Problems for the Qualifying Examination in Algebra at GSU - Prepared by F. Enescu and Y. Yao. Date: August 21 2009

(1) Let R be a ring with identity, S the ring of all $n \times n$ matrices with entries in R, and J a subset of S. Prove that J is a two-sided ideal of S if and only if J equals the set of all $n \times n$ matrices with entries in a two-sided ideal I of R.

(2) Let E be the splitting field of $X^4 + 1$ over $\mathbf{Q}$. Compute the Galois group of E over $\mathbf{Q}$. How many intermediate fields K (with $\mathbf{Q} \subseteq K \subseteq E$) are normal extensions of $\mathbf{Q}$? Please explain your answer fully.

(3) Show there is no simple group of order 48.

(4) Let M be the quotient of $\mathbb{Z}^4$ modulo the $\mathbb{Z}$-submodule generated by the column vectors of the following 4×5 matrix

$$A = \begin{pmatrix} 0 & 12 & 12 & 0 & 48 \\ -12 & 12 & 0 & -24 & 12 \\ 24 & -12 & 48 & 48 & 60 \\ 12 & -12 & 36 & 24 & 24 \end{pmatrix}.$$

Find the Smith normal form of A. Then express M as a direct sum of cyclic $\mathbb{Z}$-modules. What are the free rank, the invariant factors, and the elementary divisors of M?

(5) (i) Let $k \subseteq K$ be a field extension. Assume that every polynomial irreducible in $k[X]$ is irreducible in $K[X]$. Show that k is algebraically closed in K.
(ii) Show that any algebraic field extension of a perfect field is perfect.

(6) Let R be a (commutative) domain. Show that the intersection of all maximal ideals of $R[X]$ is the zero ideal.