ON THE VANISHING OF (CO)HOMOLOGY FOR MODULES ADMITTING CERTAIN FILTRATIONS

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ABSTRACT. We study the vanishing of (co)homology along ring homomorphisms for modules that admit certain filtrations and generalize a theorem of Celikbas-Takahashi. Our work produces new classes of rigid and test modules, particularly over local rings of prime characteristic. Additionally, it provides applications in the study of torsion in tensor products of modules, including a conjecture of Huneke-Wiegand.

1. INTRODUCTION

Throughout the paper, all rings are assumed to be commutative and Noetherian. By a local ring, we mean a ring with a unique maximal ideal. The research in this paper has been initiated by the following result:

Theorem 1.1 (Celikbas-Takahashi [10]). *Let R be a local ring and let M be a finitely generated R-module such that* $\operatorname{depth}_R(M) \geqslant 1$. *If* $\operatorname{Tor}_n^R(\mathfrak{m}^t M, N) = 0$ *for some finitely generated R-module N, where $t \geqslant 0$ and $n \geqslant 0$, then* $\operatorname{Tor}_n^R(M,N) = 0$.

The main purpose of this paper is to extend Theorem 1.1 by considering its conclusion along ring homomorphisms. Our main result, which extends Theorem 1.1, is concerned with the vanishing of (co)homology for modules (not necessarily finitely generated) that admit a certain filtration. More precisely, we have:

Theorem 1.2. *Let $f : R \to S$ be a ring homomorphism and let $M \neq 0$ be a finitely generated S-module. Assume the following hold:*

(i) *$(R, \mathfrak{m})$ is local.*
(ii) *S is Noetherian and $\mathfrak{m}S \subseteq \operatorname{Jac}(S)$.*
(iii) *There is an element of $\mathfrak{m}$ which is a non zero-divisor on M.*
(iv) *There is a filtration of M of the form $M_t \leqslant M_{t-1} \leqslant \cdots \leqslant M_1 \leqslant M_0 = M$, where each M_i is an S-module, and $\mathfrak{m}M_{i-1} \leqslant M_i \leqslant \operatorname{Jac}(S)M_{i-1}$ for all $i = 1, \ldots, t$.*

If $\operatorname{Tor}_n^R(M_i, N) = 0$*, where N is a finitely generated R-module, $n \geqslant 0$, and $1 \leqslant i \leqslant t$, then it follows*

$$\operatorname{Tor}_n^R(M_{i-1}, N) = \operatorname{Tor}_n^R(M_{i-2}, N) = \cdots = \operatorname{Tor}_n^R(M, N) = 0.$$

Theorem 1.2 is subsumed by Theorem 2.3, which is proved in section 2; see Theorem 2.4 for an Ext version of the theorem. Let us note here that it is not difficult to reprove Theorem 1.1 by using Theorem 1.2: if $R = S$ and f is the identity map, then our assumption on the filtration of M forces $M_i = \mathfrak{m}^i M$ for all $i \geqslant 0$, and hence Theorem 1.1 follows.

Let R be a ring of prime characteristic p and let F be the Frobenius map. For each R-module M, each iteration F^e defines a new R-module structure on M, denoted by ${}^eM$, where $r \cdot x = r^{p^e} x$ for $r \in R$ and $x \in M$. For a given ideal $I = (x_1, \ldots, x_r)$ of R, we define $I^{[p^e]} = (x_1^{p^e}, \ldots, x_r^{p^e})$ for each $e \geqslant 1$.

Assume $(R, \mathfrak{m})$ is local ring of prime characteristic p. Then hypothesis (iv) in Theorem 1.2 implies that $\mathfrak{m}^{[p^e]} M_{i-1} \leqslant M_i \leqslant \mathfrak{m}M_{i-1}$ for all $i = 1, \ldots, t$. One way to obtain such a filtration is to assume there are ideals $I_1, \ldots, I_t$ of R such that $\mathfrak{m}^{[p^e]} \subseteq I_j \subseteq \mathfrak{m}$ for all $j = 1, \ldots, t$, and set $M_i = (I_1 \cdots I_i)M$ for all $i = 1, \ldots, t$, which would imply $M_i = I_i M_{i-1}$. Hence we can apply Theorem 1.2 for the special case where $f = F^e$ and conclude:

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Corollary 1.3. *Let $(R,\mathfrak{m})$ be a local ring of prime characteristic p, and let M and N be nonzero finitely generated R-modules such that* $\operatorname{depth}_R(M) \geqslant 1$. *Let $I_1,\dots,I_t$ be ideals of R such that $\mathfrak{m}^{[p^e]} \subseteq I_j \subseteq \mathfrak{m}$ for all $j=1,\dots,t$, where $e \geqslant 0$ and $t \geqslant 1$. If $\operatorname{Tor}_n^R({}^e\big((I_1\cdots I_i)M\big),N)=0$ for some $n \geqslant 1$ and for some i, where $1 \leqslant i \leqslant t$, then it follows:*

$$\operatorname{Tor}_n^R({}^e\big((I_1\cdots I_i)M\big),N)=\operatorname{Tor}_n^R({}^e\big((I_1\cdots I_{i-1})M\big),N)=\cdots=\operatorname{Tor}_n^R({}^e\big(I_1M\big),N)=\operatorname{Tor}_n^R({}^eM,N)=0.$$

In Section 3, we show that each filtration module M_i in Theorem 1.2, with $1 \leqslant i \leqslant t$, is a big test module over R; see Definition 3.1 and Corollary 3.3. If M is big Tor-rigid over R, then each filtration module M_i in Theorem 1.2, with $1 \leqslant i \leqslant t$, is a big rigid-test module over R; see Corollary 3.5. We make use of this fact and obtain new classes of big test modules, for example, over rings of prime characteristic p. One such result is the following proposition, which is proved in Corollary 3.10(iii) in light of 3.8(ii).

