Minimum Vertex Degree Threshold for $\mathcal{C}_4^3$-tiling*

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Abstract: We prove that the vertex degree threshold for tiling $\mathcal{C}_4^3$ (the 3-uniform hypergraph with four vertices and two triples) in a 3-uniform hypergraph on $n \in 4\mathbb{N}$ vertices is $\binom{n-1}{2} - \binom{\frac{3}{4}n}{2} + \frac{3}{8}n + c$, where $c = 1$ if $n \in 8\mathbb{N}$ and $c = -\frac{1}{2}$ otherwise. This result is best possible, and is one of the first results on vertex degree conditions for hypergraph tiling. J. Graph Theory 00: 1–17, 2014

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1. INTRODUCTION

Given $k \geq 2$, a k-uniform hypergraph (in short, k-graph) consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$, where every edge is a k-element subset of V. Given a k-graph $\mathcal{H}$ with a set S of d vertices (where $1 \leq d \leq k-1$) we define $\deg_{\mathcal{H}}(S)$ to be the number of edges containing S (the subscript $\mathcal{H}$ is omitted if it is clear from the context). The *minimum d-degree* $\delta_d(\mathcal{H})$ of $\mathcal{H}$ is the minimum of $\deg_{\mathcal{H}}(S)$ over all d-vertex sets S in $\mathcal{H}$.

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Given a k-graph $\mathcal{G}$ of order g and a k-graph $\mathcal{H}$ of order n, a *$\mathcal{G}$-tiling* (or *$\mathcal{G}$-packing*) of $\mathcal{H}$ is a subgraph of $\mathcal{H}$ that consists of vertex-disjoint copies of $\mathcal{G}$. When g divides n, a *perfect $\mathcal{G}$-tiling* (or a *$\mathcal{G}$-factor*) of $\mathcal{H}$ is a *$\mathcal{G}$-tiling* of $\mathcal{H}$ consisting of n/g copies of $\mathcal{G}$. Define $t_d(n, \mathcal{G})$ to be the smallest integer t such that every k-graph $\mathcal{H}$ of order $n \in g\mathbb{N}$ with $\delta_d(\mathcal{H}) \geq t$ contains a perfect $\mathcal{G}$-tiling.

As a natural extension of the matching problem, tiling has been an active area in the past two decades (see surveys [15, 21]). Much work has been done on the problem for graphs ($k = 2$), see e.g. [7, 2, 12, 16]. In particular, Kühn and Osthus [17] determined $t_1(n, \mathcal{G})$, for any graph $\mathcal{G}$, up to an additive constant. Tiling problems become much harder for hypergraphs. For example, despite much recent progress [1, 5, 10, 11, 17, 24, 26], we still do not know the 1-degree threshold for a perfect matching in k-graphs for arbitrary k.

Other than the matching problem, only a few tiling thresholds are known. Let K_4^3 be the complete 3-graph on four vertices, and let $K_4^3 - e$ be the (unique) 3-graph on four vertices with three edges. Recently, Lo and Markström [19] proved that $t_2(n, K_4^3) = (1 + o(1))3n/4$, and independently Keevash and Mycroft [10] determined the exact value of $t_2(n, K_4^3)$ for sufficiently large n. In [20], Lo and Markström proved that $t_2(n, K_4^3 - e) = (1 + o(1))n/2$. Let $\mathcal{C}_4^3$ be the unique 3-graph on four vertices with two edges. This 3-graph was denoted by $K_4^3 - 2e$ in [4], and by $\mathcal{Y}$ in [9]. Here, we follow the notation in [15] and view it as a *cycle* on four vertices. Kühn and Osthus [15] showed that $t_2(n, \mathcal{C}_4^3) = (1 + o(1))n/4$, and Czygrinow et al. [4] recently determined $t_2(n, \mathcal{C}_4^3)$ exactly for large n. In this article, we determine $t_1(n, \mathcal{C}_4^3)$ for sufficiently large n. From now on, we simply write $\mathcal{C}_4^3$ as $\mathcal{C}$.

Previously we only knew $t_1(n, K_3^3)$ [11, 17] and $t_1(n, K_4^4)$ [11] exactly, and $t_1(n, K_5^5)$ [1], $t_1(n, K_3^3(m))$, and $t_1(n, K_4^4(m))$ [19] asymptotically, where K_k^k denotes a single k-edge, and $K_k^k(m)$ denotes the complete k-partite k-graph with m vertices in each part. So Theorem 1.1 below is one of the first (exact) results on vertex degree conditions for hypergraph tiling.

Theorem 1.1 (Main Result). *Suppose $\mathcal{H}$ is a 3-graph on n vertices such that $n \in 4\mathbb{N}$ is sufficiently large and*

$$\delta_1(\mathcal{H}) \geq \binom{n-1}{2} - \binom{\frac{3}{4}n}{2} + \frac{3}{8}n + c(n), \tag{1}$$

where $c(n) = 1$ if $n \in 8\mathbb{N}$ and $c(n) = -1/2$ otherwise. Then $\mathcal{H}$ contains a perfect $\mathcal{C}$-tiling.

Proposition 1.2 below shows that Theorem 1.1 is best possible. Theorem 1.1 and Proposition 1.2 together imply that $t_1(n, \mathcal{C}) = \binom{n-1}{2} - \binom{\frac{3}{4}n}{2} + \frac{3}{8}n + c(n)$.

Proposition 1.2. *For every $n \in 4\mathbb{N}$ there exists a 3-graph of order n with minimum vertex degree $\binom{n-1}{2} - \binom{\frac{3}{4}n}{2} + \frac{3}{8}n + c(n) - 1$, which does not contain a perfect $\mathcal{C}$-tiling.*

Proof. We give two constructions similar to those in [4]. Let $V = A \dot\cup B$[1] with $|A| = \frac{n}{4} - 1$ and $|B| = \frac{3n}{4} + 1$. A Steiner system $S(2, 3, m)$ is a 3-graph $\mathcal{S}$ on n vertices such

[1]Throughout the article, we write $A \dot\cup B$ for $A \cup B$ when sets A, B are disjoint.

that every pair of vertices has degree one – so $S(2,3,m)$ contains no copy of $\mathcal{C}$. It is well known that an $S(2,3,m)$ exists if and only if $m \equiv 1, 3 \bmod 6$.

Let $\mathcal{H}_0 = (V, E_0)$ be the 3-graph on $n \in 8\mathbb{N}$ vertices as follows. Let E_0 be the set of all triples intersecting A plus a Steiner system $S(2,3,\frac{3}{4}n+1)$ in B. Since for the Steiner system $S(2,3,\frac{3}{4}n+1)$, each vertex is in exactly $\frac{3}{4}n/2 = \frac{3}{8}n$ edges, we have $\delta_1(\mathcal{H}_0) = \binom{n-1}{2} - \binom{\frac{3}{4}n}{2} + \frac{3}{8}n$. Furthermore, since B contains no copy of $\mathcal{C}$, the size of the largest $\mathcal{C}$-tiling in $\mathcal{H}_0$ is $|A| = \frac{n}{4} - 1$. So $\mathcal{H}_0$ does not contain a perfect $\mathcal{C}$-tiling.

On the other hand, let $\mathcal{H}_1 = (V, E_1)$ be the 3-graph on $n \in 4\mathbb{N} \setminus 8\mathbb{N}$ vertices as follows. Let $\mathcal{G}$ be a Steiner system of order $\frac{3}{4}n + 4$. This is possible since $\frac{3}{4}n + 4 \equiv 1 \bmod 6$. Then pick an edge abc in $\mathcal{G}$ and let $\mathcal{G}'$ be the induced subgraph of $\mathcal{G}$ on $V(\mathcal{G}) \setminus \{a, b, c\}$. Finally, let E_1 be the set of all triples intersecting A plus $\mathcal{G}'$ induced on B. Since $\mathcal{G}$ is a regular graph with vertex degree $\frac{1}{2}(\frac{3}{4}n + 4 - 1) = \frac{3}{8}n + \frac{3}{2}$, we have that $\delta_1(\mathcal{G}') = \frac{3}{8}n + \frac{3}{2} - 3 = \frac{3}{8}n - \frac{3}{2}$. Thus, $\delta_1(\mathcal{H}_1) = \binom{n-1}{2} - \binom{\frac{3}{4}n}{2} + \frac{3}{8}n - \frac{3}{2}$. As in the previous case, $\mathcal{H}_1$ does not contain a perfect $\mathcal{C}$-tiling. ■

As a typical approach of obtaining exact results, we distinguish the *extremal* case from the *nonextremal* case and solve them separately. Given a 3-graph $\mathcal{H}$ of order n, we say that $\mathcal{H}$ is $\mathcal{C}$-free if $\mathcal{H}$ contains no copy of $\mathcal{C}$. In this case, clearly, every pair of vertices has degree at most one. Every vertex has degree at most $\frac{n-1}{2}$ because its link graph[2] contains no vertex of degree two.

