MATH 831 HOMEWORK SOLUTIONS – ASSIGNMENT 5

Exercise 3.1. Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Proof. Suppose that $S^{-1}M = 0$. Say that M is generated by $x_1, \ldots, x_n$ as an A-module. Then for every $i = 1, \ldots, n$, $x_i/1 = 0$ in $S^{-1}M$, which means that there is $s_i \in S$ such that $s_i x_i = 0 \in M$. Let $s = s_1 s_2 \cdots s_n$, which is in S. Then $sx_i = 0$ for all $i = 1, \ldots, n$ and therefore $xM = 0$ as M is generated by $x_1, \ldots, x_n$.

Conversely, we assume that there exists $s \in S$ such that $sM = 0$. Then for any element $m/t \in S^{-1}M$ with $m \in M$ and $t \in S$, we have $m/t = (sm)/(st) = 0/(st) = 0 \in S^{-1}M$. That is, $S^{-1}M = 0$. (Notice that this part of the proof does not rely on the fact that M is a finitely generated A-module.) □

Exercise 3.2. Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Use this result and Nakayama's lemma to give a proof of Corollary 2.5 which does not depend on determinants.

Proof. Let a/s be an arbitrary element in $S^{-1}\mathfrak{a}$ with $s \in S$ and $a \in \mathfrak{a}$. To show that a/s is in the Jacobson radical of $S^{-1}A$, it suffices to show that for any $r/t \in S^{-1}A$ with $t \in S$ and $r \in A$, $1 + (r/t)(a/s)$ is a unit in $S^{-1}A$. Indeed, $1 + (r/t)(a/s) = (ts)/(ts) + (ra)/(ts) = (ts + ra)/(ts)$. As t and s are both $\in S = 1+\mathfrak{a}$ and $ra \in \mathfrak{a}$, we easily see that ts and $ts+ra$ are both $\in 1+\mathfrak{a} = S$. Therefore we conclude that $1 + (r/t)(a/s) = (ts + ra)/(ts)$ is a unit since $(ts)/(ts + ra) \in S^{-1}A$ is its inverse.

To prove Corollary 2.5 by using the above result, we suppose that M is a finitely generated A-module and $\mathfrak{a}$ is a ideal of A such that $\mathfrak{a}M = M$. Then $S^{-1}M$ is a finitely generated $S^{-1}A$-module satisfying $S^{-1}M = S^{-1}\mathfrak{a}M$. As $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$, we conclude that $S^{-1}M = 0$ by Nakayama's lemma. Then, by the above Exercise 3.1, there exists $x \in S = 1 + \mathfrak{a}$, i.e. $x \equiv 1 \mod \mathfrak{a}$, such that $xM = 0$. □

Exercise 3.5. Let A be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_\mathfrak{p}$ is an integral domain, is A necessarily an integral domain?

Proof. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$, i.e. the nilradical of $A_\mathfrak{p}$ is 0. Let $\mathfrak{N} \subset A$ be the nilradical of A. Then, by Corollary 3.12, $(\mathfrak{N})_\mathfrak{p}$ is the nilradical of $A_\mathfrak{p}$, hence $(\mathfrak{N})_\mathfrak{p} = 0$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$. But this implies that $\mathfrak{N} = 0$ by Proposition 3.8.

To answer the second part of the exercise, let $A = k \times k$ where k is a field. Then it is easy to check that there are exactly two prime ideals, namely $\mathfrak{p} = 0 \times k$ and $\mathfrak{q} = k \times 0$, of A. We know that $\mathfrak{p}_\mathfrak{p}$ is a prime ideal of $A_\mathfrak{p}$. But it is easy to see that $(1, 0)\mathfrak{p} = 0$ and $(1, 0) \notin \mathfrak{p}$. Therefore $\mathfrak{p}_\mathfrak{p} = 0$ in $A_\mathfrak{p}$ by the second part of the proof of Exercise 3.1. Therefore $A_\mathfrak{p}$ is an integral domain. Similarly, $A_\mathfrak{q}$ is also an integral domain. But clearly $A = k \times k$ is not a domain as $(1, 0)(0, 1) = (0, 0) = 0 \in A$. □

Exercise 3.12. Let A be an integral domain and M an A-module. An element $x \in M$ is a *torsion element* of M if $\operatorname{Ann}(x) \neq 0$, that is x is killed by some non-zero element of A. Show that the torsion elements of M form a submodule of M. This submodule is called the *torsion submodule* of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

i) If M is any A-module, then $M/T(M)$ is torsion-free.
ii) If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.
iii) If $0 \to M' \to M \to M''$ is an exact sequence, then the sequence $0 \to T(M') \to T(M) \to T(M'')$ is exact.
iv) If M is any A-module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M to $K \otimes_A M$, where K is the field of fractions of A.