Proposition 1.4. *Let $(R,\mathfrak{m})$ be a local complete intersection ring of prime characteristic p with positive depth, $e \geqslant 0$, and let $t \geqslant 1$. If $\operatorname{Tor}_n^R\left({}^e(\mathfrak{m}^t),N\right)=0$ or $\operatorname{Tor}_n^R\left({}^e\Big(\big(\mathfrak{m}^{[p^e]}\big)^t\Big),N\right)=0$ for some finitely generated R-module N and $n \geqslant 1$, then $\operatorname{pd}_R(N) \leqslant n$.*

2. PROOF OF THEOREM 1.2

The following proposition is key to proving Theorem 1.2. A module M (not necessarily finitely generated) over a local ring $(R,\mathfrak{m})$ is said to have positive *grade*, denoted by $\operatorname{grade}_R(M) \geqslant 1$, if $\operatorname{Hom}_R(R/\mathfrak{m},M)=0$; when M is finitely generated, we also write *depth* instead of grade; see [5, 1.2.3].

Proposition 2.1. *Let $f: R \to S$ be a ring homomorphism such that $(R,\mathfrak{m})$ is local and S is Noetherian. Let $M \neq 0$ be a finitely generated S-module, $t \geqslant 1$, and assume the following conditions hold:*

(i) $\mathfrak{m}S \subseteq \operatorname{Jac}(S)$.
(ii) *There is a filtration of M of the form $M_t \leqslant M_{t-1} \leqslant \cdots \leqslant M_1 \leqslant M_0 = M$, where each M_i is an S-module and $\mathfrak{m}M_{i-1} \leqslant M_i \leqslant \operatorname{Jac}(S)M_{i-1}$ for all $i=1,\dots,t$.*

Let $F_\bullet$ be a minimal complex of finitely generated free R-modules:

$$F_\bullet : \qquad \cdots \longrightarrow F_{n+1} \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots$$

(1) *Assume $H_n(M_i \otimes_R F_\bullet)=0$ for some integer n and some i with $1 \leqslant i \leqslant t$. Then $1_{M_{i-1}} \otimes f_{n+1}=0$. Moreover, if $\operatorname{grade}_R(M) \geqslant 1$, then $H_n(M_{i-1}\otimes_R F_\bullet)=0$.*
(2) *If $H_n(M_i \otimes_R F_\bullet)=0$ for some integer n and M_{i-1} is faithful as an R-module for some i with $1 \leqslant i \leqslant t$, then f_{n+1} is the zero map.*
(3) *If $M_i \neq 0$ and $H_{n-1}(M_i \otimes_R F_\bullet)=H_n(M_i\otimes_R F_\bullet)=0$ for some integers n and i with $1 \leqslant i \leqslant t$, then $F_n=0$.*

Proof. It suffices to prove the case when $i=1$ (note that $\operatorname{grade}_R(M) \geqslant 1 \implies \operatorname{grade}_R(M_i) \geqslant 1$ for all $i=1,\dots,t$). Tensoring $F_\bullet$ with M over R, we obtain the complex

$$M\otimes_R F_\bullet : \qquad \cdots \longrightarrow C_{n+1} \xrightarrow{g_{n+1}} C_n \xrightarrow{g_n} C_{n-1} \longrightarrow \cdots$$

where $C_i = M\otimes_R F_i$ and $g_i = 1_M \otimes f_i$. Similarly, tensoring $F_\bullet$ with M_1 over R, we obtain the complex

$$M_1\otimes_R F_\bullet : \qquad \cdots \longrightarrow D_{n+1} \xrightarrow{h_{n+1}} D_n \xrightarrow{h_n} D_{n-1} \longrightarrow \cdots.$$

where $D_i = M_1 \otimes_R F_i$ and $h_i = 1_{M_1}\otimes f_i$.

As F_i is free over R, we may assume $D_i \leqslant C_i$ for all i and $D_\bullet$ is a subcomplex of $C_\bullet$. Since $F_\bullet$ is minimal over $(R,\mathfrak{m})$ and $\mathfrak{m}M \leqslant M_1$, we see that $\operatorname{im}(g_{n+1}) \leqslant \mathfrak{m}C_n \leqslant D_n$.

(1) Assume $H_n(M_1\otimes_R F_\bullet)=0$, that is, $\ker(h_n)=\operatorname{im}(h_{n+1})$. Notice $D_{n+1} \leqslant \operatorname{Jac}(S)C_{n+1}$ as $M_1 \leqslant \operatorname{Jac}(S)M$. Therefore $\operatorname{im}(h_{n+1}) \leqslant \operatorname{Jac}(S)\operatorname{im}(g_{n+1})$. Putting it all together, we see

$$(*) \qquad \operatorname{im}(g_{n+1}) \leqslant \ker(g_n)\cap \mathfrak{m}C_n \leqslant \ker(g_n)\cap D_n = \ker(h_n) = \operatorname{im}(h_{n+1}) \leqslant \operatorname{Jac}(S)\operatorname{im}(g_{n+1}) \leqslant \operatorname{im}(g_{n+1}),$$

which shows that $\operatorname{Jac}(S)\operatorname{im}(g_{n+1})=\operatorname{im}(g_{n+1})$. As C_i is a finitely generated S-module for each i, we conclude that $\operatorname{im}(g_{n+1})=0$ by Nakayama's lemma. This proves that the function $1_M\otimes f_{n+1}$ is zero. Moreover, we see from $(*)$ that $\ker(g_n)\cap \mathfrak{m}C_n = 0$, which implies that $\mathfrak{m}\ker(g_n) \leqslant \ker(g_n)\cap \mathfrak{m}C_n = 0$. If $\operatorname{grade}_R(M)\geqslant 1$ (so that $\operatorname{grade}_R(C_n)\geqslant 1$), we must have $\ker(g_n)=0$, and hence $H_n(M\otimes_R F_\bullet)=\ker(g_n)/\operatorname{im}(g_{n+1})=0$.

(2) Assume $H_n(M_1\otimes_R F_\bullet)=0$ for some integer n and M is faithful as an R-module. From (1) above, we see $1_M\otimes f_{n+1}=0$, which implies that the entries of a matrix representing f_{n+1} are all contained in the annihilator of M. Therefore f_{n+1} is the zero map.

