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FROBENIUS SPLITTING, STRONG F-REGULARITY, AND SMALL COHEN-MACAULAY MODULES

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Abstract. Let M be a finitely generated module over an (F-finite local) ring R of prime characteristic $p > 0$. Let ${}^eM$ denote the result of restricting scalars using the map $F^e\colon R \to R$, the eth iteration of the Frobenius endomorphism. Motivated in part by the fact that in certain circumstances the splitting of ${}^eM$ as e grows can be used to prove the existence of small (i.e., finitely generated) maximal Cohen-Macaulay modules, we study splitting phenomena for ${}^eM$ from several points of view. In consequence, we are able to prove new results about when one has such splittings that generalize results previously known only in low dimension, we give new characterizations of when a ring is strongly F-regular, and we are able to prove new results on the existence of small maximal Cohen-Macaulay modules in the multi-graded case. In addition, we study certain corresponding questions when the ring is no longer assumed F-finite and purity is considered in place of splitting. We also answer a question, raised by Datta and Smith, by showing that a regular Noetherian domain, even in dimension 2, need not be very strongly F-regular.

0. Introduction

Throughout this paper, all rings are assumed to be non-zero, commutative, with identity, Noetherian and of prime characteristic $p > 0$, unless specified otherwise explicitly. By a (graded) local ring $(R, \mathfrak{m}, k)$, we indicate that $\mathfrak{m}$ is the unique (homogeneous) maximal ideal of R and $k = R/\mathfrak{m}$. We use $\mathbb{N}$ to denote the set of all non-negative integers and use $\mathbb{N}_+$ to denote the set of all positive integers.

We often denote $q := p^e$ for varying $e \in \mathbb{N}$. For every $e \in \mathbb{N}$, there is the Frobenius map (in fact a ring homomorphism) $F^e\colon R \to R$ defined by $F^e(r) = r^q = r^{p^e}$ for all $r \in R$.

Given any R-module M and any $e \in \mathbb{N}$, there is a derived R-module structure, denoted ${}^eM$, on the same abelian group M but with its scalar multiplication determined by $r * x = r^q x = r^{p^e} x$ for all $r \in R$ and $x \in {}^eM$. It is routine to verify that $\operatorname{Ann}(M) \subseteq \operatorname{Ann}({}^eM) \subseteq \sqrt{\operatorname{Ann}(M)}$ and $\operatorname{Ass}(M) = \operatorname{Ass}({}^eM)$ for all $e \in \mathbb{N}$. Very often, for $x \in M$, we use ${}^ex$ to denote the corresponding element in ${}^eM$.

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When R is reduced, the R-module structure of ${}^eR$ is isomorphic to the natural R-module structure of $R^{1/q} := \{r^{1/p^e} \mid r \in R\}$, for every e.

Using this terminology, one can rephrase a result of E. Kunz as follows: a ring R is regular if and only if ${}^eR$ is flat over R for some $e \in \mathbb{N}_+$ if and only if ${}^eR$ is flat over R for all $e \in \mathbb{N}$ ([Ku1, Theorem 2.1]).

We say that R is *F-finite* if ${}^1R$ is finitely generated over R; this is equivalent to saying that ${}^eR$ is finitely generated over R for all (or some) $e \in \mathbb{N}_+$. By a result of E. Kunz in [Ku2], every F-finite ring is excellent.

If R is F-finite and M is a finitely generated R-module, it is easy to see that ${}^eM$ remains finitely generated over R for every $e \in \mathbb{N}$.

Moreover, if ${}^{e_0}M$ is finitely generated over R for some $e_0 \in \mathbb{N}_+$, it follows that both M and ${}^{e_0}(R/\operatorname{Ann}(M))$ are finitely generated over R. Thus $R/\operatorname{Ann}(M)$ is an F-finite ring, and hence ${}^e(R/\operatorname{Ann}(M))$ is finite over $R/\operatorname{Ann}(M)$ (or, equivalently, over R) for all $e \in \mathbb{N}$. Consequently, ${}^eM$ is finitely generated over R for all $e \in \mathbb{N}$.

For any $e \in \mathbb{N}$, the derived R-module ${}^eM$ can be roughly identified as the module structure of M over the subring $R^q := \{r^q = r^{p^e} \mid r \in R\}$. Thus, *in general*, one should expect that the "size" of ${}^eM$ will increase as $e \to \infty$.

Question 0.1. Assume that R is F-finite and M is a finitely generated R-module. Will ${}^eM$ split (as a direct sum of two non-trivial submodules) for $e \gg 0$?

In [Ho3], Hochster proved several cases where such splitting occurs: the $\mathbb{N}$-graded case and the 1-dimensional case, for example; see Theorem 3.2. In [Yao2], Yao proved the eventual splitting in the 2-dimensional case but under an assumption involving *strong F-regularity*; see Theorem 3.3.

In §3, we study the direct sum decomposability of ${}^eM$ without any dimension restriction. We prove the following splitting result that greatly strengthens those in [Ho3] and [Yao2].

Theorem (Corollary 3.7)**.** *If R is an F-finite domain (but not a perfect field) with a module-finite extension that is strongly F-regular and M is a finitely generated faithful R-module, then ${}^eM$ splits non-trivially over R for all sufficiently large e.*

In §4, we prove that if the splitting of R from ${}^eM$ occurs "frequently enough" then R is strongly F-regular.

Theorem (Corollary 4.3 and Remark 4.4)**.** *If R is F-finite or essentially of finite type over a complete excellent semilocal ring, then R is strongly F-regular if and only if for every finitely generated module M supported on* $\operatorname{Spec}(R)$*, ${}^eM$ has a direct summand isomorphic to R for all $e \gg 0$.*

It is desirable to define strong F-regularity without the assumption that the ring be F-finite. Several possible definitions have been suggested in the literature. These are all equivalent if the ring is F-finite, but, in the general case, there turn out to be two notions that are not equivalent. One of these, discussed very briefly in [HH4, Remark 5.2], is studied in [Hash], where it is called *very strong F-regularity*, and in [DaSm, §6], where it is called *F-pure regularity*. In this manuscript we shall use the former terminology (i.e., very strong F-regularity). More specifically, in this paper, very strong F-regularity of R means that, for all $c \in R \setminus \bigcup_{P \in \min(R)} P$, the R-linear map $R \to {}^eR$ such that $1 \mapsto {}^ec$ is pure over R for $e \gg 1$. Note that this property is easily shown, in the Noetherian case, to be inherited by all localizations. The second extended definition is proposed (although in differing forms, and their equivalence needs a proof) in [Sm, §7.2] and in [Ho7, p. 166], and is equivalent to the assumption that the ring is very strongly F-regular when localized at any maximal (equivalently, prime) ideal (cf. Theorem 2.11). This turns out to be weaker than very strong regularity, and is what we call *strongly F-regular* in this paper. This terminology agrees with the terminology in [Hash]. The situation is discussed in detail in §2. By [Hash, Corollary 3.35] the two notions are equivalent for algebras that are essentially of finite type over an excellent local ring (this is easily generalized to the case of algebras of finite type over an excellent semilocal ring). In §2 we recall several results about strongly F-regular and very strongly F-regular rings. For the most part, the results of §2 are available in some form in [Hash, §3], but we have included a number of relatively short proofs that give the results in the form we need, in some cases in somewhat greater generality. However, we want to acknowledge explicitly that the paper [Hash] has priority.

While all regular local rings are strongly F-regular, we show in §6 that there exists a regular Noetherian domain R of dimension two that is not very strongly regular in the sense of [DaSm, 6.3.3], that is, there exists an element $c \neq 0$ in R such that the map $R \to {}^{e}R$ with $1 \mapsto {}^{e}c$ is not pure for any $e \in \mathbb{N}_{+}$. One can already use the ring constructed in [EHo, Remarks (2), p. 159] to give a counter-example, but in §6 we construct a much larger family of examples. The proof that these rings are Noetherian is somewhat more subtle than in the example from [EHo], where the argument depends on knowing that the Krull dimension is two. These rings are not excellent. In fact, we prove that an excellent regular ring is very strongly F-regular (see Corollary 2.18). Quite recently, in [HoY], we show that every stronly F-regular excellent ring is very strongly F-regular.

Theorem 4.2 offers a more general version of the the criterion for strong F-regularity given in Corollary 4.3 that works for a large class of rings that are not necessarily F-finite.

In §5, we prove in a multi-graded situation that if the numbers of direct summands of ${}^{e}M$ are "large enough" as $e \to \infty$ then R admits a small Cohen-Macaulay module.

Theorem (Theorem 5.19). *Let $R = \oplus_{i \in \mathbb{N}^s} R_i$ be an $\mathbb{N}^s$-graded domain with R_0 being an F-finite field and $R_i \neq 0$ for all $i \in \mathbb{N}^s$. If $\dim(R) \leqslant s+2$, then there exists a small Cohen-Macaulay $\mathbb{N}^s$-graded R-module.*

1. Preliminaries

As it is stated earlier, all rings are assumed to have prime characteristic $p > 0$.

Remark 1.1. Let R be a ring, M and N be R-modules, $r \in R$, $x \in M$, $f \in \operatorname{Hom}_R(M, N)$, $P \in \operatorname{Spec}(R)$, and e, $e_1 \in \mathbb{N}$.

(1) The notation ${}^{e}x$ means the very same element $x \in M$ but considered as an element in ${}^{e}M$. In case e is understood from the context, we may just use x. Thus $r({}^{e}x) = (r^q)x = {}^{e}((r^q)x)$.
(2) The notation ${}^{e}f$ means the very same map $f \in \operatorname{Hom}_R(M, N)$ but considered as an R-linear map in $\operatorname{Hom}_R({}^{e}M, {}^{e}N)$. In case e is understood from the context, we may just use f. Thus $r({}^{e}f) = {}^{e}((r^q)f) = (r^q)f$ and ${}^{e}f({}^{e}x) = {}^{e}(f(x))$ for all $r \in R$ and $x \in M$.
(3) It is straightforward to see ${}^{e_1}({}^{e}M) = {}^{e+e_1}M$ and ${}^{e}(M_P) \cong ({}^{e}M)_P$ for all $P \in \operatorname{Spec}(R)$. So we may write ${}^{e}M_P$ without causing any confusion.

Definition 1.2. Let R be an F-finite ring. For every $P \in \operatorname{Spec}(R)$, denote

$$\alpha(R_P) := \log_p[k(P) : k(P)^p]$$

in which $k(P) := R_P/P_P$ is the residue field of R_P. This is, the field extension $k(P)^p \subseteq k(P)$ has a degree $p^{\alpha(R_P)}$. In case $(R, \mathfrak{m}, k)$ is local, clearly $\alpha(R) = \alpha(R_{\mathfrak{m}}) = \log_p[k : k^p]$.

Remark 1.3. Let R be an F-finite ring and M an R-module. (We use $\lambda_R(M)$ to denote the length of M.)

(1) If $(R, \mathfrak{m}, k)$ is local (and $\lambda_R(M) < \infty$), then $\lambda_R({}^eM) = q^{\alpha(R)}\lambda_R(M)$ for all $e \in \mathbb{N}$. This is because $\lambda_R({}^ek) = q^{\alpha(R)}$ for the R-module $k = R/\mathfrak{m}$.
(2) Thus we have $\lambda_R(0 :_{{}^eM} I) = q^{\alpha(R)}\lambda_R(0 :_M I^{[q]})$ for any ideal I of a local ring $(R, \mathfrak{m}, k)$. (The equation obviously holds if the lengths are infinite.)
(3) For every $P \in \operatorname{Spec}(R)$, it is straightforward to see $[k(P) : k(P)^q] = q^{\alpha(R_P)}$ for all $q = p^e$; Thus the torsion-free rank of ${}^e(R/P)$ over R/P is $q^{\alpha(R_P)}$.
(4) For prime ideals $P \subseteq Q$, we have $\alpha(R_P) = \alpha(R_Q) + \dim(R_Q/P_Q)$; see [Ku2, Proposition 2.3]. This also implies $\dim(R) < \infty$; also see [Ku2, Proposition 1.1].
(5) Thus $\alpha(R_P) \geqslant \dim(R/P)$ for every $P \in \operatorname{Spec}(R)$.

A very important concept in studying rings of prime characteristic $p > 0$ is tight closure, which was first studied and developed by Hochster and Huneke in the 1980s.

Definition 1.4 ([HH2])**.** Let R be a ring, I an ideal of R, and M an R-module.

(1) Denote $R^\circ := R \setminus \bigcup_{P \in \min(R)} P$.
(2) For every $q = p^e$, denote by $I^{[q]}$ the ideal generated by $\{x^q \mid x \in I\}$. Or equivalently, $I^{[q]} = I({}^eR)$.
(3) The tight closure of 0 in M, denoted 0^*_M, consists of $x \in M$ for which there exists $c \in R^\circ$ such that $0 = x \otimes {}^ec \in M \otimes_R {}^eR$ for all $e \gg 0$.
(4) If $N \subseteq M$ are R-modules, the tight closure of N in M, denoted N^*_M, is the inverse image of $0^*_{M/N}$ in M. That is, $u \in N^*_M$ if and only if its image in M/N is in $0^*_{M/N}$.
(5) We say R is weakly F-regular if $0^*_N = 0$ for all finitely generated R-modules N.
(6) We say R is F-regular if R_P is weakly F-regular for all $P \in \operatorname{Spec}(R)$.

Discussion 1.5 (**Splitting and purity**). Recall that, for R-modules M and N, we say that a map $h \in \operatorname{Hom}_R(M, N)$ is left split if there exists $g \in \operatorname{Hom}_R(N, M)$ such that $g \circ h = \mathrm{I}_M$, where I_M stands for the identity map on M. We say that $h \in \operatorname{Hom}_R(M, N)$ is pure if, for all R-modules T, the induced map $\mathrm{I}_T \otimes h \colon T \otimes_R M \to T \otimes_R N$ is injective. Left split implies pure and the converse holds if $M/h(N)$ is finitely presented. The notions are discussed and compared in [HoR1, §6], [HoR2, §5(a)]. Throughout this paper, we use $\mathrm{E}_R(N)$ to denote the injective hull of N over the ring R. The most frequent occurrence is when $(R, \mathfrak{m}, k)$ is local and $N = R/\mathfrak{m} = k$.

