MATH 831 HOMEWORK SOLUTIONS – ASSIGNMENT 8

Exercise 5.3. Let $f : B \to B'$ be a homomorphism of A-algebras, and let C be an A-algebra. If f is integral, prove that $f \otimes 1 : B \otimes_A C \to B' \otimes_A C$ is integral.

Proof. Because of Corollary 5.3, it is enough to show that, for every $x \in B'$ and $c \in C$, the element $x \otimes c \in B' \otimes_A C$ is integral over $(f \otimes 1)(B \otimes_A C)$ (since a typical element of $B' \otimes_A C$ is of the form $\sum_{i=1}^n x_i \otimes c_i$ with $x_i \in B', c_i \in C$). Then $x \in B'$ satisfies a monic polynomial equation

$$x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n = 0, \qquad \text{with } b_i \in f(B)$$

for $i = 1, 2, \ldots, n$. Now direct checking shows that

$$(x \otimes c)^n + (b_1 \otimes c)(x \otimes c)^{n-1} + \cdots + (b_{n-1} \otimes c^{n-1})(x \otimes c) + (b_n \otimes c^n) = 0,$$

in which $b_i \otimes c^{n-i} \in (f \otimes 1)(B \otimes_A C)$ for $i = 1, 2, \ldots, n$. □

Exercise 5.5. Let $A \subseteq B$ be rings, B integral over A.

i) If $x \in A$ is a unit in B, then it is a unit in A.
ii) The Jacobson radical in A is the contraction of the Jacobson radical of B.

Proof. i): Since B is integral over A, the element $x^{-1} \in B$ satisfies

$$(x^{-1})^n + a_1 (x^{-1})^{n-1} + \cdots + a_{n-1}(x^{-1}) + a_n = 0, \qquad \text{with } a_i \in A$$

for $i = 1, 2, \ldots, n$. Multiply the above equation by x^{n-1} to get an equation

$$x^{-1} + a_1 + \cdots + a_{n-1} x^{n-2} + a_n x^{n-1} = 0,$$

which implies that

$$x^{-1} = -(a_1 + \cdots + a_{n-1} x^{n-2} + a_n x^{n-1}) \in A,$$

which means x is a unit in A.

Alternatively, we can use lying-over to prove part i): If $x \in A$ is not a unit in A, then x in contained a some maximal ideal $\mathfrak{m}_A$ of A. By lying-over (Theorem 5.10), there exists a prime ideal (which has to be maximal by Corollary 5.8) $\mathfrak{m}_B$ of B such that $\mathfrak{m}_B \cap A = \mathfrak{m}_A$. Hence $x \in \mathfrak{m}_B$, which contradicts the assumption that x is a unit in B.

ii): Denote the Jacobson radicals of A and B by $\mathfrak{R}_A$ and $\mathfrak{R}_B$ respectively. We need to show that $\mathfrak{R}_A = \mathfrak{R}_B \cap A$. Choose $x \in \mathfrak{R}_B \cap A$ and an arbitrary $a \in A$. Then $1 + ax \in A$ is a unit in B by Proposition 1.9 and hence is a unit in A by part i). Therefore $x \in \mathfrak{R}_A$ by Proposition 1.9 again. Hence $\mathfrak{R}_B \cap A \subseteq \mathfrak{R}_A$. (You may also use Corollary 5.8 to prove $\mathfrak{R}_B \cap A \subseteq \mathfrak{R}_A$, which is very similar to the next portion of the proof). On the other hand, choose $x \in \mathfrak{R}_A$ and an arbitrary maximal ideal $\mathfrak{m}_B$ of B. Then $\mathfrak{m}_B \cap A$ is a maximal ideal of A (by Corollary 5.8 since B is integral over A). Therefore $x \in \mathfrak{m}_B \cap A \subseteq \mathfrak{m}_B$. Hence $x \in \mathfrak{R}_B$, which implies that $x \in \mathfrak{R}_B \cap A$. Therefore $\mathfrak{R}_A \subseteq \mathfrak{R}_B \cap A$. Thus $\mathfrak{R}_A = \mathfrak{R}_B \cap A$. □

Exercise 5.7. Let A be a subring of B, such that the set $B - A = B \setminus A$ is closed under multiplication. Show that A is integrally closed in B.

Proof. Suppose, on the contrary, that A is not integrally closed in B. Then let C be the integral closure of A in B and let n be the minimal degree of all monic polynomial equations satisfied by some elements of $C - A = C \setminus A$. Say $x \in C - A = C \setminus A$ satisfies

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0, \qquad \text{with } a_i \in A$$

for $i = 1, 2, \ldots, n$. Then clearly $n \geq 2$ since $x \notin A$. And also $x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1} \notin A$ by the minimality of n. But we have

$$x(x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1}) = -a_n \in A,$$

which contradicts the assumption that $B - A = B \setminus A$ is closed under multiplication. □

Exercise 5.8. i) Let A be a subring of an integral domain B, and let C be the integral closure of A in B. Let f, g be monic polynomials in $B[x]$ such that $fg \in C[x]$. Then f, g are in $C[x]$.

ii) Prove the same result without assuming that B (or A) is an integral domain.