Definition 1.3. *Given $\epsilon > 0$, a 3-graph $\mathcal{H}$ on n vertices is called ϵ-extremal if there is a set $S \subseteq V(\mathcal{H})$, such that $|S| \geq (1-\epsilon)\frac{3n}{4}$ and $\mathcal{H}[S]$ is $\mathcal{C}$-free.*

Theorem 1.4 (Extremal case). *There exists $\epsilon > 0$ such that for every 3-graph $\mathcal{H}$ on n vertices, where $n \in 4\mathbb{N}$ is sufficiently large, if $\mathcal{H}$ is ϵ-extremal and satisfies (1.1), then $\mathcal{H}$ contains a perfect $\mathcal{C}$-tiling.*

Theorem 1.5 (Nonextremal case). *For any $\epsilon > 0$, there exists $\gamma > 0$ such that the following holds. Let $\mathcal{H}$ be a 3-graph on n vertices, where $n \in 4\mathbb{N}$ is sufficiently large. If $\mathcal{H}$ is not ϵ-extremal and satisfies $\delta_1(\mathcal{H}) \geq \left(\frac{7}{16} - \gamma\right)\binom{n}{2}$, then $\mathcal{H}$ contains a perfect $\mathcal{C}$-tiling.*

Theorem 1.1 follows Theorems 1.4 and 1.5 immediately by choosing ϵ from Theorem 1.4. The proof of Theorem 1.4 is somewhat routine and will be presented in in Section 4.

The proof of Theorem 1.5, as the one of [4, Theorem 1.5], uses the *absorbing method* initiated by Rödl, Ruciński, and Szemerédi, e.g. [22, 23]. More precisely, we find the perfect $\mathcal{C}$-tiling by applying the Absorbing Lemma below and the $\mathcal{C}$-tiling Lemma [8, Lemma 2.15] together.

Lemma 1.6 (Absorbing Lemma). *For any $0 < \theta \leq 10^{-4}$, there exist $\beta > 0$ and integer $n_{1.6}$ such that the following holds. Let $\mathcal{H}$ be a 3-graph of order $n \geq n_{1.6}$ with $\delta_1(\mathcal{H}) \geq (\frac{1}{4} + \theta)\binom{n}{2}$. Then there is a vertex set $W \in V(\mathcal{H})$ with $|W| \in 4\mathbb{N}$ and $|W| \leq 2049\theta n$ such that for any vertex subset U with $U \cap W = \emptyset$, $|U| \in 4\mathbb{N}$ and $|U| \leq \beta n$ both $\mathcal{H}[W]$ and $\mathcal{H}[W \cup U]$ contain $\mathcal{C}$-factors.*

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[2] Given 3-graph $\mathcal{H} = (V, E)$ and $x \in V$, the link graph of x has vertex set $V \setminus \{x\}$ and the edge set $\{e \setminus \{x\} : e \in E(\mathcal{H}), x \in e\}$.

Lemma 1.7 ($\mathcal{C}$-tiling Lemma, [9]). *For any $0 < \gamma < 1$, there exists an integer $n_{1.7}$ such that the following holds. Suppose $\mathcal{H}$ is a 3-graph on $n > n_{1.7}$ vertices with*

$$\delta_1(\mathcal{H}) \geq \left(\frac{7}{16} - \gamma\right)\binom{n}{2},$$

then $\mathcal{H}$ contains a $\mathcal{C}$-tiling covering all but at most $2^{19}/\gamma$ vertices or $\mathcal{H}$ is $2^{11}\gamma$-extremal.

We postpone the proof of Lemma 1.6 to Section 3 and prove Theorem 1.5 now.

Proof of Theorem 1.5. Without loss of generality, assume $0 < \epsilon < 1$. Let $\gamma = 2^{-13}\epsilon$ and $\theta = 10^{-4}\gamma$ (thus $\theta < 10^{-4}$). We find β by applying Lemma 1.6. Choose $n \in 4\mathbb{N}$ such that $n > \max\{n_{1.6}, 2n_{1.7}, 2^{18}/(\gamma\beta)\}$. Let $\mathcal{H} = (V, E)$ be a 3-graph on n vertices. Suppose that $\mathcal{H}$ is not ϵ-extremal and $\delta_1(\mathcal{H}) \geq \left(\frac{7}{16} - \gamma\right)\binom{n}{2}$. First we apply Lemma 1.6 to $\mathcal{H}$ and find the absorbing set W with $|W| \leq 2049\theta n$. Let $\mathcal{H}' = \mathcal{H}[V \setminus W]$ and $n' = n - |W|$. Note that $2|W| < 10^4\theta n = \gamma n$ and thus $n' > n - \gamma n/2 > n_{1.7}$. Furthermore,

$$\delta_1(\mathcal{H}') \geq \delta_1(\mathcal{H}) - |W|(n-1) \geq \left(\frac{7}{16} - 2\gamma\right)\binom{n}{2} \geq \left(\frac{7}{16} - 2\gamma\right)\binom{n'}{2}.$$

Second we apply Lemma 1.7 to $\mathcal{H}'$ with parameter 2γ in place of γ and derive that either $\mathcal{H}'$ is $2^{12}\gamma$-extremal or $\mathcal{H}'$ contains a $\mathcal{C}$-tiling covering all but at most $2^{18}/\gamma$ vertices. In the former case, since

$$(1 - 2^{12}\gamma)\frac{3n'}{4} > (1 - 2^{12}\gamma)\frac{3}{4}\left(n - \frac{\gamma n}{2}\right) > (1 - 2^{13}\gamma)\frac{3n}{4} = (1 - \epsilon)\frac{3n}{4},$$

$\mathcal{H}$ is ϵ-extremal, a contradiction. In the latter case, let U be the set of uncovered vertices in $\mathcal{H}'$. Then we have $|U| \in 4\mathbb{N}$ and $|U| \leq 2^{18}/\gamma \leq \beta n$ by the choice of n. By Lemma 1.6, $\mathcal{H}[W \cup U]$ contains a perfect $\mathcal{C}$-tiling. Together with the $\mathcal{C}$-tiling provided by Lemma 1.7, this gives a perfect $\mathcal{C}$-tiling of $\mathcal{H}$. ■

The Absorbing Lemma and $\mathcal{C}$-tiling Lemma in [4] are not very difficult to prove because of the codegree condition. In contrast, our corresponding lemmas are harder. Luckily, we already proved Lemma 1.7 in [9] (as a key step for finding a loose Hamilton cycle in 3-graphs). In order to prove Lemma 1.6, we will use the Strong Regularity Lemma and an extension lemma from [3], which is a corollary of the counting lemma.

The rest of the article is organized as follows. We introduce the Regularity Lemma in Section 2, prove Lemma 1.6 in Section 3, and finally prove Theorem 1.4 in Section 4.

2. REGULARITY LEMMA FOR 3-GRAPHS

2.1. Regular Complexes

Before we can state the regularity lemma, we first define a complex. A hypergraph $\mathcal{H}$ consists of a vertex set $V(\mathcal{H})$ and an edge set $E(\mathcal{H})$, where every edge $e \in E(\mathcal{H})$ is a nonempty subset of $V(\mathcal{H})$. A hypergraph $\mathcal{H}$ is a *complex* if whenever $e \in E(\mathcal{H})$ and e' is a nonempty subset of e we have that $e' \in E(\mathcal{H})$. All the complexes considered in this article have the property that every vertex forms an edge.

For a positive integer k, a complex $\mathcal{H}$ is a k-complex if every edge of $\mathcal{H}$ consists of at most k vertices. The edges of size i are called i-edges of $\mathcal{H}$. Given a k-complex $\mathcal{H}$, for each $i \in [k]$ we denote by $\mathcal{H}_i$ the underlying i-graph of $\mathcal{H}$: the vertices of $\mathcal{H}_i$ are those of $\mathcal{H}$ and the edges of $\mathcal{H}_i$ are the i-edges of $\mathcal{H}$.

Given $s \geq k$, a (k, s)-complex $\mathcal{H}$ is an s-partite k-complex, by which we mean that the vertex set of $\mathcal{H}$ can be partitioned into sets $V_1, \ldots, V_s$ such that every edge of $\mathcal{H}$ is *crossing*, namely, meets each V_i in at most one vertex.

Given $i \geq 2$, an i-partite i-graph $\mathcal{H}$ and an i-partite $(i-1)$-graph $\mathcal{G}$ on the same vertex set, we write $\mathcal{K}_i(\mathcal{G})$ for the family of all crossing i-sets that form a copy of the complete $(i-1)$-graph $K_i^{(i-1)}$ in $\mathcal{G}$. We define the density of $\mathcal{H}$ with respect to $\mathcal{G}$ to be

$$d(\mathcal{H}|\mathcal{G}) := \frac{|\mathcal{K}_i(\mathcal{G}) \cap E(\mathcal{H})|}{|\mathcal{K}_i(\mathcal{G})|} \quad \text{if} \quad |\mathcal{K}_i(\mathcal{G})| > 0,$$

and $d(\mathcal{H}|\mathcal{G}) = 0$ otherwise. More generally, if $\mathbf{Q} = (Q_1, \ldots, Q_r)$ is a collection of r subhypergraphs of $\mathcal{G}$, we define $\mathcal{K}_i(\mathbf{Q}) := \bigcup_{j=1}^{r} \mathcal{K}_i(Q_j)$ and

$$d(\mathcal{H}|\mathbf{Q}) := \frac{|\mathcal{K}_i(\mathbf{Q}) \cap E(\mathcal{H})|}{|\mathcal{K}_i(\mathbf{Q})|} \quad \text{if} \quad |\mathcal{K}_i(\mathbf{Q})| > 0,$$

and $d(\mathcal{H}|\mathbf{Q}) = 0$ otherwise.

