Problems for the Qualifying Examination in Algebra at GSU

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(1) Let G be a group, $H \leq G$ such that $|H| < \infty$, and P is a Sylow p-subgroup of H (with p a prime divisor of $|H|$). Denote by $N(H)$ and $N(P)$ the normalizers of H and P in G respectively.
  (a) Prove that $N(H) \leq N(P)$ if P is normal in H.
  (b) Prove that $HN(P) = G$ if H is normal in G.

(2) Let R be a commutative non-zero ring with unity. Assume that every ideal I of R, with $I \neq R$, is prime. Show that R is a field.

(3) Consider the polynomial $f(x) = x^4 - 4x^2 + 1 \in \mathbb{Q}[x]$. You may use the fact that $f(x)$ is irreducible over $\mathbb{Q}$ without a proof.
  (a) Prove that $\mathbb{Q}(\sqrt{2+\sqrt{3}})$ is a splitting field of $f(x)$ over $\mathbb{Q}$.
  (b) Determine the structure of the Galois group of $\mathbb{Q}(\sqrt{2+\sqrt{3}})$ over $\mathbb{Q}$, with explanations.

(4) Consider the integer $57575 = 5^2 \cdot 7^2 \cdot 47$.
  (a) Prove or disprove: There exists a group of order 57575 that is not abelian.
  (b) Classify all groups of order 57575.

(5) Let $F \subseteq K$ be an algebraic extension of fields. Prove that the following statements are equivalent:
  (a) Every separable polynomial in $F[x]$ that is irreducible over F remains irreducible over K.
  (b) F is separably closed in K (i.e., no element $a \in K \setminus F$ is separable over F).
  Hints: A separable polynomial is a polynomial with no repeated roots. For one direction, you might want to work in a (properly chosen) extension field L of K. You may quote the fact that the separable closure of F in L, defined as $\{a \in L \mid a \text{ separable over } F\}$, is a field. Viete's formula (describing relation between roots and coefficients) could help too.

(6) Let $R = \mathbb{Q}[x]$ and M be the quotient of R^3 modulo the R-submodule generated by the columns of the 3×3 matrix $xI - A$ where I is the 3×3 identity matrix and

$$A = \begin{pmatrix} 2 & 2 & 3 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{pmatrix}.$$

  (a) Find the Smith normal form of $xI - A$ over R.
  (b) Up to isomorphism, express M as a direct sum of cyclic R-modules.
  (c) What are the free rank, the invariant factors, and the elementary divisors of M over R?
  (d) Determine the rational canonical form and the Jordan canonical form of A over $\mathbb{Q}$.