MATH 831 HOMEWORK SOLUTIONS – ASSIGNMENT 4

Exercise 2.7. Let $\mathfrak{p}$ be a prime ideal in A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, is $\mathfrak{m}[x]$ a maximal ideal in $A[x]$?

Proof. We denote the quotient ring $A/\mathfrak{p}$ by $\overline{A}$ and denote an element $a + \mathfrak{p} \in \overline{A}$ by $\overline{a}$. Then there is a ring homomorphism $\phi : A[x] \to \overline{A}[x]$ defined by $\phi(c_0 + \cdots + c_r x^r) = \overline{c_0} + \cdots + \overline{c_r} x^r$. Now notice that $\overline{A} = A/\mathfrak{p}$ is an integral domain as $\mathfrak{p}$ is a prime ideal in A. In general, if R is an integral domain, then $R[x]$ is also an integral domain (check for yourself). Therefore $\operatorname{Ker}(\phi)$, the kernel of the above map from $A[x]$ to $\overline{A}[x]$, is a prime ideal in $A[x]$ as $\overline{A}[x]$ is an integral domain. But it is easy to check that $\operatorname{Ker}(\phi)$ is exactly $\mathfrak{p}[x]$. Hence $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. Also notice that ϕ is surjective so that $A[x]/\mathfrak{p}[x] \cong \overline{A}[x]$

Now suppose that $\mathfrak{m}$ is a maximal ideal in A. Let $k = A/\mathfrak{m}$ (which is a field). Applying the last paragraph to the case of $\mathfrak{p} = \mathfrak{m}$, we get $A[x]/\mathfrak{m}[x] \cong k[x]$. As $k[x]$ is never a field (for example, $x \neq 0$ is never a unit in $k[x]$), we conclude that $\mathfrak{m}[x]$ is never a maximal ideal in $A[x]$.

Alternatively, let $\mathfrak{M} = \{f(x) \in A[x] \mid f(0) \in \mathfrak{m}\}$, i.e. $\mathfrak{M}$ consists of all polynomials $f(x)$ such that the constant terms, which can be identified as $f(0)$, are in $\mathfrak{m}$. Then direct checking will show that $\mathfrak{M}$ is an proper ideal in $A[x]$ and $\mathfrak{m}[x] \subsetneq \mathfrak{M}$. $\square$

Exercise 2.9. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are finitely generated, then so is M.

Proof. To be specific, let's say the exact sequence is $(0 \to)M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0$. That implies that $\operatorname{Im}(\phi) = \operatorname{Ker}(\psi)$ and ψ is surjective. Say M' is generated by $x_1, \ldots, x_m$ and denote $\phi(x_i)$ by y_i for $i = 1, \ldots, m$. Then $\operatorname{Ker}(\psi) = \operatorname{Im}(\phi)$ is generated by $y_1, \ldots, y_m$. Say M'' is generated by $u_1, \ldots, u_n$. Then, as ψ is surjective, there exist $v_1, \ldots, v_n \in M$ such that $\psi(v_i) = u_i$ for $i = 1, \ldots, n$. Let N be the A-submodule of M generated by $v_1, \ldots, v_n$. Then $\psi(N)$ contains $u_1 = \psi(v_1), \ldots, u_n = \psi(v_n)$, a set of generators of M''. Therefore $\psi(N) = M''$.

Now for any $z \in M$, as $\psi(z) \in M'' = \psi(N)$, there exists $v \in N$ such that $\psi(v) = \psi(z)$. That is $\psi(z - v) = 0$, i.e. $z - v \in \operatorname{Ker}(\psi) = \operatorname{Im}(\phi)$. Hence $z = (z - v) + v \in \operatorname{Im}(\phi) + N$. Therefore we conclude that $M = \operatorname{Im}(\phi) + N$ is finitely generated (e.g. by $y_1, \ldots, y_m$ and $v_1, \ldots, v_n$). $\square$

Exercise 2.10. Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

Proof. Let $L = \operatorname{Coker}(u)$, i.e. $L = N/\operatorname{Im}(u)$. Then we have an exact sequence $M \to N \to L \to 0$. Applying the right exact functor $(A/\mathfrak{a}) \otimes_A$, we get an exact sequence

$$(A/\mathfrak{a}) \otimes_A M \xrightarrow{1 \otimes_A u} (A/\mathfrak{a}) \otimes_A N \longrightarrow (A/\mathfrak{a}) \otimes_A L \longrightarrow 0.$$

By the canonical isomorphism proved in Exercise 2.2 (see Assignment 3), we have the following exact sequence

$$M/\mathfrak{a}M \xrightarrow{\overline{u}} N/\mathfrak{a}N \longrightarrow L/\mathfrak{a}L \longrightarrow 0,$$

in which $\overline{u}$ is induced from u. As $\overline{u}$ is surjective by assumption, we get that $L/\mathfrak{a}L = 0$ from above exact sequence. That is to say $L = \mathfrak{a}L$. Then, as L is finitely generated and $\mathfrak{a}$ is contained in the Jacobson radical, we conclude that $L = 0$ by Nakayama's lemma (Proposition 2.6), which in turn implies $N = \operatorname{Im}(u)$ as $L = N/\operatorname{Im}(u)$. Hence u is surjective. $\square$

Exercise 2.11. Let A be a ring $\neq 0$. Show that $A^m \cong A^n \Rightarrow m = n$.
If $\phi : A^m \to A^n$ is surjective, then $m \geq n$.
If $\phi : A^m \to A^n$ is injective, is it always the case that $m \leq n$?

