MATH 831 HOMEWORK SOLUTIONS – ASSIGNMENT 3

Exercise 2.1. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

Proof. It suffices to prove that $x \otimes y = 0$ for all $x \in \mathbb{Z}/m\mathbb{Z}$ and $y \in \mathbb{Z}/n\mathbb{Z}$, since $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ is generated by all the $x \otimes y$. Notice that $mx = 0 \in \mathbb{Z}/m\mathbb{Z}$ and $ny = 0 \in \mathbb{Z}/m\mathbb{Z}$. Therefore we have

$$m(x \otimes y) = (mx) \otimes y = 0 \otimes y = 0 \quad \text{and} \quad n(x \otimes y) = x \otimes (ny) = x \otimes 0 = 0.$$

Also notice that there exist integers $s, t \in \mathbb{Z}$ such that $1 = sm + tn$ by the assumption that m, n are coprime. Putting things together, we get that

$$x \otimes y = 1(x \otimes y) = (sm + tn)(x \otimes y) = sm(x \otimes y) + tn(x \otimes y) = 0 + 0 = 0.$$

□

Alternative proof. We use the result of Exercise 2.2, which is proved below. Denote the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ by M. Then $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} M \cong M/mM$. Then we can see that $mM = M$ using the information that m, n are coprime. (You are welcome to fill in the details.) □

Exercise 2.2. Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$.

Proof. Tensoring M with the exact sequence $0 \to \mathfrak{a} \xrightarrow{i} A \to A/\mathfrak{a} \to 0$ where i denotes the inclusion map, we get an exact sequence

$$\mathfrak{a} \otimes_A M \xrightarrow{i \otimes 1} A \otimes_A M \longrightarrow A/\mathfrak{a} \otimes_A M \longrightarrow 0$$

by Proposition 2.18. Therefore $A/\mathfrak{a} \otimes_A M \cong (A \otimes_A M)/\operatorname{Im}(i \otimes 1)$. But there is a canonical isomorphism $\phi : A \otimes_A M \cong M$ by $a \otimes x \mapsto ax$ (see Proposition 2.14) and, under this canonical isomorphism, the image of $i \otimes 1$ is identified with $\phi(\operatorname{Im}(i \otimes 1))$. Therefore we have

$$A/\mathfrak{a} \otimes_A M \cong (A \otimes_A M)/\operatorname{Im}(i \otimes 1) \cong \phi(A \otimes_A M)/\phi(\operatorname{Im}(i \otimes 1)) = M/\phi(\operatorname{Im}(i \otimes 1)).$$

As $\mathfrak{a} \otimes_A M$ is generated by all the $a \otimes x$ with $a \in \mathfrak{a}$ and $x \in M$, we conclude that $\operatorname{Im}(i \otimes 1)$ is generated by all the $a \otimes x \in A \otimes_A M$ with $a \in \mathfrak{a}$ and $x \in M$. Hence $\phi(\operatorname{Im}(i \otimes 1))$ is generated by all the $ax \in M$ with $a \in \mathfrak{a}$ and $x \in M$, i.e. $\phi(\operatorname{Im}(i \otimes 1)) = \mathfrak{a}M$. Therefore

$$A/\mathfrak{a} \otimes_A M \cong M/\phi(\operatorname{Im}(i \otimes 1)) \cong M/\mathfrak{a}M.$$

□

Alternative proof. First there is a **well-defined** map $f : A/\mathfrak{a} \times M \to M/\mathfrak{a}M$ by $((a + \mathfrak{a}), x) \mapsto ax + \mathfrak{a}M \in M/\mathfrak{a}M$ for every $a + \mathfrak{a} \in A/\mathfrak{a}, x \in M$. And it is easy to check that f is A-bilinear. Therefore there exists an A-linear map $\phi : A/\mathfrak{a} \otimes_A M \to M$ such that $\phi(\overline{a} \otimes x) = ax$ for any $\overline{a} = a + \mathfrak{a} \in A/\mathfrak{a}, x \in M$. Conversely, there is a **well-defined** map $\psi : M/\mathfrak{a}M \to A/\mathfrak{a} \otimes M$ by $x + \mathfrak{a}M \mapsto \overline{1} \otimes x$ where $\overline{1} = 1 + \mathfrak{a} \in A/\mathfrak{a}$. It is also easy to check that ψ is an A-linear map. Finally, direct checking proves that $\phi \circ \psi = 1_{M/\mathfrak{a}M}$ and $\psi \circ \phi = 1_{A/\mathfrak{a} \otimes M}$. (The last equality follows from the fact that $\psi \circ \phi(\overline{a} \otimes x) = \psi(ax) = \overline{1} \otimes (ax) = (a\overline{1}) \otimes x = \overline{a} \otimes x$ for any $\overline{a} = a + \mathfrak{a} \in A/\mathfrak{a}, x \in M$.) Therefore $A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$. □

Exercise 2.3. Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$.

Proof. Let $\mathfrak{m}$ be the maximal ideal of A and $k = A/\mathfrak{m}$ be the residue field. We may assume that $M \neq 0$ and prove that $N = 0$. By Nakayama's lemma (Proposition 2.6), we see that $\mathfrak{m}M \subsetneq M$. Therefore $A/\mathfrak{m} \otimes M \cong M/\mathfrak{m}M \cong k^n \neq 0$ is naturally a k-vector space with rank $n > 0$. Hence $M \otimes_A N = 0 \implies A/\mathfrak{m} \otimes (M \otimes_A N) = 0 \implies (k \otimes_A N)^n \cong k^n \otimes_A N \cong (A/\mathfrak{m} \otimes M) \otimes_A N = 0 \implies N/\mathfrak{m}N \cong A/\mathfrak{m} \otimes N = k \otimes_A N = 0 \implies N = \mathfrak{m}N \implies N = 0$ by Nakayama's lemma. □

Next we include Exercise 2.4, which will be used in the proof of Exercise 2.5.

