

MATH 831 HOMEWORK SOLUTIONS – ASSIGNMENT 6

Exercise 3.14. Let M be an A -module and \mathfrak{a} an ideal of A . Suppose that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \supseteq \mathfrak{a}$. Prove that $M = \mathfrak{a}M$.

Proof. Let $\overline{A} = A/\mathfrak{a}$, $\overline{M} = M/\mathfrak{a}M$ and $\overline{\mathfrak{m}} = \mathfrak{m}/\mathfrak{a}$ for all maximal ideals $\mathfrak{m} \supseteq \mathfrak{a}$. Then \overline{M} is naturally an \overline{A} -module and it suffices to show $\overline{M} = 0$ to conclude that $M = \mathfrak{a}M$. As the maximal ideals of \overline{A} are exactly the ideals of the form $\overline{\mathfrak{m}} = \mathfrak{m}/\mathfrak{a}$ where \mathfrak{m} are maximal ideals of A such that $\mathfrak{m} \supseteq \mathfrak{a}$, it is enough to prove that $\overline{M}_{\overline{\mathfrak{m}}} = 0$ for all maximal ideals $\mathfrak{m} \supseteq \mathfrak{a}$ by Proposition 3.8. But

$$\overline{M}_{\overline{\mathfrak{m}}} \cong \frac{M_{\mathfrak{m}}}{\mathfrak{a}M_{\mathfrak{m}}} \quad (\text{check for yourself}).$$

Therefore $\overline{M}_{\overline{\mathfrak{m}}} = 0$ as $M_{\mathfrak{m}} = 0$. □

Alternative proof. Suppose, on the contrary, that $M \not\subseteq \mathfrak{a}M$. Choose $x \in M \setminus \mathfrak{a}M$. Then, clearly, we have $\mathfrak{a} \subseteq (\mathfrak{a}M :_A x) \subsetneq (1)$ where $(\mathfrak{a}M :_A x) := \{a \in A \mid ax \in \mathfrak{a}M\}$ (which is an ideal). Choose a maximal ideal \mathfrak{m} of A such that $\mathfrak{m} \supseteq (\mathfrak{a}M :_A x) \supseteq \mathfrak{a}$. Then the assumption that $M_{\mathfrak{m}} = 0$ implies that $x/1 = 0 \in M_{\mathfrak{m}}$, which implies that there exists $s \in A \setminus \mathfrak{m}$ such that $sx = 0$. But this definitely contradicts our choice that $\mathfrak{m} \supseteq (\mathfrak{a}M :_A x)$. Hence $M = \mathfrak{a}M$. □

Note that, in the above two proofs of Exercise 3.14, all we need is that $M_{\mathfrak{m}} = \mathfrak{a}M_{\mathfrak{m}}$ for all maximal ideals \mathfrak{m} of A such that $\mathfrak{m} \supseteq \mathfrak{a}$.

Exercise 3.15. Let A be a ring, and let F be the A -module A^n . Show that every set of n generators of F is a basis of F .

Deduce that every set of generators of F has at least n elements.

Proof. Let $x_1, x_2, \dots, x_n \in F = A^n$ be a set of n generators of F . Say $x_i = (a_{i1}, a_{i2}, \dots, a_{in})$ for $1 \leq i \leq n$. Let $e_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$ be the i -th standard basis element of $F = A^n$. Then, as x_1, x_2, \dots, x_n generate F , there exist $b_{i1}, b_{i2}, \dots, b_{in}$ such that $e_i = b_{i1}x_1 + b_{i2}x_2 + \dots + b_{in}x_n$.

Let $U = (a_{ij})_{n \times n}$ and $V = (b_{ij})_{n \times n}$ be the two resulted matrices. Then we have $VU = E_n$ where E_n is the $n \times n$ identity matrix. Hence $UV = E_n$. (Indeed, $VU = E_n$ implies that $\det(U)$ is a unit in A therefore U^{-1} exists and therefore $V = VUU^{-1} = U^{-1}$ just as in linear algebra.)

Now it is clear that x_1, \dots, x_n are linearly independent over A . Indeed, for any $c_1, c_2, \dots, c_n \in A$ such that $c_1x_1 + c_2x_2 + \dots + c_nx_n = 0$, we have $(c_1, c_2, \dots, c_n)U = 0$. Thus $(c_1, c_2, \dots, c_n) = (c_1, c_2, \dots, c_n)UV = 0$. Hence x_1, x_2, \dots, x_n form a basis of F .