(3) Assume $M_1\neq 0$ and $H_{n-1}(M_1\otimes_R F_\bullet)=H_n(M_1\otimes_R F_\bullet)=0$ for some integer n. It follows from part (1) that $1_M\otimes f_n=0$ and $1_M\otimes f_{n+1}=0$. Since $D_\bullet$ is viewed as a subcomplex of $C_\bullet$, we obtain $1_{M_1}\otimes f_n=0$ and $1_{M_1}\otimes f_{n+1}=0$. Now, from the assumption that $H_n(M_1\otimes_R F_\bullet)=0$, we see

$$M_1\otimes_R F_n=\ker(1_{M_1}\otimes f_n)=\operatorname{im}(1_{M_1}\otimes f_{n+1})=0.$$

As F_n is free over R and $M_1\neq 0$, this implies that $F_n=0$. □

Remark 2.2. The general idea used to prove the first two parts of Proposition 2.1 stems from the work of Levin and Vascencelos [20, Lemma, page 316] (see also [10, the proof of 1.2 and 2.2]). Moreover, our proof in part (3) of the proposition is similar to some of the arguments in the proof of [13, 2.5].

Theorem 1.2, which is advertised in the introduction, is subsumed by the next result.

Theorem 2.3. *Let $f: R\to S$ be a ring homomorphism such that $(R,\mathfrak{m})$ is local and S is Noetherian. Let $M\neq 0$ be a finitely generated S-module, $t\geqslant 1$, and assume the following conditions hold:*

(i) $\mathfrak{m}S\subseteq \operatorname{Jac}(S)$.

(ii) *There is a filtration of M of the form $M_t\leqslant M_{t-1}\leqslant\cdots\leqslant M_1\leqslant M_0=M$, where each M_i is an S-module and $\mathfrak{m}M_{i-1}\leqslant M_i\leqslant \operatorname{Jac}(S)M_{i-1}$ for all $i=1,\dots,t$.*

Then, for each $n\geqslant 0$, and $1\leqslant i\leqslant t$, and for each finitely generated R-module N, we have the following:

(a) *If $\operatorname{grade}_R(M)\geqslant 1$ and $\operatorname{Tor}_n^R(M_i,N)=0$, then $\operatorname{Tor}_n^R(M_{i-1},N)=0$.*

(b) *If $\operatorname{Tor}_n^R(M_i,N)=0$ and M_{i-1} is a faithful R-module, then $\operatorname{pd}_R(N)\leqslant n$.*

(c) *Assume $M_i\neq 0$ and $\operatorname{Tor}_{n-1}^R(M_i,N)=\operatorname{Tor}_n^R(M_i,N)=0$. Then $\operatorname{pd}_R(N)\leqslant n-1$. In the case where $n=0$ (i.e., $M_i\otimes_R N=0$), we get $N=0$.*

Proof. The claims follow from Proposition 2.1 by letting $F_\bullet$ be a minimal free resolution of N over R. In part (c), when $n=0$, we get $F_0=0$, which implies $N=0$. □

One can obtain a variant version of Theorem 2.3 in terms of the Ext modules.

Theorem 2.4. *Assume the same setup as in Theorem 2.3. Then, for each $n\geqslant 0$, and $1\leqslant i\leqslant t$, and for each finitely generated R-module N, we have the following:*

(a) *If $\operatorname{grade}_R(M)\geqslant 1$ and $\operatorname{Ext}_R^n(N,M_i)=0$, then $\operatorname{Ext}_R^n(N,M_{i-1})=0$.*

(b) *If $\operatorname{Ext}_R^n(N,M_i)=0$ and M_{i-1} is a faithful R-module, then $\operatorname{pd}_R(N)\leqslant n-1$.*

(c) *Assume $M_i\neq 0$ and $\operatorname{Ext}_R^n(N,M_i)=\operatorname{Ext}_R^{n+1}(N,M_i)=0$. Then $\operatorname{pd}_R(N)\leqslant n-1$. In the case where $n=0$ (i.e., $\operatorname{Hom}_R(N,M_i)=\operatorname{Ext}_R^1(N,M_i)=0$), we get $N=0$.*

Proof. Let $G_\bullet$ be the minimal free resolution of N over R. Then $\operatorname{Hom}_R(G_\bullet,M)\cong \operatorname{Hom}_R(G_\bullet,R)\otimes_R M$ as complexes. As $\operatorname{Ext}_R^n(N,M_i)=\mathrm{H}_{-n}(\operatorname{Hom}_R(G_\bullet,R)\otimes_R M_i)$, we can apply Proposition 2.1 with the complex $\operatorname{Hom}_R(G_\bullet,R)$ and establish the claims. □

In the next section, we establish some corollaries of Theorem 2.3 and extend several results from the literature concerning rigid and test modules.

3. On rigid and test modules

Test and rigid modules are defined for finitely generated modules in the literature. However, our arguments work with modules that are not necessarily finitely generated. Therefore, motivated by big Cohen-Macaulay modules [5, page 323], we define "big" versions of test and rigid modules.

Definition 3.1. Let R be a ring and let $M\neq 0$ be an R-module (not necessarily finitely generated).

(i) M is called a *big test* module (or a big pd-test module) provided that the following condition holds: if N is a finitely generated R-module and $\operatorname{Tor}_i^R(M,N)=0$ for all $i \gg 0$, then $\operatorname{pd}_R(N)<\infty$; cf. [6, 1.1].
(ii) M is called a *big Tor-rigid* module provided that the following condition holds: if N is a finitely generated R-module and $\operatorname{Tor}_n^R(M,N)=0$ for some $n \geqslant 0$, then $\operatorname{Tor}_i^R(M,N)=0$ for all $i \geqslant n$; cf. [1].
(iii) M is called a *big rigid-test* module provided that M is both test and Tor-rigid; cf. [21, 2.3].
(iv) M is called a *big strongly-rigid* module provided that the following condition holds: if N is a finitely generated R-module and $\operatorname{Tor}_n^R(M,N)=0$ for some $n \geqslant 1$, then $\operatorname{pd}_R(N)<\infty$; cf. [12, 2.1].

