MATH 831 HOMEWORK SOLUTIONS – ASSIGNMENT 2

Exercise 1.15. Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

i) If $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$;
ii) $V(0) = X$, $V(1) = \emptyset$;
iii) If $(E_i)_{i\in I}$ is a family of subsets of A, then
$$V\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} V(E_i);$$
iv) $V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called *Zariski topology.* The topological space X is called the *prime spectrum of* A, and is written $\operatorname{Spec}(A)$.

Proof. i): This is evident since for any prime ideal $\mathfrak{p}$ of A,
$$E \subseteq \mathfrak{p} \iff \mathfrak{a} \subseteq \mathfrak{p} \iff r(\mathfrak{a}) \subseteq \mathfrak{p}.$$

ii): $V(0) = X$ is evident since $0 \subseteq \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of A. Similarly, $V(1) = \emptyset$ is evident as there is no prime ideal $\mathfrak{p}$ of A such that $1 \subseteq \mathfrak{p}$.

iii): This follows from the fact that for any prime ideal $\mathfrak{p}$ of A, we have
$$\begin{aligned}
\mathfrak{p} \in V\left(\bigcup_{i\in I} E_i\right) \iff \bigcup_{i\in I} E_i \subseteq \mathfrak{p} &\iff E_i \subseteq \mathfrak{p} && \text{for all } i \in I\\
&\iff \mathfrak{p} \in V(E_i) && \text{for all } i \in I\\
&\iff \mathfrak{p} \in \bigcap_{i\in I} V(E_i).
\end{aligned}$$

iv) For any prime ideal $\mathfrak{p}$ of A, we have a circle of implication
$$\begin{aligned}
\mathfrak{p} \in V(\mathfrak{a}\cap\mathfrak{b}) &\implies \mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{p}\\
&\implies \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{p} \text{ i.e. } \mathfrak{p} \in V(\mathfrak{a}\mathfrak{b}) \implies \mathfrak{a} \subseteq \mathfrak{p} \text{ or } \mathfrak{b} \subseteq \mathfrak{p} \qquad \text{since } \mathfrak{p} \text{ is prime}\\
&\implies \mathfrak{p} \in V(\mathfrak{a}) \text{ or } \mathfrak{p} \in V(\mathfrak{b}) \quad \text{that is} \quad \mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})\\
&\implies \mathfrak{a} \subseteq \mathfrak{p} \text{ or } \mathfrak{b} \subseteq \mathfrak{p} \implies \mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{p} \implies \mathfrak{p} \in V(\mathfrak{a}\cap\mathfrak{b}),
\end{aligned}$$
which proves $V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$. □

Exercise 1.17. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

i) $X_f \cap X_g = X_{fg}$;
ii) $X_f = \emptyset \iff f$ is nilpotent;
iii) $X_f = X \iff f$ is a unit;
iv) $X_f = X_g \iff r((f)) = r((g))$;
v) X is quasi-compact (that is, every open covering has a finite subcovering);
vi) More generally, each X_f is quasi-compact;
vii) An open subset of X is quasi-compact if and only if it is a finite union of X_f.

The sets X_f are called *basic open sets* of $X = \operatorname{Spec}(A)$.

Proof. To show that $(X_f)_{f\in A}$ form a basis of open sets for the Zariski topology, it suffices to show tha any open subset O of X can be written as a union of open subsets of the form X_f. Indeed, as O is open in X, there exists a subset E of A such that $O = X \setminus V(E)$. By Exercise 1.15, part iii), $V(E) = \cap_{f\in E} V(f)$. Therefore

$$O = X \setminus V(E) = X \setminus \bigcap_{f\in E} V(f) = \bigcup_{f\in E} (X \setminus V(f)) = \bigcup_{f\in E} X_f.$$

i): This follows immediately from the fact that, for any prime ideal $\mathfrak{p}$,

$$\mathfrak{p} \in X_f \cap X_g \iff \mathfrak{p} \in X_f \text{ and } \mathfrak{p} \in X_g \iff f \notin \mathfrak{p} \text{ and } g \notin \mathfrak{p} \iff fg \notin \mathfrak{p} \qquad \text{since } \mathfrak{p} \text{ prime.}$$

ii) $X_f = \emptyset \iff V(f) = X \iff f \in \mathfrak{p}$ for all prime ideals $\mathfrak{p} \iff f$ is a nilpotent by Proposition 1.8.

