MATH 831 HOMEWORK SOLUTIONS – ASSIGNMENT 9

Exercise 5.20. Let A be a subring of an integral domain B such that B is finitely generated over A. Show that there exists $s \neq 0$ in A and elements $y_1, \ldots, y_n$ in B, algebraically independent over A and such that B_s is integral over B'_s, where $B' = A[y_1, \ldots, y_n]$.

Proof. Let $S = A \setminus 0$ and let $K = S^{-1}A$ be the fraction field of A. Say $B = A[b_1, \ldots, b_m]$ with $b_1, \ldots, b_m \in B$. Then $S^{-1}B = K[b_1, \ldots, b_m]$ is a finitely generated algebra over K. By Noether Normalization, there exist $y_1/s_1, \ldots, y_n/s_n$ in $S^{-1}B$ with $y_1, \ldots, y_n \in B$, $s_1, \ldots, s_n \in S$ such that $y_1/s_1, \ldots, y_n/s_n$ are algebraically independent over K and $S^{-1}B$ is integral over $K[y_1/s_1, \ldots, y_n/s_n]$. Then it is easy to see that $y_1, \ldots, y_n$ are algebraically independent over K and $S^{-1}B$ is integral over $K[y_1, \ldots, y_n]$. The fact that $b_1, \ldots, b_m \in B \subseteq S^{-1}B$ are integral over $K[y_1, \ldots, y_n]$ implies that there are equations

$$b_i^{r_i} + \frac{a_{(i,1)}}{s} b_i^{r_i - 1} + \cdots + \frac{a_{(i,r_i-1)}}{s} b_i + \frac{a_{(i,r_i)}}{s} = 0$$

with $a_{(i,j)} \in A[y_1, \ldots, y_n] =: B'$ and $s \in S$ for $i = 1, 2, \ldots, m$. (Here notice that we can always make sure that the coefficients have the same denominator.) Clearly the above equations imply that $b_1, \ldots, b_m$ are all integral over B'_s. Since $B_s = B'_s[b_1, \ldots, b_m]$, we conclude that B_s is integral over B'_s. □

Exercise 5.23. Let A be a ring. Show that the following are equivalent:

i) Every prime ideal is an intersection of maximal ideals;
ii) In every homomorphic image of A the nilradical is equal to the Jacobson radical;
iii) Every prime ideal in A which is not maximal is equal to the intersection of the prime ideals which contain it strictly.

A ring A with the three equivalent properties above is called a *Jacobson ring*.

Proof. i) $\Rightarrow$ ii): Let B be an arbitrary ring that is a homomorphic image of A. Then clearly every prime ideal of B is an intersection of maximal ideals. This implies that the nilradical of B, which is the intersection of all prime ideals of B, is an intersection of certain (hence all) maximal ideals of B. Therefore the nilradical of B contains the Jacobson radical of B. But, on the other hand, the nilradical of B is always contained in the Jacobson radical of B. Hence the nilradical of B is equal to the Jacobson radical of B.

ii) $\Rightarrow$ iii): Let $\mathfrak{p}$ be an arbitrary prime ideal of A that is not maximal and let $B = A/\mathfrak{p}$. Then the nilradical of B, which is 0, is equal to the Jacobson radical of B. That is to say that the intersection of all the maximal ideals of B is 0. Then, since the maximal ideals of $B = A/\mathfrak{p}$ correspond to the maximal ideals of A that contain $\mathfrak{p}$, we conclude that $\mathfrak{p}$ is an intersection of all the maximal ideals of A containing $\mathfrak{p}$, in which all the containments of $\mathfrak{p}$ in the maximal ideals are automatically strict. Hence $\mathfrak{p}$ is equal to the intersection of the prime ideals which contain it strictly.

iii) $\Rightarrow$ i): Suppose that there is a prime ideal $\mathfrak{p}$ of A that is not an intersection of maximal ideals. Let $B = A/\mathfrak{p}$. Then B is a domain and its zero ideal is not the intersection of all maximal ideals of B, i.e. the Jacobson radical of B is not zero. Choose $f \neq 0$ in the Jacobson radical of B. Then B_f is a non-zero ring since f is not nilpotent. Choose a maximal ideal $\mathfrak{m}$ of B_f and let $\mathfrak{q} = \mathfrak{m} \cap B$. Then $\mathfrak{q}$ is a prime ideal of B and $\mathfrak{q}$ is not maximal in B since otherwise f would be in $\mathfrak{q}$. Moreover, by our choice of $\mathfrak{q}$, every prime ideal of B that strictly contains $\mathfrak{q}$ has non-empty intersection with the multiplicatively closed set $\{f^n \mid n \geq 0\}$ and therefore has to contain f. Hence f is contained in the intersection of the prime ideals which contain $\mathfrak{q}$ strictly. But condition iii) implies that every prime ideal (in particular $\mathfrak{q}$) in B which is not maximal is equal to the intersection of the prime ideals which contain it strictly. Thus we get a contradiction. □

Exercise 9.2. Let A be a Dedekind domain. If $f = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial with coefficients in A, the *content* of f is the ideal $c(f) = (a_0, \ldots, a_n)$ in A. Prove *Gauss's lemma* that $c(fg) = c(f)c(g)$.