A very useful criterion for purity for maps from $(R, \mathfrak{m}, k)$ is the following (see, for example [HoR1, Proposition 6.11]).

Proposition 1.6. *Let $(R, \mathfrak{m}, k)$ be local and let M be any R-module. Then the map $h : R \to M$ is pure if and only if the induced map $\mathrm{E}_R(k) \cong \mathrm{E}_R(k) \otimes_R R \to \mathrm{E}_R(k) \otimes_R M$ is injective, i.e., if and only if the copy of $k \cong ku$, where u is a socle generator in $\mathrm{E}_R(k)$, does not map to 0.* □

Another important notion for rings of prime characteristic $p > 0$ is the *Hilbert-Kunz multiplicity.*

Definition 1.7. Let $(R, \mathfrak{m}, k)$ be a local ring with $\dim(R) = d$, I an $\mathfrak{m}$-primary ideal of R, and M a finitely generated R-module. The *Hilbert-Kunz multiplicity* of M with respect to I, denoted $\mathrm{e_{HK}}(I, M)$, is defined as

$$\mathrm{e_{HK}}(I, M) = \lim_{e\to\infty} \frac{\lambda_R(M/I^{[q]}M)}{q^d}.$$

The existence of the (finite) limit is due to Monsky [Mo]. We often write $\mathrm{e_{HK}}(\mathfrak{m}, M)$ as $\mathrm{e_{HK}}(M)$.

Remark 1.8. Let R, I and M be as in Definition 1.7 above.

(1) By considering M as a module over $R/\operatorname{Ann}(M)$, we see that
$$\lim_{e\to\infty} \frac{\lambda_R(M/I^{[q]}M)}{q^{\dim(M)}}$$
exists (and is positive as long as $M \neq 0$).
(2) Thus there exists a positive number $C \in \mathbb{R}$ such that $\lambda_R(M/I^{[q]}M) \leqslant Cq^{\dim(M)}$ for all q.
(3) Consequently, $\mathrm{e_{HK}}(I, M) = 0 \iff \dim(M) < \dim(R)$.

Discussion 1.9. **The Γ construction.** We give a brief recap of the Γ construction developed in [HH4, §6], which gives a very well behaved method of transition from algebras essentially finite type over a complete local ring to F-finite rings. Let A be a complete local ring of prime characteristic $p > 0$, and let k be a coefficient field for A. Let Γ denote a cofinite subset of a p-base Λ for k, i.e., $\Lambda \setminus \Gamma$ is finite. For each $q := p^e$, Let k_e denote $k[\gamma^{1/p^e} \mid \gamma \in \Gamma]$. Define $A^\Gamma := \bigcup_{e\in\mathbb{N}_+} k_e \widehat{\otimes}_k A$, where $\widehat{\otimes}_k$ indicates the complete tensor product. Then A^Γ is a faithfully flat purely inseparable local extension of A, and the maximal ideal of A extends to the maximal ideal of A^Γ. The construction depends on the choice of coefficient field, although this choice is not indicated in the notation. If A is the formal power series ring $k[[X_1, \dots, X_n]]$, then $A^\Gamma = \bigcup_{e\in\mathbb{N}_+} k_e[[X_1, \dots, X_n]]$. When B is complete local ring module-finite over A with the same coefficient field k, $B^\Gamma \cong B \otimes_A A^\Gamma$.

One has a natural isomorphism $(A/J)^\Gamma \cong (A^\Gamma/JA^\Gamma)$ for every ideal J of A. If R is essentially of finite type over A, we define $R^\Gamma := A^\Gamma \otimes_A R$, and R^Γ is faithfully flat and purely inseparable over R. If S is essentially of finite type over R, $S^\Gamma \cong R^\Gamma \otimes_R S$, by the associativity of tensor. in particular, if W is any multipliciative system in R, we have $W^{-1}(R^\Gamma) \cong (W^{-1}R)^\Gamma$, and we may simply write $W^{-1}R^\Gamma$.

Note also that if $\Gamma \subseteq \Gamma'$ then $R^\Gamma \subseteq R^{\Gamma'}$ is faithfully flat.

One may identify $\operatorname{Spec}(R^\Gamma)$ with $\operatorname{Spec}(R)$ using the maps obtained from contraction and taking the radical of the extension. *It is of great importance that the rings A^Γ and R^Γ are F-finite.* If J is prime or radical ideal in R, then for all sufficiently small choices of the cofinite set Γ, JR^Γ has the same property in R^Γ.

2. Strong and very strong F-regularity

In this section, we discuss what the notion of strong F-regularity should be for a Noetherian ring of characteristic p that is not necessarily F-finite. Two notions will emerge: one we shall refer to as "strong F-regularity" and the other as "very strong F-regularity." The results given here will be needed in later sections. A number of these results are implicitly or explicitly in the literature, especially in [Hash, §3] and in [DaSm, §6] (where the term "F-pure regularity" is used instead of "very strong F-regularity"). We follow the terminology of [Hash]. However, in a number of cases we have given somewhat different proofs, which are short, and we state some results in greater generality or more precisely than in the earlier papers. We do want to emphasize that [Hash] establishes most of what we need, and §3 of that paper, by and large, can be substituted for what we present here.

The notion of *strong F-regularity* was first defined in [HH1] for F-finite rings. See also [HH4, Definition 5.1] for the case of F-finite rings, as well as [Sm, §7.1.2], and [Ho7, p. 166] for suggestions of a general definition. It turns out that these generalized definitions are equivalent. A different condition is given in [HH4, Remark 5.3] and this idea is pursued in [DaSm, §6], but the term "F-pure regular" is used. These issues are addressed further below in Discussion 2.3. In this section we explore the properties of both notions and prove several results about classes of rings for which they are equivalent. In Examples 6.7 we show that, in general, they are different. When R has prime characteristic p and $c \in R$, we will have frequent occasion to consider the map $R \to {}^eR$ defined by the condition that $1 \mapsto {}^ec$. In particular, we shall often be interested in whether this map is left split or whether it is pure over R. In the main case, when R is reduced and $c \in R^\circ$, one can think instead about whether the injection $R \cong Rc^{1/p^e} \to R^{1/p^e}$ left splits over R, or is pure as a map of R-modules. Alternatively, if $R^{p^e} = \{r^{p^e} \mid r \in R\}$, one can ask, equivalently, whether the map $R^{p^e} \to R$ such that $1 \mapsto c$ left splits, or is pure, over R^{p^e}. We note:

Remark 2.1 ([HH4, Remarks 5.4]). Let $h\colon R \to S$ be a ring homomorphism (with R and S being rings of any characteristic) and M an S-module.

(1) If there exists $f \in \operatorname{Hom}_R(R, M)$ that is pure (left split) then $h\colon R \to S$ is pure (left split).
(2) When R and S have prime characteristic $p > 0$, we apply (1) to the map $F^e \circ h\colon R \to S \to S$ to obtain the following special case: if there exists $f \in \operatorname{Hom}_R(R, {}^{e_0}M)$ that is pure (left split) for some $e_0 \in \mathbb{N}_+$ then $F^{e_0} \circ h\colon R \to S$ is pure (left split) thus $h\colon R \to S$ is pure (left split). If furthermore $R = S$ and h is the identity map, then $F^e\colon R \to R$ is pure (left split) for all $e \in \mathbb{N}$, i.e., R is F-pure (in particular, R is reduced).

Definition 2.2. We define *strong F-regularity* of a ring R more generally as follows: we say that R is strongly F-regular if, for every local ring $R_\mathfrak{m}$ of R at a maximal ideal $\mathfrak{m}$ and every $c \in (R_\mathfrak{m})^\circ$, the $R_\mathfrak{m}$-linear map $R_\mathfrak{m} \to {}^e(R_\mathfrak{m})$ defined by $1 \mapsto {}^ec$ (hence $r \mapsto {}^e(r^qc)$ for all $r \in R$) is pure (or, equivalently, is left split in the F-finite case) for some $e \in \mathbb{N}_+$ or, equivalently, for all $e \gg 0$.

Evidently, this agrees with the earlier definitions in [HH1, HH4] in the F-finite case, which simply required that the map be left split. That property passes to localizations. Moreover, in the F-finite case, if one has a splitting when one localizes at $\mathfrak{m}$, one also has it in a Zariski open neighborhood of $\mathfrak{m}$, so that the existence of a splitting locally implies that there is a splitting globally.

A particularly noteworthy result is the equivalence of weak and strong F-regularity for finitely generated $\mathbb{N}$-graded algebras over a field of characteristic $p > 0$, proved in [LS1]. It remains an open question whether this equivalence holds for locally excellent rings, and this question is open even for affine algebras over an algebraically closed field.

Discussion 2.3. The problem of defining strong F-regularity when the ring is Noetherian but not F-finite has a long history.

(1) Smith [Sm, Definition 7.1.3] proposed that in the local case, the property be defined by the condition that 0 be tightly closed in an injective hull of the residue class field, and that the property be defined globally by the requirement that all local rings at maximal ideals be strongly F-regular in this sense.
(2) Hochster [Ho7, p. 166] proposed that the condition be defined by requiring that every submodule of every module be tightly closed.
(3) In [HH4, Remark 5.2] the possibility is suggested of defining strong F-regularity by requiring that for all $c \in R^\circ$, the map $R \to {}^eR$ such that $1 \mapsto {}^ec$ be pure over R for $e \gg 1$. This idea is pursued in [Hash, Definition 3.4] and in [DaSm, §6]. In [DaSm] this condition is considered even when R is not Noetherian (since the authors are interested in valuation rings that may not be Noetherian). This property is called *very strong F-regularity* in [Hash] while Datta and Smith say that R is *F-pure regular* when this condition holds. We use the term "very strong F-regularity" here.

Thus, our definition of strongly F-regular for Noetherian rings can be rephrased as requiring that all local rings of R at maximal ideals are very strongly F-regular, and the condition of very strong F-regularity then turns out to hold for *all* local rings of R. In §6 we construct regular Noetherian domains, even in dimension two, that are not very strongly F-regular, which answers a question raised in [DaSm, 6.3.3]. In [HoY], we show that every stronly F-regular excellent ring is very strongly F-regular.

Below we discuss the equivalence of Definition 2.2 here, of the definition proposed in [Sm, Definition 7.1.3], and of the definition proposed in [Ho7, p. 166]. See Theorem 2.11]. All of these are also equivalent to very strong F-regularity if the ring is F-finite. The condition of very strong F-regularity is strictly stronger by the results of §6, but is equivalent if the ring is F-finite or local; the local case is proved in [Sm, 7.1.2], in [LS2, Proposition 2.9], and in [DaSm, Proposition 6.3.2].

Note that in the definition of F-pure regularity in [DaSm, §6] the authors work with the assumption that c is not a zerodivisor rather than that c is an element of R°. However, this does not matter, since their condition implies that R is reduced, and in a Noetherian ring R (the only case considered here), if R is reduced then R° is precisely the set of nonzerodivisors.

Henceforth, when we use the term "R is strongly F-regular" we mean that R is a Noetherian ring of prime characteristic $p > 0$ satisfying Definition 2.2. When we use the term "R is very strongly F-regular" we mean that R is a Noetherian ring of prime characteristic $p > 0$ such that for every $c \in R^\circ$ for all $e \gg 1$ the map $\theta_e : R \to {}^eR$ such that $1 \mapsto {}^ec$ is pure, as indicated in Discussion 2.3. Note that the condition that θ_e is pure implies that c is a nonzerodivisor. Hence, in defining the notion of very strongly F-regular ring, one may require instead that for every nonzerodivisor $c \in R$, the map $R \to {}^eR$ such that $1 \mapsto {}^ec$ is pure for all $e \gg 1$. Both conditions imply that R is reduced, so that R° is the set of nonzerodivisors.

We next give several results that establish the equivalences stated in Discussion 2.3 and show that strongly F-regular rings in the more general sense, i.e., without the condition that the ring be F-finite, have behavior very similar to what one has in the F-finite case.

Remark 2.4. Let S be an R-algebra. If $\varphi : R \to S$ mapping $1 \mapsto c$ is pure (as a map of R-modules), N is an R-module, $u \in N$, and $u \otimes c = 0$ in $N \otimes_R S$, then $u = 0$ in N, since the induced map $N \cong N \otimes_R R \to N \otimes_R S$ sending

$$u \mapsto u \otimes 1 \mapsto u \otimes c$$

is injective.

The following is also proved in [Hash, §3] and in [DaSm, §6], but since the argument is very brief we include it here.

Remark 2.5. If R is very strongly F-regular, then every submodule of every R-module is tightly closed. It suffices to see that if u is in the tight closure of 0 in N then $u = 0$. Choose $c \in R^\circ$ such that $u \otimes {}^ec = 0$ in $N \otimes_R {}^eR$ for all $e \gg 0$. Then choose $e \gg 0$ such that $R \to {}^eR$ with $1 \to {}^ec$ is pure. Now Remark 2.4 gives the result.

Remark 2.6. If W is any multiplicative system in R, then R° maps into $(W^{-1}R)^\circ$, and every element of $(W^{-1}R)^\circ$ is a unit times the image of an element in R°: in fact, if $w \in W$, $c \in R$, and $w^{-1}c \in (W^{-1}R)^\circ$, then $c/1$ is the image of an element of R°. To see this, choose an element d of R that is precisely in those minimal primes of R to which c does not belong. The $d/1$ is in every minimal prime of $W^{-1}R$ (since $c/1$ is not in any of these) and so is nilpotent in $W^{-1}R$. We may replace d by a power so that $d/1$ is 0 in $W^{-1}R$. Then $c + d \mapsto (c + d)/1 = c/1$ has the required property.