iii): Follows from the more general statement iv) below with $g = 1$. (Remember that $X_g = X$ if $g = 1$ and $r(f) = r(1) = (1)$ implies f is a unit.)

iv): $X_f = X_g \implies V(f) = V(g) \implies r(f) = \cap_{\mathfrak{p}\in V(f)}\mathfrak{p} = \cap_{\mathfrak{p}\in V(g)}\mathfrak{p} = r(g)$ by Proposition 1.14 $\implies V(f) = V(g) \implies X_f = X_g$.

v): Follows from the more general statement vi) below with $f = 1$. ($X_f = X$ if $f = 1$.)

vi): As $(X_g)_{g\in A}$ form a basis of open sets for X, it suffices to show that if $X_f \subseteq \cup_{g\in E} X_g$ for some subset E of A, there exist a finitely many element $g_1, \dots, g_n \in E$ such that $X_f \subseteq \cup_{i=1}^n X_{g_i}$. Since $\cup_{g\in E} X_g = \cup_{g\in E}(X \setminus V(g)) = X \setminus V(E)$, we get that $X_f subseteq \cup_{g\in E} X_g \implies V(E) \subseteq V(f) \implies V(\mathfrak{a}) \subseteq V(f)$ by Exercise 1.15, part i), where $\mathfrak{a}$ is the ideal generated by E. But $V(\mathfrak{a}) \subseteq V(f)$ implies that $f \in \cap_{\mathfrak{p}\in V(f)}\mathfrak{p} \subseteq \cap_{\mathfrak{p}\in V(\mathfrak{a})}\mathfrak{p} = r(\mathfrak{a})$ by Proposition 1.14. Therefore $f^r \in \mathfrak{a}$ for some $r \in \mathbb{N}$, i.e. there exist $g_1, \dots, g_n \in E$ and $h_1, \dots, h_n \in A$ such that $g_1h_1 + \cdots + g_nh_n = f^r$, which implies that $f \in r(\mathfrak{b})$ where $\mathfrak{b}$ is the ideal generated by the subset $F = \{g_1, \dots, g_n\} \subseteq E$. But then, as $\mathfrak{p} \supseteq F \iff \mathfrak{p} \supseteq \mathfrak{b} \implies \mathfrak{p} \ni f^r \iff \mathfrak{p} \ni f$ for every prime ideal $\mathfrak{p}$ of A, we have $V(F) = V(\mathfrak{b}) \subseteq V(f)$, which inturn implies that $\cup_{i=1}^n X_{g_i} = X \setminus (\cap_{i=1}^n V(g_i)) = X \setminus V(F) \supseteq X \setminus V(f) = X_f$.

vii) If an open subset O of X is a finite union of X_f, then O is evidently quasi-compact since each X_f is quasi-compact. Conversely, we suppose that an open subset O of X is quasi-compact. Then, as all the X_f form a basis of open sets, there exist a subset E of A such that $O = \cup_{f\in E} X_f$. Therefore there exist finitely many $f_1, \dots, f_n \in E$ such that $O = \cup_{i=1}^n X_{f_i}$ by the quasi-compactness. □

Exercise 1.18. For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \operatorname{Spec}(A)$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$ (logically, of course, it is the same thing). Show that

i) The set $\{x\}$ is closed (we say that x is a "closed point") in $\operatorname{Spec}(A) \iff \mathfrak{p}_x$ is maximal;
ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$;
iii) $y \in \overline{\{x\}} \iff \mathfrak{p}_x \subseteq \mathfrak{p}_y$;
iv) X is a T_0-space (this means that if x, y are distinct points of X, then either there is a neighborhood of x which does not contain y, or there is a neighborhood of y which does not contain x).