Proof. Since equality is a local property, we may assume that A is a discrete valuation ring without loss of generality. As the case in which A is a field in easy, we may as well assume that A is a discrete valuation ring of dimension one. Write

$$f = a_0 + a_1x + \cdots + a_nx^n \qquad \text{and} \qquad g = b_0 + b_1x + \cdots + b_mx^m,$$

with $a_i, b_j \in A$. It is always true that $c(fg) \subseteq c(f)c(g)$ (without any assumption on A). So we only need to prove that $c(fg) \supseteq c(f)c(g)$. Say the maximal ideal of A is generated by x, $c(f) = x^sA$ and $c(g) = x^tA$. That is to say that $v(x) = 1, \min\{v(a_0), \ldots, v(a_n)\} = s$ and $\min\{v(b_0), \ldots, v(b_m)\} = t$. (Here recall that, for any $a \in A$, $v(a)$ is defined to be $\max\{n \mid a \in (x^n)\}$.) Then there exist $0 \leq n_0 \leq n, 0 \leq m_0 \leq m$ such that $v(a_{n_0}) = s, v(b_{m_0}) = t$ and $v(a_i) > s, v(b_j) > t$ for all $0 \leq i < n_0, 0 \leq j < m_0$. Also we can see that the coefficient of $x^{n_0+m_0}$ in polynomial fg is

$$\begin{aligned} c_{n_0+m_0} &= \sum_{i+j=n_0+m_0} a_ib_j \\ &= a_0b_{n_0+m_0} + \cdots + a_{n_0-1}b_{m_0+1} + a_{n_0}b_{m_0} + a_{n_0+1}b_{m_0-1} + \cdots + a_{n_0+m_0}b_0, \end{aligned}$$

in which we agree that $a_i = b_j = 0$ if $i > n$ or $j > m$. Now it is easy to check that $v(a_{n_0}b_{m_0}) = s+t$ and all the other terms have valuations strictly larger than $s+t$ by our choice of n_0, m_0. Therefore $v(c_{n_0+m_0}) = s + t$, which implies that $c(fg) \supseteq c_{n_0+m_0}A = x^{s+t}A = c(f)c(g)$. □

Exercise 9.4. Let A be a local domain which is not a field and in which the maximal ideal $\mathfrak{m}$ is principal and $\cap_{n=1}^{\infty}\mathfrak{m}^n = 0$. Prove that A is a discrete valuation ring.

Proof. By assumption, there exists $x \in \mathfrak{m}$ such that $\mathfrak{m} = (x)$. Then, by Proposition 9.2, it is enough to prove that every non zero ideal $\mathfrak{a}$ of A is of the form (x^k) for some $k \geq 0$. Since $\cap_{n=1}^{\infty}\mathfrak{m}^n = 0$, there exists $k \geq 0$ such that $(x^{k+1}) \not\supseteq \mathfrak{a} \subseteq (x^k)$. Then the ideal $(\mathfrak{a} : x^k)$ is either the unit ideal (1) or contained in $\mathfrak{m} = (x)$ and, furthermore, direct checking shows that $\mathfrak{a} = (\mathfrak{a} : x^k)(x^k)$. If $(\mathfrak{a} : x^k) \subseteq (x)$, then we would get $\mathfrak{a} = (\mathfrak{a} : x^k)(x^k) \subseteq (x)(x^k) = (x^{k+1})$ which is a contradiction to our choice of k. Hence $(\mathfrak{a} : x^k) = (1)$ and therefore $\mathfrak{a} = (\mathfrak{a} : x^k)(x^k) = (x^k)$. □

Exercise 9.8. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be three ideals in a Dedekind domain. Prove that

$$\begin{aligned} \mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) &= (\mathfrak{a} \cap \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{c}) \\ \mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c}) &= (\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c}) \end{aligned}$$

Proof. As equality is a local property and, furthermore, sum and intersection of ideals both commute with localization, we may assume that A is a discrete valuation ring without loss of generality. Then the maximal ideal $\mathfrak{m}$ of A is principal, say $\mathfrak{m} = (x)$, and every ideal of A is of the form of (x^k) for some $k \geq 0$. In particular, say $\mathfrak{a} = (x^a), \mathfrak{b} = (x^b)$ and $\mathfrak{c} = (x^c)$ with $a, b, c \in \mathbb{N}$.

Then it is easy to check that

$$\begin{aligned}
\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) &= (x^a) \cap (x^{\min(b,c)}) = (x^{\max(a,\min(b,c))}) \\
\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c}) &= (x^a) + (x^{\max(b,c)}) = (x^{\min(a,\max(b,c))}) \\
(\mathfrak{a} \cap \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{c}) &= (x^{\max(a,b)}) + (x^{\max(a,c)}) = (x^{\min(\max(a,b),\max(a,c))}) \\
(\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c}) &= (x^{\min(a,b)}) \cap (x^{\min(a,c)}) = (x^{\max(\min(a,b),\, \min(a,c))}).
\end{aligned}$$

So it suffices to prove that

$$(*) \qquad \begin{aligned}
\max(a, \min(b, c)) &= \min(\max(a, b), \max(a, c)) \\
\min(a, \max(b, c)) &= \max(\min(a, b),\, \min(a, c))
\end{aligned}$$

for any $a, b, c \in \mathbb{N}$. But the equalities in $(*)$ can be proved directly. Hence the proof is finished. □

Note: The exercises are from '**Introduction to Commutative Algebra**' by M. F. Atiyah and I. G. Macdonald. All the quoted results are from the textbook unless different sources are quoted explicitly. For the convenience of the readers, the number of the chapter is included when a particular exercise is numbered. For example, **Exercise m.n** means the **Exercise n** from **Chapter m**.