Remark 2.7. Let R be a Noetherian ring, let N be any R-module, not necessarily finitely generated, and let $u \in N$ be a nonzero element that is contained in every nonzero submodule of N. Then u is killed by a maximal ideal $\mathfrak{m}$ of R, and N injects to an injective hull $\mathrm{E}_R(k)$ of $k = R/\mathfrak{m}$, for some maximal ideal $\mathfrak{m}$ of R, over R. (Let $\mathfrak{m}$ be any maximal ideal of R such that $\mathrm{Ann}(u) \subseteq \mathfrak{m}$. If $\mathrm{Ann}(u) \subsetneq \mathfrak{m}$ then $u \notin \mathfrak{m}u$, which implies $\mathfrak{m}u = 0$, a contradiction. So $Ru \cong R/\mathfrak{m}$ and the extension $k \cong Ru \subseteq N$ must then be essential).

Remark 2.8. Let R be a Noetherian ring of prime characteristic $p > 0$. It follows from Remark 2.7 that every submodule of every R-module is tightly closed if and only if 0 is tightly closed in $\mathrm{E}_R(R/\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R. (If u is in in $W_N^* \setminus W$ for some R-modules $W \leqslant N$, we can replace W by a submodule W' of N maximal with respect to containing W but not containing u, by using Zorn's lemma, and then we may replace u by its image in N/W', which injects into $\mathrm{E}_R(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of R). Of course, if R is local this condition need only be placed on the unique maximal ideal.

Note also that $\mathrm{E}_R(R/\mathfrak{m}) \cong \mathrm{E}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m})$ and that 0 is tightly closed in this module over R if and only if 0 is tightly closed in this module over $R_\mathfrak{m}$. ("If" is clear, since $c \in R^\circ$ maps to $c/1 \in (R_\mathfrak{m})^\circ$. "Only if" follows from Remark 2.6.)

The following result is proved, essentially, in [Sm, §7.1.2], although a splitting is used when only purity is needed. It is also proved in [LS2, Proposition 2.9], and in [DaSm, 6.3.2].

Proposition 2.9. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p > 0$. Then 0 is tightly closed in the injective hull $\mathrm{E}_R(k)$ if and only if for every $c \in R^\circ$, there exists $e \in \mathbb{N}_+$ such that the map $R \to {}^eR$ with $1 \mapsto {}^ec$ is pure over R.* □

Proposition 2.10. *Assume that given rings are Noetherian of prime characteristic $p > 0$.*

1. *The localization of a strongly F-regular ring at any multiplicative system is strongly F-regular.*
2. *A finite product of Noetherian rings is strongly F-regular if and only if each factor ring is strongly F-regular.*
3. *If R is strongly F-regular, then every R-submodule of every R-module is tightly closed, even if the modules are not finitely generated. In particular, R is weakly F-regular and, hence, normal. Thus, R is a finite product of normal domains, each of which is strongly F-regular.*
4. *Strongly F-regular rings are F-regular.*

Proof. (1) It suffices to show that R_P is strongly F-regular for all primes P: we may then apply this to the primes disjoint from W. Choose a maximal ideal $\mathfrak{m}$ containing P. Then $R_\mathfrak{m}$ is strongly F-regular, and we may replace R by $R_\mathfrak{m}$ without loss of generality. Now assume that R is local and let $W = R \setminus P$, with P prime. Given an element $c/w \in (W^{-1}R)^\circ$, choose $c + d \in R^\circ$ as in Remark 2.6 so that $c + d \mapsto c/1$ in $W^{-1}R$. Then there exists e so that $R \to {}^eR$ with $1 \mapsto {}^e(c + d)$ is pure. Then $W^{-1}R \to W^{-1}({}^eR) \cong {}^e(W^{-1}R)$ with $1/1 \mapsto {}^e((c + d)/1)$ is pure, and this map has the same image as $W^{-1}R \to {}^e(W^{-1}R)$ with $1 \mapsto {}^e(c/w)$.

(2) This is immediate from the local nature of the definition.

(3) Suppose that R is strongly F-regular. For the first statement, by Remark 2.8, it suffices to show that 0 is tightly closed in $\mathrm{E}_R(R/\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$. Suppose that a nonzero element $u \in \mathrm{E}_R(R/\mathfrak{m})$ is in the tight closure of 0 over R. Then this remains true over $R_\mathfrak{m}$, contradicting Proposition 2.9. The second statement is clear: it is the case of the first statement where the modules are finitely generated. The last statement is clear from (2).

(4) This follows at once from (1) and (3). □

Now we are ready to show the equivalence among some of the definitions of 'strong F-regularity' in the literature. In Theorem 2.11 below, (4) is how the notion is defined in [Sm] while (5) is how the notion is defined in [Ho7]. We have:

Theorem 2.11. *For a ring R of prime characteristic p, the following are equivalent:*

1. *R is strongly F-regular.*
2. *R_P is strongly F-regular for all $P \in \mathrm{Spec}(R)$.*
3. *$R_\mathfrak{m}$ is strongly F-regular for all $\mathfrak{m} \in \mathrm{Max}(R)$.*

(4) $0^*_M = 0$ *over* $R_{\mathfrak{m}}$ *for all* $\mathfrak{m} \in \mathrm{Max}(R)$ *and for all* $R_{\mathfrak{m}}$*-modules* M *or, equivalently for* $M = \mathrm{E}_R(R/\mathfrak{m})$*. (Note that* M *is not necessarily finitely generated.)*
(5) $0^*_M = 0$ *for all* R*-modules* M*. (Again, note that* M *is not necessarily finitely generated.)*

Proof. (1) ⇒ (2): This follows from Proposition 2.10(1).

(2) ⇒ (3) is clear.

(3) ⇒ (4): Apply Proposition 2.9 and Remark 2.8.

(4) ⇒ (5): The fact that 0 is tightly closed in $\mathrm{E}_R(R/\mathfrak{m}) \cong \mathrm{E}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}R_m)$ over $R_{\mathfrak{m}}$ implies that this holds over R as well, and we may then apply the first paragraph of Remark 2.8.

(5) ⇒ (1): For every $\mathfrak{m} \in \mathrm{Max}(R)$, we have $0^*_{\mathrm{E}_R(R/\mathfrak{m})} = 0$ (over R, and, hence, over $R_{\mathfrak{m}}$ by the second paragraph of Remark 2.8) which implies that $R_{\mathfrak{m}}$ is strongly F-regular by Proposition 2.9. □

If $(R, \mathfrak{m}, k)$ is an F-finite regular local ring, then ${}^eR$ is free over R for all e by [Ku1, Theorem 2.1]. Thus for any $c \in R^\circ$, the R-linear map $R \to {}^eR$ sending 1 to c splits if and only if e is large enough such that $c \notin \mathfrak{m}^{[p^e]} = \mathfrak{m}({}^eR)$. This shows that every F-finite regular ring is strongly F-regular. In fact every regular local ring (F-finite or not) is strongly F-regular, since $0^*_M = 0$ for all R-modules M. We also note:

Proposition 2.12. *Let* $h : R \to S$ *be a homomorphism of rings of prime characteristic* p*. If* h *is pure over* R*,* $c \in R$ *and* $\beta : S \to {}^eS$ *with* $\beta(1) = {}^e(h(c))$ *is pure over* R *(e.g., over* S*), then* $\alpha : R \to {}^eR$ *with* $1 \mapsto {}^ec$ *is pure over* R*.*

Proof. The map $R \to {}^eS$ which may be thought of as either of the composites ${}^eh \circ \alpha$ or $\beta \circ h$ is pure (because the second composition is). Since ${}^eh \circ \alpha$ is pure, α must be pure. □

Corollary 2.13. *If* $R \to S$ *is pure,* R° *maps into* S°*, and* S *is strongly F-regular (respectively, very strongly F-regular), then* R *is strongly F-regular (respectively, very strongly F-regular). In particular, if* S *is faithfully flat over* R *and* S *is strongly F-regular (respectively, very strongly F-regular), then* R *is strongly F-regular (respectively, very strongly F-regular).*

We have the following criterion of strong F-regularity.

Theorem 2.14 ([HH4, Theorem 5.9], [HH1, Theorem 3.3])**.** *Let* R *be an F-finite ring and* $c \in R^\circ$ *such that* R_c *is strongly F-regular. Then* R *is strongly F-regular if and only if there exists* $e \in \mathbb{N}_+$ *such that the* R*-linear map* $R \to {}^eR$ *determined by* $1 \mapsto {}^ec$ *is left split.*

The above criterion of strong F-regularity can be considerably generalized in certain cases where one does not have F-finiteness, including, for example, to all rings essentially of finite type over an excellent semilocal rings. Such rings are quite well-behaved with respect to the property of strong F-regularity. We shall prove below that for such rings, strong F-regularity is equivalent to very strong F-regularity and that the strongly F-regular locus is open. Note that the results for the case when the semilocal base is local are proved in [Hash, §3].

We first need the lemma just below. We will use results on flat local extensions $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ from [HH4, Lemma 7.10] which depends on [Mat, Corollary 20.F]. Note that in [HH4, Lemma 7.10(b)] the restriction that M be finitely generated is not needed: if $z_1, \ldots, z_d \in \mathfrak{n}$ have images that are a regular sequence in $S/\mathfrak{m}S$, then they are a regular sequence on *every* module of the form $M \otimes_R S$: this follows by a direct limit argument, since M is the directed union of its finitely generated submodules.

Lemma 2.15. *Let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a flat ring homomorphism of local rings such that $S/\mathfrak{m}S$ is regular. For any $c \in R$ and $e \in \mathbb{N}$, if the R-linear map $f\colon R \to {}^eR$ sending 1 to ${}^ec$ is pure then the S-linear map $g\colon S \to {}^eS$ sending 1 to ${}^e\varphi(c)$ is pure.*

Proof. Since S is faithfully flat over R, we may think of R as a subring of S and assume that φ is an inclusion map. Let $\underline{z} = z_1, \ldots, z_d$ be elements of S whose images in $S/\mathfrak{m}S$ form a regular system of parameters. Let $z = z_1 \cdots z_d$. Then $\mathrm{H}^d_{(\underline{z})}(S) \cong \varinjlim_t (S/\mathfrak{A}_t)$, where $\mathfrak{A}_t := (z_1^t, \ldots, z_d^t)S$.

Let $E = \mathrm{E}_R(R/\mathfrak{m})$ be the injective hull of $R/\mathfrak{m}$, and let u be a generator of the socle of E. By [HH4, Lemma 7.10] we may identify $\mathrm{E}_S(S/\mathfrak{n})$ with

$$E_S := E \otimes_R \mathrm{H}^d_{(\underline{z})}(S) \cong \varinjlim_t \big(E \otimes_R (S/\mathfrak{A}_t)\big),$$

and a socle generator in each $E \otimes_R (S/\mathfrak{A}_t)$ is represented by $\eta_t := u \otimes z^{t-1}$. These all map to the socle element in $E \otimes_R \mathrm{H}^d_{(\underline{z})}(S)$, which we denote η. (In the case where $S/\mathfrak{m}S$ is a field, i.e., when $\mathfrak{n} = \mathfrak{m}A$, this is [Yao1, Remark 2.3(3)].) Note that the maps between consecutive terms in the direct limit system are induced by multiplication by z.

By Proposition 1.6, it will be sufficient to show that the induced S-linear map $\theta : E_S \to E_S \otimes_S {}^eS$ does not kill $u \otimes \eta$, and we may think of θ as the induced map

$$\theta : E \otimes_S \mathrm{H}^d_{(\underline{z})}(S) \to (E \otimes_S \mathrm{H}^d_{(\underline{z})}(S)) \otimes {}^eS.$$

Note that $\theta(u \otimes \eta) = u \otimes \eta \otimes {}^ec$. By a direct limit argument, it suffices to show that for every t, the induced map $\theta_t : E \otimes_S (S/\mathfrak{A}_t) \to (E \otimes_S (S/\mathfrak{A}_t)) \otimes {}^eS$ does not kill $u \otimes \tilde{z}^{t-1}$, where $\tilde{z}$ is the image of z in $S/\mathfrak{A}_t$. Thus, we need to prove that $\theta_t(u \otimes \tilde{z}^{t-1}) = u \otimes \tilde{z}^{t-1} \otimes {}^ec \in E \otimes_R (S/\mathfrak{A}_t) \otimes_S {}^eS$, is not 0. With $q = p^e$, as usual, we have that

$$E \otimes_R (S/\mathfrak{A}_t) \otimes_S {}^eS \cong E \otimes_R {}^e(S/\mathfrak{A}_{qt}) \cong (E \otimes_R {}^eR) \otimes_{{}^eR} {}^e(S/\mathfrak{A}_{qt}),$$

and $\theta_t(u \otimes \tilde{z}^{t-1})$ may be identified with $(u \otimes {}^ec) \otimes_{{}^eR} \otimes {}^e\overline{z}^{q(t-1)}$, where $\overline{z}$ denotes the image of z in $(S/\mathfrak{A}_{qt})$. Since the z_i form a regular sequence on $(E \otimes_R {}^eR) \otimes_{{}^eR} {}^eS$, if $(u \otimes {}^ec) \otimes \overline{z}^{q(t-1)}$ were 0 in $(E \otimes_R {}^eR) \otimes_{{}^eR} {}^e(S/\mathfrak{A}_{qt})$ then $(u \otimes {}^ec) \otimes 1 \in \mathfrak{A}_q((E \otimes_R {}^eR) \otimes_{{}^eR} {}^eS)$, which translates to $(u \otimes {}^ec) \otimes 1 = 0$ in $(E \otimes {}^eR) \otimes_{{}^eR} {}^e(S/\mathfrak{A}_q)$. Since ${}^e(S/\mathfrak{A}_q)$ is faithfully flat, and therefore pure, over ${}^eR$, the map $E \otimes_R {}^eR \to (E \otimes {}^eR) \otimes_{{}^eR} {}^e(S/\mathfrak{A}_q)$ such that $v \mapsto v \otimes 1$ is injective. Thus, if $\theta_t(u \otimes {}^ec \otimes \tilde{z}^{t-1}) = 0$, we have that $u \otimes {}^ec = 0$ in $E \otimes_R {}^eR$, contradicting our hypothesis. $\square$

