

MATH 831 HOMEWORK SOLUTIONS – ASSIGNMENT 7

- Exercise 6.1.** i) Let M be a Noetherian A -module and $u : M \rightarrow M$ a module homomorphism. If u is surjective, then u is an isomorphism.
 ii) If M is Artinian and u is injective, then again u is an isomorphism.

Proof. i): It is easy to check that we have an ascending chain

$$\text{Ker}(u) \subseteq \text{Ker}(u^2) \subseteq \cdots \subseteq \text{Ker}(u^n) \subseteq \text{Ker}(u^{n+1}) \subseteq \cdots$$

of submodules of M . Since M is Noetherian, there exists $k \in \mathbb{N}$ such that $\text{Ker}(u^k) = \text{Ker}(u^{k+1})$. Notice that $u^n : M \rightarrow M$ is surjective for all $n \in \mathbb{N}$ since u is surjective. In particular, u^k is surjective.

For any $x \in \text{Ker}(u)$, there exists $y \in M$ such that $u^k(y) = x$. Then $u^{k+1}(y) = u(u^k(y)) = u(x) = 0$, which implies that $y \in \text{Ker}(u^{k+1}) = \text{Ker}(u^k)$. Therefore $x = u^k(y) = 0$. Hence $\text{Ker}(u) = 0$ since x is arbitrary in $\text{Ker}(u)$, which means u is injective. Therefore u is an isomorphism.

ii): It is easy to check that we have a descending chain

$$\text{Im}(u) \supseteq \text{Im}(u^2) \supseteq \cdots \supseteq \text{Im}(u^n) \supseteq \text{Im}(u^{n+1}) \supseteq \cdots$$

of submodules of M . Since M is Artinian, there exists $k \in \mathbb{N}$ such that $\text{Im}(u^k) = \text{Im}(u^{k+1})$. Notice that $u^n : M \rightarrow M$ is injective for all $n \in \mathbb{N}$ since u is injective. In particular, u^k is injective.

For any $x \in M$, there exists $y \in M$ such that $u^{k+1}(y) = u^k(x)$ since $u^k(x) \in \text{Im}(u^k) = \text{Im}(u^{k+1})$. But then $u^k(u(y) - x) = u(u^k(y)) - u^k(x) = u^{k+1}(y) - u^k(x) = 0$, which implies that $u(y) - x = 0$, i.e. $u(y) = x$, since u^k is injective. Hence $\text{Im}(u) = M$ since x is arbitrary in M , which means u is surjective. Therefore u is an isomorphism. \square

Exercise 7.2. Let A be a Noetherian ring and let $f = \sum_{n=0}^{\infty} a_n x^n \in A[[x]]$. Prove that f is nilpotent if and only if each a_n is nilpotent.

Proof. The ‘only if’ part is very similar to the proof of Exercise 1.2 ii), which was assigned. Indeed to prove that a_n are nilpotent for all $n \geq 0$, it suffices to show $a_n \in \mathfrak{p}$ for every prime ideal \mathfrak{p} of A . For any arbitrarily chosen and then fixed prime ideal \mathfrak{p} , we denote the quotient ring A/\mathfrak{p} by \bar{A} and denote an element $a + \mathfrak{p} \in \bar{A}$ by \bar{a} . Then there is a ring homomorphism $\phi : A[[x]] \rightarrow \bar{A}[[x]]$ defined by $\phi(c_0 + \cdots + c_r x^r + \cdots) = \bar{c}_0 + \cdots + \bar{c}_r x^r + \cdots$. Since $f = a_0 + a_1 x + \cdots + a_n x^n + \cdots$ is a nilpotent in $A[[x]]$, $\phi(f) = \bar{a}_0 + \bar{a}_1 x + \cdots + \bar{a}_n x^n + \cdots$ is a nilpotent in $\bar{A}[[x]]$. Now notice that $\bar{A} = A/\mathfrak{p}$ is an integral domain. In general, if R is an integral domain, then $R[[x]]$ is also a integral domain (check yourself), in particular $R[[x]]$ has no non-zero nilpotent. Therefore $\phi(f) = \bar{a}_0 + \bar{a}_1 x + \cdots + \bar{a}_n x^n + \cdots = 0$, i.e. $\bar{a}_n = a_n + \mathfrak{p}$ are zero in $\bar{A} = A/\mathfrak{p}$ for all $n \geq 0$, which is the same as saying $a_n \in \mathfrak{p}$ for all $n \geq 0$. Since the prime ideal \mathfrak{p} is arbitrary, we conclude that a_n are nilpotent for all $n \geq 0$. (Alternatively, you may induct on n to show directly that a_n is nilpotent for all n .) (Also notice that we do not need A to be Noetherian for the ‘only if’.)

Now we prove the ‘if’ part. Let \mathfrak{N} be the nilradical of A . Then, as \mathfrak{N} is finitely generated, there exists an integer $k \in \mathbb{N}$ such that $\mathfrak{N}^k = 0$. (Notice that here is the only place where the assumption that A is Noetherian is used.) Then it is easy to check that $f^k = (\sum_{n=0}^{\infty} a_n x^n)^k = 0 \in A[[x]]$ if each a_n is nilpotent, i.e. $a_n \in \mathfrak{N}$. \square

Exercise 7.9. Let A be a ring such that

- (1) for each maximal ideal \mathfrak{m} of A , the local ring $A_{\mathfrak{m}}$ is Noetherian;
- (2) for each $x \neq 0$ in A , the set of maximal ideals of A which contain x is finite.

Show that A is Noetherian.