Proof. We are going to prove part ii) once and for all since it is more general than part i). We first claim that there is an extension ring $D \supseteq B$ such that both f and g can be factorized as products of degree one monic polynomials. Assume the claim and say

$$f(x) = (x - d_1)(x - d_2)\cdots(x - d_m) \qquad \text{and} \qquad g(x) = (x - d'_1)(x - d'_2)\cdots(x - d'_n),$$

in which $d_i, d'_j \in D$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then we have that the monic polynomial

$$f(x)g(x) = (x - d_1)(x - d_2)\cdots(x - d_m)(x - d'_1)(x - d'_2)\cdots(x - d'_n) \in C[x],$$

which implies that d_i, d'_j are all integral over C and hence are integral over A by Corollary 5.4. Since the coefficients of f and g can be expressed as polynomials in d_i and d'_j, we conclude that the coefficients are all integral over A. Finally, since $f, g \in B[x]$ and C is the integral closure of A in B, we conclude that the coefficients are all in C, i.e. $f, g \in C[x]$.

To complete the proof, we need to prove the claim. It is enough to find an extension ring $D \supseteq B$ such that $f(x) \in B[x]$ can be factorized as a product of degree one monic polynomials. We prove it by induction on $\deg f(x)$. It is trivially true if $\deg f(x) = 1$. If $\deg f(x) > 1$, then let $D_1 := \frac{B[t]}{f(t)B[t]}$, which is a ring. For any $h(t) \in B[t]$, denote the coset $h(t) + f(t)B[t] \in D_1 := \frac{B[t]}{f(t)B[t]}$ by $\overline{h(t)}$. Then the ring B is naturally embedded into D_1 by $b \mapsto \bar{b}$. So we may as well consider B as a subring of D_1 by identifying $b \in B$ with $\bar{b} \in D_1$. Keeping this identification in mind, we can directly check that $\bar{t} \in D_1$ is a root of $f(x) = 0$ since $\overline{f(t)} = 0 \in D_1$. Therefore $f(x) = (x - \bar{t})f_1(x) \in D_1[x]$, in which $\deg f_1(x) < \deg f(x)$. By induction hypothesis, there exists a extension ring $D \supseteq D_1$ such that $f_1(x) \in D_1[x]$ can be factorized as a product of degree one monic polynomials. Thus $f(x) = (x - \bar{t})f_1(x)$ can be factorized as a product of degree one monic polynomials in $D_1[x]$. (Looks familiar? Find a algebra book containing field theory and look up the proof of existence of splitting fields.) □

Exercise 5.9. Let A be a subring of a ring B and let C be the integral closure of A in B. Prove that $C[x]$ is the integral closure of $A[x]$ in $B[x]$.

Proof. By Exercise 5.3, $C[x] = C \otimes_A A[x]$ is integral over $A[x] = A \otimes_A A[x]$.

To show that $C[x]$ is the the integral closure of $A[x]$ in $B[x]$, we follow the hint provided by the textbook. (Yon may look up Eisenbud's book for another proof.) For any $f \in B[x]$ that is integral over $A[x]$, it satisfy

$$f^m + g_1f^{m-1} + \cdots + g_{m-1}f + g_m = 0 \qquad \text{with } g_i \in A[x].$$

Let r be an integer larger than the degrees of $f, g_1, g_2, \ldots, g_m$, and let $f_1 = f - x^r$, so that

$$(f_1 + x^r)^m + g_1(f_1 + x^r)^{m-1} + \cdots + g_{m-1}(f_1 + x^r) + g_m = 0.$$

Expanding the powers $(f_1 + x^r)^i$ out and then collecting the terms in terms of the powers of f_1, we get an equation of the following form

$$f_1^m + h_1f_1^{m-1} + \cdots + h_{m-1}f_1 + h_m = 0,$$

in which each h_i is a polynomial expression of x^r and the g_j. In particular,

$$h_m = (x^r)^m + g_1(x^r)^{m-1} + \cdots + g_{m-1}x^r + g_m \in A[x],$$

which gives

$$-f_1(f_1^{m-1} + h_1 f_1^{m-2} + \cdots + h_{m-1}) = h_m \in A[x].$$

Notice that, by our choice of r, both $-f_1 = x^r - f$ and h_m are monic, which implies that $f_1^{m-1} + h_1 f_1^{m-2} + \cdots + h_{m-1}$ is also monic. Now apply Exercise 5.8 to conclude that both $-f_1 = x^r - f$ and $f_1^{m-1} + h_1 f_1^{m-2} + \cdots + h_{m-1}$ are in $C[x]$. But $-f_1 = x^r - f \in C[x]$ implies that $f \in C[x]$. □

Note: The exercises are from '**Introduction to Commutative Algebra**' by M. F. Atiyah and I. G. Macdonald. All the quoted results are from the textbook unless different sources are quoted explicitly. For the convenience of the readers, the number of the chapter is included when a particular exercise is numbered. For example, **Exercise m.n** means the **Exercise n** from **Chapter m**.