We say that $\mathcal{H}$ is (d, δ, r)*-regular with respect to* $\mathcal{G}$ if every r-tuple $\mathbf{Q}$ with $|\mathcal{K}_i(\mathbf{Q})| > \delta|\mathcal{K}_i(\mathcal{G})|$ satisfies $|d(\mathcal{H}|\mathbf{Q}) - d| \leq \delta$. Instead of $(d, \delta, 1)$-regularity we simply refer to (d, δ)-regularity.

Given a $(3, 3)$-complex $\mathcal{H}$, we say that $\mathcal{H}$ is $(d_3, d_2, \delta_3, \delta, r)$-regular if the following conditions hold:

(1) For every pair K of vertex classes, $\mathcal{H}_2[K]$ is (d_2, δ)-regular with respect to $\mathcal{H}_1[K]$ unless $e(\mathcal{H}_2[K]) = 0$, where $\mathcal{H}_i[K]$ is the restriction of $\mathcal{H}_i$ to the union of all vertex classes in K.
(2) $\mathcal{H}_3$ is (d_3, δ_3, r)-regular with respect to $\mathcal{H}_2$ unless $e(\mathcal{H}_3) = 0$.

2.2. Statement of the Regularity Lemma

In this section, we state the version of the regularity lemma due to Rödl and Schacht [26] for 3-graphs, which is almost the same as the one given by Frankl and Rödl [7]. We need more notation. Suppose that V is a finite set of vertices and $\mathcal{P}^{(1)}$ is a partition of V into sets $V_1, \ldots, V_t$, which will be called *clusters*. Given any $j \in [3]$, we denote by $\mathrm{Cross}_j = \mathrm{Cross}_j(\mathcal{P}^{(1)})$ the set of all crossing j-subsets of V. For every set $A \subseteq [t]$ we write Cross_A for all the crossing subsets of V that meet V_i precisely when $i \in A$. Let $\mathcal{P}_A$ be a partition of Cross_A. We refer to the partition classes of $\mathcal{P}_A$ as *cells*. Let $\mathcal{P}^{(2)}$ be the union of all $\mathcal{P}_A$ with $|A| = 2$ (so $\mathcal{P}^{(2)}$ is a partition of Cross_2). We call $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ *a family of partitions* on V.

Given $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ and $K = v_i v_j v_k$ with $v_i \in V_i$, $v_j \in V_j$ and $v_k \in V_k$, the *polyad* $P(K)$ is the 3-partite 2-graph on $V_i \cup V_j \cup V_k$ with edge set $C(v_i v_j) \cup C(v_i v_k) \cup C(v_j v_k)$, where e.g. $C(v_i v_j)$ is the cell in $\mathcal{P}_{i,j}$ that contains $v_i v_j$. We say that $P(K)$ is (d_2, δ)-regular if all $C(v_i v_j), C(v_i v_k), C(v_j v_k)$ are (d_2, δ)-regular with respect to their underlying sets. We let $\hat{\mathcal{P}}^{(2)}$ be the family of all $P(K)$ for $K \in \mathrm{Cross}_3$.

Now we are ready to state the regularity lemma for 3-graphs.

Theorem 2.1 (Rödl and Schacht [26], Theorem 17). *For all $\delta_3 > 0, t_0 \in \mathbb{N}$ and all functions $r : \mathbb{N} \to \mathbb{N}$ and $\delta : \mathbb{N} \to (0, 1]$, there are $d_2 > 0$ such that $1/d_2 \in \mathbb{N}$ and integers T, n_0 such that the following holds for all $n \geq n_0$ that are divisible by $T!$. Let $\mathcal{H}$ be a*

3-graph of order n. Then there exists a family of partitions $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ *of the vertex set* V *of* $\mathcal{H}$ *such that*

(1) $\mathcal{P}^{(1)} = \{V_1, \ldots, V_t\}$ *is a partition of* V *into* t *clusters of equal size, where* $t_0 \le t \le T$,
(2) $\mathcal{P}^{(2)}$ *is a partition of* $Cross_2$ *into at most* T *cells,*
(3) *for every* $K \in Cross_3$, $P(K)$ *is* $(d_2, \delta(T))$*-regular,*
(4) $\sum |\mathcal{K}_3(P)| \le \delta_3 |V|^3$, *where the summation is over all* $P \in \hat{\mathcal{P}}^{(2)}$ *such that* $\mathcal{H}$ *is not* $(d, \delta_3, r(T))$*-regular with respect to* P *for any* $d > 0$.

2.3. The Reduced 3-graph and the Extension Lemma

Given $t_0 \in \mathbb{N}$ and $\delta_3 > 0$, we choose functions $r : \mathbb{N} \to \mathbb{N}$ and $\delta : \mathbb{N} \to (0, 1]$ such that the output of Theorem 2.1 satisfies the following hierarchy:

$$\frac{1}{n_0} \ll \left\{\frac{1}{r}, \delta\right\} \ll \left\{\delta_3, d_2, \frac{1}{T}\right\}, \tag{2}$$

where $r = r(T)$ and $\delta = \delta(T)$. Let $\mathcal{H}$ be a 3-graph on V of order $n \ge n_0$ such that $T!$ divides n. Suppose that $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ satisfies Properties (1)–(4) given in Theorem 2.1. For any $d > 0$, the *reduced 3-graph* $\mathcal{R} = \mathcal{R}(\mathcal{H}, \mathcal{P}, d)$ is defined as the 3-graph whose vertices are clusters $V_1, \ldots, V_t$ and three clusters V_i, V_j, V_k form an edge of $\mathcal{R}$ if there is some polyad P on $V_i \cup V_j \cup V_k$ such that $\mathcal{H}$ is (d', δ_3, r)-regular with respect to P for some $d' \ge d$.

Fact 2.2. *Let* $\mathcal{R} = \mathcal{R}(\mathcal{H}, \mathcal{P}, d)$ *be the reduced 3-graph defined above. If* $V_iV_jV_k \in E(\mathcal{R})$, *then there exists a (3,3)-complex* $\mathcal{H}^*$ *on* $V_i \cup V_j \cup V_k$ *such that* $\mathcal{H}^*_3$ *is a subhypergraph of* $\mathcal{H}$ *and* $\mathcal{H}^*$ *is* $(d', d_2, \delta_3, \delta, r)$*-regular for some* $d' \ge d$.

Proof. Since $V_iV_jV_k \in E(\mathcal{R})$, there exists a polyad P on $V_i \cup V_j \cup V_k$ such that $\mathcal{H}$ is (d', δ_3, r)-regular with respect to P for some $d' \ge d$. Let $\mathcal{H}^*_2 = P$ and $\mathcal{H}^*_3 = E(\mathcal{H}) \cap \mathcal{K}_3(P)$. By Theorem 2.1, $\mathcal{H}^*$ is a $(d', d_2, \delta_3, \delta, r)$-regular (3,3)-complex. ∎

The following lemma says that the reduced 3-graph almost inherits the minimum degree condition from $\mathcal{H}$. Its proof is almost identical to the one of [13, Lemma 4.3], which gives the corresponding result on codegree. We thus omit the proof.

Lemma 2.3. *In addition to* (2), *suppose that*

$$\delta_3, 1/t_0 \ll d \ll \theta \ll \mu < 1.$$

Let $\mathcal{H}$ *be a 3-graph of order* $n \ge n_0$ *such that* $T!$ *divides* n *and* $\delta_1(\mathcal{H}) \ge (\mu + \theta)\binom{n}{2}$. *Then in the reduced 3-graph* $\mathcal{R} = \mathcal{R}(\mathcal{H}, \mathcal{P}, d)$, *all but at most* θt *vertices* $v \in V(\mathcal{R})$ *satisfy* $\deg_{\mathcal{R}}(v) \ge \mu\binom{t}{2}$.

Suppose that $\mathcal{H}$ is a (3,3)-complex with vertex classes V_1, V_2, V_3, and $\mathcal{G}$ is a (3,3)-complex with vertex classes X_1, X_2, X_3. A subcomplex $\mathcal{H}'$ of $\mathcal{H}$ is called a *partition-respecting copy of* $\mathcal{G}$ if $\mathcal{H}'$ is isomorphic to $\mathcal{G}$ and for each $i \in [3]$ the vertices corresponding to those in X_i lie within V_i. We write $|\mathcal{G}|_{\mathcal{H}}$ for the number of (labeled) partition-respecting copies of $\mathcal{G}$ in $\mathcal{H}$.

Roughly speaking, the Extension Lemma [3], Lemma 5] says that if $\mathcal{G}'$ is an induced subcomplex of $\mathcal{G}$, and $\mathcal{H}$ is suitably regular, then almost all copies of $\mathcal{G}'$ in $\mathcal{H}$ can be extended to a large number of copies of $\mathcal{G}$ in $\mathcal{H}$. Below we only state it for (3,3)-complexes.