Proof. It is enough to show that an arbitrary ascending chain of ideals of A

$$(*) \quad \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_j \subseteq \mathfrak{a}_{j+1} \subseteq \cdots$$

eventually stabilizes. If $\mathfrak{a}_i = 0$ for all $i \geq 1$, then the chain $(*)$ obviously stabilizes. Therefore we may assume that $\mathfrak{a}_i \neq 0$ for some $i \geq 1$. But then, by dropping the zero ideals (if any) at the beginning of the chain, we may assume that $\mathfrak{a}_1 \neq 0$. Choose $0 \neq x \in \mathfrak{a}_1$. By assumption (2), there are only finitely many maximal ideals of A , say $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_r$, that contain x .

For each $i = 1, 2, \dots, r$, we get an ascending chain

$$(*_{\mathfrak{m}_i}) \quad (\mathfrak{a}_1)_{\mathfrak{m}_i} \subseteq (\mathfrak{a}_2)_{\mathfrak{m}_i} \subseteq \cdots \subseteq (\mathfrak{a}_j)_{\mathfrak{m}_i} \subseteq (\mathfrak{a}_{j+1})_{\mathfrak{m}_i} \subseteq \cdots$$

of ideals of local ring $A_{\mathfrak{m}_i}$. By assumption (1), each local ring $A_{\mathfrak{m}_i}$ is Noetherian. Hence, for each $1 \leq i \leq r$, there is $n_i \in \mathbb{N}$ such that $(\mathfrak{a}_{n_i})_{\mathfrak{m}_i} = (\mathfrak{a}_{n_i+1})_{\mathfrak{m}_i} = \cdots$. Let $n = \max\{n_i \mid 1 \leq i \leq r\}$. So we have

$$(\mathfrak{a}_n)_{\mathfrak{m}_i} = (\mathfrak{a}_{n+1})_{\mathfrak{m}_i} = \cdots = (\mathfrak{a}_{n+k})_{\mathfrak{m}_i} = (\mathfrak{a}_{n+k+1})_{\mathfrak{m}_i} = \cdots$$

for all $i = 1, 2, \dots, r$.

Let \mathfrak{m} be an arbitrary maximal ideal such that $x \notin \mathfrak{m}$. Then we have $(\mathfrak{a}_j)_{\mathfrak{m}} = A_{\mathfrak{m}}$ for all $j \geq 1$ as $x \in \mathfrak{a}_j$.

Therefore we have

$$(\mathfrak{a}_n)_{\mathfrak{m}} = (\mathfrak{a}_{n+1})_{\mathfrak{m}} = \cdots = (\mathfrak{a}_{n+k})_{\mathfrak{m}} = (\mathfrak{a}_{n+k+1})_{\mathfrak{m}} = \cdots$$

for all maximal ideals \mathfrak{m} of A . Thus we conclude that

$$\mathfrak{a}_n = \mathfrak{a}_{n+1} = \cdots = \mathfrak{a}_{n+k} = \mathfrak{a}_{n+k+1} = \cdots,$$

i.e. $(*)$ stabilizes, by Proposition 3.8. So A is Noetherian.

(Notice that, after choosing $0 \neq x \in \mathfrak{a}_1$, we may proceed by passing to the ring A/xA and show the resulted chain of ideals in A/xA stabilizes eventually.) \square

Exercise 7.11. Let A be a ring such that each local ring $A_{\mathfrak{p}}$ is Noetherian. Is A necessarily Noetherian?

Proof. Let $A := \prod_{i=1}^{\infty} \mathbb{Z}_2$, a direct product of infinitely many copies of field $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$. It is easy to check that every element $x \in A$ satisfies the property $x^2 = x$.

Let \mathfrak{p} be an arbitrary prime ideal of A . Then $A_{\mathfrak{p}}$ is a local ring with its unique maximal ideal equal to $\mathfrak{p}_{\mathfrak{p}}$. We claim that $\mathfrak{p}_{\mathfrak{p}} = 0$. Indeed, for any element $x \in \mathfrak{p}$, it is easy to see that $1 - x \notin \mathfrak{p}$ (as otherwise the identity element 1 would be in \mathfrak{p}) and $(1 - x)x = x - x^2 = 0$. Thus $A_{\mathfrak{p}}$ has to be a field, which is Noetherian.

Evidently A is not Noetherian. (Why? Can you find a strictly ascending chain of ideals of A ?) \square

Exercise 8.2. Let A be a Noetherian ring. Prove that the following are equivalent:

- i) A is Artinian;
- ii) $\text{Spec}(A)$ is discrete and finite;
- iii) $\text{Spec}(A)$ is discrete.

Proof. i) \Rightarrow ii): If A is Artinian, then, by Proposition 8.1, every prime ideal $\mathfrak{p} \in \text{Spec}(A)$ is maximal, i.e. $\{\mathfrak{p}\}$ is closed in $\text{Spec}(A)$ by Exercise 1.18 i). Also there are finitely many maximal ideals of A by Proposition 8.3, hence $\text{Spec}(A)$ is finite. As a result, every subset of $\text{Spec}(A)$ is closed, i.e. $\text{Spec}(A)$ is discrete.

ii) \Rightarrow iii): This is obvious.

iii) \Rightarrow i): Since $\text{Spec}(A)$ is discrete, every point-set $\{\mathfrak{p}\}$ is closed, hence every prime ideal \mathfrak{p} of A is maximal by Exercise 1.18 i). Therefore the Krull dimension of A is zero. Finally every zero-dimensional Noetherian ring is Artinian by Theorem 8.5. \square

Note: The exercises are from ‘**Introduction to Commutative Algebra**’ by M. F. Atiyah and I. G. Macdonald. All the quoted results are from the textbook unless different sources are quoted explicitly. For the convenience of the readers, the number of the chapter is included when a particular exercise is numbered. For example, **Exercise m.n** means the **Exercise n** from **Chapter m**.