant t_r(k)n^2 + z_t(n) + \frac{t-1}{2}(k-a-1)n - C^* < g_2(n, r, k, t)$. Otherwise $an + n = z_1 + w_1$. Since $b = 1$, this forces that, for all $i \geqslant 2$, we have $W_i' = \varnothing$ and, by (Q2),$\tilde{Z}_i = Z_i$. Now, by Claim 4.9, we have $\sum_{i=2}^{r} e_G(\tilde{U}_i) \leqslant (r-1)\lfloor \frac{t-1}{2} an \rfloor$ and consequently, $e(G) \leqslant t_r(k)n^2 + z_t(n) + (r-1)\lfloor \frac{t-1}{2} an \rfloor$, as desired.

The case when $a = 1$**.** Let $L_1 \cup L_2 \cup L_3$ be a partition of $[2, r]$ such that $i \in L_1$ if and only if $|\tilde{Z}_i| < n$, $i \in L_2$ if and only if $|\tilde{Z}_i| = n$, and $i \in L_3$ if and only if $|\tilde{Z}_i| > n$. The following properties hold for L_1, L_2 and L_3.

($R1$) If $i \in L_1$, then $Z_i^j \neq \varnothing$ for some $j \neq i$. By ($Q2$), we have $i \in L$ and, by Claim 4.12, $\tilde{Z}_i = Z_i^i \subsetneq V_i$ and $W_i' \subseteq V_{q_i}$.

($R2$) If $i \in L_2$, then $\tilde{Z}_i = Z_i^i = V_i$. Otherwise $\tilde{Z}_i \neq Z_i^i$, then $|Z_i^i| < n$ and $Z_i^j \neq \varnothing$ for some $j \neq i$. By ($Q2$) and (4.12), we have $\tilde{Z}_i = Z_i^i$, a contradiction.

($R3$) If $i \in L_3$, then $\tilde{Z}_i \nsubseteq Z_i$. By Claim 4.12, we have $|W_i| < 2\varepsilon n$, which implies that $Z_i^j = \varnothing$ for $j \neq i$ by ($Q2$). Thus, $Z_i^i = V_i \subsetneq \tilde{Z}_i$.

By ($R1$), for every $i \in L_1$, $G[\tilde{U}_i]$ is a bipartite graph of maximum degree at most $t-1$, and thus $e_G(\tilde{U}_i) \leqslant (t-1)\min\{z_i, w_i\}$. For $i \in L_2$, since $\tilde{Z}_i = V_i$ is an independent set, we have $e_G(\tilde{U}_i) \leqslant (t-1)w_i + e_G(W_i') \leqslant (t-1)w_i + \alpha_i$. For $i \in L_3$, using (4.5) and the independence of Z_j^i for all $j \in [r]$, we have

$$\begin{aligned} e_G(\tilde{U}_i) &= e_G((\tilde{Z}_i \backslash V_i) \cup W_i') + e_G((\tilde{Z}_i \backslash V_i) \cup W_i', Z_i^i) \\ &\leqslant \alpha_i + (t-1)(|\tilde{Z}_i \backslash V_i| + |W_i'|) = \alpha_i + (t-1)(z_i - n + w_i). \end{aligned}$$

Using $\sum_{i \in L_1 \cup L_3}(z_i - n) = n - z_1$ and $\sum_{i=2}^{r} w_i = bn - w_1$, we derive that

$$\begin{aligned} &\sum_{i \in L_1} \min\{z_i, w_i\} + \sum_{i \in L_2} w_i + \sum_{i \in L_3}(z_i - n + w_i) \\ =& bn - w_1 + \sum_{i \in L_1}(\min\{z_i, w_i\} - w_i) + \sum_{i \in L_3}(z_i - n) \\ =& bn - w_1 + \sum_{i \in L_1}(z_i - n + \min\{n - w_i, n - z_i\}) + \sum_{i \in L_3}(z_i - n) \\ =& bn - w_1 + n - z_1 + \sum_{i \in L_1} \min\{n - w_i, n - z_i\}. \end{aligned}$$

Therefore,

$$\sum_{i=2}^{r} e_G(\tilde{U}_i) \leqslant \sum_{i\in L_1}(t-1)\min\{z_i,w_i\} + \sum_{i\in L_2}((t-1)w_i+\alpha_i) + \sum_{i\in L_3}((t-1)(z_i-n+w_i)+\alpha_i)$$
$$= (t-1)(bn-w_1+n-z_1) + \sum_{i\in L_1}(t-1)\min\{n-w_i,n-z_i\} + \sum_{i\in L_2\cup L_3}\alpha_i.$$

Recall that $e(G') \leqslant t_r(k)n^2 + \sum_{i\geqslant 2}(\beta_i-\alpha_i)$ by Claim 4.4 and (4.16). For $i \in L_1 \cup L_2$, we have $\tilde{Z}_i = Z_i^i$ and thus $\beta_i = 0$ by (4.25). It follows that (note that $L \cap L_3 = \varnothing$)

$$\sum_{i\geqslant 2}\beta_i = \sum_{i\in L_3}\beta_i = \sum_{i\in L_3}\sum_{j\in L\backslash\{i\}}|Z_j^i|(|W_j'|-n) = \sum_{j\in L}\sum_{i\in L_3\backslash\{j\}}|Z_j^i|(|W_j'|-n) = \sum_{j\in L}(n-z_j)(w_j-n)$$

