Qualifying Examination in Algebra at GSU – Prepared by F. Enescu and Y. Yao

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Please write your name on every page. Show your work.

(1) Let k be a field and $k(X) = \{f(X)/g(X) : f(X) \in k[X], 0 \neq g(X) \in k[X]\}$. Show that $k(X)$ is not an algebraically closed field. Also, find a strict subfield of $k(X)$ isomorphic to $k(X)$.

(2) Let $E \subseteq \mathbb{C}$ such that E is the splitting field of $(x^2 - 3)(x^2 - 5)$ over $\mathbb{Q}$.
  (a) Find the Galois group of E over $\mathbb{Q}$.
  (b) Prove $E = \mathbb{Q}[2\sqrt{3} + \sqrt{5}]$.

(3) Consider the integer $5929 = 7^2 \cdot 11^2$.
  (a) Prove or disprove: Every group of order 5929 is abelian.
  (b) Classify all groups of order 5929.

(4) Let $R = \mathbb{Q}[x]$ and M be the quotient of R^3 modulo the R-submodule generated by the columns of the 3×3 matrix $xI - A$ where I is the 3×3 identity matrix and

$$A = \begin{pmatrix} -9 & -10 & -1 \\ 7 & 8 & 1 \\ 3 & 2 & -1 \end{pmatrix}.$$

  (a) Find the Smith normal form of $xI - A$ over R.
  (b) Up to isomorphism, express M as a direct sum of cyclic R-modules.
  (c) What are the free rank, the invariant factors, and the elementary divisors of M over R?
  (d) Determine the rational canonical form and the Jordan canonical form of A over $\mathbb{Q}$.

(5) Let G be a finite group and H a subgroup of G with $H \neq G$. Prove $G \neq \bigcup_{g \in G} (gHg^{-1})$.

(6) Let A be a unique factorization domain. Show that any minimal nonzero prime ideal of A is principal. (An ideal P is called a minimal nonzero prime ideal if P is nonzero prime ideal such that there is no nonzero prime ideal strictly contained in P.)