## MATH 831 HOMEWORK SOLUTIONS – ASSIGNMENT 10

**Exercise 9.7.** Let $A$ be a Dedekind domain and $\mathfrak{a} \neq 0$ an ideal in $A$. Show that every ideal in $A/\mathfrak{a}$ is principal.

Deduce that every ideal in $A$ can be generated by at most 2 elements.

*Proof.* The fact that $A$ has dimension one and $\mathfrak{a} \neq 0$ implies that $A/\mathfrak{a}$ has Krull dimension zero, which, together with the implicit assumption that $A$ (hence $A/\mathfrak{a}$) is Noetherian, implies that $A/\mathfrak{a}$ is Artinian by Theorem 8.5. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ be all the minimal prime ideals over $\mathfrak{a}$. Then $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ are all the maximal ideals of $A$ that contain $\mathfrak{a}$ (since $\dim(A) = 1$) and therefore they account for all the prime ideals of $A/\mathfrak{a}$. By the structure theorem of Artinian rings, we know that

$$A/\mathfrak{a} \cong \prod_{i=1}^{n} (\frac{A}{\mathfrak{a}})_{\mathfrak{p}_i} \cong \prod_{i=1}^{n} \frac{A_{\mathfrak{p}_i}}{\mathfrak{a}_{\mathfrak{p}_i}}.$$

Notice that $A_{\mathfrak{p}_i}$ is a discrete valuation ring (hence principal) for every $1 \leq i \leq n$. This implies that $\frac{A_{\mathfrak{p}_i}}{\mathfrak{a}_{\mathfrak{p}_i}}$ is principal for every $1 \leq i \leq n$ and therefore $A/\mathfrak{a} \cong \prod_{i=1}^{n} \frac{A_{\mathfrak{p}_i}}{\mathfrak{a}_{\mathfrak{p}_i}}$ is principal. (In general, any direct product of finitely many principal rings is still principal.)

It remains to show that every ideal $\mathfrak{b}$ of $A$ can be generated by at most 2 elements. If $\mathfrak{b} = 0$, then there is nothing to prove. If $\mathfrak{b} \neq 0$, then choose $0 \neq x \in \mathfrak{b}$ and let $\mathfrak{a} = (x)$. Then what we have just proved shows that $A/(x)$ is principal and therefore $\mathfrak{b}/(x)$ can be generated by one element. Say $\mathfrak{b}/(x)$, as an ideal in $A/(x)$, is generated by $y + (x)$. Then it is easy to check that $\mathfrak{b} = (x, y)$ which is generated by two elements. □

**Exercise 9.9.** (Chinese Remainder Theorem). Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals and let $x_1, \ldots, x_n$ be elements in a Dedekind domain $A$. Then the system of congruences $x \equiv x_i \mod \mathfrak{a}_i$ $(1 \leq i \leq n)$ has a solution $x$ in $A$ $\iff$ $x_i \equiv x_j \mod \mathfrak{a}_i + \mathfrak{a}_j$ whenever $i \neq j$.

*Proof.* $\Rightarrow$: The system of congruences $x \equiv x_i \mod \mathfrak{a}_i$ $(1 \leq i \leq n)$ has a solution $x$ in $A$ simply says that there exists $x \in A$ such that $x - x_i \in \mathfrak{a}_i$ for $1 \leq i \leq n$. Therefore $x_i - x_j = (x - x_j) - (x - x_i) \in \mathfrak{a}_j + \mathfrak{a}_i = \mathfrak{a}_i + \mathfrak{a}_j$ whenever $i \neq j$.

$\Leftarrow$: We prove this direction by induction on $n$. There is nothing to prove when $n = 1$. For $n = 2$, the assumption that $x_1 \equiv x_2 \mod \mathfrak{a}_1 + \mathfrak{a}_2$ simply says that $x_1 - x_2 \in \mathfrak{a}_1 + \mathfrak{a}_2$, i.e. $x_1 - x_2 = a_1 + a_2$ with $a_1 \in \mathfrak{a}_1, a_2 \in \mathfrak{a}_2$. Then it is easy to check that $x := x_1 - a_1 = x_2 + a_2$ is a solution of the system of congruences $x \equiv x_i \mod \mathfrak{a}_i$ $(1 \leq i \leq 2)$.