Proof. ii): First we have $x = \mathfrak{p}_x \in V(\mathfrak{p}_x)$ as $\mathfrak{p}_x \subseteq \mathfrak{p}_x$, which implies that $\overline{\{x\}} \subseteq V(\mathfrak{p}_x)$ by the definition of closure. On the other hand, $\overline{\{x\}} = V(E)$ for some subset E of A by the definition of closed sets in $\operatorname{Spec}(A)$. Then $\mathfrak{p}_x = x \in \overline{\{x\}} = V(E)$ implies that $E \subseteq \mathfrak{p}_x$. Therefore for every $\mathfrak{p} \in V(\mathfrak{p}_x)$, we have $\mathfrak{p} \supseteq \mathfrak{p}_x \supseteq E$, which implies that $\mathfrak{p} \in V(E) = \overline{\{x\}}$. Hence $\overline{\{x\}} = V(\mathfrak{p}_x)$.

i): By part ii), the set $\{x\}$ is closed $\iff \{x\} = V(\mathfrak{p}_x) \iff \mathfrak{p}_x = x$ is the only prime ideal containing $\mathfrak{p}_x \iff \mathfrak{p}_x$ is maximal.

iii): By part ii) and the definition of $V(E)$ in Exercise 1.15, $y \in \overline{\{x\}} \iff y \in V(\mathfrak{p}_x) \iff \mathfrak{p}_x \subseteq \mathfrak{p}_y$.

iv): Let $x \neq y \in X$ be two arbitrary points of X. Then either $\mathfrak{p}_x \nsubseteq \mathfrak{p}_y$ or $\mathfrak{p}_y \nsubseteq \mathfrak{p}_x$. Say $\mathfrak{p}_x \nsubseteq \mathfrak{p}_y$. Then $y \notin \overline{\{x\}}$ by part iii), which reads that $X \setminus \overline{\{x\}}$ is a neighborhood of y which does not contain x. □

Exercise 1.19. A topological space X is said to be *irreducible* if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that Spec(A) is irreducible if and only if the nilradical of A is a prime ideal.

Proof. We first prove the 'if' part. Suppose that the nilradical $\mathfrak{N}$ is a prime ideal. Let O_1, O_2 be two arbitrary nonempty open subsets of $X = \mathrm{Spec}(A)$. Then, as the X_f form a basis of open sets for Spec(A), there exist $f_i \in A$ such that $\emptyset \neq X_{f_i} \subseteq O_i$ for $i = 1, 2$. Then $f_1 \notin \mathfrak{N}$ and $f_2 \notin \mathfrak{N}$ by Exercise 1.17. Therefore $f_1 f_2 \notin \mathfrak{N}$ since $\mathfrak{N}$ is a prime ideal. But this implies that $X_{f_1} \cap X_{f_2} = X_{f_1 f_2} \neq \emptyset$ by Exercise 1.17 again. Finally we have $O_1 \cap O_2 \neq \emptyset$ as $X_{f_1} \cap X_{f_2} \subseteq O_1 \cap O_2$.

To prove the 'only if' direction, let us suppose that Spec(A) is irreducible. For any two arbitrary elements $f, g \in A$, we have $f, g \notin \mathfrak{N} \implies X_f$ and X_g are non-empty open sets by Exercise 1.17 $\implies X_f \cap X_g = X_{fg} \neq \emptyset \implies fg \notin \mathfrak{N}$ by Exercise 1.17. Hence $\mathfrak{N}$ is a prime ideal. □

Exercise 1.21. Let $\phi : A \to B$ be a ring homomorphism. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, i.e. a point of X. Hence ϕ induces a mapping $\phi^* : Y \to X$. Show that

i) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ^* is continuous;
ii) If $\mathfrak{a}$ is an ideal of A, then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$;
iii) If $\mathfrak{b}$ is an ideal of B, then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$;
iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\ker(\phi))$ of X. (In particular, Spec(A) and Spec$(A/\mathfrak{N})$ (where $\mathfrak{N}$ is the nilradical of A) are naturally homeomorphic;)
v) If ϕ is injective, then $\phi^*(Y)$ is dense in X. More precisely, $\phi^*(Y)$ is dense in $X \iff \ker(\phi) \subseteq \mathfrak{N}$;
vi) Let $\psi : B \to C$ be another ring homeomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$;
vii) Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \to B$ by $\phi(x) = (\overline{x}, x)$, where $\overline{x}$ is the image of x in $A/\mathfrak{p}$. Show that ϕ^* is bijective but not a homeomorphism.