Proposition 2.16. *Let R be a Noetherian ring of prime characteristic $p > 0$. If there is a Zariski open cover $\{D(f_\lambda) \,|\, \lambda \in \Lambda\}$ of $\mathrm{Spec}(R)$ such that every R_{f_λ} is very strongly F-regular for every $\lambda \in \Lambda$, then R is very strongly F-regular.*

Proof. First, pass to a finite subcover, say $\{D(f_{\lambda_i}) \,|\, 1 \le i \le n\}$. Let $c \in R^\circ$ be given. Choose $e_i \in \mathbb{N}_+$ such that $R_{f_{\lambda_i}} \to {}^eR_{f_{\lambda_i}}$ with $1 \mapsto {}^ec/1$ is pure, and let e denote the supremum of the e_i. Then $R \to {}^eR$ with $1 \mapsto {}^ec$ is pure. $\square$

Theorem 2.17. *Let R be a Noetherian ring. Suppose that for every prime ideal $\mathfrak{p}$ of R,* $\operatorname{Spec}(R/\mathfrak{p})$ *contains a non-empty open set on which there is a finite bound for the multiplicities of all local rings; this holds if the singular locus of* $\operatorname{Spec}(R/\mathfrak{p})$ *is not dense, and, in particular, this holds whenever R is excellent. Then the multiplicity function* $\mathrm{e}\big((R/I)_P\big)$ *is bounded on* $\operatorname{Spec}(R/I)$ *for every ideal I of R. In particular, the multiplicity function* $\mathrm{e}\big(R_P\big)$ *is bounded on* $\operatorname{Spec}(R)$.

Proof. We use Noetherian induction on I. If $I \subseteq P$, and $\mathcal{S}(P)$ is the set of minimal primes $\mathfrak{p}$ of I contained in P such that the dimension of $R_P/\mathfrak{p}R_P$ is equal to the dimension of R_P, then

$$\mathrm{e}(R_P/I_P) = \sum_{\mathfrak{p}\in\mathcal{S}(P)} \ell(R_\mathfrak{p}/IR_\mathfrak{p})\,\mathrm{e}(R_P/\mathfrak{p}R_P).$$

Consequently if we have a bound $B_\mathfrak{p}$ for multiplicities of the local rings of $R/\mathfrak{p}$ for every minimal prime $\mathfrak{p}$ of I we get the bound $\sum_{\mathfrak{p}\in\operatorname{Min}(I)} \ell(R_\mathfrak{p}/IR_\mathfrak{p})B_\mathfrak{p}$. Thus, we may assume that $I = \mathfrak{p}$ is prime. By assumption, there exists $f \notin \mathfrak{p}$ such that we have a bound B for the multiplicities of the local rings of $(R/\mathfrak{p})_f$. For primes P not containing f, the multiplicity of $(R/\mathfrak{p})_P$ is at most B. For primes P containing f, we have $\mathrm{e}\left((R/\mathfrak{p})_P\right) \leqslant \mathrm{e}\left((R/(\mathfrak{p}+fR))_P\right)$, which is bounded by the hypothesis of Noetherian induction applied to $\mathfrak{p}+fR$. $\square$

The following result, with the hypothesis that the singular locus is not dense in every homomorphic image domain of the regular ring R, also appears with a different proof in [DET, Theorem 7.1.4].

Corollary 2.18. *Let R be an regular ring of prime characteristic $p > 0$. Suppose that the multiplicity function* $\mathrm{e}\big(R_P\big)$ *is bounded on* $\operatorname{Spec}(R)$*; this holds, for example, if for every prime ideal $\mathfrak{p}$ of R, the singular locus in* $\operatorname{Spec}(R/\mathfrak{p})$ *is not dense. Then R is very strongly F-regular. In particular, excellent regular rings are very strongly F-regular.*

Proof. Let $c \in R^\circ$ be given. By Theorem 2.17 we can choose e so that p^e is a strict upper bound for the multiplicities of all the local rings of R/cR. We shall show that the map $R \to {}^eR$ such that $1 \to {}^ec$ is pure.

It suffices to check this after localizing at a prime P. Therefore we may assume that $(R,\mathfrak{m})$ is regular local. Since $\mathrm{e}(R/cR) < p^e$, we see that $c \notin \mathfrak{m}^{p^e}$, and so $c \notin \mathfrak{m}^{[p^e]}$. Since $\widehat{R}$ is faithfully flat over R, it suffices to show that the corresponding map of completions is pure. We may then make a faithfully flat extension of $\widehat{R}$, enlarging the residue class field to an algebraically closed field. In this F-finite case, ${}^eR$ is module-finite and free over R, and ${}^ec$ is part of a free basis, since the expansion of $\mathfrak{m}$ to ${}^eR$ is ${}^e\big(\mathfrak{m}^{[p^e]}\big)$. $\square$

In our next major result, Theorem 2.21, we make essential use of the Γ construction 1.9. When Λ is a set, we use the terminology "for all $\Gamma \lll_{\mathrm{cof}} \Lambda$" to mean "for all sufficiently small cofinite subsets of Λ." In other words, a statement holds for all $\Gamma \lll_{\mathrm{cof}} \Lambda$ precisely if there exists Γ_0 cofinite in Λ such that the statement holds for all $\Gamma \subseteq \Gamma_0$ that are cofinite in Λ. The following fact about the Γ construction is proved in [HoJ, Theorem 5.5(i)].

Theorem 2.19. *Let R be a ring of prime characteristic $p > 0$ essentially of finite type over an complete local ring A, and let Λ denote a p-base for a coefficient field*

k for A. Then for all $\Gamma \lll_{\mathrm{cof}} \Lambda$, the singular locus in R^Γ maps homeomorphically to the singular locus of R under the restriction of the canonical homeomorphism $\mathrm{Spec}(R^\Gamma) \approx \mathrm{Spec}(R)$.

Note that an excellent local ring is strongly F-regular if and only if its F-signature is positive (cf. [AL]). We will need the following result from [Yao1, Theorem 5.6]

Theorem 2.20. *If $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is flat local with regular closed fiber, then R and S have the same F-signature.*

Part (2) of the result that follows is parallel to Theorem 2.19.

Theorem 2.21. *Let R be a ring essentially of finite type over a complete local ring $(A, \mathfrak{m})$ of prime characteristic $p > 0$, let $k \subseteq A$ be a coefficient field, so that $k \cong A/\mathfrak{m}$, and let Λ be a p-base for k. Then:*

1. *For each prime P of R, R_P is strongly F-regular (equivalently, very strongly F-regular) if and only if for all $\Gamma \lll_{\mathrm{cof}} \Lambda$, R^Γ_P is strongly F-regular.*
2. *For all $\Gamma \lll_{\mathrm{cof}} \Lambda$, the strongly F-regular locus of R is the same as the strongly F-regular locus of R^Γ, and is open.*
3. *R is strongly F-regular if and only if for all $\Gamma \lll_{\mathrm{cof}} \Lambda$ we have that R^Γ is strongly F-regular.*
4. *R is strongly F-regular if and only if R is very strongly F-regular.*

Proof. (1) follows from (2) by reduction to the F-finite case (the parenthetical comment holds because we are in the local case), while (3) is a special case of (2).

(2) Since the strongly F-regular locus of R^Γ (which is open, because of the F-finite condition) can only increase as the cofinite subset Γ shrinks, the locus stabilizes for all $\Gamma \lll_{\mathrm{cof}} \Lambda$. But this stabilized open locus (which is a subset of the strong F-regular locus of R) must agree with the strongly F-regular locus of R. If not, there is a prime $P \in \mathrm{Spec}(R^\Gamma)$ such that R^Γ_P is not strongly F-regular but $R_{P \cap R}$ is. But then, if the cofinite subset Γ is taken sufficiently smaller, the fiber of $R \to R^\Gamma$ at P is a field and, hence P is in the strongly F-regular locus of R^Γ by Theorem 2.20, a contradiction.

(4) The "if" implication is clear. For the "only if" implication, note that the strong F-regularity of R implies the same for R^Γ (for all $\Gamma \lll_{\mathrm{cof}} \Lambda$), which then implies the very strong F-regularity of R^Γ, and this in turn implies the same for R in light of Corollary 2.13. □

Remark 2.22. The following results in the case where B is local are essentially in [Hash, §3]. We also note that Theorem 2.23(2) is proved in the local case in [DaMuSm, Remark 3.2.2(2)] with the weaker assumption that the local ring B is a G-ring instead an excellent local ring.

Theorem 2.23. *Let R be a Noetherian ring of prime characteristic $p > 0$ that is essentially of finite type over an excellent semilocal ring B. Let $\widehat{B}$ be the completion of B with respect to its Jacobson radical, and let $\widetilde{R} := \widehat{B} \otimes_B R$.*

1. *R is strongly F-regular if and only if $\widetilde{R}$ is strongly F-regular.*
2. *R is strongly F-regular if and only if it is very strongly F-regular, and the same holds for $\widetilde{R}$.*
3. *The strongly F-regular locus of R is open. Moreover, if J is the radical ideal that defines the complement of the strongly F-regular locus of $\widetilde{R}$, and I is the*

radical ideal that defines the complement of the strongly F-regular locus of R*, we have that* $I = J \cap R$ *and* $J = \mathrm{Rad}(I\widetilde{R})$.

Proof. Note that if the local rings of B at its maximal ideals are $A_1, \ldots, A_h$, then $\widehat{B} \cong \prod_{i=1}^{h} \widehat{A_i}$. Let $\widetilde{R}_i := \widehat{A_i} \otimes_B R$. Then $\widetilde{R} \cong \prod_{i=1}^{h} \widetilde{R}_i$.

(1) "If" follows because $\widetilde{R}$ is faithfully flat over R. Now assume that R is strongly F-regular. Any prime ideal Q of $\widetilde{R}$ lies over a prime ideal of P of R, and $\widetilde{R}_Q \cong (\widetilde{R}_P)_{Q\widetilde{R}_P}$. Thus, it suffices to prove the result when R is replaced by R_P, and we may first replace R by $R_{\mathfrak{m}}$ where $\mathfrak{m}$ is maximal in R. Thus, by Proposition 2.10(2), there is no loss of generality in replacing B by its localization A_i at a maximal ideal, and we henceforth assume that $B = A$ is local. Suppose that $\widetilde{R}$ has a maximal ideal $\mathcal{M}$ at which the localization is not strongly F-regular. Let P be the contraction of $\mathcal{M}$ to R. Since A is excellent, $A \to \widehat{A}$ is geometrically regular, and, hence, so is $R \to \widetilde{R}$. Consequently, $R \to \widetilde{R}_{\mathcal{M}}$ has geometrically regular fibers, which implies that the closed fiber of the map $R_P \to \widetilde{R}_{\mathcal{M}}$ is regular. It follows from Theorem 2.20 that $\widetilde{R}_{\mathcal{M}}$ is strongly F-regular after all, a contradiction.

(2) By part (1), if R is strongly F-regular then so is $\widetilde{R}$. But then $\widetilde{R}$ is very strongly F-regular by Theorem 2.21(4) applied to the factors R_i, and hence R is very strongly F-regular because $R \to \widetilde{R}$ is faithfully flat, see Corollary 2.13.

(3) Now let J be the defining radical ideal of the complement of the strongly F-regular locus of $\widetilde{R}$, which is closed by Theorem 2.21(2) applied to the factors R_i, and let I be its contraction to R. We want to show that I defines the complement of the strongly F-regular locus of R (which proves that it is closed), and that $J = \mathrm{Rad}(I\widetilde{R})$. Let P be a prime ideal of R. Then R_P is strongly F-regular if and only if $\widetilde{R}_P := (R \setminus P)^{-1}\widetilde{R}$ is by part (1), and the condition that $\widetilde{R}_P$ be strongly F-regular is equivalent to the condition that the multiplicative system $R \setminus P$ meets J. This says exactly that $J \cap R = I$ is not contained in P, which says that that $P \notin V(I)$ Thus, I defines the complement of the strongly F-regular locus of R.

It remains only to show that J is the radical of $I\widetilde{R}$. Otherwise, there exists a prime Q of $\widetilde{R}$ that contains IS but not J. Then Q contracts to a prime P of R with $I \subseteq P$, which forces that R_P is not strongly F-regular. But since Q does not contain J, $\widetilde{R}_Q$ is strongly F-regular, a contradiction, since it is faithfully flat over R_P. □

Recall that a Noetherian ring is called *locally excellent* if its localization at each maximal ideal (equivalently, every prime ideal) is excellent. Now we are ready to generalize Theorem 2.14 in a way that substantially relaxes the hypothesis of F-finiteness on the ring. This generalization is used in §4.

Theorem 2.24. *Let* R *be a locally excellent ring and* $c \in R^\circ$ *be such that* R_c *is strongly F-regular. If the map* $\theta_e : R \to {}^eR$ *such that* $1 \mapsto {}^ec$ *is pure for some* e *over* R*, then* R *is strongly F-regular. If* R *is very strongly F-regular, the converse holds: if there exists* $c \in R^\circ$ *such that* R_c *is strongly F-regular, there must exist* e *such that* θ_e *is pure over* R.