When M is finitely generated, we drop the word big. For example, we call a finitely generated big test module a *test* module.

There are various classes of test and rigid modules in the literature. For example, the residue field is rigid-test and strongly-rigid over all local rings. Also, finitely generated modules are Tor-rigid over regular local rings, and finitely generated modules of infinite projective dimension are test over hypersurface rings; see, [18, 1.9] for the details. Note that, by definition, each strongly-rigid module is a test module. Similarly, each rigid-test module is strongly-rigid, but we do not know if the converse is true. More precisely, we do not know if each strongly-rigid module is Tor-rigid, in general. On the other hand, in light of the Auslander-Buchsbaum formula, if R is a local ring of depth at most one and M is a finitely generated R-module, then M is strongly-rigid if and only if M is rigid-test.

Our aim is to make use of Theorem 1.2 and obtain new classes of big rigid and big test modules, and extend some known results in this direction. Let us note that Theorem 1.2 allows one to generalize many results (for example results from [21]) along ring homomorphisms; here we obtain only a few such results to demonstrate some useful applications of Theorem 1.2.

We first turn our attention to the following beautiful result of Levin and Vascencelos:

3.2 (Levin-Vascencelos, [20, 1.1 and Lemma, page 316])**.** *Let $(R,\mathfrak{m})$ be a local ring and M and N are finitely generated R-modules such that $\mathfrak{m}M \neq 0$. If $\operatorname{Tor}_n^R(\mathfrak{m}M,N)=\operatorname{Tor}_{n+1}^R(\mathfrak{m}M,N)=0$ for some $n \geqslant 0$, then $\operatorname{pd}_R(N) \leqslant n$. Therefore, $\mathfrak{m}M \neq 0$ is a test module; see also* [8, 2.9].

As mentioned in the introduction, for the special case where $R=S$ and f is the identity map, our assumption on the filtration of M in Theorem 1.2 forces $M_i=\mathfrak{m}^iM$ for each $i \geqslant 0$. Therefore, the next corollary yields an extension of the result of Levin and Vascencelos recorded in 3.2. The corollary also shows that certain characterizations of local rings in terms of test modules carry over along ring homomorphisms:

Corollary 3.3. *Let $f: R \to S$ be a ring homomorphism such that $(R,\mathfrak{m})$ is local, S is Noetherian, and $\mathfrak{m}S \subseteq \operatorname{Jac}(S)$. Let M be a finitely generated S-module. Assume there is a filtration of M of the form*

$$0 \neq M_t \leqslant M_{t-1} \leqslant \cdots \leqslant M_1 \leqslant M_0 = M,$$

where each M_i is a nonzero S-module and $\mathfrak{m}M_{i-1} \leqslant M_i \leqslant \operatorname{Jac}(S)M_{i-1}$ for all $i=1,\ldots,t$. Let $H=M_j$, for some j, where $1 \leqslant j \leqslant t$. Then:

(1) *H is a big test module over R. In fact, if $\operatorname{Tor}_{n-1}^R(H,N)=\operatorname{Tor}_n^R(H,N)=0$ for some finitely generated R-module N and some integer $n \geqslant 0$, then $\operatorname{pd}_R(N) \leqslant n-1$.*
(2) *The following statements are equivalent:*
   (i) *R is regular.*
   (ii) $\operatorname{pd}_R(H)<\infty$.
   (iii) $\operatorname{Tor}_n^R(T,H)=\operatorname{Tor}_{n+1}^R(T,H)=0$ *for some $n \geqslant 0$ and (finitely generated) test module T over R.*
   (iv) $\operatorname{id}_R(H)<\infty$.
   (v) $\operatorname{Ext}_R^n(T,H)=\operatorname{Ext}_R^{n+1}(T,H)=0$ *for some $n \geqslant 0$ and (finitely generated) test module T over R.*

Proof. (1) This follows from Definition 3.1(i) and Theorem 2.3(c).

(2) It is clear that part (i) implies part (ii), and part (ii) implies part (iii). If part (iii) holds, then (1) implies that $\operatorname{pd}_R(T)<\infty$, so that R is regular since $\operatorname{Tor}_i^R(T,R/\mathfrak{m})=0$ for all $i \gg 0$. This shows the equivalence of parts (i), (ii), and (iii). As part (i) implies part (iv), and part (iv) implies part (v), it suffices to prove that part (v) implies part (i), which follows from Theorem 2.4(c). ☐

The following observation is used in Corollaries 3.5 and 3.7, and Lemma 4.3.

3.4. *Let $f:(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a local ring homomorphism of local rings and let M be a finitely generated S-module (thus a not necessarily finitely generated R-module via f). Assume M admits a filtration of the form*

$$0\neq M_t\leqslant M_{t-1}\leqslant\cdots\leqslant M_1\leqslant M_0=M,$$

where each M_i is a nonzero S-module and $\mathfrak{m}M_{i-1}\leqslant M_i$ for all $i=1,\ldots,t$. Then, for each integer i with $1\leqslant i\leqslant t$, M_{i-1}/M_i is annihilated by $\mathfrak{m}$ so that there is a short exact sequence of R-modules

$$0\to M_i\to M_{i-1}\to k^{\oplus r_i}\to 0 \qquad (3.4.1)$$

for some cardinality r_i, where $0\leqslant r_i\leqslant\infty$ and $k=R/\mathfrak{m}$. Note that, if $M_i\neq M_{i-1}$ (e.g., $M_i\leqslant\mathfrak{n}M_{i-1}$), then $M_{i-1}/M_i\neq 0$, and hence $1\leqslant r_i\leqslant\infty$.