Proof. i): For a prime ideal $\mathfrak{q} \in Y = \mathrm{Spec}(B)$, we have

$$\mathfrak{q} \in \phi^{*-1}(X_f) \iff \phi^*(\mathfrak{q}) = \mathfrak{q}^c \in X_f \iff f \notin \mathfrak{q}^c = \phi^{-1}(\mathfrak{q}) \iff \phi(f) \notin \mathfrak{q} \iff \mathfrak{q} \in Y_{\phi(f)},$$

which proves that $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ^* is continuous since X_f form a basis of open sets for X.

ii): Similarly, for a prime ideal $\mathfrak{q} \in Y = \mathrm{Spec}(B)$, we have $\mathfrak{q} \in \phi^{*-1}(V(\mathfrak{a})) \iff \phi^*(\mathfrak{q}) = \mathfrak{q}^c \in V(\mathfrak{a}) \iff \mathfrak{a} \subseteq \mathfrak{q}^c = \phi^{-1}(\mathfrak{q}) \iff \phi(\mathfrak{a}) \subseteq \mathfrak{q} \iff \mathfrak{a}^e \subseteq \mathfrak{q} \iff \mathfrak{q} \in V(\mathfrak{a}^e)$, which proves that $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$, and, once again, that ϕ^* is continuous.

iii): For any $\mathfrak{p} \in \phi^*(V(\mathfrak{b}))$, there exists a prime ideal $\mathfrak{q} \in V(\mathfrak{b}) \subseteq \mathrm{Spec}(B)$ (i.e. $\mathfrak{b} \subseteq \mathfrak{q}$) such that $\phi^*(\mathfrak{q}) = \mathfrak{q}^c = \mathfrak{p}$, which implies that $\mathfrak{b}^c \subseteq \mathfrak{q}^c = \mathfrak{p}$, i.e. $\mathfrak{p} \in V(\mathfrak{b}^c) \subseteq \mathrm{Spec}(A)$. Therefore $\phi^*(V(\mathfrak{b})) \subseteq V(\mathfrak{b}^c)$, which implies that $\overline{\phi^*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c)$ as $V(\mathfrak{b}^c)$ is closed.

On the other hand, as $\overline{\phi^*(V(\mathfrak{b}))}$ is closed, we have $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A. Using the above result of ii), we get $V(\mathfrak{a}^e) = \phi^{*-1}(V(\mathfrak{a})) = \phi^{*-1}\left(\overline{\phi^*(V(\mathfrak{b}))}\right) \supseteq \phi^{*-1}(\phi^*(V(\mathfrak{b}))) \supseteq V(\mathfrak{b})$.

Therefore

$$\mathfrak{a}^e \subseteq \bigcap_{\mathfrak{q}\in V(\mathfrak{a}^e)} \mathfrak{q} \subseteq \bigcap_{\mathfrak{q}\in V(\mathfrak{b})} \mathfrak{q} = r(\mathfrak{b}).$$

So for any $x \in \mathfrak{a} \subseteq A$, we have $\phi(x) \in \mathfrak{a}^e \subseteq r(\mathfrak{b})$, which means that $\phi(x^n) = (\phi(x))^n \in \mathfrak{b}$ for some integer $n \in \mathbb{N}$, which implies that $x^n \in \phi^{-1}(\mathfrak{b}) = \mathfrak{b}^c$. Therefore $\mathfrak{a} \subseteq r(\mathfrak{b}^c)$, which proves that $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{a}) \supseteq V(r(\mathfrak{b}^c)) = V(\mathfrak{b}^c)$ by Exercise 1.15. Hence $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

iv): By Proposition 1.1 (generalized to the case of $\phi : A \to B$ where ϕ is surjective), we know that $\phi^*(Y) = V(\ker(\phi))$ and ϕ^* induces a bijective map from $Y = \mathrm{Spec}(B)$ to the closed subspace $V(\ker(\phi))$ of $X = \mathrm{Spec}(A)$ (which is still called ϕ^* by abuse of notation). We already know that ϕ^* is continuous by part i). To show that $\phi^* : Y \to V(\ker(\phi))$ is a homeomorphism, we only need to show that $\phi^*(Y')$ is closed in X (hence in $V(\ker(\phi))$) for every closed subsets Y' of Y. For every closed subset Y' of Y, there exists an ideal $\mathfrak{b}$ of B such that $Y' = V(\mathfrak{b})$ (see Exercise 1.15). Let $\mathfrak{a} = \phi^{-1}(\mathfrak{b})$. Then, for a prime ideal $\mathfrak{p}$ of A, we have $\mathfrak{p} \in \phi^*(Y') = \phi^*(V(\mathfrak{b})) \iff \mathfrak{p} = \phi^*(\mathfrak{q})$ for some $\mathfrak{q} \in V(\mathfrak{b})$ (i.e. $\mathfrak{b} \subseteq \mathfrak{q}$) $\iff \mathfrak{p} = \phi^{-1}(\mathfrak{q}) \supseteq \phi^{-1}(\mathfrak{b}) = \mathfrak{b}^c \iff \mathfrak{p} \in V(\mathfrak{b}^c)$, which means that $\phi^*(Y') = V(\mathfrak{b}^c)$ is closed in X (hence in $V(\ker(\phi))$). Therefore $\phi^* : Y \to V(\ker(\phi))$ is a homeomorphism.

