Problems for the Qualifying Examination in Algebra at GSU - Prepared by F. Enescu and Y. Yao. Date: January 14, 2010

(1) Let $\mathbb{Q}$ be the field of all rational numbers, R the ring of all 4×4 matrices with entries in $\mathbb{Q}$, and J a two-sided ideal of R such that $A \in J$ with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Determine whether the 4×4 identity matrix I_4 is contained in J. Justify your claim fully.
(b) Determine whether J coincides with R. Explain why.

(2) Let R be a commutative ring (with identity). Assume that R has only three distinct ideals: 0, I, R. Prove that
(a) If $a \in I$ then $1 - a$ is invertible in R.
(b) If a, b are nonzero elements in I then $ab = 0$.

(3) Show there is no simple group of order 72.

(4) Let $R = \mathbb{Q}[x]$ and M be the quotient of R^3 modulo the R-submodule generated by the columns of the 3×3 matrix $xI - A$ where I is the 3×3 identity matrix and

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{pmatrix}.$$

(a) Find the Smith normal form of $xI - A$.
(b) Up to isomorphism, express M as a direct sum of cyclic R-modules.
(c) What are the free rank, the invariant factors, and the elementary divisors of M?
(d) Determine the rational canonical form and the Jordan canonical form of A.

(5) Compute the Galois group of $\mathbf{Q}(\sqrt{5}, \sqrt{3})$ over $\mathbf{Q}$.

(6) Prove that $\mathbf{F}_p(x^p, y^p) \subseteq \mathbf{F}_p(x, y)$ is not a simple extension. (Here p is a prime number.)