MATH 831 HOMEWORK SOLUTIONS – ASSIGNMENT 1

Exercise 1.1. Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Proof. Say $x^n = 0$ for some $n > 0$. Then, by direct computation, we see that

$$(1+x)(1-x+x^2+\cdots+(-1)^n x^n) = 1+(-1)^n x^{n+1} = 1+0 = 1,$$

i.e. $1+x$ is a unit in A.

Next let $a \in A$ be a unit (therefore a^{-1} exists) and $x \in A$ a nilpotent (i.e. $x^n = 0$ for some $n > 0$). We need to show that $a + x = a(1 + a^{-1}x)$ is a unit. Notice that $x^n = 0$ implies $(a^{-1}x)^n = 0$. Therefore $a^{-1}x$ is a nilpotent and hence $1 + a^{-1}x$ is a unit by the first part of the proof. Finally we deduce that $a + x = a(1 + a^{-1}x)$ is a unit as any product of units is still a unit. □

Exercise 1.2. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

i) f is a unit in $A[x]$ $\iff$ a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent;
ii) f is nilpotent $\iff$ $a_0, a_1, \ldots, a_n$ are nilpotent;
iii) f is a zero-divisor $\iff$ there exists $a \neq 0$ in A such that $af = 0$;
iv) f is said to be primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\iff$ f and g are primitive.

Proof. i): If f is a unit, then there exist $g = b_0 + \cdots + b_mx^m \in A[x]$ such that $fg = 1$. Direct computation shows that $a_0b_0 = 1$, which proves that a_0 is a unit in A. To prove that a_i are nilpotent for all $i = 1, 2, \ldots, n$, it suffices to show $a_i \in \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of A. For any arbitrarily chosen and then fixed prime ideal $\mathfrak{p}$, we denote the quotient ring $A/\mathfrak{p}$ by $\overline{A}$ and denote an element $a + \mathfrak{p} \in \overline{A}$ by $\overline{a}$. Then there is a ring homomorphism $\phi : A[x] \to \overline{A}[x]$ defined by $\phi(c_0 + \cdots + c_rx^r) = \overline{c_0} + \cdots + \overline{c_r}x^r$. Since $f = a_0 + a_1x + \cdots + a_nx^n$ is a unit in $A[x]$, $\phi(f) = \overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n$ is a unit in $\overline{A}[x]$. Now notice that $\overline{A} = A/\mathfrak{p}$ is an integral domain. In general, if R is an integral domain, then every unit in $R[x]$ has to have degree 0 (check for yourself). Therefore our polynomial $\phi(f) = \overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n$ has to have degree 0, i.e. $\overline{a_i} = a_i + \mathfrak{p}$ are zero in $\overline{A} = A/\mathfrak{p}$ for all $i = 1, 2, \ldots, n$, which is the same as saying $a_i \in \mathfrak{p}$ for all $i = 1, 2, \ldots, n$. Since the prime ideal $\mathfrak{p}$ is arbitrary, we conclude that a_i are nilpotent for all $i = 1, 2, \ldots, n$.

Conversely, we assume a_0 is a unit and $a_1, \ldots, a_n$ are nilpotent. Then $a_i^m = 0$ for a large enough $m \in \mathbb{N}$ and all $i = 1, 2, \ldots, n$. Then direct computation gives that $(a_1x+\cdots+a_nx^n)^{n(m-1)+1} = 0$. That is to say $(a_1x + \cdots + a_nx^n)$ is nilpotent in $A[x]$. Finally $f = a_0 + (a_1x + \cdots + a_nx^n)$ is a unit by Exercise 1.1.

ii): If f is nilpotent, so is $xf = a_0x + a_1x^2 + \cdots + a_nx^{n+1} \in A[x]$. Then $1 + xf$ is a unit in $A[x]$ by Exercise 1.1. As $1+xf = 1+a_0x+a_1x^2+\cdots+a_nx^{n+1}$, we conclude that $a_0, a_1, \ldots, a_n$ are all nilpotent by part i).