In particular, if $\mathcal{C}$ *is a class of locally excellent rings for which we know that the conditions strongly F-regular and very strongly regular are equivalent, and* $c \in R^\circ$ *is such that* R_c *is strongly F-regular, then the purity of* θ_e *for some* e *is necessary and sufficient for strong F-regularity.*

Proof. The condition that θ_e be pure for some e over R is obviously necessary for very strong F-regularity. We only need to show the 'if' direction, for which it suffices to prove that R_P is strongly F-regular for all $P \in \operatorname{Spec}(R)$ such that $c \in P$. Thus, without loss of generality, we may assume that $(R, \mathfrak{m}, k)$ is an excellent local ring with $c \in \mathfrak{m}$. If we replace R by its completion, R_c remains strongly F-regular, by Theorem 2.23(1), where we may take $R = B$. It suffices to show that $\widetilde{R} = \widehat{R}$ is F-regular. Thus, we have reduced to proving the result when R is complete. Choose a p-base Λ for k. By Lemma 2.15, $R^\Gamma \to {}^eR^\Gamma$ such that $1 \mapsto {}^ec$ remains pure for all $\Gamma \lll_{\mathrm{cof}} \Lambda$: one only needs the $\mathfrak{m}R^\Gamma$ remain prime. By Theorem 2.14, R^Γ is strongly F-regular, and, hence, so is R. □

Remark 2.25. We already know that we can take $\mathcal{C}$ to include all F-finite rings and all rings essentially of finite type over an excellent semilocal ring. We do not know whether the conditions of strong F-regularity and very strong F-regularity are equivalent for all excellent rings.

3. From strong F-regularity to the splitting of ${}^eM$

Let R be an F-finite ring and M be a finitely generated R-module. We investigate whether and when ${}^eM$ splits non-trivially for $e \gg 0$.

The case when $\lambda_R(M) < \infty$ is well understood. For simplicity (and in fact without loss of generality), we assume $(R, \mathfrak{m}, k)$ is local and M is an R-module of finite length. There exists $e_0 \in \mathbb{N}_+$ such that $\mathfrak{m}^{[p^{e_0}]} \subseteq \operatorname{Ann}_R(M)$. Then ${}^{e_0}M$ is annihilated by $\mathfrak{m}$ and hence ${}^{e_0}M \cong k^{n_0}$ for some $n_0 \in \mathbb{N}_+$. In fact $n_0 = \lambda({}^{e_0}M) = p^{e_0\alpha(R)}\lambda(M)$. Thus ${}^eM \cong k^{q^{\alpha(R)}}$ for all $e \geqslant e_0$. In fact the following is true for the case $\dim(M) = 0$.

Observation 3.1. Let $M \neq 0$ be a module over an (F-finite or not) ring R such that $\lambda(M) < \infty$. Then ${}^eM$ is indecomposable for all $e \in \mathbb{N}_+$ if and only if $\lambda(M) = 1$ and $R/\operatorname{Ann}(M)$ is a perfect field.

For the case $\dim(M) = 1$ or 2, there are the following results of Hochster and Yao:

Theorem 3.2 ([Ho3, Theorem 5.16(2)])**.** *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring and M a finitely generated R-module with $\dim(M) = 1$. Fix any $P \in \operatorname{Ass}(M)$ with $\dim(R/P) = 1$ and let $A = \overline{R/P}$ be the integral closure of R/P in its fraction field $k(P)$. Then, for any $n \in \mathbb{N}_+$, ${}^eM$ has a direct summand isomorphic to A^n for all $e \gg 0$.*

Theorem 3.3 ([Yao2, Theorem 1.8])**.** *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring and M a finitely generated R-module with $\dim(M) = 2$. If, for some $P \in \operatorname{Ass}_R(M)$ with $\dim(R/P) = 2$, there exists a module-finite extension A of R/P such that A is strongly F-regular, then for every $n \in \mathbb{N}_+$, ${}^eM$ has a direct summand isomorphic to A^n for all $e \gg 0$.*

When $\dim(M) > 2$, some cases of the splitting of ${}^eM$ are proved in [Ho3, Fact 5.14, Theorem 5.16] and [Yao2, Theorem 1.11].

The goal of this section is to generalize Theorem 3.3 to all positive dimensions. We start with some preparation.

For any F-finite ring R, any finitely generated R-module M, and any $e \in \mathbb{N}_+$, denote by $\sharp({}^eM)$ the largest rank of all free direct summands of ${}^eM$. If moreover

$(R, \mathfrak{m}, k)$ is local, Tucker showed in [Tu] that the following limits exist

$$\lim_{e\to\infty} \frac{\sharp({}^eR)}{q^{\dim(R)+\alpha(R)}} = \mathrm{s}(R) \in \mathbb{R} \quad \text{and} \quad \lim_{e\to\infty} \frac{\sharp({}^eM)}{q^{\dim(R)+\alpha(R)}} = r\,\mathrm{s}(R) \in \mathbb{R},$$

in which r is the torsion-free rank of M (assuming R is a domain). The limit $\mathrm{s}(R)$ is called the *F-signature* of R; see [HL]. Also, Aberbach and Leuschke proved in [AL] that $(R, \mathfrak{m}, k)$ is strongly F-regular if and only if $\mathrm{s}(R) > 0$. So for a finite generated faithful module over a strongly F-regular F-finite local ring $(R, \mathfrak{m}, k)$, $\sharp({}^eM)$ grows at the same magnitude as $q^{\dim(R)+\alpha(R)}$ when $e \to \infty$. Without the local assumption, at least the following properties hold:

Proposition 3.4.[1] *Let R be an F-finite strongly F-regular domain and M a finitely generated R-module supported on* $\mathrm{Spec}(R)$*. Denote by F the fraction field of R.*

1. *For every $x \in M$ such that $\mathrm{Ann}_R(x) = 0$, there exists $e_0 \in \mathbb{N}_+$ such that the R-linear map from R to ${}^{e_0}M$ defined by $1 \mapsto {}^{e_0}x$ is left split.*
2. *The sequence $\{\sharp({}^eM)\}_{e\in\mathbb{N}_+}$ is non-decreasing.*
3. *If $\alpha(F) > 0$ (e.g., $\dim(R) > 0$) then $\lim_{e\to\infty} \sharp({}^eM) = \infty$.*

Proof. (1) There exists $h \in \mathrm{Hom}_R(M, R)$ such that $h(x) = c \in R^\circ$. Then there exists $e_0 \in \mathbb{N}_+$ such that the R-linear map h_c from R to ${}^{e_0}R$ sending 1 to ${}^{e_0}c$ is left split. Note that $h_c = h \circ h_x$ where h_x is the R-linear map from R to ${}^{e_0}M$ sending 1 to ${}^{e_0}x$. Thus h_x is left split, as required.

(2) Say $\sharp({}^eM) = a$ for $e \in \mathbb{N}_+$. Then ${}^eM \cong R^a \oplus N$, hence ${}^{e+1}M \cong ({}^1R)^a \oplus {}^1N$. By Remark 2.1, ${}^1R$ has a free direct summand of rank (at least) 1. Thus ${}^{e+1}M$ has a free direct summand of rank (at least) a, meaning $\sharp({}^{e+1}M) \geqslant a = \sharp({}^eM)$. This proves that the sequence $\{\sharp({}^eM)\}_{e\in\mathbb{N}_+}$ is non-decreasing.

(3) There exists $x \in M$ such that $\mathrm{Ann}_R(x) = 0$. By (1), $\sharp({}^{e_0}M) \geqslant 1$ for some $e_0 \in \mathbb{N}_+$. This implies $\sharp({}^{e_0+e}M) \geqslant \sharp({}^eR)$ for all $e \in \mathbb{N}_+$. Thus it suffices to prove $\lim_{e\to\infty} \sharp({}^eR) = \infty$, that is, to prove that for every $n \in \mathbb{N}_+$, there exists $e_n \in \mathbb{N}_+$ such that $\sharp({}^eM) \geqslant n$ for all $e \geqslant e_n$.

The existence of e_1 is immediate, since $\sharp({}^eR) \geqslant 1$ for all $e \geqslant 1$ by Remark 2.1. Assume that there exists e_n such that $\sharp({}^eM) \geqslant n$ for all $e \geqslant e_n$. We may further assume $e_n > (\log_p n)/\alpha(F)$. Write ${}^{e_n}R \cong R^n \oplus k$. Note that ${}^{e_n}R$ has torsion-free rank $p^{e_n\alpha(F)} > n$; see Remark 1.3. Thus k has a positive torsion-free rank. By (1), there exists $e' \in \mathbb{N}_+$ such that $\sharp({}^{e'}k) \geqslant 1$. Let $e_{n+1} = e_n + e'$, so that

$${}^{e_n+e'}R \cong {}^{e'}(R^n \oplus k) = ({}^{e'}R)^n \oplus {}^{e'}k.$$

Thus $\sharp({}^{e_{n+1}}M) \geqslant n + 1$; see Remark 2.1. By (2), $\sharp({}^eM) \geqslant n + 1$ for all $e \geqslant e_{n+1}$. This completes the proof. □

If R is strongly F-regular, then R is a direct product of strongly F-regular domains. Thus Proposition 3.4 has a global version as follows.

Theorem 3.5. *Let R be an F-finite strongly F-regular ring and M a finitely generated R-module supported on* $\mathrm{Spec}(R)$*. Write $R = R_1 \times \cdots \times R_m$ with each R_i an F-finite strongly F-regular domain. Denote by F_i the fraction field of R_i.*

1. *For every $x \in M$ such that $\mathrm{Ann}_R(x) = 0$, there exists $e_0 \in \mathbb{N}_+$ such that the R-linear map from R to ${}^{e_0}M$ defined by $1 \mapsto {}^{e_0}x$ is left split.*
2. *The sequence $\{\sharp({}^eM)\}_{e\in\mathbb{N}_+}$ is non-decreasing.*

[1]See [DSPY] for a more detailed investigation about the growth of $\sharp({}^eM)$ as $e \to \infty$.

(3) *If* $\alpha(F_i) > 0$ *(e.g.,* $\dim(R_i) > 0$*) for all* $i \in \{1, \ldots, m\}$*, then* $\lim_{e\to\infty} \sharp({}^eM) = \infty$.

Proof. This follows from Proposition 3.4 via component-wise consideration over R_i. □

We are ready to generalize Theorem 3.2 and Theorem 3.3 without the restriction on dimensions or the local assumption. We use $\mathrm{Min}_R(M)$ to denote the set of minimal primes of an R-module M.

Theorem 3.6. *Let R be an F-finite ring and M a finitely generated R-module. If there exist distinct $P_1, \ldots, P_m \in \mathrm{Min}_R(M)$ such that, for every $i \in \{1, \ldots, m\}$, $\alpha(R_{P_i}) > 0$ (e.g., $\dim(R/P_i) > 0$) and there is a module-finite domain extension A_i of R/P_i such that A_i is strongly F-regular, then for every $n \in \mathbb{N}_+$ there exists $e_n \in \mathbb{N}_+$ such that, for every $e \geqslant e_n$, ${}^{e_n}M$ admits a direct summand that is isomorphic to $(\oplus_{i=1}^m A_i)^n$ over R.*

Proof. Replacing M by ${}^eM$ for $e \gg 0$, we may assume $\sqrt{\mathrm{Ann}_R(M)} = \mathrm{Ann}_R(M)$. Then replacing R by $R/\mathrm{Ann}_R(M)$, we may assume that R is reduced and M is faithful over R. Thus for each $i \in \{1, \ldots, m\}$, $P_i \in \mathrm{Min}(M) = \mathrm{Min}(R)$ and $R_{P_i} \cong (R/P_i)_{P_i} = k(P_i)$ is a field.

Let $W = R \setminus (\bigcup_{i=1}^m P_i)$. Then $W^{-1}R \cong R_{P_1} \times \cdots \times R_{P_m}$, a direct product of fields. Moreover, $W^{-1}M \cong M_{P_1} \times \cdots \times M_{P_m}$ is a finitely generated module over $W^{-1}R$. For each i, as $M_{P_i} \neq 0$, we see $M_{P_i} \cong (R_{P_i})^{s_i} = k(P_i)^{s_i}$ with $s_i \in \mathbb{N}_+$. Then

$$\begin{aligned} W^{-1}({}^eM) \cong {}^e(W^{-1}M) &\cong {}^e[(R_{P_1})^{s_1} \times \cdots \times (R_{P_m})^{s_m}] \\ &\cong ({}^eR_{P_1})^{s_1} \times \cdots \times ({}^eR_{P_m})^{s_m} \\ &\cong (R_{P_1})^{s_1 p^{e\alpha(R_{P_1})}} \times \cdots \times (R_{P_m})^{s_m p^{e\alpha(R_{P_m})}} \end{aligned}$$

for all $e \in \mathbb{N}_+$; see Remark 1.3.

Regarding each A_i as an R-module, we see $W^{-1}A_i \cong A_{P_i}$, which is a finitely generated module over the field R_{P_i}. So $A_{P_i} \cong (R_{P_i})^{r_i}$ for some $r_i \in \mathbb{N}_+$.

Let $A := \prod_{i=1}^m A_i$, which is naturally an R-algebra. It is easy to see

$$W^{-1}A \cong (R_{P_1})^{r_1} \times \cdots \times (R_{P_m})^{r_m}.$$

Since $\alpha(R_{P_i}) \geqslant 1$ for all i, there exists $e' \in \mathbb{N}_+$ such that $s_i p^{e'\alpha(R_{P_i})} \geqslant r_i$ for all i. Thus $W^{-1}({}^{e'}M)$ admits a direct summand isomorphic to $W^{-1}A$ over $W^{-1}R$. Lifting the relevant $W^{-1}R$-linear maps, we obtain R-linear maps $\varphi \in \mathrm{Hom}_R(A, {}^{e'}M)$ and $\psi \in \mathrm{Hom}_R({}^{e'}M, A)$ such that $\psi \circ \varphi = c\,\mathrm{I}_A \in \mathrm{Hom}_R(A, A)$ for some $c \in W$, i.e., the map $\psi \circ \varphi$ is multiplication by c. Trivially, for all $e \in \mathbb{N}_+$, we have R-linear maps ${}^e\varphi \in \mathrm{Hom}_R({}^eA, {}^{e'+e}M)$ and ${}^e\psi \in \mathrm{Hom}_R({}^{e'+e}M, {}^eA)$ such that ${}^e\psi \circ {}^e\varphi = {}^e(c\,\mathrm{I}_A) \in \mathrm{Hom}_R({}^eA, {}^eA)$; see Remark 1.1.