Proof. As A is a non-zero ring, there exists a maximal ideal $\mathfrak{m}$ in A. Denote the quotient field $A/\mathfrak{m}$ by k.

Suppose that $A^m \cong A^n$. Then $k \otimes_A (A^m) \cong k \otimes_A (A^n)$. By the canonical isomorphisms described in Proposition 2.14, iii) and iv), we get $k^m \cong k^n$, which implies that $m = n$ by the uniqueness of rank of a vector space.

Suppose that $\phi : A^m \to A^n$ is surjective. Applying $k \otimes_A$ to the exact sequence $A^m \xrightarrow{\phi} A^n \to 0$, we get an exact sequence $k \otimes_A (A^m) \xrightarrow{1 \otimes_A \phi} k \otimes_A (A^n) \to 0$, in which $1 \otimes_A \phi$ can be viewed as a k-homomorphism. By the canonical isomorphisms described in Proposition 2.14, iii) and iv), we get an exact sequence

$$k^m \longrightarrow k^n \longrightarrow 0$$

of k-vector spaces. Then k^n is a homomorphic image of k^m implies that $m \geq n$.

Actually, if $\phi : A^m \to A^n$ is injective, then $m \leq n$ is always true. Let $e_i \in A^m$ be the i-th standard basis element, $(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$, of A^m and let $\phi(e_i) = (a_{i1}, a_{i2}, \ldots, a_{in}) \in A^n$ for $i = 1, 2, \ldots, m$. Then we denote the $m \times n$ matrix $(a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ by D. By getting rid of the easy-to-prove trivial cases, we may assume that $m, n > 0$ and $D \neq 0$. Let r be the largest integer in $\{1, 2, \ldots, \min\{m, n\}\}$ such that there exists a non-zero $r \times r$ minor of D. By possibly rearranging the orders of the basis elements of A^m and A^n, we may assume that the non-zero $r \times r$ minor is at the upper-left coner of D, i.e. $\left|(a_{ij})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq r}}\right| \neq 0$.

Suppose, on the contrary, that $m > n$. Then $m \geq r + 1$ and we denote the $(r+1) \times r$ block matrix at the upper-left coner of D by D'. That is $D' = (a_{ij})_{\substack{1 \leq i \leq r+1 \\ 1 \leq j \leq r}}$. For each $i = 1, 2, \ldots, r+1$, let $a_i \in A$ be the $r \times r$ minor of D' by removing its i-th row and let $b_i = (-1)^i a_i$. Then $b_{r+1} \neq 0$ by our construction (choice). For each $j = 1, 2, \ldots, n$, $\sum_{i=1}^{r+1} a_{ij} b_i$ can be realized as the determinant of a $(r+1) \times (r+1)$ matrix. If $1 \leq j \leq r$, then the matrix has two identical columns (up to ± 1). If $r + 1 \leq j \leq n$, then the determinant of the matrix is an $(r+1) \times (r+1)$ minor (up to ± 1) of D . Therefore we have $\sum_{i=1}^{r+1} a_{ij} b_i = 0$ for all $1 \leq j \leq n$. But this means that

$$\phi(b_1, \ldots, b_{r+1}, 0, \ldots, 0) = \left(\sum_{i=1}^{r+1} a_{i1} b_i, \ldots, \sum_{i=1}^{r+1} a_{in} b_i \right) = (0, \ldots, 0) \in A^n,$$

which is a contradiction to the injectivity of ϕ since $(b_1, \ldots, b_{r+1}, 0, \ldots, 0) \neq (0, \ldots, 0) \in A^m$.

Therefore an injective A-linear map $\phi : A^m \to A^n$ implies that $m \leq n$. $\square$

Exercise 2.13. Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.

Proof. It is evident that the map $g : N \to N_B$ defined by $y \mapsto 1 \otimes y$ is A-linear and is also B-linear if we the scalar multiplication of B on $N_B = B \otimes_A N$ is induced by the B-module structure of N (i.e. $b(\sum_{i=1}^n b_i \otimes y_i) = \sum_{i=1}^n b_i \otimes (by_i)$ for any $b, b_i \in B$ and $y_i \in N$).

As the mapping $(b, y) \mapsto by$ is an A-bilinear map from $B \times N$ to N, we have an A-linear map $p : N_B = B \otimes_A N \to N$ such that $p(b \otimes y) = by$ for all $b \in B$ and $y \in N$. Notice that p is also B-linear (for each of the two natural B-module structures on $B \otimes_A N$).

It is easy to check that the composition map $p \circ g : N \to N$ is the identity map on N: Indeed, $p \circ g(y) = p(g(y)) = p(1 \otimes y) = 1y = y$ for any $y \in N$. But the fact that $p \circ g = 1_N$ implies that g is injective and $N_B = g(N) \oplus \mathrm{Ker}(p)$. Therefore $g(N)$ is a direct summand of N_B if both are considered as A-modules (or as B-modules if the the B-module structure of N_B is as described in the first paragraph). (Recall that the 'supposed' B-module structure of $N_B = B \otimes_A N$ obtained from A-module N by scalar extension is inherited from B, i.e. $b(\sum_{i=1}^n b_i \otimes y_i) = \sum_{i=1}^n (bb_i) \otimes y_i$ for any $b, b_i \in B$ and $y_i \in N$.) $\square$

Note: The exercises are from '**Introduction to Commutative Algebra**' by M. F. Atiyah and I. G. Macdonald. All the quoted results are from the textbook unless different sources are quoted explicitly. For the convenience of the readers, the number of the chapter is included when a particular exercise is numbered. For example, **Exercise m.n** means the **Exercise n** from **Chapter m**.