Lemma 2.4 (Extension Lemma [3]). *Let r, b, b', m_0 be positive integers, where $b' < b$, and let $c, \theta, d_2, d_3, \delta, \delta_3$ be positive constants such that $1/d_2 \in \mathbb{N}$ and*

$$1/m_0 \ll \{1/r, \delta\} \ll c \ll \min\{\delta_3, d_2\} \le \delta_3 \ll \theta, d_3, 1/b.$$

Then the following holds for all integers $m \ge m_0$. Suppose that $\mathcal{G}$ is a (3,3)-complex on b vertices with vertex classes X_1, X_2, X_3 and let $\mathcal{G}'$ be an induced subcomplex of $\mathcal{G}$ on b' vertices. Suppose also that $\mathcal{H}^*$ is a $(d_3, d_2, \delta_3, \delta, r)$-regular $(3,3)$-complex with vertex classes V_1, V_2, V_3, all of size m and $e(\mathcal{H}^*) > 0$. Then all but at most $\theta|\mathcal{G}'|_{\mathcal{H}^*}$ labeled partition-respecting copies of $\mathcal{G}'$ in $\mathcal{H}^*$ are extendible to at least $cm^{b-b'}$ labeled partition-respecting copies of $\mathcal{G}$ in $\mathcal{H}^*$.

3. PROOF OF LEMMA 1.6

In this section, we prove Lemma 1.6 by using the lemmas introduced in Section 2. We remark that the constant $\frac{1}{4}$ in Lemma 1.6 is best possible because if $\mathcal{H}$ consists of two disjoint cliques of order $n/2$ each, then $\delta_1(\mathcal{H})$ is about $\frac{1}{4}\binom{n}{2}$ and any 4-vertex set that intersects both cliques can not be absorbed.

For $\alpha > 0$, $i \in \mathbb{N}$ and two vertices $u, v \in V$, we say that u is (α, i)-reachable to v if and only if there are at least αn^{4i-1} $(4i-1)$-sets W such that both $\mathcal{H}[u \cup W]$ and $\mathcal{H}[v \cup W]$ contain $\mathcal{C}$-factors. In this case, we call W a reachable set for u and v. Similar definitions for absorbing method can be found in [18, 19]. Suppose that

$$1/n_0 \ll \{1/r, \delta\} \ll c \ll \min\{\delta_3, 1/T, d_2\} \le \delta_3, 1/t_0 \ll d \ll \theta \le 10^{-4},$$

and $n_0 \ge 4T!/\theta$. Let $\mathcal{H}$ be a 3-graph on $n \ge n_0 + T!$ vertices with $\delta_1(\mathcal{H}) \ge (\frac{1}{4} + \theta)\binom{n}{2}$. We will prove that almost all pairs of vertices of $\mathcal{H}$ are $(\beta_0, 2)$-reachable to each other, where $\beta_0 = c^2/(5T^7)$.

Claim 3.1. *There are at most $4\theta n^2$ pairs $u, v \in \binom{V}{2}$ such that u is not $(\beta_0, 2)$-reachable to v.*

Proof. Let $n' \in \mathbb{N}$ such that $n - n' < T!$ and $T!$ divides n'. Then $n' \ge n_0 \ge 4T!/\theta$. As $\theta \le 10^{-4}$, we have $n' \ge \frac{40000}{40001}n$.

Let $\mathcal{H}'$ be an induced subhypergraph of $\mathcal{H}$ on any n' vertices. Since $n \ge 4T!/\theta$, we have

$$\delta_1(\mathcal{H}') \ge \left(\frac{1}{4} + \theta\right)\binom{n}{2} - T!(n-1) \ge \left(\frac{1}{4} + \frac{\theta}{2}\right)\binom{n'}{2}.$$

We apply Theorem 2.1 to $\mathcal{H}'$, and let $\mathcal{P}$ be the the family of partitions, with clusters $V_1, \dots, V_t$. Let $m = n'/t$ be the size of each cluster. Define the reduced 3-graph $\mathcal{R} = \mathcal{R}(\mathcal{H}', \mathcal{P}, d)$ on these clusters as in Section 2.3.

Let I be the set of $i \in [t]$ such that $\deg_{\mathcal{R}}(V_i) < (\frac{1}{4} + \frac{\theta}{4})\binom{t}{2}$ and let $V_I = \bigcup_{i \in I} V_i$. By Lemma 2.3, we have $|I| \le \theta t/4$ and thus $|V_I| \le (\theta t/4) \cdot m = \theta n'/4$. Let $N(i)$ be the set of vertices $V_j \in V(\mathcal{R}) \setminus \{V_i\}$ such that $\{V_i, V_j\} \subseteq e$ for some $e \in \mathcal{R}$. For any $i \in [t] \setminus I$,

$$\left(\frac{1}{4} + \frac{\theta}{4}\right)\binom{t}{2} \le \deg_{\mathcal{R}}(V_i) \le \binom{|N(i)|}{2}$$

implies that $|N(i)| \ge (\frac{1}{2} + \frac{\theta}{8})t$. Thus $|N(i) \cap N(j)| \ge \frac{\theta}{4}t$ for any $i, j \in [t] \setminus I$.

Fix two not necessarily distinct $i, j \notin I$ and $V_k \in N(i) \cap N(j)$. We pick $V_{i'}$ and $V_{j'}$ such that $V_iV_kV_{i'}, V_jV_kV_{j'} \in \mathcal{R}$. Note that it is possible to have $i' = j'$ or $i' = j$ or $j' = i$. Let $\mathcal{H}^*$ be the $(d_3, d_2, \delta_3, \delta, r)$-regular $(3, 3)$-complex with vertex classes $V_i, V_k, V_{i'}$ provided by Fact 2.2, where $d_3 \geq d$.

Let $\mathcal{G}$ be the (3,3)-complex on $X_1 = \{x, u\}$, $X_2 = \{y, v\}$, $X_3 = \{w\}$ such that $\mathcal{G}_3 = \{xvw, uyw, uvw\}$ and $\mathcal{G}_2$ is the family of all 2-subsets of the members of $\mathcal{G}_3$. Note that in $\mathcal{G}_3$ both $\{x, u, v, w\}$ and $\{y, u, v, w\}$ span copies of $\mathcal{C}$. Let $\mathcal{G}'$ be the induced subcomplex of $\mathcal{G}$ on $\{u, v\}$. Since $\mathcal{H}^*_3$, the highest level of the complex $\mathcal{H}^*$, is not empty, by Lemma 2.4, all but at most θm^2 ordered pairs $(v_i, v_k) \in V_i \times V_k$ are extendible to at least cm^3 labeled copies of $\mathcal{G}$ in $\mathcal{H}^*$, which implies that v_i is $(cm^3n^{-3}, 1)$-reachable to v_k. By averaging, all but at most $3\theta m$ vertices $v_i \in V_i$ are $(cm^3n^{-3}, 1)$-reachable to at least $\frac{2}{3}m$ vertices of V_k. We apply the same argument on $V_j, V_k, V_{j'}$ and obtain that for all but at most $3\theta m$ vertices $v_j \in V_j$, v_j is $(cm^3n^{-3}, 1)$-reachable to at least $\frac{2}{3}m$ vertices of V_k. Thus for those v_i and v_j, there are $\frac{1}{3}m$ vertices $v_k \in V_k$ such that both v_i and v_j are $(cm^3n^{-3}, 1)$-reachable to v_k. Fix such v_i, v_j, v_k. There are at least $cm^3 - m^2$ reachable 3-sets for v_i and v_k from $(V_i, V_{i'}, V_k)$ avoiding v_j.[3] Fix one such 3-set, the number of 3-sets from $(V_j, V_{j'}, V_k)$ intersecting its three vertices is at most $3m^2$. So the number of reachable 7-sets for v_i, v_j is at least

$$\frac{m}{3} \cdot (cm^3 - m^2) \cdot (cm^3 - 3m^2) > \frac{c^2}{4}m^7 \geq \frac{c^2}{4}\left(\frac{n'}{T}\right)^7 > \frac{c^2}{5}\frac{n^7}{T^7} = \beta_0 n^7,$$

which means that v_i is $(\beta_0, 2)$-reachable to v_j, where the last inequality holds because $(\frac{n'}{n})^7 \geq (\frac{40000}{40001})^7 > \frac{4}{5}$. Note that this is true for all but at most $2.3\theta m \cdot m = 6\theta m^2$ pairs of vertices in (V_i, V_j). Since there are at most $\binom{t}{2} + t$ choices for V_i and V_j, $|V_I| \leq \theta n'/4$, and $T! \leq \theta n'/4$, there are at most

$$6\theta m^2\left(\binom{t}{2} + t\right) + (|V_I| + T!)(n-1) \leq 3\theta m^2(t^2 + t) + \frac{\theta}{2}n'(n-1) \leq 4\theta n^2$$

pairs u, v in $V(\mathcal{H})$ such that u is not $(\beta_0, 2)$-reachable to v. ■

Proof of Lemma 1.6. Let $\beta = \beta_0^{10}$. Let V' be the set of vertices $v \in V$ such that at least $\frac{n}{64}$ vertices are not $(\beta_0, 2)$-reachable to v. By Claim 3.1, $|V'| \leq 512\theta n$.

There are two steps in our proof. In the first step, we build an absorbing family $\mathcal{F}'$ such that for any small portion of vertices in $V(\mathcal{H}) \setminus V'$, we can absorb them using members of $\mathcal{F}'$. In the second step, we put the vertices in V' not covered by any member of $\mathcal{F}'$ into a set $\mathcal{A}$ of copies of $\mathcal{C}$. Thus, the union of $\mathcal{F}'$ and $\mathcal{A}$ gives the desired absorbing set.