Consider $\overline{c} := (c + P_1, \ldots, c + P_m) \in \prod_{i=1}^m (R/P_i)^\circ \subseteq \prod_{i=1}^m A_i^\circ = A^\circ$. Since A is strongly F-regular, there exists $e'' \in \mathbb{N}_+$ such that the A-linear map $h_{\overline{c}} \in \mathrm{Hom}_A(A, {}^{e''}A)$ sending 1 to $\overline{c}$ is left split; that is, there exists $g \in \mathrm{Hom}_A({}^{e''}A, A)$ such that $g \circ h_{\overline{c}} = \mathrm{I}_A$.

Let $h_1 \in \mathrm{Hom}_A(A, {}^{e''}A)$ be defined by sending 1 to 1 (hence $a \mapsto a^{p^{e''}}$ for all $a \in A$). Clearly ${}^{e''}\psi \circ {}^{e''}\varphi \circ h_1 \in \mathrm{Hom}_R(A, {}^{e''}A)$ and ${}^{e''}\psi \circ {}^{e''}\varphi \circ h_1(1) = \overline{c}$, hence

${}^{e''}\psi \circ {}^{e''}\varphi \circ h_1 = h_{\overline{c}}$. Therefore

$$g \circ {}^{e''}\psi \circ {}^{e''}\varphi \circ h_1 = g \circ h_{\overline{c}} = \mathrm{I}_A,$$

which shows that ${}^{e''}\varphi \circ h_1 \in \mathrm{Hom}_R(A, {}^{e'+e''}M)$ is left split. Let $e_0 = e' + e''$. Then ${}^{e_0}M$ admits a direct summand isomorphic to A as an R-module.

Without affecting the claim, we replace M by ${}^{e_0}M$. Write $M \cong A \oplus N$, so that ${}^{e}M \cong {}^{e}A \oplus {}^{e}N$ for all $e \in \mathbb{N}_+$. Applying Theorem 3.5 to the F-finite strongly F-regular domains A, we see that for every $n \in \mathbb{N}_+$ there exists $e_n \in \mathbb{N}_+$ such that ${}^{e}A$ admits a direct summand isomorphic to A^n over A for every $e \geqslant e_n$. Thus ${}^{e}M$ admits a direct summand isomorphic to $A^n \cong (\oplus_{i=1}^{m} A_i)^n$ over R for every $e \geqslant e_n$. Now the proof is complete. □

Corollary 3.7. *Let R be an F-finite ring and M a finitely generated R-module. If there exists $P \in \mathrm{Min}(M)$ such that $\alpha(R_P) > 0$ (e.g., $\dim(R/P) > 0$) and there is a module-finite domain extension A of R/P such that A is strongly F-regular, then ${}^{e}M$ splits non-trivially over R for all $e \gg 0$.*

Theorem 3.6 clearly generalizes Theorem 3.3. Also note that in Theorem 3.2, $A = \overline{R/P}$ is regular and hence strongly F-regular. Thus Theorem 3.6 recovers Theorem 3.2.

Corollary 3.8. *Let R be an F-finite ring and M a finitely generated R-module. If there exists $P \in \mathrm{Min}(M)$ such that $\dim(R/P) \leqslant 1$ and $\alpha(R_P) > 0$, then ${}^{e}M$ splits non-trivially over R for all $e \gg 0$.*

Proof. In case $\dim(R/P) = 0$, Corollary 3.7 applies with $A = R/P$.

In case $\dim(R/P) = 1$ (so that $\alpha(R_P) > 0$ automatically), Corollary 3.7 applies with $A = \overline{R/P}$, the integral closure of R/P in its fraction field. □

One might wonder what happens if there exist $P \in \mathrm{Min}(M)$ such that $\alpha(R_P) = 0$ (which is equivalent to R/P being a perfect field). We make the following remarks concerning this case.

Remark 3.9. Let R be an F-finite ring and M be a finitely generated R-module. Assume that there exists $\mathfrak{m} \in \mathrm{Min}(M)$ with $\dim(R/\mathfrak{m}) = 0$. Let $H = \bigcup_{i=1}^{\infty}(0 :_M \mathfrak{m}^i)$ and $K = \ker(M \to M_{\mathfrak{m}})$. Let $r = \lambda_R(H)$, so that $1 \leqslant r < \infty$. Then

(1) An elementary proof should verify the following direct sum decompositions:

$$M = H \oplus K \qquad \text{hence} \qquad {}^{e}M = {}^{e}H \oplus {}^{e}K \text{ for all } e \in \mathbb{N}_+.$$

(2) For all $e \gg 0$ (such that $\mathfrak{m}({}^{e}H) = \mathfrak{m}^{[p^e]}H = 0$), we see ${}^{e}H \cong (R/\mathfrak{m})^{rp^{e\alpha(R_{\mathfrak{m}})}}$.

(3) Therefore, ${}^{e}M$ remains indecomposable for all $e \in \mathbb{N}_+$ if and only if $K = 0$, $r = 1$ and $\alpha(R_{\mathfrak{m}}) = 0$ if and only if $M \cong R/\mathfrak{m}$ and $R/\mathfrak{m}$ is a perfect field.

4. From the splitting of ${}^{e}M$ to strong F-regularity

In light of Theorem 3.6, with R being F-finite, we wonder whether the eventual splitting of R off ${}^{e}M$ would imply the strong F-regularity of R. In fact, we can ask the question without assuming that R is F-finite, as follows.

Question 4.1. Let R be such that, for every finitely generated faithful R-module M, there exists $e \in \mathbb{N}_+$ such that ${}^{e}M$ admits a pure map $R \to {}^{e}M$. Is R strongly F-regular? (This is a partial converse of Theorem 3.6.)

We are able to answer the question positively for locally excellent rings, which establishes a characterization of strong F-regularity. For any ring R, its total fraction ring is $Q(R) = W^{-1}R$ with W consisting of all nonzerodivisors of R. By the *integral closure* of R, denoted $\overline{R}$, we mean the integral closure of R in $Q(R)$. We say that R is *normal* if R is reduced and $R = \overline{R}$. Thus, normal Noetherian rings are finite products of normal integral domains.

In the following theorem, we consider modules over both R and $\overline{R}$ that may not be finitely generated, as well as finitely generated modules. To make sure that this is clear, we describe modules that may not be finitely generated as *arbitrary*.

Theorem 4.2. *Let R be a locally excellent ring. For the eight statements below we have* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8). *Moreover, if R is in a class of rings for which very strongly F-regular and strongly F-regular are equivalent, such as F-finite rings or rings essentially of finite type over an excellent semilocal ring, then* (8) $\Rightarrow$ (1) *and all of the statements below are equivalent.*

(1) *R is very strongly F-regular.*
(2) *For every module M that is finitely generated over R or over $\overline{R}$ and supported on all of* $\operatorname{Spec}(R)$, *${}^eM$ (as an R-module) admits a pure R-submodule isomorphic to R for all $e \gg 0$.*
(3) *For every module M that is finitely generated over either R or $\overline{R}$ and that is faithful over R, there exists $e \in \mathbb{N}_+$ such that ${}^eM$ admits a pure submodule isomorphic to R.*
(4) *There exists an arbitrary module N over $\overline{R}$ such that, for every $M \in \{N\} \cup \{P \in \operatorname{Spec}(R) \mid \dim(R_P) \geqslant 2\}$, there exists $e \in \mathbb{N}_+$ such that ${}^eM$ admits a pure submodule isomorphic to R.*
(5) *There exists an arbitrary module N over $\overline{R}$ and there exists $e_1 \in \mathbb{N}_+$ such that ${}^{e_1}N$ admits a pure submodule isomorphic to R, and for every $P \in \operatorname{Spec}(R)$ such that $\dim(R_P) \geqslant 2$, there exists $e \in \mathbb{N}_+$ such that ${}^eP_P$, as an R_P-module, admits a pure submodule isomorphic to R_P.*
(6) *There exists $e_1 \in \mathbb{N}_+$ such that ${}^{e_1}\overline{R}$ admits a pure submodule isomorphic to R, and for every $P \in \operatorname{Spec}(R)$ such that $\dim(R_P) \geqslant 2$, there exists $e \in \mathbb{N}_+$ such that ${}^eP_P$, as an R_P-module, admits a pure submodule isomorphic to R_P.*
(7) *The ring R is normal, and for every $P \in \operatorname{Spec}(R)$ such that $\dim(R_P) \geqslant 2$, there exists $e \in \mathbb{N}_+$ such that ${}^eP_P$, as an R_P-module, admits a pure submodule isomorphic to R_P.*
(8) *R is strongly F-regular.*

Proof. (1) $\Rightarrow$ (2): Since R is very strongly F-regular, R is normal, i.e., $R = \overline{R}$. Let M be a finitely generated R-module supported on $\operatorname{Spec}(R)$. As R is reduced, there exist $x \in M$ and an R-linear map $h\colon M \to R$ such that $h(x) = c \in R^\circ$. Then there exists $e \in \mathbb{N}_+$ such that the R-linear map $h_c\colon R \to {}^eR$ sending 1 to ${}^ec$ is pure. Note that $h_c = h \circ h_x$ where h_x is the R-linear map from R to ${}^eM$ sending 1 to ${}^ex$. Thus h_x is a pure map, as required. (This is similar to the proof of Proposition 3.4(1).)

(2) $\Rightarrow$ (3): This is clear.

(3) $\Rightarrow$ (4): This implication holds because, for example, $\overline{R}$ is finitely generated over $\overline{R}$ and faithful over R while every $M \in \{P \in \operatorname{Spec}(R) \mid \dim(R_P) \geqslant 2\}$ is finitely generated and faithful over R.

(4) $\Rightarrow$ (5): This is immediate, since pure maps localize.

(5) $\Rightarrow$ (6): The pure map $R \to {}^{e_1}N$ factors through ${}^{e_1}\overline{R}$ by Remark 2.1.

(6) ⇒ (7): Since ${}^{e_1}\overline{R}$ admits a pure submodule isomorphic to R, we see that R is reduced and the embedding $R \subseteq \overline{R}$ is pure by Remark 2.1, which implies that $R = \overline{R}$, i.e., that R is normal.

(7) ⇒ (8): The hypothesis of (6) passes to R_P for all $P \in \operatorname{Spec}(R)$. To prove R is strongly F-regular, it suffices to show R_P is strongly F-regular for all $P \in \operatorname{Spec}(R)$ (cf. [HH4, Theorem 5.5]). Thus we may assume that $(R, \mathfrak{m}, k)$ is local without loss of generality, so that $(R, \mathfrak{m}, k)$ is an excellent, local, normal domain. (This the the only implication where the assumption that R is locally excellent is needed.)

We are going to prove the strong F-regularity of $(R, \mathfrak{m}, k)$ by induction on $\dim(R)$. When $\dim(R) \leqslant 1$, the normality of R implies that R is regular (hence strongly F-regular). For the rest of the proof, let $\dim(R) \geqslant 2$. By assumption, there exist $e \in \mathbb{N}_+$ and $c \in \mathfrak{m} \setminus \{0\}$ such that the R-linear map $R \to {}^e\mathfrak{m}$ determined by $1 \mapsto {}^e c$ is pure; that is, the embedding $R({}^e c) \subseteq {}^e\mathfrak{m}$ is pure.

Next, we prove that the embedding $R({}^e c) \subseteq {}^e R$ is pure. For this, it suffices to show that the embedding $R({}^e c) \subseteq N$ is pure for every finitely generated R-module N such that $R({}^e c) \subseteq N \subseteq {}^e R$. For any such N, consider the short exact sequence

$$0 \longrightarrow N \cap {}^e\mathfrak{m} \xrightarrow{\iota} N \longrightarrow N/(N \cap {}^e\mathfrak{m}) \longrightarrow 0,$$

in which ι is the embedding map. This induces the following exact sequence

$$\cdots \longrightarrow \operatorname{Hom}_R(N, R) \xrightarrow{\iota^*} \operatorname{Hom}_R(N \cap {}^e\mathfrak{m}, R) \longrightarrow \operatorname{Ext}^1_R(N/(N \cap {}^e\mathfrak{m}), R) \longrightarrow \cdots$$

with ι^* being the restriction map. Since the R-linear map $h_c \colon R \to N \cap {}^e\mathfrak{m}$ determined by $h_c(1) = {}^e c$ is pure (hence is left split), there exist $g \in \operatorname{Hom}_R(N \cap {}^e\mathfrak{m}, R)$ such that $g \circ h_c = \mathrm{I}_R$. Note that $N/(N \cap {}^e\mathfrak{m}) \cong (N + {}^e\mathfrak{m})/{}^e\mathfrak{m} \subseteq {}^e(R/\mathfrak{m})$, which implies $\mathfrak{m} \subseteq \operatorname{Ann}(N/(N \cap {}^e\mathfrak{m}))$. Since R is normal, the $\mathbf{S}_2$ property indicates that $\mathfrak{m}$ contains an R-regular sequence of length (at least) two. Hence $\operatorname{Ext}^1_R(N/(N \cap {}^e\mathfrak{m}), R) = 0$ and therefore the restriction map ι^* is surjective onto $\operatorname{Hom}_R(N \cap {}^e\mathfrak{m}, R)$. In particular, there exists $f \in \operatorname{Hom}_R(N, R)$ such that

$$f|_{N \cap {}^e\mathfrak{m}} = \iota^*(f) = g \in \operatorname{Hom}(N \cap {}^e\mathfrak{m}, R).$$

This implies

$$f \circ h_c = \mathrm{I}_R,$$

which proves that the R-linear map $R \to N$ sending 1 to ${}^e c$ is left split. As mentioned earlier, this shows that the embedding $R({}^e c) \subseteq {}^e R$ is pure.