Note that $1 \in L$ by (4.19) and $(n-z_1)(w_1-n) \leqslant 0$ by Claim 4.8. Furthermore, since $n - z_j = 0$ for $j \in L_2$, it follows that $\sum_{i\geqslant 2}\beta_i = \sum_{j\in L_1}(n-z_j)(w_j-n)$. Consequently, $e(G') \leqslant t_r(k)n^2 + \sum_{j\in L_1}(n-z_j)(w_j-n) - \sum_{i\geqslant 2}\alpha_i$.

Recall that $e_G(\tilde{U}_1) = f_1 \leqslant z_t(z_1,w_1)$. By (E3), we have $z_t(z_1,w_1)+(t-1)(n-z_1+n-w_1) \leqslant z_t(n)$. Thus, combining these estimates together, we get

$$e(G) = e(G') + \sum_{i=1}^{r} e_G(\tilde{U}_i) \leqslant t_r(k)n^2 + z_t(n) + (t-1)(b-1)n + y,$$

where $y := \sum_{i\in L_1}((t-1)\min\{n-w_i,n-z_i\} - (n-z_i)(n-w_i))$. For each $i \in L_1$, let $y_i := \min\{n-w_i,n-z_i\}$ and $y_i' := \max\{n-w_i,n-z_i\}$. Then $y_i \leqslant y_i'$ and thus,

$$(t-1)\min\{n-w_i,n-z_i\} - (n-z_i)(n-w_i) = y_i(t-1-y_i') \leqslant \lfloor (t-1)^2/4\rfloor.$$

Since $L_1 \subseteq L\backslash\{1\}$, we have $|L_1| \leqslant b-1$. Consequently, $e(G) \leqslant t_r(k)n^2 + z_t(n) + (t-1)(b-1)n + (b-1)\lfloor (t-1)^2/4\rfloor = g_2(n,r,k,t)$. □

References

[1] B. Bollobás, P. Erdős, and E. Szemerédi, *On complete subgraphs of r-chromatic graphs*, Discrete Math. **13** (1975), no. 2, 97–107. MR0389639 ↑1, 1

[2] W. G. Brown, *On graphs that do not contain a Thomsen graph*, Canad. Math. Bull. **9** (1966), 281–285. MR200182 ↑1

[3] W. Chen, C. Lu, and L. Yuan, *A stability theorem for multi-partite graphs*, ArXiv ePrint 2208:13995 (2023). ↑1

[4] P. Erdős, *On some new inequalities concerning extremal properties of graphs*, Theory of Graphs (Proc. Colloq., Tihany, 1966), 1968, pp. 77–81. MR232703 ↑1

[5] P. Erdős, P. Frankl, and V. Rödl, *The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent*, Graphs Combin. **2** (1986), no. 2, 113–121. MR932119 ↑2

[6] P. Erdős, A. Rényi, and V. T. Sós, *On a problem of graph theory*, Studia Sci. Math. Hungar. **1** (1966), 215–235. MR223262 ↑1

[7] P. Erdős and M. Simonovits, *An extremal graph problem*, Acta Math. Acad. Sci. Hungar. **22** (1971/72), 275–282. MR292707 ↑1, 1, 1

[8] P. Erdös and A. H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091. MR18807 ↑1

[9] P. E. Haxell, *A note on vertex list colouring*, Combin. Probab. Comput. **10** (2001), no. 4, 345–347. MR1860440 ↑1

[10] P. Haxell and T. Szabó, *Odd independent transversals are odd*, Combin. Probab. Comput. **15** (2006), no. 1-2, 193–211. MR2195582 ↑1

[11] G. P. Jin, *Complete subgraphs of r-partite graphs*, Combin. Probab. Comput. **1** (1992), no. 3, 241–250. MR1208805 ↑1

[12] T. Kövari, V. T. Sós, and P. Turán, *On a problem of K. Zarankiewicz*, Colloq. Math. **3** (1954), 50–57. MR65617 ↑1, 4

[13] A. Lo, A. Treglown, and Y. Zhao, *Complete subgraphs in a multipartite graph*, Combin. Probab. Comput. **31** (2022), no. 6, 1092–1101. MR4496024 ↑1

[14] W. Mantel, *Problem 28*, Wiskundige Opgaven **10** (1907), 60–61. ↑1

[15] M. Simonovits, *A method for solving extremal problems in graph theory, stability problems*, Theory of Graphs (Proc. Colloq., Tihany, 1966), 1968, pp. 279–319. MR233735 ↑1

[16] T. Szabó and G. Tardos, *Extremal problems for transversals in graphs with bounded degree*, Combinatorica **26** (2006), no. 3, 333–351. MR2246152 ↑1

[17] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452. MR0018405 ↑1

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