We say that a set A absorbs another set B if $A \cap B = \emptyset$ and both $\mathcal{H}[A]$ and $\mathcal{H}[A \cup B]$ contains $\mathcal{C}$-factors. Fix any 4-set $S = \{v_1, v_2, v_3, v_4\} \in V \setminus V'$, we will show that there are many 24-sets absorbing S. First, we find vertices u_2, u_3, u_4 such that

- $v_1u_2u_3u_4$ spans a copy of $\mathcal{C}$,
- u_i is $(\beta_0, 2)$-reachable to v_i, for i=2,3,4.

---

[3]Recall that it is possible to have $v_j \in V_i$ or $v_j \in V_{i'}$ (when $j = i$ or $j = i'$).

For the first condition, consider the link graph $\mathcal{H}_{v_1}$ of v_1, which contains at least $(\frac{1}{4}+\theta)\binom{n}{2}$ edges. By convexity, the number of paths of length two in $\mathcal{H}_{v_1}$ is

$$\sum_{x\in V\setminus\{v_1\}}\binom{\deg_{\mathcal{H}_{v_1}}(x)}{2} \geq (n-1)\binom{\frac{1}{n-1}\sum_{x\in V\setminus\{v_1\}}\deg_{\mathcal{H}_{v_1}}(x)}{2}$$

$$\geq (n-1)\binom{(\frac{1}{4}+\theta)n}{2} > \frac{1}{32}n^3,$$

where the last inequality holds because $\theta n \gg 1$. Since $v_1u_2u_3u_4$ spans a copy of $\mathcal{C}$ if $u_2u_3u_4$ is a path of length two in $\mathcal{H}_{v_1}$, then there are at least $\frac{1}{32}n^3$ choices for such $u_2u_3u_4$. Moreover, the number of triples violating the second condition is at most $3\cdot\frac{n}{64}\cdot\binom{n}{2} < \frac{3}{128}n^3$. Thus, there are at least $\frac{1}{128}n^3$ such $u_2u_3u_4$ satisfying both of the conditions.

Second, we find reachable 7-sets C_i for u_i and v_i, for i=2,3,4, which is guaranteed by the second condition above. Since in each step we need to avoid at most 21 previously selected vertices, there are at least $\frac{\beta_0}{2}n^7$ choices for each C_i. In total, we get $\frac{1}{128}n^3\cdot(\frac{\beta_0}{2}n^7)^3 > \beta_0^4 n^{24}$ 24-sets $F = C_1\cup C_2\cup C_3\cup\{u_2,u_3,u_4\}$ (because $\beta_0 < c^2 < 10^{-8}$). It is easy to see that F absorbs S. Indeed, $\mathcal{H}[F]$ has a $\mathcal{C}$-factor since $C_i\cup\{u_i\}$ spans two copies of $\mathcal{C}$ for $i=2,3,4$. In addition, $\mathcal{H}[F\cup S]$ has a $\mathcal{C}$-factor since $v_1u_2u_3u_4$ spans a copy of $\mathcal{C}$ and $C_i\cup\{v_i\}$ spans two copies of $\mathcal{C}$ for $i=2,3,4$.

Now we choose a family $\mathcal{F}\subset\binom{v}{24}$ of 24-sets by selecting each 24-set randomly and independently with probability $p=\beta_0^5 n^{-23}$. Then $|\mathcal{F}|$ follows the binomial distribution $B(\binom{n}{24},p)$ with expectation $\mathbb{E}(|\mathcal{F}|)=p\binom{n}{24}$. Furthermore, for every 4-set S, let $f(S)$ denote the number of members of $\mathcal{F}$ that absorb S. Then $f(S)$ follows the binomial distribution $B(N,p)$ with $N\geq\beta_0^4n^{24}$ by previous calculation. Hence $\mathbb{E}(f(S))\geq p\beta_0^4n^{24}$. Finally, since there are at most $\binom{n}{24}\cdot 24\cdot\binom{n}{23} < \frac{1}{2}n^{47}$ pairs of intersecting 24-sets, the expected number of the intersecting pairs of 24-sets in $\mathcal{F}$ is at most $p^2\cdot\frac{1}{2}n^{47}=\beta_0^{10}n/2$.

Applying Chernoff's bound on the first two properties and Markov's bound on the last one, we know that, with positive probability, $\mathcal{F}$ satisfies the following properties:

- $|\mathcal{F}|\leq 2p\binom{n}{24} < \beta_0^5 n$,
- for any 4-set S, $f(S)\geq\frac{p}{2}\cdot\beta_0^4n^{24}=\beta_0^9n/2$,
- the number of intersecting pairs of elements in $\mathcal{F}$ is at most $\beta_0^{10}n$.

Thus, by deleting one member from each intersecting pair and the non-absorbing members from $\mathcal{F}$, we obtain a family $\mathcal{F}'$ consisting of at most β_0^5n 24-sets and for each 4-set S, at least $\beta_0^9n/2-\beta_0^{10}n > \beta_0^{10}n=\beta n$ members in $\mathcal{F}'$ absorb S.

At last, we will greedily build $\mathcal{A}$, a collection of copies of $\mathcal{C}$ to cover the vertices in V' not already covered by any member of $\mathcal{F}'$. Indeed, assume that we have built $a < |V'| \leq 512\theta n$ copies of $\mathcal{C}$. Together with the vertices in $\mathcal{F}'$, there are at most $4a + 24\beta_0^5n < 2,049\theta n$ vertices already selected. Then at most $2,049\theta n^2$ pairs of vertices intersect these vertices. So for any remaining vertex $v\in V'$, there are at least

$$\deg(v)-2,049\theta n^2 \geq \left(\frac{1}{4}+\theta\right)\binom{n}{2}-2,049\theta n^2 > n/2$$

edges containing v and not intersecting the existing vertices, where the last inequality follows from $\theta \le 10^{-4}$. So there is a path of length two in the link graph of v not intersecting the existing vertices, which gives a copy of $\mathcal{C}$ containing v.

Combining the vertices covered by $\mathcal{A}$ and $\mathcal{F}'$ together, we get the desired absorbing set W satisfying $|W| \le 4 \cdot 512\theta n + 24\beta_0^5 n < 2049\theta n$. ■

4. PROOF OF THEOREM 1.4

In this section, we prove Theorem 1.4. Our proof is similar to the one of [4], Theorem 1.4]. First let us start with some notation. Fix a 3-graph $\mathcal{H} = (V, E)$. Recall that the link graph of a vertex $v \in V$ is a 2-graph on $V \setminus \{v\}$. Then for a set $\mathcal{E}$ of pairs in $\binom{V}{2}$ (which can be viewed as a 2-graph), let $\deg_{\mathcal{H}}(v, \mathcal{E}) = |N_{\mathcal{H}}(v) \cap \mathcal{E}|$. When $\mathcal{E} = \binom{X}{2}$ for some $X \subseteq V$, we write $\deg_{\mathcal{H}}(v, \binom{X}{2})$ as $\deg_{\mathcal{H}}(v, X)$ for short. Let $\overline{\deg}_{\mathcal{H}}(v, \mathcal{E}) = |\mathcal{E} \cap \binom{V\setminus\{v\}}{2}| - \deg_{\mathcal{H}}(v, \mathcal{E})$. Given not necessarily disjoint subsets X, Y, Z of V, define

$$\begin{aligned} e_{\mathcal{H}}(XYZ) \quad &= \{xyz \in E(\mathcal{H}) : x \in X, y \in Y, z \in Z\} \\ \overline{e}_{\mathcal{H}}(XYZ) &= \left\{xyz \in \tbinom{V}{3} \setminus E(\mathcal{H}) : x \in X, y \in Y, z \in Z\right\}. \end{aligned}$$

We often omit the subscript $\mathcal{H}$ if it is clear from the context.

The following fact is the only place where we need the *exact* degree condition (1.1).

Fact 4.1. *Let $\mathcal{H}$ be a 3-graph on n vertices with $n \in 4\mathbb{N}$ satisfying* (1). *If $S \subseteq V(\mathcal{H})$ spans no copy of $\mathcal{C}$, then $|S| \le \frac{3}{4}n$.*

Proof. Assume to the contrary, that $S \subseteq V(\mathcal{H})$ spans no copy of $\mathcal{C}$ and is of size at least $\frac{3}{4}n + 1$. Take $S_0 \subseteq S$ with size exactly $\frac{3}{4}n + 1$. Then for any $v \in S_0$, $\deg(v, S_0) \le \frac{|S_0|-1}{2} = \frac{3}{8}n$. We split into two cases.

Case 1 $n \in 8\mathbb{N}$.

In this case, for any $v \in S_0$, since $\deg(v, S_0) \le \frac{3}{8}n$, we have that

$$\deg(v) = \deg(v, S_0) + \deg\left(v, \binom{V}{2}\binom{S_0}{2}\right) \le \frac{3}{8}n + \binom{n-1}{2} - \binom{\frac{3}{4}n}{2} < \delta_1(\mathcal{H}),$$

contradicting (1.1).