Conversely, if $a_0, a_1, \ldots, a_n$ are all nilpotent, there exists a large enough $m \in \mathbb{N}$ such that $a_i^m = 0$ for all $i = 0, 1, \ldots, n$. Similarly we can show that $f^{(n+1)(m-1)+1} = (a_0 + a_1x + \cdots + a_nx^n)^{(n+1)(m-1)+1} = 0$ by direct computation.

iii): The direction $\Longleftarrow$ is evident. To prove the $\Longrightarrow$ direction, we choose a particular $g \in \{h \in A[x] \mid h \neq 0, hf = 0\} \neq \emptyset$ with minimal degree. Say $g = b_0 + b_1x + \cdots + b_mx^m$ with

$b_m \neq 0$. We claim that $b_m f = 0$. If not, there exists an largest integer $r \in \{0, 1, \ldots, n\}$ such that $b_m a_r \neq 0$ (so that $b_m a_i = 0$ for all $i = r+1, \ldots, n$). Then, for every $i = r+1, \ldots, n$, we have $a_i g f = 0$ and the degree of $a_i g = a_i(b_0 + b_1 x + \cdots + b_m x^m) = a_i b_0 + a_i b_1 x + \cdots + a_i b_{m-1} x^{m-1}$ less than the degree of g. By our choice of g, we know that $a_i g = 0$ for all $i = r+1, \ldots, n$. But then we have $0 = fg = (a_0 + \cdots + a_r x^r + a_{r+1} x^{r+1} + \cdots + a_n x^n) g = (a_0 + \cdots + a_r x^r) g = (a_0 + \cdots + a_r x^r)(b_0 + \cdots + b_m x^m)$, which forces $a_r b_m = 0$, which is a contradiction.

iv): First let us observe that an ideal $I = (1) \iff I \not\subseteq \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of A. For any such maximal ideal $\mathfrak{m}$, we denote the quotient ring $A/\mathfrak{m}$ by $\overline{A}$, an element $a + \mathfrak{m} \in \overline{A}$ by $\overline{a}$ and the natural ring homomorphism $A[x] \to \frac{A}{\mathfrak{m}}[x] = \overline{A}[x]$ by $\phi_{\mathfrak{m}}$.

Second we observe that a polynomial $f = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$ is primitive $\iff (a_0, a_1, \ldots, a_n) = (1) \iff$ for every maximal ideal $\mathfrak{m}$, there exists at least one i such that $\overline{a_i} \neq 0 \in \overline{A} \iff \phi_{\mathfrak{m}}(f) \neq 0 \in \overline{A}[x]$ for every maximal ideal $\mathfrak{m}$.

Third, just as in part i), we notice that $\overline{A}[x]$ is an integral domain for every maximal ideal $\mathfrak{m}$.

Now we have fg is primitive $\iff \phi_{\mathfrak{m}}(fg) = \phi_{\mathfrak{m}}(f)\phi_{\mathfrak{m}}(g) \neq 0 \in \overline{A}[x]$ for every maximal ideal $\mathfrak{m} \iff \phi_{\mathfrak{m}}(f) \neq 0$ and $\phi_{\mathfrak{m}}(g) \neq 0$ for every maximal ideal $\mathfrak{m} \iff f$ and g are both primitive. □

Exercise 1.3. Generalize the results of Exercise 1.2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.

Solution. We state the generalizations without providing proof.

First let us set up notations. Every polynomial $f \in A[x_1, \ldots, x_r]$ can be written as

$$f = \sum_{n_1, \ldots, n_r \in \mathbb{N}} a_{(n_1, \ldots, n_r)} x_1^{n_1} \cdots x_r^{n_r},$$

in which $a_{(n_1, \ldots, n_r)} \neq 0 \in A$ for only finitely many $(n_1, \ldots, n_r)$. Notice $A[x_1, \ldots, x_r]$ can be naturally viewed as $A[x_1][x_2, \ldots, x_r]$, a polynomial ring over $A[x_1]$ with $r-1$ indeterminates $x_2, \ldots, x_r$ (so we may use induction). Accordingly, we can rearrange the terms of the above polynomial f so that

$$f = \sum_{n_2, \ldots, n_r \in \mathbb{N}} \left(\sum_{n_1 \in \mathbb{N}} a_{(n_1, \ldots, n_r)} x_1^{n_1} \right) x_2^{n_2} \cdots x_r^{n_r}.$$

Furthermore we say that f is primitive if the set $\{a_{(n_1, \ldots, n_r)} \mid n_1, \ldots, n_r \in \mathbb{N}\}$ generates the unit ideal (1).