In particular, let $\mathfrak{N}$ be the nilradical of A. Then there is a natural surjective homomorphism $\phi : A \to A/\mathfrak{N}$. Therefore $\phi^* : \mathrm{Spec}(A/\mathfrak{N}) \to V(\ker(\phi)) = V(\mathfrak{N})\,\mathrm{Spec}(A)$ is a homeomorphism.

v): We prove the general statement that "$\phi^*(Y)$ is dense in $X \Leftrightarrow \ker(\phi) \subseteq \mathfrak{N}$". By the above part iii), we have

$$\overline{\phi^*(Y)} = \overline{\phi^*(V(0))} = V(0^c) = V(\ker(\phi)).$$

Therefore $\phi^*(Y)$ is dense in $X \iff \overline{\phi^*(Y)} = V(\ker(\phi)) = X \iff \ker(\phi) \subseteq \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of $A \iff \ker(\phi) \subseteq \mathfrak{N}$.

vi): For any prime ideal $\mathfrak{q} \in \mathrm{Spec}(C)$ (i.e. $\mathfrak{q}$ is a prime ideal of C), we have $(\psi\circ\phi)^*(\mathfrak{q}) = (\psi\circ\phi)^{-1}(\mathfrak{q})$ and $\phi^*\circ\psi^*(\mathfrak{q}) = \phi^{-1}\left(\psi^{-1}(\mathfrak{q})\right)$. Then the desired result $(\psi\circ\phi)^* = \phi^*\circ\psi^*$ follows immediately from the fact that $(\psi\circ\phi)^*(\mathfrak{q}) = \phi^{-1}\left(\psi^{-1}(\mathfrak{q})\right)$.

vii): By assumption, A has exactly two prime ideals, 0 and $\mathfrak{p}$. Therefore $\mathfrak{p}$ is a maximal ideal of A, which implies $A/\mathfrak{p}$ is a field. Hence we conclude that the ring $B = (A/\mathfrak{p}) \times K$ also has exactly two ideals, namely $\mathfrak{q}_1 = \{(\overline{x}, 0) \mid x \in A\}$ and $\mathfrak{q}_2 = \{(\overline{0}, k) \mid k \in K\}$ (please check that they are prime ideals and there is no other prime ideal of B). It is easy to see that $\phi : A \to B$ defined by $\phi(x) = (\overline{x}, x)$ is a ring homomorphism. Direct checking will show that $\phi^*(\mathfrak{q}_1) = \phi^{-1}(\mathfrak{q}_1) = 0$ and $\phi^*(\mathfrak{q}_2) = \phi^{-1}(\mathfrak{q}_2) = \mathfrak{p}$. Therefore ϕ^* is bijective (and is always continuous).

But ϕ^* is not a homeomorphism. Indeed, in the topological space $\mathrm{Spec}(B) = \{\mathfrak{q}_1, \mathfrak{q}_2\}$, we have $\{\mathfrak{q}_1\} = V(\mathfrak{q}_1)$ is closed as $\mathfrak{q}_1 \not\subseteq \mathfrak{q}_2$. But $\phi^*(\{\mathfrak{q}_1\}) = \{0\}$ is not closed in $\mathrm{Spec}(A)$ since 0 is not a maximal ideal of A (see Exercise 1.18, part i). □

Note: The exercises are from '**Introduction to Commutative Algebra**' by M. F. Atiyah and I. G. Macdonald. All the quoted results are from the textbook unless different sources are quoted explicitly. For the convenience of the readers, the number of the chapter is included when a particular exercise is numbered. For example, **Exercise m.n** means the **Exercise n** from **Chapter m**.