Case 2 $n \in 4\mathbb{N} \setminus 8\mathbb{N}$

In this case, for any $v \in S_0$, $\deg(v, S_0) \le \frac{3}{8}n$ implies that $\deg(v, S_0) \le \frac{3}{8}n - \frac{1}{2}$ because $n \in 4\mathbb{N} \setminus 8\mathbb{N}$. So we have

$$3e(S_0) = \sum_{v \in S_0} \deg(v, S_0) \le \left(\frac{3}{8}n - \frac{1}{2}\right)\left(\frac{3}{4}n + 1\right) = \frac{3n-4}{8} \cdot \frac{3n+4}{4}.$$

However, neither $\frac{3n-4}{8}$ or $\frac{3n+4}{4}$ is a multiple of 3. Thus $\sum_{v \in S_0} \deg(v, S_0) < \frac{3n-4}{8} \cdot \frac{3n+4}{4}$, which implies that there exists $v_0 \in S_0$ such that $\deg(v_0, S_0) < \frac{3}{8}n - \frac{1}{2}$. Consequently,

$$\deg(v_0) = \deg(v_0, S_0) + \deg\left(v_0, \binom{V}{2} \setminus \binom{S_0}{2}\right) < \frac{3}{8}n - \frac{1}{2} + \binom{n-2}{2} - \binom{\frac{3}{4}n}{2} \le \delta_1(\mathcal{H}),$$

contradicting (1.1). ■

Proof of Theorem 1.4. Take $\epsilon = 10^{-18}$ and let n be sufficiently large. We write $\alpha = \epsilon^{1/3} = 10^{-6}$. Let $\mathcal{H} = (V, E)$ be a 3-graph of order n satisfying (1) that is ϵ-extremal, namely, there exists a set $S \subseteq V(\mathcal{H})$ such that $|S| \geq (1-\epsilon)\frac{3n}{4}$ and $\mathcal{H}[S]$ is $\mathcal{C}$-free.

Let $C \subseteq V$ be a maximum set for which $\mathcal{H}[C]$ is $\mathcal{C}$-free. Define

$$A = \left\{ x \in V \setminus C : \deg(x, C) \geq (1-\alpha)\binom{|C|}{2} \right\}, \tag{3}$$

and $B = V \setminus (A \cup C)$. We first claim the following bounds of $|A|$, $|B|$, $|C|$.

Claim 4.2. $|A| > \frac{n}{4}(1 - 4\alpha^2)$, $|B| < \alpha^2 n$ *and* $\frac{3n}{4}(1-\epsilon) \leq |C| \leq \frac{3n}{4}$.

Proof. The estimate on $|C|$ follows from our hypothesis and Fact 4.1. We now estimate $|B|$. For any $v \in C$, we have $\deg(v, C) \leq \frac{|C|-1}{2}$, which gives $\overline{\deg}(v, C) \geq \binom{|C|-1}{2} - \frac{|C|-1}{2}$. By (1.1), $\overline{\deg}(v) \leq \binom{\frac{3}{4}n}{2} - \frac{3}{8}n + \frac{1}{2}$. Thus

$$\begin{aligned} \overline{\deg}\left(v, \binom{V}{2} \setminus \binom{C}{2}\right) &\leq \binom{\frac{3}{4}n}{2} - \frac{3}{8}n + \frac{1}{2} - \binom{|C|-1}{2} + \frac{|C|-1}{2} \\ &\leq \binom{\frac{3}{4}n}{2} - \binom{|C|-1}{2} \quad \text{because } |C| \leq \frac{3}{4}n \\ &= \left(\frac{3}{4}n - |C| + 1\right) \cdot \frac{1}{2}\left(\frac{3}{4}n + |C| - 2\right). \end{aligned}$$

The estimate on $|C|$ gives $\frac{3}{4}n \leq \frac{|C|}{1-\epsilon} < (1+2\epsilon)(|C|-1)$. Hence

$$\begin{aligned} \overline{\deg}\left(v, \binom{V}{2} \setminus \binom{C}{2}\right) &< \left(\frac{3}{4}n - |C| + 1\right) \cdot \frac{1}{2}\Big((1+2\epsilon)(|C|-1) + |C| - 1\Big) \\ &= \left(\frac{3}{4}n - |C| + 1\right) \cdot (1+\epsilon)(|C|-1) \end{aligned} \tag{4}$$

$$\leq \left(\tfrac{3}{4}\epsilon n + 1\right) \cdot (1+\epsilon)(|C|-1) < \epsilon n \cdot (|C|-1). \tag{5}$$

Consequently $\bar{e}(CC(A \cup B)) < \frac{1}{2}|C| \cdot \epsilon n \cdot (|C|-1) = \epsilon n \cdot \binom{|C|}{2}$. Together with the definition of A and B, we have

$$(|A \cup B| - \epsilon n)\binom{|C|}{2} < e(CC(A \cup B)) \leq (1-\alpha)\binom{|C|}{2}|B| + \binom{|C|}{2}|A|,$$

so that $|A \cup B| - \epsilon n < |A| + |B| - \alpha|B|$. Since A and B are disjoint, we get that $|B| < \alpha^2 n$. Finally, $|A| = n - |B| - |C| > n - \alpha^2 n - \frac{3}{4}n = \frac{n}{4}(1 - 4\alpha^2)$. ■

In the rest of the section, we will build four vertex-disjoint $\mathcal{C}$-tilings $\mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}$ whose union is a perfect $\mathcal{C}$-tiling of $\mathcal{H}$. In particular, when $|A| = n/4$, $B = \emptyset$ and $|C| = 3n/4$, we have $\mathcal{Q} = \mathcal{R} = \mathcal{S} = \emptyset$ and the perfect $\mathcal{C}$-tiling $\mathcal{T}$ of $\mathcal{H}$ will be provided by Lemma 4.4. The purpose of $\mathcal{C}$-tilings $\mathcal{Q}, \mathcal{R}, \mathcal{S}$ is covering the vertices of B and adjusting the sizes of A and C such that we can apply Lemma 4.4 after $\mathcal{Q}, \mathcal{R}, \mathcal{S}$ are removed.

The $\mathcal{C}$-tiling $\mathcal{Q}$. Let $\mathcal{Q}$ be a largest $\mathcal{C}$-tiling in $\mathcal{H}$ on $B \cup C$ and $q = |\mathcal{Q}|$. We claim that $|B|/4 \leq q \leq |B|$. Since C contains no copy of $\mathcal{C}$, every element of $\mathcal{Q}$ contains at least one vertex of B and consequently $q \le |B|$. On the other hand, suppose that $q < |B|/4$, then $(B \cup C) \setminus V(\mathcal{Q})$ spans no copy of $\mathcal{C}$ and has order

$$|B| + |C| - 4q > |B| + |C| - |B| = |C|,$$

which contradicts the assumption that C is a maximum $\mathcal{C}$-free subset of $V(\mathcal{H})$.

Claim 4.3. $q + |A| \ge \frac{n}{4}$.

Proof. Let $l = \frac{n}{4} - |A|$. There is nothing to show if $l \le 0$. If $l = 1$, we have $|B \cup C| = \frac{3}{4}n + 1$, and thus Fact 4.1 implies that $\mathcal{H}[B \cup C]$ contains a copy of $\mathcal{C}$. Thus $q \ge 1 = l$ and we are done. We thus assume $l \ge 2$ and $l > q \ge |B|/4$, which implies that $|B| \le 4(l-1)$. In this case $|B| \ge 2$ because $|C| \le \frac{3}{4}n$.

For any $v \in C$, by (4), we have $\overline{\deg}(v, BC) < \left(\frac{3}{4}n - |C| + 1\right) \cdot (1+\epsilon)(|C|-1)$. By definition, $\frac{3}{4}n - |C| = |A| + |B| - \frac{n}{4} = |B| - l$. So we get

$$\overline{e}(BCC) < \frac{1}{2}|C|\left(\frac{3}{4}n - |C| + 1\right) \cdot (1+\epsilon)(|C|-1) = (1+\epsilon)(|B| - l + 1)\binom{|C|}{2}.$$

Together with $|B| \le 4(l-1)$, this implies

$$\begin{aligned} e(BCC) &> (|B| - (1+\epsilon)(|B| - l + 1))\binom{|C|}{2} = ((1+\epsilon)(l-1) - \epsilon|B|)\binom{|C|}{2} \\ &\ge ((1+\epsilon)(l-1) - 4\epsilon(l-1))\binom{|C|}{2} = (1-3\epsilon)(l-1)\binom{|C|}{2}. \qquad (6) \end{aligned}$$

On the other hand, we want to bound $e(BCC)$ from above and then derive a contradiction. Assume that $\mathcal{Q}'$ is the maximum $\mathcal{C}$-tiling of size q' such that each element of $\mathcal{Q}'$ contains exactly one vertex in B and three vertices in C. Note that $q' \ge 1$ because C is a maximum $\mathcal{C}$-free set and $B \ne \emptyset$. Write $B_{\mathcal{Q}'}$ for the set of vertices of B covered by $\mathcal{Q}'$ and $C_{\mathcal{Q}'}$ for the set of vertices of C covered by $\mathcal{Q}'$. Clearly, $|B_{\mathcal{Q}'}| = q'$, $|C_{\mathcal{Q}'}| = 3q'$ and $q' \le q \le l - 1$. For any vertex $v \in B \setminus B_{\mathcal{Q}'}$, $\deg(v, C) \le 3q'(|C|-1) + \frac{1}{2}|C| < 4q'|C|$. Together with the definition of B and Claim 4.2, we get

$$\begin{aligned} e(BCC) &= e(B_{\mathcal{Q}'}CC) + e((B \setminus B_{\mathcal{Q}'})CC) \\ &\le q'(1-\alpha)\binom{|C|}{2} + |B| \cdot 4q'|C| < q'(1-\alpha)\binom{|C|}{2} + 4\alpha^2 nq'|C|. \qquad (7) \end{aligned}$$