Next, we extend [10, 2.5] and observe that the filtration modules enjoy further properties if the module considered in Theorem 1.2 has positive grade:

Corollary 3.5. *Let $f:(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a local ring homomorphism and let M be a finitely generated S-module. Assume the following hold:*

(i) *$M\neq 0$ and* $\operatorname{grade}_R(M)\geqslant 1$.
(ii) *There is a filtration of M of the form*

$$M_t\leqslant M_{t-1}\leqslant\cdots\leqslant M_1\leqslant M_0=M,$$

where each M_i is an S-module and $\mathfrak{m}M_{i-1}\leqslant M_i\leqslant\mathfrak{n}M_{i-1}$ for all $i=1,\ldots,t$.

Then the following hold:

(a) *If M_{i-1} is big strongly-rigid over R for some i, where $1\leqslant i\leqslant t$, then M_i is big strongly-rigid over R.*
(b) *If M_{i-1} is big Tor-rigid over R for some i, where $1\leqslant i\leqslant t$, and $\operatorname{Tor}_n^R(M_i,N)=0$ for some $n\geqslant 0$ and for some finitely generated R-module N, then $\operatorname{pd}_R(N)\leqslant n$ so that $\operatorname{Tor}_j^R(M_i,N)=0$ for all $j\geqslant n$.*
(c) *If M_{i-1} is big Tor-rigid over R for some i, where $1\leqslant i\leqslant t$, then M_i is big strongly-rigid and big Tor-rigid over R.*
(d) *It follows that:*

$$M \text{ is big Tor-rigid over } R\Longrightarrow M_1 \text{ is big rigid-test over } R\Longrightarrow\cdots\Longrightarrow M_t \text{ is big rigid-test over } R.$$

Proof. Since $M\neq 0$, $\operatorname{grade}_R(M)\geqslant 1$ and $\mathfrak{m}M_{i-1}\leqslant M_i\leqslant\mathfrak{n}M_{i-1}$, we observe that $0\neq M_i\neq M_{i-1}$ for all $i=1,\ldots,t$. [1] Part (c) is an immediate consequence of part (b), and part (d) is a consequence of parts (a) and (c). To prove parts (a) and (b), we assume $\operatorname{Tor}_n^R(M_i,N)=0$ for some $n\geqslant 0$ and for some finitely generated R-module N, so Theorem 2.3(a) implies that $\operatorname{Tor}_n^R(M_{i-1},N)=0$ since M has positive grade.

(a) If M_{i-1} is big strongly-rigid, then the vanishing of $\operatorname{Tor}_n^R(M_{i-1},N)$ implies that $\operatorname{pd}_R(N)<\infty$.

(b) Assume M_{i-1} is big Tor-rigid. As $\operatorname{Tor}_n^R(M_{i-1},N)=0$, it follows that $\operatorname{Tor}_j^R(M_{i-1},N)=0$ for all $j\geqslant n$. Hence (3.4.1) yields the exact sequence

$$0=\operatorname{Tor}_{n+1}^R(M_{i-1},N)\to\operatorname{Tor}_{n+1}^R(k^{\oplus r_i},N)\to\operatorname{Tor}_n^R(M_i,N)=0,$$

which implies $\operatorname{pd}_R(N)\leqslant n$ and the vanishing of $\operatorname{Tor}_j^R(M_i,N)$ for all $j\geqslant n$. □

Remark 3.6. Under the setup of Corollary 3.5, if M is Tor-rigid over R (so that finitely generated over R) and $\operatorname{Tor}_n^R(M_i,M_j)=0$ for some $i\geqslant 1$, $j\geqslant 1$, and $n\geqslant 1$, M_i and M_j have finite projective dimension and this forces R to be regular. See Corollary 3.3(2).

Corollary 3.7. *Let $f:(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a local ring homomorphism and let M be a finitely generated S-module. Assume the following hold:*

(i) *M is big Tor-rigid as an R-module (so $M\neq 0$) and* $\operatorname{grade}_R(M)\geqslant 1$.

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[1] Thus M_i is a big test module over R for each $i=1,\ldots,t$; see Corollary 3.3(1).

(ii) *There is a filtration of M of the form*

$$M_t \leqslant M_{t-1} \leqslant \cdots \leqslant M_1 \leqslant M_0 = M,$$

where each M_i is an S-module and $\mathfrak{m}M_{i-1} \leqslant M_i \leqslant \mathfrak{n}M_{i-1}$ for all $i = 1, \ldots, t$.

Consider M_i for some $i \geqslant 1$. Given a finitely generated R-module N, we have:

(a) *If $\operatorname{Tor}_n^R(N, M_i) = 0$ for some $n \geqslant 1$, then $\operatorname{pd}_R(N) \leqslant n$.*
(b) *If M_i is finitely generated over R and $\operatorname{Ext}_R^n(N, M_i) = 0$ for some $n \geqslant 1$, then $\operatorname{pd}_R(N) \leqslant n-1$.*
(c) *Assume $\operatorname{Ext}_R^n(M_j, M_i) = 0$ or $\operatorname{Tor}_n^R(M_j, M_i) = 0$ for some $j \geqslant 1$ and $n \geqslant 1$ and further assume that $M_i \oplus M_j$ is finitely generated over R. Then $\operatorname{pd}_R(M_j) < \infty$ and so R is regular.*

Proof. As M is big Tor-rigid, Corollary 3.5(d) implies that each M_i is a big rigid-test, and hence a big Tor-rigid, module over R for every $i = 1, \ldots, t$. Also observe that $0 \neq M_i \neq M_{i-1}$ for all $i = 1, \ldots, t$.

(a) If $\operatorname{Tor}_n^R(N, M_i) = 0$, we conclude from Corollary 3.5(b) that $\operatorname{pd}_R(N) \leqslant n$.

(b) Note that $\operatorname{depth}_R(M_i) = 1$ for all $i = 1, \ldots, t$; see the short exact sequence (3.4.1). Thus, if $\operatorname{Ext}_R^n(N, M_i) = 0$, it follows from [21, 6.1(ii)] that $\operatorname{pd}_R(N) \leqslant n-1$.