Proof. The fact that $T(M)$ is actually an A-submodule follows from part iv). But as it is not hard to prove it directly, we attach a direct proof: Let $x, y \in T(M)$ and $r \in A$, it is enough to show that $x + y \in T(M)$ and $rx \in T(M)$. There exist non-zero elements a, b such that $ax = 0$ and $by = 0$. Then $ab \neq 0$ as A is an integral domain. And clearly $ab(x + y) = 0$ and $a(rx) = 0$ and therefore $x + y \in T(M)$ and $rx \in T(M)$.

i): Let $x+T(M) \in T(M/T(M))$ be any torsion element of $M/T(M)$, where $x \in M$. That is to say that there exists $a \neq 0 \in A$ such that $a(x+T(M)) = ax+T(M) = 0+T(M) \in M/T(M)$, i.e. $ax \in T(M)$. But $ax \in T(M)$ means that there exists $b \neq 0 \in A$ such that $b(ax) = 0 \in M$, which implies that $(ba)x = 0$. Hence, as $ba \neq 0 \in A$, we get $x \in T(M)$, i.e. $x + T(M) = 0 \in M/T(M)$. Therefore $T(M/T(M)) = 0$, that is, $M/T(M)$ is torsion-free.

ii): For any $y \in f(T(M))$, there exists $x \in T(M)$ such that $y = f(x)$. There exists $a \neq 0 \in A$ such that $ax = 0$. Therefore $ay = af(x) = f(ax) = f(0) = 0 \in N$, which implies that $y \in T(N)$. Hence $f(T(M)) \subseteq T(N)$.

iii): To be specific, let us say we have an exact sequence $0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M''$. By part ii), we have a sequence

$$0 \longrightarrow T(M') \xrightarrow{\phi'} T(M) \xrightarrow{\psi'} T(M''),$$

in which ϕ' and ψ' are the restrictions of the homomorphisms ϕ and ψ respectively. Therefore it is evident that $\phi' : T(M') \to T(M)$ is injective and $\psi' \circ \phi' : T(M') \to T(M'')$ is a zero map. Now all we need to show is that $\mathrm{Ker}(\psi') \subseteq \mathrm{Im}(\phi')$. Let $x \in \mathrm{Ker}(\psi') \in T(M) \subseteq M$. That is, there exists $a \neq 0 \in A$ such that $ax = 0$ and $0 = \psi'(x) = \psi(x)$, i.e. $x \in \mathrm{Ker}(\psi)$. But $\mathrm{Ker}(\psi) = \mathrm{Im}(\phi)$ by assumption. Therefore $x = \phi(y)$ for some $y \in M'$. But then we have $\phi(ay) = a\phi(y) = ax = 0$, which implies that $ay = 0 \in M'$ as ϕ is injective. Therefore $y \in T(M')$ and hence $x = \phi'(y) \in \phi'(T(M')) = \mathrm{Im}(\phi')$. All done.

iv): Let $S = A\setminus 0$. Then $K = S^{-1}A$. By the canonical isomorphism $K \otimes_A M = S^{-1}A \otimes_A M \cong S^{-1}M$, the kernel of the mapping $x \mapsto 1 \otimes x$ of M to $K \otimes_A M$ is exactly the kernel of the mapping $x \mapsto x/1$ of M to $S^{-1}M$. But we have that $x \in \mathrm{Ker}(M \to S^{-1}M) \iff x/1 = 0 \in S^{-1}M \iff ax = 0$ for some $a \in S$ (i.e. $a \neq 0 \in A$) $\iff x \in T(M)$. □

Exercise 3.13. Let S be a multiplicatively closed subset of an integral domain A. In the notation of Exercise 3.12, show that $T(S^{-1}M) = S^{-1}T(M)$. Deduce that the following are equivalent:

i) M is torsion-free.
ii) $M_\mathfrak{p}$ is torsion-free for all prime ideals $\mathfrak{p}$.
iii) $M_\mathfrak{m}$ is torsion-free for all prime ideals $\mathfrak{m}$.

Proof. i) ⇒ ii): Let $\mathfrak{p}$ be any prime ideal of A and $x/s \in T(M_\mathfrak{p})$ in which $x \in M, s \in A \setminus \mathfrak{p}$. That is to say that there exists $a/t \neq 0 \in A_\mathfrak{p}$ such that $0 = (a/t)(x/s) = (ax)/(ts) \in M_\mathfrak{p}$. But $(ax)/(ts) = 0 \in M_\mathfrak{p}$ means there exists $r \in A \setminus \mathfrak{p}$ such that $0 = r(ax) = (ra)x$, which implies that $x \in T(M)$ as $ra \neq 0 \in A$. Hence $x = 0$ since M is torsion-free. Therefore $x/s = 0$. So $T(M_\mathfrak{p}) = 0$, i.e. $M_\mathfrak{p}$ is torsion-free.

ii) ⇒ iii): Evident.

iii) $\Rightarrow$ i): Suppose M is not torsion-free. Then choose $x \neq 0 \in T(M)$ so that $0 \subsetneq \text{Ann}(x) \subsetneq (1)$. That is, there exist $a \neq 0 \in A$ and a maximal ideal $\mathfrak{m}$ of A such that $ax = 0$ and $\text{Ann}(x) \subseteq \mathfrak{m}$, the latter of which implies that $x/1 \neq 0$ in $M_{\mathfrak{m}}$ (as in the proof of Proposition 3.8). But we also have $(a/1)(x/1) = (ax)/1 = 0/1 = 0$ in $M_{\mathfrak{m}}$ and $a/1 \neq 0 \in A_{\mathfrak{m}}$ as A is an integral domain. Thus $x/1$ is a non-zero torsion element of $M_{\mathfrak{m}}$, which is a contradiction. Hence M is torsion-free. □

Note: The exercises are from '**Introduction to Commutative Algebra**' by M. F. Atiyah and I. G. Macdonald. All the quoted results are from the textbook unless different sources are quoted explicitly. For the convenience of the readers, the number of the chapter is included when a particular exercise is numbered. For example, **Exercise m.n** means the **Exercise n** from **Chapter m**.