By the induction hypothesis, R_P is strongly F-regular for all $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$, hence R_c is strongly F-regular. Thus R is strongly F-regular by Theorem 2.24, which completes the proof. □

In the case where R is F-finite or essentially of finite type over a excellent semilocal ring, Theorem 4.2 takes a simpler form, since very strong F-regularity is equivalent to strong F-regularity and it is always true that $\overline{R}$ is module-finite over R:

Corollary 4.3. *Let R be a ring that is F-finite or essentially of finite type over an excellent semilocal ring. The following statements are equivalent:*

1. *R is strongly F-regular.*
2. *For every finitely generated R-module M supported on all of* $\operatorname{Spec}(R)$*, ${}^e M$ admits a pure submodule isomorphic to R for all $e \gg 0$.*

(3) *For every finitely generated faithful R-module M, there exists $e \in \mathbb{N}_+$ such that ${}^eM$ admits a pure submodule isomorphic to R.*
(4) *There exists an arbitrary module N over $\overline{R}$ such that, for every $M \in \{N\} \cup \{P \in \operatorname{Spec}(R) \mid \dim(R_P) \geqslant 2\}$, there exists $e \in \mathbb{N}_+$ such that ${}^eM$ admits a pure submodule isomorphic to R.*
(5) *There exists an arbitrary module N over $\overline{R}$ and there exists $e_1 \in \mathbb{N}_+$ such that ${}^{e_1}N$ admits a pure submodule isomorphic to R, and for every $P \in \operatorname{Spec}(R)$ such that $\dim(R_P) \geqslant 2$, there exists $e \in \mathbb{N}_+$ such that ${}^eP_P$, as an R_P-module, admits a pure submodule isomorphic to R_P.*
(6) *There exists $e_1 \in \mathbb{N}_+$ such that ${}^{e_1}\overline{R}$ admits a pure submodule isomorphic to R, and for every $P \in \operatorname{Spec}(R)$ such that $\dim(R_P) \geqslant 2$, there exists $e \in \mathbb{N}_+$ such that ${}^eP_P$, as an R_P-module, admits a pure submodule isomorphic to R_P.*
(7) *R is normal, and for every $P \in \operatorname{Spec}(R)$ such that $\dim(R_P) \geqslant 2$, there exists $e \in \mathbb{N}_+$ such that ${}^eP_P$ admits a pure submodule isomorphic to R_P.*

Remark 4.4. When R is F-finite or essentially of finite type over a *complete* semilocal ring A, then we can use "direct summand" to replace "pure submodule". In the latter case, we can use the product structure of the complete base ring to reduce to the case where the base ring $(A, \mathfrak{m})$ is complete and local. Choose a coefficient field k for A and a p-base Λ for k. Suppose $R \to M$ is pure with $1 \mapsto u$. For sufficiently small cofinite Γ, the closed fiber of $A \to A^\Gamma$ is a field and hence $R^\Gamma \to R^\Gamma \otimes M$ is pure over R^Γ (by an argument similar to the proof of Lemma 2.15). Since R^Γ is F-finite, we get a splitting of $R^\Gamma \to R^\Gamma \otimes M$ over R^Γ. Restricting this splitting to M, we get an R-linear map $M \to R^\Gamma$ with $u \mapsto 1$. Thus, it will suffice to get a splitting of $R \subseteq R^\Gamma$ over R. For this purpose, it suffices to split $A \to A^\Gamma$: we can then tensor with R. But since this map is pure, we get an injective A-linear map $\mathrm{E}_A(A/\mathfrak{m}) \to \mathrm{E}_A(A/\mathfrak{m}) \otimes_A A^\Gamma$. Now that A is complete, we get a splitting map by Matlis duality.

5. Splitting and small maximal Cohen-Macaulay modules

Let $(R, \mathfrak{m})$ be a local ring with and M a finitely generated R-module of Krull dimenson d. We say M is *Cohen-Macaulay* if there exists an proper M-regular sequence of length d, which means that there exist $x_1, \ldots, x_d \in \mathfrak{m}$ such that $\sum_{i=1}^d x_iM \neq M$ and $(\sum_{i=1}^{j-1} x_iM :_M x_j) = \sum_{i=1}^{j-1} x_iM$ for all $j \in \{1, \ldots, d\}$. If M is a finitely generated Cohen-Macaulay module over $(R, \mathfrak{m})$ then M_P is Cohen-Macaulay over R_P for all $P \in \operatorname{Supp}(M)$, where $\operatorname{Supp}(M) := \{P \in \operatorname{Spec}(R) \mid M_P \neq 0\}$. In this case, $x_1, \ldots, x_d \in \mathfrak{m}$ form a regular sequence on M if and only if they are a system of parameters for $R/\operatorname{Ann}_R M$.

In general (i.e., without assuming that R is local), we say that a finitely generated R-module $M \neq 0$ is Cohen-Macaulay if M_P is Cohen-Macaulay over R_P for all $P \in \operatorname{Supp}(M)$.

A finitely generated module over a local ring R is called a *maximal* Cohen-Macaulay module over R or a small Cohen-Macaulay module over R if M is Cohen-Macaulay and has Krull dimension equal to $\dim(R)$.

Here is question concerning the existence of small Cohen-Macaulay modules.

Question 5.1 (Hochster)**.** Do small Cohen-Macaulay modules exist over complete local rings?

Note that by replacing R by a quotient by a minimal prime, this question reduces to the case of a complete local domain.

In the equicharacteristic case, the existence of big (i.e., not necessarily finitely generated) Cohen-Macaulay modules was established in [Ho2], and the existence of big Cohen-Macaulay algebras was proved in [Ho5, HH2, HH4]. There is an exposition with further motivation for the problem in [Ho3, Ho4]. The case of dimension 3 in mixed characteristic was settled in [Ho6] using the results of [He]. Relatively recent breakthroughs using almost mathematics and perfectoid geometry have settled the issue in mixed characteristic in general [And, Bha, HeMa].

In this section, we investigate the existence of small Cohen-Macaulay R-modules via the splitting of ${}^eM$ for any finitely generated module M over any F-finite ring R.

Hochster used the splitting of ${}^eM$ in proving the existence of small maximal Cohen-Macaulay modules (cf. [Ho3, Proposition 5.11]). The results concern $\mathbb{N}$-graded rings. A form of Theorem 5.2 below was first proved by R. Hartshorne (the ring was assumed to have an isolated singularity) and later, independently, by Peskine and Szpiro (unpublished). There are also results on the existence of small Cohen-Macaulay modules in [Ha1, Ha2, Scho1, Scho2].

Hanes used a related idea to construct graded small Cohen-Macaulay modules with negative $\mathfrak{a}$-invariant in the $\mathbb{N}$-graded case in certain instances, which enabled him to construct small Cohen-Macaulay modules over Segre products.

Theorem 5.2. *Let $R = \oplus_{i=0}^{\infty} R_i$ be an equidimensional $\mathbb{N}$-graded ring (of prime characteristic $p > 0$) with R_0 being a perfect field and $\dim(R) = d$.*

(1) *If $d = 3$, then there exists a graded small Cohen-Macaulay R-module.*
(2) *If there is a finitely generated graded R-module M with $\dim(M) = d$ whose non-Cohen-Macaulay locus consists of the maximal homogeneous ideal only, then there exists a graded small maximal Cohen-Macaulay R-module.*

We are going to extend the above results to a greater generality. The general idea is that *sufficient splitting of ${}^eM$ implies sufficient depth.* To make this idea precise, we need to define the *splitting number* of a (graded) module over a (graded) ring.

For simplicity, we only study rings $R = \oplus_{i\in\mathbb{N}^s} R_i$ graded by the semigroup $(\mathbb{N}^s, +)$ where $0 \leqslant s \in \mathbb{Z}$. By abuse of notation, we write 0 for the zero element of $(\mathbb{N}^s, +)$. When $s = 0$, we agree that $\mathbb{N}^0 = \{0\}$; thus an $\mathbb{N}^0$-graded ring is simply a ring $R = R_0$.

We call a ring R to be $\mathbb{N}^s$*-graded local* if R is $\mathbb{N}^s$-graded with $(R_0, \mathfrak{m}_0, k)$ local; and if this is the case, we write the $\mathbb{N}^s$-graded local ring as $(R, \mathfrak{m}, k)$ with $\mathfrak{m}$ being the maximal homogeneous ideal. Clearly, an $\mathbb{N}^0$-graded local ring is simply a local ring.

In the sequel, we are going to encounter modules graded by $\frac{1}{p^e}\mathbb{N}^s$ or $\frac{1}{p^e}\mathbb{Z}^s$ for various $e \geqslant 0$ (see Remark 5.4). To embrace them all, we describe $\mathbb{Q}^s$-graded modules as follows: We say that a module M over an $\mathbb{N}^s$-graded ring $R = \oplus_{i\in\mathbb{N}^s} R_i$ is $\mathbb{Q}^s$*-graded* if

$$M = \oplus_{j\in\mathbb{Q}^s} M_j \text{ over } \mathbb{Z} \qquad \text{and} \qquad R_i M_j \subseteq M_{i+j}$$

for all $i \in \mathbb{N}^s$ and $j \in \mathbb{Q}^s$. Clearly, this covers all modules graded by $\frac{1}{p^e}\mathbb{N}^s$ or $\frac{1}{p^e}\mathbb{Z}^s$. When $s = 0$, an $\mathbb{Q}^0$-graded module (over an $\mathbb{N}^0$-graded ring R) is simply an ordinary

R-module. Also, for any $k \in \mathbb{Q}^s$, we use $M(k)$ to denote the 'shifted' $\mathbb{Q}^s$-graded R-module in which the j-graded piece of $M(k)$ is M_{k+j}, i.e., $M(k)_j = M_{k+j}$.

Remark 5.3. Let $R = \oplus_{i\in\mathbb{N}^s} R_i$ be a $\mathbb{N}^s$-graded local ring (of any characteristic) and $M = \oplus_{j\in\mathbb{Q}^s} M_j$ a $\mathbb{Q}^s$-graded R-module, with $s \geqslant 0$.

(1) There is the following direct sum decomposition of M over R induced by the given grading:
$$M = \bigoplus_{i\in[0,\,1)^s\cap\mathbb{Q}^s} M_{[i]} \qquad \text{where} \qquad M_{[i]} := \bigoplus_{j\in i+\mathbb{Z}^s} M_j,$$
in which $[0,\,1) := \{r \in \mathbb{R} \mid 0 \leqslant r < 1\}$ and $[i] := i+\mathbb{Z}^s$ is the coset represented by i in the quotient group $\mathbb{Q}^s/\mathbb{Z}^s$. Note that each $M_{[i]}(i)$ is $\mathbb{Z}^s$-graded.

(2) Assume that M is indecomposable over R. Then there exists $i_0 \in [0,\,1)^s$ such that $M = M_{[i_0]}$, so that $M(i_0)$ is $\mathbb{Z}^s$-graded. If, furthermore, M is finitely generated over R, then there exists $i \in [i_0]$ such that $M(i)$ is $\mathbb{N}^s$-graded.

Remark 5.4. Let $R = \oplus_{i\in\mathbb{N}^s} R_i$ be a $\mathbb{N}^s$-graded local ring (of prime characteristic $p > 0$) and $M = \oplus_{j\in\mathbb{Q}^s} M_j$ a $\mathbb{Q}^s$-graded R-module, with $s \in \mathbb{N}$.

(1) For every $e \in \mathbb{N}$, the R-module ${}^e\!M$ is a $\mathbb{Q}^s$-graded R-module in following sense: for all $x \in M$ and $i \in \mathbb{Q}^s$, the element ${}^e\!x \in {}^e\!M$ is homogeneous of degree $i \in \mathbb{Q}^s$ if and only if $x \in M$ is homogeneous of degree $p^e i \in \mathbb{Q}^s$.

(2) Applying Remark 5.3(1) to the graded module ${}^e\!M$, we see the following direct sum decomposition of ${}^e\!M$ over R induced by grading:
$${}^e\!M = \bigoplus_{i\in[0,\,p^e)^s\cap\mathbb{Q}^s} {}^e(M_{i+p^e\mathbb{Z}^s}) \qquad \text{where} \qquad M_{i+p^e\mathbb{Z}^s} := \bigoplus_{j\in i+p^e\mathbb{Z}^s} M_j.$$
Note that $({}^e(M_{i+p^e\mathbb{Z}^s}))(\frac{i}{p^e}) = {}^e(M_{i+p^e\mathbb{Z}^s}(i))$ is $\mathbb{Z}^s$-graded over R.

(3) In particular, if M is an $\mathbb{N}^s$-graded R-module, then ${}^e\!M$ is $\frac{1}{q}\mathbb{N}^s$-graded over R. (In this case, we sometimes still say that ${}^e\!M$ is $\mathbb{Q}^s$-graded for simplicity.) We see the following direct sum decomposition of ${}^e\!M$ over R induced by grading:
$${}^e\!M = \bigoplus_{i\in[0,\,p^e)^s\cap\mathbb{N}^s} {}^e(M_{i+p^e\mathbb{N}^s}) \qquad \text{where} \qquad M_{i+p^e\mathbb{N}^s} := \bigoplus_{j\in i+p^e\mathbb{N}^s} M_j.$$
Note that $({}^e(M_{i+p^e\mathbb{N}^s}))(\frac{i}{p^e}) = {}^e(M_{i+p^e\mathbb{N}^s}(i))$ is $\mathbb{N}^s$-graded over R.

Definition 5.5. Let R be a $\mathbb{N}^s$-graded local ring (of any characteristic) and $M \neq 0$ a finitely generated $\mathbb{Q}^s$-graded R-module, with $s \in \mathbb{N}$.