Consequently, every set of generators of F has at least n elements. If, on the contrary, F is generated by y_1, \dots, y_m for some $m < n$, then clearly F is also generated by $y_1, \dots, y_m, y_{m+1} = 0, \dots, y_n = 0$. But then by the above result, $y_1, \dots, y_m, y_{m+1} = 0, \dots, y_n = 0$ form a basis of F , which is impossible. □

Exercise 6.3. Let M be an A -module and let N_1, N_2 be submodules of M . If M/N_1 and M/N_2 are Noetherian, so is $M/(N_1 \cap N_2)$. Similarly with Artinian in place of Noetherian.

Proof. Define $f : M \rightarrow \frac{M}{N_1} \oplus \frac{M}{N_2}$ by $f(x) = (x + N_1, x + N_2)$ for any $x \in M$. It is easy to check that f is an A -linear homomorphism. Therefore f induces an injective A -homomorphism $\overline{f} : \frac{M}{\text{Ker}(f)} \rightarrow \frac{M}{N_1} \oplus \frac{M}{N_2}$. As M/N_1 and M/N_2 are Noetherian (Artinian), so is $\frac{M}{N_1} \oplus \frac{M}{N_2}$ by Corollary 6.4, and so is $\frac{M}{\text{Ker}(f)}$ by Proposition 6.3.

Hence it is enough to show that $\text{Ker}(f) = N_1 \cap N_2$. Indeed, for any $x \in M$, we have that $x \in \text{Ker}(f) \iff f(x) = (x + N_1, x + N_2) = (0, 0) \in \frac{M}{N_1} \oplus \frac{M}{N_2} \iff x \in N_1$ and $x \in N_2 \iff x \in N_1 \cap N_2$. \square

Exercise 6.4. Let M be a Noetherian A -module and let \mathfrak{a} be the annihilator of M in A . Prove that A/\mathfrak{a} is a Noetherian ring.

If we replace “Noetherian” by “Artinian” in this result, is it still true?

Proof. As M is Noetherian, it is finitely generated. Say M is generated by x_1, x_2, \dots, x_n . Define

$f : A \rightarrow \overbrace{M \oplus \dots \oplus M}^{n \text{ copies}} = M^n$ by $f(a) = (ax_1, ax_2, \dots, ax_n)$ for any $a \in A$. It is easy to see that f is an A -linear map. As M is Noetherian, so is M^n by Corollary 6.4, and so is $A/\text{Ker}(f)$ by Proposition 6.3.

Hence it is enough to show that $\text{Ker}(f) = \mathfrak{a}$, the annihilator of M . Indeed, for any $a \in A$, we have that $a \in \text{Ker}(f) \iff f(a) = (ax_1, ax_2, \dots, ax_n) = (0, 0, \dots, 0) \in M^n \iff ax_1 = ax_2 = \dots = ax_n = 0 \iff aM = 0$, i.e. $a \in \{a \in A \mid aM = 0\} =: \mathfrak{a}$.

However, the result is not true if we replace “Noetherian” by “Artinian”. For example, let $A = \mathbb{Z}$ and $M = G$ as in Example 3) on page 74. That is, G is the subgroup of \mathbb{Q}/\mathbb{Z} consisting of all elements whose order is a power of fixed prime number p . Then G is an Artinian \mathbb{Z} -module as shown in the textbook. Moreover, the annihilator of G is 0: Indeed, for any $0 \neq n \in \mathbb{Z}$, there is $\frac{1}{p^{|n|}} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, which is in G , such that $n(\frac{1}{p^{|n|}} + \mathbb{Z}) = \frac{n}{p^{|n|}} + \mathbb{Z} \neq 0 \in G$ (since $\frac{n}{p^{|n|}} \notin \mathbb{Z}$). But $A/0 \cong A = \mathbb{Z}$ is not Artinian as shown in Example 2) on page 74. \square

Exercise 7.1. Let A be a non-Noetherian ring and let Σ be the set of ideals in A which are not finitely generated. Show that Σ has maximal elements and that the maximal elements of Σ are prime ideals.

Hence a ring in which every prime ideal is finitely generated is Noetherian (I. S. Cohen).