Exercise 2.4. Let $M_i (i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\iff$ each M_i is flat.

Sketch of proof. The module $M = \oplus_{i \in I} M_i$ is flat if an only if the sequence

$$0 \longrightarrow N_1 \otimes_A M \longrightarrow N_2 \otimes_A M \longrightarrow N_3 \otimes_A M \longrightarrow 0 \qquad \text{i.e.}$$

$$0 \longrightarrow N_1 \otimes_A \big(\oplus_{i \in I} M_i\big) \longrightarrow N_2 \otimes_A \big(\oplus_{i \in I} M_i\big) \longrightarrow N_3 \otimes_A \big(\oplus_{i \in I} M_i\big) \longrightarrow 0$$

is exact for every exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$, if and only if the sequence

$$0 \longrightarrow \bigoplus_{i \in I} (N_1 \otimes_A M_i) \longrightarrow \bigoplus_{i \in I} (N_2 \otimes_A M_i) \longrightarrow \bigoplus_{i \in I} (N_3 \otimes_A M_i) \longrightarrow 0$$

is exact for every exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ by Proposition 2.14 iii), if and only if the sequence

$$0 \longrightarrow N_1 \otimes_A M_i \longrightarrow N_2 \otimes_A M_i \longrightarrow N_3 \otimes_A M_i \longrightarrow 0$$

is exact for every exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ and for each M_i, if and only if each M_i is flat. □

Exercise 2.5. Let $A[x]$ be the polynomial ring in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra.

Proof. First we observe (or recall) that A, as an A-module, is flat because of the canonical isomorphism $M \otimes_A A \cong M$.

Second we observe that the polynomial ring $A[x]$, considered as an module over A, is isomorphic to $\oplus_{i=0}^{\infty} A = A \oplus A \oplus \cdots$. Indeed a natural one-to-one correspondence is defined by

$$a_0 + a_1 x + \cdots + a_n x^n \longleftrightarrow (a_0, a_1, \ldots a_n, 0, 0, \ldots).$$

Finally we conclude that $A[x]$ is a flat A-module by Exercise 2.4. That is to say $A[x]$ is a flat A-algebra. □

Exercise 2.8. i) If M and N are flat A-modules, then so is $M \otimes_A N$;

ii) If B is a flat A algebra and N is a flat B-module, then N is flat as an A-module.

Proof. Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be an arbitrary exact sequence of A-modules.

i): Since M is a flat A-module, the sequence

$$0 \longrightarrow N_1 \otimes_A M \longrightarrow N_2 \otimes_A M \longrightarrow N_3 \otimes_A M \longrightarrow 0$$

is exact. But N is also flat over A. Therefore the sequence

$$0 \longrightarrow (N_1 \otimes_A M) \otimes_A N \longrightarrow (N_2 \otimes_A M) \otimes_A N \longrightarrow (N_3 \otimes_A M) \otimes_A N \longrightarrow 0$$

is exact. By Proposition 2.14 ii), $(N_i \otimes_A M) \otimes_A N$ and $N_i \otimes_A (M \otimes_A N)$ are naturally isomorphic for each $i = 1, 2, 3$. Therefore the sequence

$$0 \longrightarrow N_1 \otimes_A (M \otimes_A N) \longrightarrow N_2 \otimes_A (M \otimes_A N) \longrightarrow N_3 \otimes_A (M \otimes_A N) \longrightarrow 0$$

is exact. From this we conclude that $M \otimes_A N$ is a flat A-module.

ii): Since B is a flat A-algebra, the sequence

$$0 \longrightarrow N_1 \otimes_A B \longrightarrow N_2 \otimes_A B \longrightarrow N_3 \otimes_A B \longrightarrow 0$$

is an exact sequence of B-modules. As N is flat over B, the sequence

$$0 \longrightarrow (N_1 \otimes_A B) \otimes_B N \longrightarrow (N_2 \otimes_A B) \otimes_B N \longrightarrow (N_3 \otimes_A B) \otimes_B N \longrightarrow 0$$

is exact. By Exercise 2.15 on page 27, $(N_i \otimes_A B) \otimes_B N$ and $N_i \otimes_A (B \otimes_B N)$ are naturally isomorphic for each $i = 1, 2, 3$. Therefore the sequence

$$0 \longrightarrow N_1 \otimes_A (B \otimes_B N) \longrightarrow N_2 \otimes_A (B \otimes_B N) \longrightarrow N_3 \otimes_A (B \otimes_B N) \longrightarrow 0$$

is exact. From this we conclude that $B \otimes_B N$ is flat as an A-module. Finally, as $B \otimes_B N \cong N$ as A-modules (and as B-modules), N is flat as an A-module. □

Note: The exercises are from '**Introduction to Commutative Algebra**' by M. F. Atiyah and I. G. Macdonald. All the quoted results are from the textbook unless different sources are quoted explicitly. For the convenience of the readers, the number of the chapter is included when a particular exercise is numbered. For example, **Exercise m.n** means the **Exercise n** from **Chapter m**.