Then the following statements are true:

i) f is a unit in $A[x_1, \ldots, x_r] \iff a_{(0, \cdots 0)}$ is a unit in A and $a_{(n_1, \ldots, n_r)}$ are nilpotent for all $(n_1, \ldots, n_r) \neq (0, \cdots 0)$;
ii) f is nilpotent $\iff a_{(n_1, \ldots, n_r)}$ are nilpotent for all $(n_1, \ldots, n_r)$;
iii) f is a zero-divisor $\iff$ there exists $a \neq 0$ in A such that $af = 0$;
iv) For $f, g \in A[x_1, \ldots, x_r]$, fg is primitive $\iff f$ and g are primitive.

□

Sketch of proof. i): Prove by induction on r, the number of the indeterminates. Use the results in Exercise 1.2, part i) and ii).

ii): Prove by induction on r, the number of the indeterminates. Use the results in Exercise 1.2, part ii).

iii): The direction $\Leftarrow$ is evident. To prove the $\Rightarrow$ direction, we use induction on r, the number of the indeterminates. Then we may mimic the proof of Exercise 1.2, part iii) to find $a \neq 0$ in A such that $af = 0$.

iv): We may mimic the proof of Exercise 1.2, part iv) almost word by word. The only difference is that our ring now has several indeterminates instead of one. □

Exercise 1.4. In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Proof. The nilradical $\mathfrak{N}$ is always contained in the Jacobson radical $\mathfrak{R}$ as $\mathfrak{N}$ is the intersection of all prime ideals while $\mathfrak{R}$ is the intersection of all maximal ideals. This is true for any ring.

On the other hand, if $f = a_0 + a_1 x + \cdots + a_n x^n$ is in the Jacobson radical $\mathfrak{R}$ of $A[x]$, then $1 + xf$ is a unit in $A[x]$ (see Proposition 1.9). As $1 + xf = 1 + a_0 x + a_1 x^2 + \cdots + a_n x^{n+1}$, we conclude that $a_0, a_1, \ldots, a_n$ are all nilpotent by Exercise 1.2, part i). But then $f = a_0 + a_1 x + \cdots + a_n x^n$ is a nilpotent by Exercise 1.2, part ii), i.e. $f \in \mathfrak{N}$. Therefore $\mathfrak{R} = \mathfrak{N}$ in $A[x]$. □

Exercise 1.7. Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Proof. Let $\mathfrak{p}$ be an arbitrary prime ideal of A. We need to prove that the only ideal of A properly containing $\mathfrak{p}$ is the unit ideal. Let $\mathfrak{a}$ be an such ideal, i.e. $\mathfrak{p} \subsetneq \mathfrak{a}$. Then there exists an element $x \in \mathfrak{a} \setminus \mathfrak{p}$. By assumption, $x^n = x$ for some $n > 1$. That is to say that $x(1 - x^{n-1}) = 0 \in \mathfrak{p}$, which implies $(1 - x^{n-1}) \in \mathfrak{p} \subsetneq \mathfrak{a}$ since $\mathfrak{p}$ is prime and $x \notin \mathfrak{p}$. But then we have $1 = (1 - x^{n-1}) + x^{n-1} \in \mathfrak{a}$ as $n - 1 \geq 1$. Hence $\mathfrak{a} = (1)$ is the unit ideal. □

Alternative proof of Exercise 1.7. Let $\mathfrak{p}$ be an arbitrary prime ideal of A. To show that $\mathfrak{p}$ is maximal, it enough to prove that $A/\mathfrak{p}$ is a field, i.e. every non-zero element of $A/\mathfrak{p}$ is a unit. Since there is a surjective ring homomorphism from A to $A/\mathfrak{p}$, every element $\overline{x} \in A/\mathfrak{p}$ satisfies $\overline{x}^n = \overline{x}$ for some $n > 1$ (depending on $\overline{x}$). But if $\overline{x} \neq 0 \in A/\mathfrak{p}$, then $\overline{x}^n = \overline{x}$ implies $\overline{x}^{n-1} = 1 \in A/\mathfrak{p}$ since $A/\mathfrak{p}$ is an integral domain, which implies $\overline{x}$ is a unit as $n - 1 \geq 1$. □

Note: The exercises are from '**Introduction to Commutative Algebra**' by M. F. Atiyah and I. G. Macdonald. All the quoted results are from the textbook unless different sources are quoted explicitly. For the convenience of the readers, the number of the chapter is included when a particular exercise is numbered. For example, **Exercise m.n** means the **Exercise n** from **Chapter m**.