Putting (6) and (7) together and using $q' \le l - 1$ and $|C| > n/2$, we get

$$1 - 3\alpha^3 = 1 - 3\epsilon < 1 - \alpha + \frac{8\alpha^2 n}{|C| - 1} < 1 - \alpha + 16\alpha^2 < 1 - \frac{\alpha}{2},$$

which is a contradiction since $\alpha = 10^{-6}$. ■

Let B_1 and C_1 be the vertices in B and C not covered by $\mathcal{Q}$, respectively. By Claim 4.2,

$$|C_1| \ge |C| - 3q \ge |C| - 3|B| > \frac{3}{4}n(1-\epsilon) - 3\alpha^2 n > \frac{3}{4}n - 4\alpha^2 n + 1. \qquad (8)$$

The $\mathcal{C}$-tiling $\mathcal{R}$. Next we will build our $\mathcal{C}$-tiling $\mathcal{R}$ which covers B_1 such that every element in $\mathcal{R}$ contains one vertex from A, one vertex from B_1 and two vertices from

C_1. Since $\mathcal{Q}$ is a maximum $\mathcal{C}$-tiling on $B \cup C$, for every vertex $v \in B_1$, we have that $\deg(v, C_1) \le \frac{|C_1|}{2}$. Together with (8), this implies that

$$\overline{\deg}(v, C_1) \ge \binom{|C1|}{2} - \frac{|C_1|}{2} = \frac{|C_1|(|C_1|-2)}{2} > \frac{(\frac{3}{4}n - 4\alpha^2 n)^2 - 1}{2}.$$

Together with (1), we get that for every $v \in B_1$,

$$\begin{aligned}\overline{\deg}(v, AC_1) &< \binom{\frac{3}{4}n}{2} - \frac{3}{8}n + \frac{1}{2} - \frac{(\frac{3}{4}n - 4\alpha^2 n)^2 - 1}{2} \\ &= \frac{1}{2}\left(\frac{3}{2}n - 4\alpha^2 n\right)4\alpha^2 n - \frac{3}{4}n + 1 < 3\alpha^2 n^2.\end{aligned}$$

By Claim 4.2 and (8), we have that $|A||C_1| > (1-4\alpha^2)\frac{n}{4} \cdot (\frac{3}{4} - 4\alpha^2)n > \frac{3}{17}n^2$. Thus, $\overline{\deg}(v, AC_1) < 3\alpha^2 n^2 < 17\alpha^2|A||C_1|$, equivalently, $\deg(v, AC_1) > (1 - 17\alpha^2)|A||C_1|$. For every $v \in B_1$, we greedily pick a copy of $\mathcal{C}$ containing v by picking a path of length two with center in A and two ends in C_1 from the link graph of v. Suppose we have found $i < |B_1|$ copies of $\mathcal{C}$, then for any remaining vertex $v \in B_1$, by Claim 4.2, the number of pairs not intersecting the existing vertices is at least

$$\deg(v, AC_1) - 3i \cdot (|A| + |C_1|) > (1 - 17\alpha^2)|A||C_1| - 3|B_1| \cdot 2|C_1| > |A|,$$

which guarantees a path of length two centered at A, so a copy of $\mathcal{C}$ containing v.

Now all vertices of B are covered by $\mathcal{Q}$ or $\mathcal{R}$. Let A_2 denote the set of vertices of A not covered by $\mathcal{Q}$ or $\mathcal{R}$ and define C_2 similarly. By the definition of $\mathcal{Q}$ and $\mathcal{R}$, we have $|A_2| = |A| - |B_1|$ and $|C_2| = |B| + |C| - 4q - 3|B_1|$. Define $s = \frac{1}{4}(3|A_2| - |C_2|)$. Then

$$s = \frac{1}{4}(3|A| - 3|B_1| - |B| - |C| + 4q + 3|B_1|) = \frac{1}{4}(4|A| - n + 4q) = q + |A| - \frac{n}{4}.$$

Thus $s \in \mathbb{Z}$, and $s \ge 0$ by Claim 4.3. Since $q \le |B|$, by Claim 4.2,

$$s = q + |A| - \frac{n}{4} \le |B| + |A| - \frac{n}{4} = \frac{3}{4}n - |C| \le \frac{3}{4}\epsilon n. \tag{9}$$

The definition of $\mathcal{Q}$ and $\mathcal{R}$ also implies that $|C \setminus C_2| \le 3|B|$ and

$$|C_2| \ge |C| - 3|B| > |C| - 3 \cdot 2\alpha^2|C| = (1 - 6\alpha^2)|C|, \tag{10}$$

where the second inequality follows from $|B| < \alpha^2 n < 2\alpha^2|C|$.

The $\mathcal{C}$-tiling $\mathcal{S}$. Next we will build our $\mathcal{C}$-tiling $\mathcal{S}$ of size s such that every element of $\mathcal{S}$ contains two vertices in A_2 and two vertices in C_2. Note that for any vertex $v \in A_2$, by (3) and (10),

$$\overline{\deg}(v, C_2) \le \alpha\binom{|C|}{2} \le \alpha\binom{\frac{1}{1-6\alpha^2}|C_2|}{2} < 2\alpha\binom{|C_2|}{2}.$$

Suppose that we have found $i < s$ copies of $\mathcal{C}$ of the desired type. We next select two vertices a_1, a_2 in A_2 and note that they have at least $(1-4\alpha)\binom{|C_2|}{2}$ common neighbors in C_2. By (9) and(10),

$$(1-4\alpha)\binom{|C_2|}{2} - 2s|C_2| \ge (1-4\alpha)\binom{|C_2|}{2} - \frac{3}{2}\epsilon n|C_2| \ge (1-5\alpha)\binom{|C_2|}{2} > 0.$$

So we can pick a common neighbor c_1c_2 of a_1 and a_2 from unused vertices of C_2 such that $\{a_1, a_2, c_1, c_2\}$ spans a copy of $\mathcal{C}$.

Let A_3 be the set of vertices of A not covered by $\mathcal{Q}, \mathcal{R}, \mathcal{S}$ and define C_3 similarly. Then $|A_3| = |A_2| - 2s = \frac{1}{2}(|C_2| - |A_2|)$ and $|C_3| = |C_2| - 2s = \frac{3}{2}(|C_2| - |A_2|)$, so $|C_3| = 3|A_3|$. Furthermore, by (9) and (10), we have

$$|C_3| = |C_2| - 2s \geq (1 - 6\alpha^2)|C| - \frac{3}{2}\epsilon n > (1 - 6\alpha^2)|C| - 3\epsilon|C| > (1 - 7\alpha^2)|C|.$$

Hence, for every vertex $v \in A_3$,

$$\overline{\deg}(v, C_3) \leq \alpha\binom{|C|}{2} \leq \alpha\binom{\frac{1}{1-7\alpha^2}|C_3|}{2} < 2\alpha\binom{|C_3|}{2}.$$

Since $|C_3| \geq (1 - 7\alpha^2)|C| \geq (1 - 7\alpha^2)(1 - \epsilon)\frac{3}{4}n$, by (5), we know that for any vertex $v \in C_3$,

$$\overline{\deg}(v, A_3C_3) < \epsilon n \cdot (|C| - 1) < 2\epsilon|C_3|^2 = 6\epsilon|A_3||C_3|.$$

The $\mathcal{C}$-tiling $\mathcal{T}$. Finally, we use the following lemma to find a $\mathcal{C}$-tiling $\mathcal{T}$ covering A_3 and C_3 such that every element of $\mathcal{T}$ contains one vertex in A_3 and three vertices in C_3. Note that in [4], this was done by applying a general theorem of Pikhurko [20, Theorem 3] (but impossible here because we do not have the codegree condition).