Now let $n > 2$ and assume that the cases of less than $n$ ideals are all proved. Remember that we need to show that the system of congruences $x \equiv x_i \mod \mathfrak{a}_i$ $(1 \leq i \leq n)$ has a solution $x$ in $A$ provided that $x_i \equiv x_j \mod \mathfrak{a}_i + \mathfrak{a}_j$ whenever $i \neq j$. By induction hypothesis (for $n-1$ ideals), there exists $y \in A$ satisfying $y \equiv x_i \mod \mathfrak{a}_i$ $(1 \leq i \leq n-1)$. Next let us look at the following system of congruences (for 2 ideals)

$$(*) \qquad \begin{cases} x \equiv y & \mod \cap_{i=1}^{n-1} \mathfrak{a}_i \\ x \equiv x_n & \mod \mathfrak{a}_n. \end{cases}$$

Note that $y - x_n = (y - x_i) + (x_i - x_n) \in \mathfrak{a}_i + (\mathfrak{a}_i + \mathfrak{a}_n) = \mathfrak{a}_i + \mathfrak{a}_n$ for all $1 \leq i \leq n-1$. Hence $y - x_n \in \cap_{i=1}^{n-1}(\mathfrak{a}_i + \mathfrak{a}_n) = (\cap_{i=1}^{n-1}\mathfrak{a}_i) + \mathfrak{a}_n$ by Exercise 9.8 (assigned last time). Now, by induction hypothesis (for 2 ideals), the system of congruences $(*)$ has a solution $x$ in $A$.

Finally it is straightforward to check that $x$ is a solution of the system of congruences $x \equiv x_i \mod \mathfrak{a}_i$ ($1 \le i \le n$) by our choices of $y$ and system $(*)$. $\square$

**Exercise 10.3.** Let $A$ be a Noetherian ring, $\mathfrak{a}$ an ideal and $M$ a finitely generated $A$-module. Using Krull's Theorem and Exercise 14 of Chapter 3, prove that

$$\bigcap_{n=1}^{\infty} \mathfrak{a}^n M = \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \operatorname{Ker}(M \to M_{\mathfrak{m}}),$$

where $\mathfrak{m}$ runs over all maximal ideals containing $\mathfrak{a}$.

Deduce that

$$\widehat{M} = 0 \Leftrightarrow \operatorname{Supp}(M) \cap V(\mathfrak{a}) = \emptyset \qquad (\text{in } \operatorname{Spec}(A)),$$

where $\widehat{M}$ is the $\mathfrak{a}$-adic completion of $M$.

*Proof.* By Krull's Theorem (Theorem 10.17), the submodule $E = \cap_{n=1}^{\infty} \mathfrak{a}^n M$ is annihilated by some element of the form $1 + a$ with $a \in \mathfrak{a}$. Hence $E_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ containing $\mathfrak{a}$ since $1+a$ is a unit in $A_{\mathfrak{m}}$ if $\mathfrak{a} \subseteq \mathfrak{m}$. Therefore $\cap_{n=1}^{\infty} \mathfrak{a}^n M = E \subseteq \cap_{\mathfrak{m} \supseteq \mathfrak{a}} \operatorname{Ker}(M \to M_{\mathfrak{m}})$. Conversely, let $K = \cap_{\mathfrak{m} \supseteq \mathfrak{a}} \operatorname{Ker}(M \to M_{\mathfrak{m}})$. Then $K_{\mathfrak{m}} = 0$ for all maximal ideals containing $\mathfrak{a}$, which implies that $K = \mathfrak{a}K$ by Exercise 3.14 (assigned). Hence $K = \mathfrak{a}K = \mathfrak{a}^2 K = \cdots = \mathfrak{a}^n = \cdots \subseteq \cap_{n=1}^{\infty} \mathfrak{a}^n M$. Therefore $\cap_{n=1}^{\infty} \mathfrak{a}^n M = \cap_{\mathfrak{m} \supseteq \mathfrak{a}} \operatorname{Ker}(M \to M_{\mathfrak{m}})$.

To prove the remaining part of the proof, first notice that $\widehat{M} = 0 \overset{\text{NAK}}{\iff} \widehat{M} = \widehat{\mathfrak{a}}\widehat{M} \overset{\text{Proposition 10.15}}{\iff} M = \mathfrak{a}M \iff M = \cap_{n=1}^{\infty} \mathfrak{a}^n M$, i.e. $M = \cap_{\mathfrak{m} \supseteq \mathfrak{a}} \operatorname{Ker}(M \to M_{\mathfrak{m}})$, i.e. $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ containing $\mathfrak{a} \iff \operatorname{Supp}(M) \cap V(\mathfrak{a})$ contains no maximal ideal $\iff \operatorname{Supp}(M) \cap V(\mathfrak{a}) = \emptyset$. $\square$

**Exercise 10.4.** Let $A$ be a Noetherian ring, $\mathfrak{a}$ an ideal in $A$, and $\widehat{A}$ the $\mathfrak{a}$-adic completion. For any $x \in A$, let $\widehat{x}$ be the image of $x$ in $\widehat{A}$. Show that

$$x \text{ not a zero-divisor in } A \Rightarrow \widehat{x} \text{ not a zero-divisor in } \widehat{A}$$

Does this imply that

$$A \text{ is an integral domain} \Rightarrow \widehat{A} \text{ is an integral domain?}$$

*Proof.* The assumption that $x$ not a zero-divisor in $A$ says that the sequence $0 \to A \xrightarrow{x} A$ is exact. Since $\widehat{A}$ is flat over $A$ (see Proposition 10.14), we have an exact sequence $0 \to \widehat{A} \xrightarrow{\widehat{x}} \widehat{A}$, which says that $\widehat{x}$ is not a zero-divisor in $\widehat{A}$.