(c) From (a) and (b), we see that $\operatorname{pd}_R(M_j) < \infty$. Now the regularity of R follows from Corollary 3.3(2). □

We also need the following results of Avramov-Miller [4], Funk-Marley [14, 15], and Koh-Lee [19] in the sequel:

3.8. *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p.*

(i) *If $\dim(R) = 1$, then ${}^eR$ is a big rigid-test module for each $e \gg 0$; see* [19, 2.6] *and* [14, 15, 3.2].
(ii) *If R is a complete intersection, then ${}^eR$ is a big rigid-test module for each $e \geqslant 1$;* [4, Thm].

Corollary 3.9. *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be the Frobenius map F^e for some $e \geqslant 1$, where $R = S$ has prime characteristic p and R is an F-finite local complete intersection ring of positive depth. Let $v \geqslant 0$ be an integer. Assume there is a filtration of $M = {}^vS$ of the form*

$$M_t \leqslant M_{t-1} \leqslant \cdots \leqslant M_1 \leqslant M_0 = M,$$

where each M_i is an S-module and $\mathfrak{m}M_{i-1} \leqslant M_i \leqslant \mathfrak{n}M_{i-1}$ for all $i = 1, \ldots, t$. [2] *If $\operatorname{Ext}_R^n({}^eN, M_j) = 0$ or $\operatorname{Tor}_n^R({}^eN, M_j) = 0$ for some nonzero finitely generated S-module N, $j \geqslant 1$, and $n \geqslant 1$, then R is regular.*

Proof. Note that M is a Tor-rigid R-module by 3.8(ii). Given the vanishing of Ext or Tor, it follows from Corollary 3.7 that $\operatorname{pd}_R({}^eN) < \infty$. Hence R is regular due to [3, 1.1]. □

In the following corollary, we give some specific examples of filtration modules that are big strongly-rigid modules or big rigid-test modules.

Corollary 3.10. *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p, and let $M \neq 0$ and $N \neq 0$ be finitely generated R-modules such that $\operatorname{depth}_R(M) \geqslant 1$. Let $e \geqslant 0$ and $t \geqslant 1$ be integers.*

(i) *If $\operatorname{Tor}_n^R\left({}^e(\mathfrak{m}^tM), N\right) = 0$ or $\operatorname{Tor}_n^R\left({}^e\!\left(\left(\mathfrak{m}^{[p^e]}\right)^tM\right), N\right) = 0$ for some $n \geqslant 1$, then $\operatorname{Tor}_n^R({}^eM, N) = 0$.*
(ii) *If ${}^eM$ is a big strongly-rigid module, then ${}^e(\mathfrak{m}^tM)$ and ${}^e\!\left(\left(\mathfrak{m}^{[p^e]}\right)^tM\right)$ are big strongly-rigid modules.*
(iii) *Assume that ${}^eM$ is a big Tor-rigid module. Then ${}^e(\mathfrak{m}^tM)$ and ${}^e\!\left(\left(\mathfrak{m}^{[p^e]}\right)^tM\right)$ are big rigid-test modules.*

In fact, if $\operatorname{Tor}_n^R\left({}^e(\mathfrak{m}^tM), N\right) = 0$ or $\operatorname{Tor}_n^R\left({}^e\!\left(\left(\mathfrak{m}^{[p^e]}\right)^tM\right), N\right) = 0$ for some $n \geqslant 1$, then $\operatorname{pd}_R(N) \leqslant n$.

Proof. The claim in part (i) is a direct consequence of Theorem 2.3 because ${}^e(\mathfrak{m}^tM)$ and ${}^e\!\left(\left(\mathfrak{m}^{[p^e]}\right)^tM\right)$ are specific examples of filtration modules M_i of the module considered in the theorem. Consequently, both part (ii) and part (iii) follow from Corollary 3.5(a)(b)(d). □

[2] This implies that, as an S-module, M_1 satisfies ${}^v\!\left(\mathfrak{n}^{[p^{e+v}]}S\right) = \mathfrak{n}^{[p^e]}({}^vS) = \mathfrak{m}({}^vS) \leqslant M_1 \leqslant \mathfrak{n}({}^vS) = {}^v\!\left(\mathfrak{n}^{[p^v]}S\right)$.

Remark 3.11. Let R be a ring of prime characteristic p, $M = T \oplus \Omega_R T$ for some finitely generated R-module $T \neq 0$, and let $e \geqslant 1$. There is an exact sequence of R-modules $0 \to {}^e(\Omega_R T) \to {}^eF \to {}^eT \to 0$, where $F \neq 0$ is a free R-module. Thus, if $\operatorname{Tor}_n^R({}^eM, N) = 0$ for some $n \geqslant 1$, then $\operatorname{Tor}_n^R({}^e(\Omega_R T), N) = \operatorname{Tor}_n^R({}^eT, N) = 0$ so that $\operatorname{Tor}_n^R({}^eF, N) = 0$, which implies that $\operatorname{Tor}_n^R({}^eR, N) = 0$. In particular, if ${}^eR$ is a big strongly-rigid module over R (respectively, a big test module over R), then so is ${}^eM$.

Corollary 3.12. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of prime characteristic p, and let $T \neq 0$ be a finitely generated R-module such that $\operatorname{depth}_R(T) \geqslant 1$. Let $M = T \oplus \Omega_R T$, $e \geqslant 1$, and $t \geqslant 1$. Let X denote ${}^e(\mathfrak{m}^t M)$ or ${}^e\Big(\big(\mathfrak{m}^{[p^e]}\big)^t M\Big)$. Assume at least one of the following conditions:*

(i) $\dim(R) = 1$ *and* $e \gg 0$.
(ii) *R is a complete intersection.*

Then both ${}^eM$ and X are big strongly-rigid modules.