Proof. The existence of maximal elements of Σ is a consequence of Zorn’s lemma. Indeed, let $\Omega \subseteq \Sigma$ be a totally ordered subset of Σ (under containment). Let $\mathfrak{b} = \cup_{\mathfrak{a} \in \Omega} \mathfrak{a}$. Then \mathfrak{b} is an ideal of A since Ω is totally ordered. If \mathfrak{b} is finitely generated, say $\mathfrak{b} = (b_1, b_2, \dots, b_r)$, then $b_i \in \mathfrak{a}_i$ for some $\mathfrak{a}_i \in \Omega$. We may assume that $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots \subseteq \mathfrak{a}_r$ since Ω is totally ordered. Therefore $\mathfrak{b} = (b_1, b_2, \dots, b_r) \subseteq \cup_{i=1}^r \mathfrak{a}_i = \mathfrak{a}_r \subseteq \cup_{\mathfrak{a} \in \Omega} \mathfrak{a} = \mathfrak{b}$, which forces $\mathfrak{a}_r = (b_1, b_2, \dots, b_r)$, which is contrary to the choice that $\mathfrak{a}_r \in \Omega \subseteq \Sigma$. Therefore \mathfrak{b} is not finitely generated. Hence $\mathfrak{b} \in \Sigma$ is an upper bound for Ω . So Σ has at least one maximal element.

Suppose that there is a maximal element \mathfrak{a} of Σ such that \mathfrak{a} is not a prime ideal of A . Then there exist $x, y \in A \setminus \mathfrak{a}$ such that $xy \in \mathfrak{a}$.

As $x \notin \mathfrak{a}$, i.e. $\mathfrak{a} \subsetneq \mathfrak{a} + (x)$, and \mathfrak{a} is a maximal element of Σ , we have $\mathfrak{a} + (x) \notin \Sigma$, i.e. $\mathfrak{a} + (x)$ is finitely generated. Say $\mathfrak{a} + (x) = (c_1, c_2, \dots, c_n)$. Write $c_i = a_i + b_i x$ where $a_i \in \mathfrak{a}, b_i \in A$ for each $i = 1, 2, \dots, n$ and let $\mathfrak{a}_0 = (a_1, a_2, \dots, a_n)$. Then clearly we have $\mathfrak{a}_0 \subset \mathfrak{a}$ and therefore $\mathfrak{a} + (x) = (c_1, c_2, \dots, c_n) \subseteq \mathfrak{a}_0 + (x) \subseteq \mathfrak{a} + (x)$, which forces $\mathfrak{a}_0 + (x) = \mathfrak{a} + (x)$.

Let $(\mathfrak{a} : x) = \{a \in A \mid ax \in \mathfrak{a}\}$. Then obviously $\mathfrak{a}_0 + x(\mathfrak{a} : x) \subseteq \mathfrak{a}$. On the other hand, for any $a \in \mathfrak{a}$, we can write $a = a_0 + bx$ with $a_0 \in \mathfrak{a}_0, b \in A$. Therefore $bx = a - a_0 \in \mathfrak{a}$, i.e. $b \in (\mathfrak{a} : x)$. Hence $a = a_0 + bx \in \mathfrak{a}_0 + x(\mathfrak{a} : x)$. So we have $\mathfrak{a} = \mathfrak{a}_0 + x(\mathfrak{a} : x)$.

By our choice of x and y , we have $y \in (\mathfrak{a} : x)$ and $y \notin \mathfrak{a}$. Together with the easy fact that $\mathfrak{a} \subset (\mathfrak{a} : x)$, we know that $(\mathfrak{a} : x)$ is strictly larger than \mathfrak{a} and hence is finitely generated, say by z_1, z_2, \dots, z_m .

But then $\mathfrak{a} = \mathfrak{a}_0 + x(\mathfrak{a} : x) = (a_1, \dots, a_n) + x(z_1, \dots, z_m) = (a_1, \dots, a_n, xz_1, \dots, xz_m)$ is finitely generated, which is a contradiction. Hence \mathfrak{a} has to be a prime ideal of A .

Hence a ring in which every prime ideal is finitely generated is Noetherian. \square

Note: The exercises are from ‘**Introduction to Commutative Algebra**’ by M. F. Atiyah and I. G. Macdonald. All the quoted results are from the textbook unless different sources are quoted explicitly. For the convenience of the readers, the number of the chapter is included when a particular exercise is numbered. For example, **Exercise m.n** means the **Exercise n** from **Chapter m**.