Lemma 4.4. *Suppose that $0 < \rho \leq 2 \cdot 10^{-6}$ and $n_{4.4}$ is sufficiently large. Let $\mathcal{H}$ be a 3-graph on $n \geq n_{4.4}$ vertices with $V(\mathcal{H}) = X\dot\cup Z$ such that $|Z| = 3|X|$. Further, assume that for every vertex $v \in X$, $\overline{\deg}(v, Z) \leq \rho\binom{|Z|}{2}$ and for every vertex $v \in Z$, $\overline{\deg}(v, XZ) \leq \rho|X||Z|$. Then $\mathcal{H}$ contains a perfect $\mathcal{C}$-tiling.*

Applying Lemma 4.4 with $X = A_3$, $Z = C_3$, $\rho = 2\alpha$ finishes the proof of Theorem 1.4. ■

Proof of Lemma 4.4. Let us outline the proof first. Let $X = \{x_1, \ldots, x_{|X|}\}$. Our goal is to partition the vertices of Z into $|X|$ triples $\{Q_1, \ldots, Q_{|X|}\}$ such that for every $i = 1, \ldots, |X|$, $\{x_i\} \cup Q_i$ spans a copy of $\mathcal{C}$ – in this case we say Q_i and x_i are *suitable* for each other. From our assumptions, every $x \in X$ is suitable for most triples of $\mathcal{C}$, and most triples of $\mathcal{C}$ are suitable for most vertices of X. However, once we partition $\mathcal{C}$ into a particular set of triples $\{Q_1, \ldots, Q_{|X|}\}$, we can not guarantee that every vertex in X is suitable for most Q_i's. To handle this difficulty, we use the absorbing method – first find a small number of triples that can absorb any small(er) amount of vertices of X and then extend it to a partition $\{Q_1, \ldots, Q_{|X|}\}$ covering Z, and finally apply the greedy algorithm and the Marriage Theorem to find a perfect matching between X and $\{Q_1, \ldots, Q_{|X|}\}$. Note that a similar approach was outlined in [11] to prove the extremal case.

We now start our proof. Let G be the graph of all pairs uv in Z such that $\deg(uv, X) \geq (1 - \sqrt{\rho})|X|$. We claim that

$$\delta(G) \geq |Z| - \sqrt{\rho}|Z| - 1. \tag{11}$$

Otherwise, some vertex $v \in Z$ satisfies $\deg_G(v) < |Z| - \sqrt{\rho}|Z| - 1$, equivalently, $\overline{\deg}_G(v) > \sqrt{\rho}|Z|$. As each $u \notin N_G(v)$ satisfies $\overline{\deg}(uv, X) > \sqrt{\rho}|X|$, we have

$$\overline{\deg}(v, XZ) > \sqrt{\rho}|Z| \cdot \sqrt{\rho}|X| = \rho|Z||X|,$$

contradicting our assumption. We call a triple $z_1 z_2 z_3$ in Z *good* if $G[z_1z_2z_3]$ contains at least two edges, otherwise *bad*. Since a bad triple contains at least two nonedges of G, by (11), the number of bad triples in Z is at most

$$\sum_{x\in Z}\binom{\overline{\deg}_G(x)}{2} \le |Z|\binom{\sqrt{\rho}|Z|}{2} \le 3\rho\binom{|Z|}{3}.$$

If $z_1z_2z_3$ is good, then by the definition of G, it is suitable for at least $(1-2\sqrt{\rho})|X|$ vertices of X. On the other hand, for any $x \in X$, consider the link graph $\mathcal{H}_x$ of x on Z, which contains at least $(1-\rho)\binom{|Z|}{2}$ edges. By convexity, the number of triples $z_1z_2z_3$ suitable for x is at least

$$\frac{1}{3}\sum_{z\in Z}\binom{\deg_{\mathcal{H}_x}(z)}{2} \ge \frac{1}{3}|Z|\binom{(1-\rho)(|Z|-1)}{2} > (1-2\rho)\binom{|Z|}{3},$$

where the last inequality holds because $|Z|$ is large enough. Thus, the number of good triples $z_1z_2z_3$ suitable for x is at least $(1-2\rho-3\rho)\binom{|Z|}{3} = (1-5\rho)\binom{|Z|}{3}$.

Let $\mathcal{F}_0$ be the set of good triples in Z. We want to form a family of disjoint good triples in Z such that for any $x \in X$, many triples from this family are suitable for x. To achieve this, we choose a subfamily $\mathcal{F}$ from $\mathcal{F}_0$ by selecting each member randomly and independently with probability $p = 2\rho^{1/4}|Z|^{-2}$. Then $|\mathcal{F}|$ follows the binomial distribution $B(|\mathcal{F}_0|, p)$ with expectation $\mathbb{E}(|\mathcal{F}|) = p|\mathcal{F}_0| \le p\binom{|Z|}{3}$. Furthermore, for every $x \in X$, let $f(x)$ denote the number of members of $\mathcal{F}$ that are suitable for x. Then $f(x)$ follows the binomial distribution $B(N, p)$ with $N \ge (1-5\rho)\binom{|Z|}{3}$ by previous calculation. Hence $\mathbb{E}(f(x)) \ge p(1-5\rho)\binom{|Z|}{3}$. Finally, since there are at most $\binom{|Z|}{3}\cdot 3\cdot\binom{|Z|-1}{2} < \frac{1}{4}|Z|^5$ pairs of intersecting triples, the expected number of the intersecting triples in $\mathcal{F}$ is at most $p^2\cdot\frac{1}{4}|Z|^5 = \rho^{1/2}|Z|$.

By applying Chernoff's bound on the first two properties below and Markov's bound on the last one, we can find a family $\mathcal{F} \subseteq \mathcal{F}_0$ that satisfies

- $|\mathcal{F}| \le 2p\binom{|Z|}{3} < \frac{2}{3}\rho^{1/4}|Z|$,
- for any vertex $x \in X$, at least $\frac{1}{2}p\cdot(1-5\rho)\binom{|Z|}{3} > \frac{1}{6}\rho^{1/4}(1-6\rho)|Z|$ triples in $\mathcal{F}$ are suitable for x,
- the number of intersecting pairs of triples in $\mathcal{F}$ is at most $2\rho^{1/2}|Z|$.

After deleting one triple from each of the intersecting pairs from $\mathcal{F}$, we obtain a subfamily $\mathcal{F}'$ consisting of at most $\frac{2}{3}\rho^{1/4}|Z|$ disjoint good triples in Z and for each $x \in X$, at least

$$\frac{\rho^{1/4}}{6}(1-6\rho)|Z| - 2\rho^{1/2}|Z| > \frac{\rho^{1/4}}{12}|Z| \tag{12}$$

members of $\mathcal{F}'$ are suitable for x, where the inequality holds because $\rho \le 2\cdot 10^{-6}$.

Denote $\mathcal{F}'$ by $\{Q_1, \ldots, Q_q\}$ for some $q \le \frac{2}{3}\rho^{1/4}|Z|$. Let Z_1 be the set of vertices of Z not in any element of $\mathcal{F}'$. Define $G' = G[Z_1]$. Note that $|Z_1| = |Z| - 3q$. For every $v \in Z_1$, $\overline{\deg}_{G'}(v) \le \overline{\deg}_G(v) \le \sqrt{\rho}|Z|$ by (11). Thus

$$\delta(G') \ge |Z_1| - \sqrt{\rho}|Z| = |Z| - 3q - \sqrt{\rho}|Z| > \frac{|Z_1|}{2},$$

because $\rho \leq 2 \cdot 10^{-6}$. By Dirac's Theorem, G' is Hamiltonian. We thus find a Hamiltonian cycle of G', denoted by $b_1b_2 \cdots b_{3m}b_1$, where $m = |X| - q$. Let $Q_{q+i} = b_{3i-2}b_{3i-1}b_{3i}$ for $1 \leq i \leq m$. Then $Q_{q+1}, \ldots, Q_{|X|}$ are good triples.

Now consider the bipartite graph Γ between X and $\mathcal{Q} := \{Q_1, Q_2, \ldots, Q_{|X|}\}$, such that $x \in X$ and $Q_i \in \mathcal{Q}$ are adjacent if and only if Q_i is suitable for x. For every Q_i, since it is good, $\deg_\Gamma(Q_i) \geq (1 - 2\sqrt{\rho})|X|$. Let $\mathcal{Q}_2 = \{Q_{q+1}, \ldots, Q_{|X|}\}$. Let X_0 be the set of $x \in X$ such that $\deg_\Gamma(x, \mathcal{Q}_2) \leq |\mathcal{Q}_2|/2$. Then

$$|X_0|\frac{|\mathcal{Q}_2|}{2} \leq \bar{e}_\Gamma(X, \mathcal{Q}_2) \leq 2\sqrt{\rho}|X| \cdot |\mathcal{Q}_2|,$$

which implies that $|X_0| \leq 4\sqrt{\rho}|X| = \frac{4}{3}\sqrt{\rho}|Z|$.

We now find a perfect matching between X and $\mathcal{Q}$ as follows.

Step 1. Each vertex $x \in X_0$ is matched to a different member of $\mathcal{F}'$ that is suitable for x – this is possible because of (12) and $|X_0| \leq \frac{4}{3}\sqrt{\rho}|Z| \leq \frac{1}{12}\rho^{1/4}|Z|$ since $\rho \leq 2 \cdot 10^{-6}$.

Step 2. Each of the unused triples in $Q_1Q_2 \cdots Q_q$ is matched to a suitable vertex in $X \setminus X_0$ – this is possible because $\deg_\Gamma(Q_i) \geq (1 - 2\sqrt{\rho})|X| \geq q$.

Step 3. Let X_1 be the set of the remaining vertices in X. Then $|X_1| = |X| - q = |\mathcal{Q}_2|$. Now consider $\Gamma' = \Gamma[X_1, \mathcal{Q}_2]$. It is easy to check that $\delta(\Gamma') \geq |X_1|/2$ – thus Γ' contains a perfect matching by the Marriage Theorem.

The perfect matching between X and $\mathcal{Q}$ gives rise to the desired perfect $\mathcal{C}$-tiling of $\mathcal{H}$ as outlined in the beginning of the proof. ■

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