Proof. Since $\operatorname{depth}_R(T) \geqslant 1$ and R is Cohen-Macaulay, we see $\operatorname{depth}_R(M) \geqslant 1$ and hence $\operatorname{grade}_R({}^eM) \geqslant 1$. Thus both Corollary 3.5 and Corollary 3.10 apply. The claim that ${}^eM$ is a big strongly-rigid module follows from Remark 3.11 and 3.8. Now, applying either Corollary 3.5(a) or Corollary 3.10(ii), we conclude that X is a big strongly-rigid module. ☐

Remark 3.13. Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p, $M \neq 0$ be a finitely generated R-module such that $\operatorname{depth}_R(M) = 0$, $t \geqslant 1$ and $e \gg 0$. In this case, ${}^eM$ is a big rigid-test module by [19, 2.6].

Let X denote ${}^e(\mathfrak{m}^t M)$ or ${}^e\Big(\big(\mathfrak{m}^{[p^e]}\big)^t M\Big)$. Since $\operatorname{depth}_R(M) = 0$, we cannot apply our results such as Corollary 3.5 and Corollary 3.10 to deduce that X is a big rigid-test module. However, if $X \neq 0$ and $\operatorname{grade}_R(X) = 0$, then X is a big rigid-test module due to [19, 2.6].

4. On torsion in tensor products of modules

In this section, we consider the following conjecture of Huneke-Wiegand. Set $(-)^* = \operatorname{Hom}_R(-, R)$.

Conjecture 4.1 ([17, page 473-474])**.** Let R be a one-dimensional local ring and let M be a finitely generated R-module which has rank. If $M \otimes_R M^*$ is nonzero and torsion-free, then M is free.

Recall that a finitely generated module M over a ring R is said to have *rank* if there is a nonnegative integer r such that $M_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus r}$ for each associated prime ideal $\mathfrak{p}$ of R. For example, if M has finite projective dimension over a local ring, or R is a domain, then M has rank. Conjecture 4.1 fails if the module considered does not have rank, or the ring in question has dimension at least two; see [11, 8.5 and page 447] for example. Note that, in Conjecture 4.1, it suffices to additionally assume that M is a torsion-free module which has positive constant rank.

Conjecture 4.1 is wide open in general. It turns out that, over Gorenstein rings, the conjecture is a special case of a celebrated conjecture of Auslander and Reiten [2] on the vanishing of cohomology that stems from the representation theory of finite dimensional algebras; this is one of the main motivations to study Conjecture 4.1; see also [11, 8.6].

Huneke and Wiegand [17, 3.1] proved that Conjecture 4.1 holds over hypersurface rings. There are also several other cases where the conjecture holds, for example, if R is a Cohen-Macaulay local ring of minimal multiplicity [16, 3.6], or M is an integrally closed $\mathfrak{m}$-primary ideal [7, 2.17]. Note that Conjecture 4.1 holds if the module M in question is strongly-rigid [7, 2.15]. Therefore, Corollary 3.12 yields new classes of modules establishing the conjecture over local rings of prime characteristic.

If M is a finitely generated module over a local ring $(R, \mathfrak{m})$ such that M has rank and $0 \neq M \otimes_R M^*$ is torsion-free, then $\operatorname{Supp}_R(M) = \operatorname{Spec}(R)$; see, for example, [9, 1.3]. Therefore, the following result of Dey and Kobayashi [13] establishes Conjecture 4.1 for nonzero modules that are of the form $\mathfrak{m}N$.

Theorem 4.2 ([13, 1.5(1)])**.** *Let $(R, \mathfrak{m})$ be a local ring of depth one and let $M = \mathfrak{m}N$ for some finitely generated R-module N. Assume $\operatorname{Supp}_R(M) = \operatorname{Spec}(R)$ and $M_\mathfrak{p}$ is free over $R_\mathfrak{p}$ for each associated prime $\mathfrak{p}$ of R. If $M \otimes_R M^*$ is nonzero and torsion-free, then M is free and R is regular.*

Lemma 4.3. *Let $(R,\mathfrak{m})$ be a local ring of positive depth and let M be a nonzero finitely generated R-module. Assume there is a filtration of M of the form*

$$0 \neq M_t \leqslant M_{t-1} \leqslant \cdots \leqslant M_1 \leqslant M_0 = M,$$

where each M_i is a nonzero R-module and $\mathfrak{m}M_{i-1} \leqslant M_i$ for all $i = 1, \ldots, t$. Then it follows

$$M \text{ satisfies } (\mathscr{P}) \Longrightarrow M_1 \text{ satisfies } (\mathscr{P}) \Longrightarrow \cdots \Longrightarrow M_t \text{ satisfies } (\mathscr{P}),$$

where the property $(\mathscr{P})$ denotes one of the following: (a) being not torsion, (b) being faithful, (c) being locally free on the set of associated primes of R, or (d) having rank.

Proof. Assume M satisfies $(\mathscr{P})$. Note that it is enough to show M_1 satisfies $(\mathscr{P})$.

For property (a), if M_1 is torsion, then $xM_1 = 0$ for some non zero-divisor $x \in \mathfrak{m}$. Since $xM \subseteq \mathfrak{m}M \subseteq M_1$, we see that $x^2M = 0$ and hence M is torsion. This shows that M_1 is not torsion.

For property (b), since $\mathfrak{m}M \subseteq M_1$, it is enough to show that $\mathfrak{m}M$ is faithful. Let $y \in \operatorname{Ann}_R(\mathfrak{m}M)$. Then $y\mathfrak{m}M = 0$ and hence $y\mathfrak{m} = 0$ since M is faithful. This implies $y = 0$ as $\mathfrak{m}$ contains a non zero-divisor.

For properties (c) and (d), we see from the exact sequence (3.4.1) that $M_{\mathfrak{p}} \cong (M_1)_{\mathfrak{p}}$ for each associated prime $\mathfrak{p}$ of R. This gives the required conclusions. □

We recall the following basic facts for the proof of Theorem 4.5; note that $(-)^* = \operatorname{Hom}_R(-,R)$.