We have an example of an integral domain whose completion is not an integral domain. This example is taken from Eisenbud's book *Commutative Algebra,* page 187–188. Let $R = k[x, y]$ where $k$ is a field of characteristic zero and $\mathfrak{m} = (x, y)$. Then the completion of $R$ with respect to $\mathfrak{m}$ is $\widehat{R} \cong k[[x, y]]$. Then let $A = k[x, y]/(y^2 - x^2 - x^3)$ and $\overline{\mathfrak{m}} = (\overline{x}, \overline{y}) \subset A$ where $\overline{x}, \overline{y}$ are the corresponding elements in $A$. Then the completion of $A$ with respect to $\overline{\mathfrak{m}}$ is $\widehat{A} \cong A \otimes_R \widehat{R} \cong k[[x, y]]/(y^2 - x^2 - x^3)$. First it easy to see that $y^2 - x^2 - x^3$ is irreducible in $R$ and therefore it is a prime element as $R$ is a UFD. Hence $A$ is an integral domain. To see that $\widehat{A} \cong k[[x, y]]/(y^2 - x^2 - x^3)$ is not an integral domain, we make a claim that there exists $f \in k[[x, y]]$ such that $f^2 = 1 + x$. Assuming the claim, we have that

$$y^2 - x^2 - x^3 = y^2 - x^2(1 + x) = y^2 - (xf)^2 = (y + xf)(y - xf),$$

which shows that $y^2 - x^2 - x^3$ is not a prime in $k[[x, y]]$. Therefore $\widehat{A}$ is not an integral domain.

It remains to show that there exists $f \in k[[x, y]]$ such that $f^2 = 1 + x$. To that end, it suffices to find $a_0, a_1, \ldots \in k$ such that

$$(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)^2 = 1 + x.$$

To make it explicit, we need $a_0^2 = 1, 2a_0a_1 = 1, 2a_0a_2 + a_1^2 = 0, \dots$ etc. But it is easy to solve this inductively: $a_0 = 1, a_1 = 1/2, a_2 = -1/8, \dots$ on and on. Therefore there exists $f = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \in k[[x]] \subset k[[x, y]]$ satisfying $f^2 = 1 + x$. □

**Exercise 10.6.** Let $A$ be a Noetherian ring and $\mathfrak{a}$ an ideal in $A$. Prove that $\mathfrak{a}$ is contained in the Jacobson radical of $A$ if and only if every maximal ideal of $A$ is closed for the $\mathfrak{a}$-topology.

*Proof.* Recall that, for an element $x \in A$ and an ideal $\mathfrak{a}$ of $A$, the coset $x + \mathfrak{a} = \{x + a \mid a \in \mathfrak{a}\}$ is a subset of $A$.

Suppose that $\mathfrak{a}$ is contained in the Jacobson radical of $A$. Let $\mathfrak{m}$ be an arbitrary maximal ideal in $A$. Then $\mathfrak{a} \subseteq \mathfrak{m}$. For any $x \notin \mathfrak{m}$, it is straightforward to check that $(x + \mathfrak{a}) \cap \mathfrak{m} = \emptyset$. As $x + \mathfrak{a}$ is a open subset of $A$ containing $x$, we conclude that $x$ in not in the closure of $\mathfrak{m}$. Hence $\mathfrak{m}$ is closed.

Suppose that $\mathfrak{a}$ is not contained in the Jacobson radical of $A$. Then there exists a maximal ideal $\mathfrak{m}$ of $A$ such that $\mathfrak{a} \not\subseteq \mathfrak{m}$. Hence $\mathfrak{a}^n \not\subseteq \mathfrak{m}$ for all $n \geq 0$. This forces $\mathfrak{a}^n + \mathfrak{m} = (1)$ for all $n \geq 0$. Hence there exist $a_n \in \mathfrak{a}^n, m_n \in \mathfrak{m}$ such that $a_n + m_n = 1$, i.e. $1 - a_n = m_n \in \mathfrak{m}$, i.e. $(1 + \mathfrak{a}^n) \cap \mathfrak{m} \neq \emptyset$ for every $n \geq 0$. Since $\{1 + \mathfrak{a}^n \mid n \geq 0$ forms a neighborhood basis for element 1, we conclude that 1 is in the closure of $\mathfrak{m}$. So $\mathfrak{m}$ is not closed. □

**Note:** The exercises are from '**Introduction to Commutative Algebra**' by M. F. Atiyah and I. G. Macdonald. All the quoted results are from the textbook unless different sources are quoted explicitly. For the convenience of the readers, the number of the chapter is included when a particular exercise is numbered. For example, **Exercise m.n** means the **Exercise n** from **Chapter m**.