4.4. *Let R be a ring and let M and N be finitely generated R-modules.*

(i) *Let $\top_R(M)$ denote the torsion submodule of M and set $\overline{M} = M/\top_R(M)$. Assume $0 \neq N$ and $M \otimes_R N$ is torsion-free. Then $M \otimes_R N \cong \overline{M} \otimes_R N$, and M is free if and only if $\overline{M}$ is free; see* [17, 1.1].

(ii) *If N is* torsionless*, that is, if the natural map $N \to N^{**}$ is injective, then N embeds into a free R-module so that it is torsion-free; see, for example,* [5, 1.4.19]. *Also if N is torsion-free and locally free on each associated prime of R, then the kernel of $N \to N^{**}$ is zero when localized at each associated prime of R, hence the kernel is torsion so that it vanishes and N is torsionless.*

Theorem 4.5. *Let $f: R \to S$ be a ring homomorphism, where $(R,\mathfrak{m})$ is a local ring of positive depth, S is Noetherian, and $\mathfrak{m}S \subseteq \operatorname{Jac}(S)$. Let M be a finitely generated S-module. Assume:*

(i) *There is a filtration of M of the form*

$$0 \neq M_t \leqslant M_{t-1} \leqslant \cdots \leqslant M_1 \leqslant M_0 = M,$$

where each M_i is a nonzero S-module and $\mathfrak{m}M_{i-1} \leqslant M_i \leqslant \operatorname{Jac}(S)M_{i-1}$ for all $i = 1, \ldots, t$.

(ii) *M is finitely generated as an R-module via f and $\operatorname{Supp}_R(M) = \operatorname{Spec}(R)$.*

(iii) *$M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for each associated prime $\mathfrak{p}$ of R.*

Then the following hold:

(a) *If $0 \neq N$ is a finitely generated torsionless R-module and $M_i \otimes_R N$ is a torsion-free R-module for some $i \in \{1, \ldots, t\}$, then N is free and M_i is torsion-free over R.*

(b) *If $M_i \otimes_R M_j$ is a torsion-free R-module for some $i, j \in \{1, \ldots, t\}$, then both M_i and M_j are free over R and R is regular.*

(c) *Assume* $\operatorname{depth}(R) = 1$. *If $0 \neq M_i \otimes_R M_i^*$ is a torsion-free R-module for some $i \in \{1, \ldots, t\}$, then M_i is free over R and R is regular.*

Proof. (a) There is an exact sequence of finitely generated R-modules $0 \to N \to F \to C \to 0$ where F is free; see 4.4(ii). Tensoring this sequence with M_i over R, we obtain an injection $\operatorname{Tor}_1^R(M_i,C) \hookrightarrow M_i \otimes_R N$. By Lemma 4.3(c), for each associated prime $\mathfrak{p}$ of R, we have that $(M_i)_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ and so $\operatorname{Tor}_1^R(M_i,C)_{\mathfrak{p}} = 0$. This implies that $\operatorname{Tor}_1^R(M_i,C)$ is a torsion R-module and hence it vanishes as $M_i \otimes_R N$ is torsion-free over R. Also M is a faithful R-module; see, for example, [13, 2.12]. So Lemma 4.3(b) shows that M_{i-1} is faithful. As $\operatorname{Tor}_1^R(M_i,C) = 0$, Theorem 2.3(b) implies that $\operatorname{pd}_R(C) \leqslant 1$. Now the sequence $0 \to N \to F \to C \to 0$ shows that $N \neq 0$ is free. Consequently, M_i is torsion-free over R because $M_i \otimes_R N$ is torsion-free over R.

(b) Note that $(M_j)_{\mathfrak{p}} \cong (\overline{M_j})_{\mathfrak{p}}$ for each associated prime $\mathfrak{p}$ of R. Hence Lemma 4.3(c) implies that $(\overline{M_j})_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for each associated prime $\mathfrak{p}$ of R. Therefore, $\overline{M_j}$ is a torsionless R-module; see 4.4(ii). It follows that $0 \neq M_i \otimes_R M_j \cong M_i \otimes_R \overline{M_j}$; see 4.4(i). Thus $\overline{M_j} \neq 0$, so part (a) implies that $\overline{M_j}$ is free over R. As $M_i \neq 0$, we conclude from 4.4(i) that M_j is free over R, which implies that R is regular due to Corollary 3.3. By interchanging the roles of M_i and M_j, we see that M_i is free over R as well.

(c) As $0 \neq M_i^*$ is a torsionless R-module, part (a) implies that M_i^* is free over R and M_i is torsion-free over R. Then M_i is free over R, given that $\operatorname{depth}(R) = 1$; see [7, 2.13]. So R is regular by Corollary 3.3. ☐

We finish this section with the following corollary of Theorem 4.5, which establishes Conjecture 4.1 for certain filtration modules.

Corollary 4.6. *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a finite local ring homomorphism, where R is a one-dimensional domain, and let $0 \neq M$ be a finitely generated S-module that is torsion-free as an R-module. Assume that there is a filtration of M of the form*

$$M_t \leqslant M_{t-1} \leqslant \cdots \leqslant M_1 \leqslant M_0 = M,$$

where each M_i is an S-module and $\mathfrak{m}M_{i-1} \leqslant M_i \leqslant \mathfrak{n}M_{i-1}$ for all $i = 1, \dots, t$. If $M_j \otimes_R M_j^*$ is a torsion-free R-module for some j, where $1 \leqslant j \leqslant t$, then M_j is free over R.

Proof. Since M is torsion-free over R and $\mathfrak{m}^i M \subseteq M_i$, we see that $0 \neq M_i$ is torsion-free over R. Thus $M_i^* \neq 0$, which implies $M_i \otimes_R M_i^* \neq 0$. On the other hand, as $M \neq 0$ is a torsion-free R-module and R is a domain, we have $\operatorname{Supp}_R(M) = \operatorname{Spec}(R)$. Consequently, if $M_j \otimes_R M_j^*$ is a torsion-free R-module for some $j \in \{1, \dots, t\}$, then Theorem 4.5(c) implies that M_j is free over R and R is regular (hence M is free over R as well). ☐

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