(1) The *splitting number* of M, denote $\Theta(M)$, is the supremum of all $n \in \mathbb{N}_+$ such that M can be written as a direct sum of n non-zero graded R-submodules, that is,
$$\Theta(M) = \sup\{n \in \mathbb{N}_+ \mid M = \oplus_{i=1}^n M_i,\ M_i \neq 0,\ M_i \text{ is } \mathbb{Q}^s\text{-graded}\} < \infty.$$

(2) Using Card$(-)$ to denote cardinality, we also define
$$\theta(M) = \mathrm{Card}\{i \in [0,\,1)^s \cap \mathbb{Q}^s \mid M_{[i]} \neq 0\},$$
which is simply the number of non-zero direct summands in the direct sum decomposition described in Remark 5.3(1).

Remark 5.6. Let R and M be as in Definition 5.5 above. Then

(1) It is obvious that $\Theta(M) \geqslant \theta(M)$.

(2) In case $s = 0$, $\Theta(M)$ simply means the maximum number $n \in \mathbb{N}_+$ such that M can be written as a direct sum of n non-zero submodules over the local ring R.
(3) Further assume that R is F-finite of prime characteristic $p > 0$. Then, for every $e \in \mathbb{N}$, we can study $\Theta({}^eM)$ and $\theta({}^eM)$ since ${}^eM$ is finitely generated and $\mathbb{Q}^s$-graded over R in light of Remark 5.4(1). For every $e \in \mathbb{N}$, it is routine to see the following

$$\theta({}^eM) = \operatorname{Card}\{i \in [0, p^e)^s \cap \mathbb{Q}^s \mid M_{i+p^e\mathbb{N}^s} \neq 0\}.$$

In particular, if M is $\mathbb{N}^s$-graded over R (as in Remark 5.4(3)), then

$$\theta({}^eM) = \operatorname{Card}\{i \in [0, p^e)^s \cap \mathbb{N}^s \mid M_{i+p^e\mathbb{N}^s} \neq 0\}.$$

Remark 5.7. Let R be a ring (of prime characteristic $p > 0$), I an ideal of R, and M an R-module. Then

(1) There are natural isomorphisms $\mathrm{H}^i_I({}^eM) \cong {}^e\mathrm{H}^i_I(M)$ for all $i \in \mathbb{Z}$ and all $e \in \mathbb{N}_+$.
(2) Assume R is F-finite and M is finitely generated over R. Then there are natural isomorphisms ${}^e\big(\widehat{M}^I\big) \cong \widehat{({}^eM)}^I$ over $\widehat{R}^I$ for all $e \in \mathbb{N}_+$, where $\widehat{N}^I$ stands for the I-adic completion of any R-module N.

For any (Noetherian) ring R, any ideal I, and any finitely generated R-module M such that $IM \neq M$, the depth of M with respect to I is

$$\operatorname{depth}(I, M) := \min\{i \in \mathbb{N} \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\},$$

which agrees with the maximum of the lengths of all M-regular sequences in I. Also, we say that M satisfies $\mathbf{S}_t$, with $t \in \mathbb{N}$, if

$$\operatorname{depth}(P, M) \geqslant \min\{\dim(M_P), t\}$$

for all $P \in \operatorname{Spec}(R)$ (or equivalently, $\operatorname{depth}(P_P, M_P) \geqslant \min\{\dim(M_P), t\}$ for all $P \in \operatorname{Spec}(R)$. (We agree that the dimension of the zero module is $-\infty$.)

The following results are well-known, see [BH] for example. In [BH], the results are stated for $\mathbb{N}$-graded rings/modules. In our case (the case below with general s), we can first make M a $\mathbb{Z}^s$-graded R-module by scaling degrees (multiplying all degrees used for M and R by a suitable positive integer) and then shifting degrees by adding a vector of positive constants, so that both M and R become $\mathbb{N}^s$-graded, and then regard both the ring and the module as $\mathbb{N}$-graded by taking the degree of a form of multi-degree $i = (i_1, \ldots, i_s) \in \mathbb{N}^s$ to be its total degree $|i| := i_1 + \cdots + i_s$.

Remark 5.8. Let $(R, \mathfrak{m}, k)$ be an $\mathbb{N}^s$-graded local ring and $M \neq 0$ a finitely generated $\mathbb{Q}^s$-graded R-module.

(1) We have $\operatorname{depth}(\mathfrak{m}, M) = \operatorname{depth}(\mathfrak{m}R_\mathfrak{m}, M_\mathfrak{m}) = \min\{i \in \mathbb{N} \mid \mathrm{H}^i_\mathfrak{m}(M) \neq 0\}$.
(2) Also, M is small Cohen-Macaulay over R if and only if $M_\mathfrak{m}$ is small Cohen-Macaulay over $R_\mathfrak{m}$.
(3) Furthermore, M satisfies $\mathbf{S}_t$ as an R-module if and only if $M_\mathfrak{m}$ satisfies $\mathbf{S}_t$ as an $R_\mathfrak{m}$-module.

Note that, in the next theorem, when $s = 0$, an $\mathbb{N}^0$-graded local ring is none other than a local ring and a $\mathbb{Q}^0$-graded R-module is none other than an R-module.

Theorem 5.9. *Let $(R, \mathfrak{m}, k)$ be an F-finite $\mathbb{N}^s$-graded local ring (of prime characteristic $p > 0$) and $M \neq 0$ a finitely generated $\mathbb{Q}^s$-graded R-module, with $s \in \mathbb{N}$. For some fixed $m \in \{0, \dots, \dim(M) - 1\}$, let*

$$K := \operatorname{Hom}_R(\oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(M),\, \operatorname{E}(R/\mathfrak{m})) = \operatorname{Hom}_{\widehat{R}^{\mathfrak{m}}}(\oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(M),\, \operatorname{E}(R/\mathfrak{m})).$$

Assume that, for some $e \in \mathbb{N}$,

$$\Theta({}^eM) > \lambda_{\widehat{R}^{\mathfrak{m}}}\left(\frac{K}{\mathfrak{m}^{[q]}K}\right) q^{\alpha(R_{\mathfrak{m}})}.$$

Then there exists a finitely generated $\mathbb{N}^s$-graded non-zero R-module N such that $\operatorname{depth}(\mathfrak{m}, N) \geqslant m+1$. *In fact, N can be chosen as a graded direct summand of ${}^eM$ subject to a proper shift.*

Proof. Let $u = \Theta({}^eM)$ and write down a direct sum decomposition as follows

$${}^eM = M_1 \oplus \cdots \oplus M_u$$

in which $M_i \neq 0$ are $\mathbb{Q}^s$-graded R-submodules of ${}^eM$.

We claim that, for some $j \in \{1, \dots, u\}$, $\operatorname{depth}(\mathfrak{m}, M_j) \geqslant m + 1$, or equivalently, $\oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(M_j) = 0$ by Remark 5.8. To prove by contradiction, suppose $\oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(M_j) \neq 0$ for all $j \in \{1, \dots, u\}$. Then

$$\operatorname{Hom}_R(R/\mathfrak{m},\, \oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(M_j)) \neq 0,$$

which implies

$$\lambda_R(\operatorname{Hom}_R(R/\mathfrak{m},\, \oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(M_j))) \geqslant 1 \quad \text{for all} \quad j \in \{1, \dots, u\}.$$

By Matlis duality, Remark 1.3 and Remark 5.7, we have

$$\begin{aligned}
\lambda_{\widehat{R}^{\mathfrak{m}}}\left(\frac{K}{\mathfrak{m}^{[q]}K}\right) q^{\alpha(R_{\mathfrak{m}})} &= \lambda_R(\operatorname{Hom}_R(R/\mathfrak{m}^{[q]},\, \oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(M)))q^{\alpha(R_{\mathfrak{m}})} \\
&= \lambda_R(\operatorname{Hom}_R(R/\mathfrak{m},\, {}^e(\oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(M)))) \\
&= \lambda_R(\operatorname{Hom}_R(R/\mathfrak{m},\, \oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}({}^eM))) \\
&= \lambda_R(\operatorname{Hom}_R(R/\mathfrak{m},\, \oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(\oplus_{j=1}^u M_j))) \\
&= \lambda_R(\oplus_{j=1}^u \operatorname{Hom}_R(R/\mathfrak{m},\, \oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(M_j))) \\
&= \sum_{j=1}^u \lambda_R(\operatorname{Hom}_R(R/\mathfrak{m},\, \oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(M_j))) \geqslant u = \Theta({}^eM),
\end{aligned}$$

which is a contradiction. So there exists $j \in \{1, \dots, u\}$ such that $\operatorname{depth}(\mathfrak{m}, M_j) \geqslant m + 1$.

By Remark 5.3, there exists $i \in \mathbb{Q}^s$ such that $N := M_j(i)$ is an $\mathbb{N}^s$-graded R-module. Clearly, $\operatorname{depth}(\mathfrak{m}, N) = \operatorname{depth}(\mathfrak{m}, M_j) \geqslant m+1$. The proof is complete. □

Corollary 5.10. *Let $(R, \mathfrak{m}, k)$, M, m and K be as in Theorem 5.9. Assume that there exists $n \in \mathbb{N}$ such that*

$$\dim_{\widehat{R}^{\mathfrak{m}}}(K) \leqslant n \qquad \text{and} \qquad \limsup_{e \to \infty} \frac{\Theta({}^eM)}{q^{n+\alpha(R_{\mathfrak{m}})}} = \infty.$$

Then there exists a finitely generated $\mathbb{N}^s$-graded non-zero R-module N such that $\operatorname{depth}(\mathfrak{m}, N) \geqslant m + 1$. *In fact, N can be chosen as an $\mathbb{N}^s$-graded direct summand of ${}^eM$, for some $e \in \mathbb{N}$, up to a shift.*

Proof. Since $\dim_{\widehat{R}^{\mathfrak{m}}}(K) \leqslant n$, there exists $0 < C \in \mathbb{R}$ such that $\lambda_{\widehat{R}^{\mathfrak{m}}}\left(\frac{K}{\mathfrak{m}^{[q]}K}\right) \leqslant Cq^n$ for all $q = p^e$, see Remark 1.8. So $\lambda_{\widehat{R}^{\mathfrak{m}}}\left(\frac{K}{\mathfrak{m}^{[q]}K}\right) q^{\alpha(R_{\mathfrak{m}})} \leqslant Cq^{n+\alpha(R_{\mathfrak{m}})}$ for all $q = p^e$. Because $\limsup \frac{\Theta({}^eM)}{q^{n+\alpha(R_{\mathfrak{m}})}} = \infty$, there exists $e \in \mathbb{N}$ such that

$$\Theta({}^eM) > \lambda_{\widehat{R}^{\mathfrak{m}}}\left(\frac{K}{\mathfrak{m}^{[q]}K}\right) q^{\alpha(R_{\mathfrak{m}})}.$$

Now Theorem 5.9 applies, which completes the proof. □

Corollary 5.11. *Let $(R, \mathfrak{m}, k)$ be an F-finite $\mathbb{N}^s$-graded local ring with $\dim(R) = d$ and M a finitely generated $\mathbb{Q}^s$-graded R-module.*

(1) *Assume that, for some $n \in \mathbb{N}$, we have*

$$\limsup_{e\to\infty} \frac{\Theta({}^eM)}{q^{n+\alpha(R_{\mathfrak{m}})}} = \infty.$$

Then there exists a finitely generated $\mathbb{N}^s$-graded non-zero R-module N such that $\operatorname{depth}(\mathfrak{m}, N) \geqslant n + 1$. In fact, N can be chosen as a graded direct summand of ${}^eM$, for some $e \in \mathbb{N}$, up to a shift.

(2) *Assume that*

$$\limsup_{e\to\infty} \frac{\Theta({}^eM)}{q^{d-1+\alpha(R_{\mathfrak{m}})}} = \infty.$$

Then there exists a finitely generated $\mathbb{N}^s$-graded non-zero R-module N that is Cohen-Macaulay. In fact, N can be chosen as a graded direct summand of ${}^eM$, for some $e \in \mathbb{N}$, up to a shift.

(3) *In particular, if there is an upper bound B for the Hilbert-Kunz multiplicities (with respect to $\mathfrak{m}$) of all indecomposable $\mathbb{N}^s$-graded R-modules that are direct summands of ${}^eM$, then there exists a small Cohen-Macaulay $\mathbb{N}^s$-graded R-module.*

Proof. (1) Let $K := \operatorname{Hom}_R(\oplus_{i=0}^n \mathrm{H}^i_{\mathfrak{m}}(M), \mathrm{E}(R/\mathfrak{m}))$, which can be naturally identified with $\operatorname{Hom}_{\widehat{R}^{\mathfrak{m}}}(\oplus_{i=0}^n \mathrm{H}^i_{\mathfrak{m}}(M), \mathrm{E}(R/\mathfrak{m}))$. It is known that $\dim_{\widehat{R}^{\mathfrak{m}}}(K) \leqslant n$ (which holds quite generally, by localization and local duality, cf. [BH, 8.1.1]); so the claim follows from Corollary 5.10.

(2) This is a special case of (1), when $n = d - 1$.

(3) Observe that $\mathrm{e_{HK}}(\mathfrak{m}, {}^eR) = \mathrm{e_{HK}}(\mathfrak{m}, R)q^{d+\alpha(R_{\mathfrak{m}})}$. Thus

$$\Theta({}^eR) \geqslant \frac{\mathrm{e_{HK}}(\mathfrak{m}, R)q^{d+\alpha(R_{\mathfrak{m}})}}{B}, \quad \text{which implies} \quad \lim_{e\to\infty} \frac{\Theta({}^eM)}{q^{d-1+\alpha(R_{\mathfrak{m}})}} = \infty.$$

Now the claim follows from (2). □

Review 5.12. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring (of any characteristic) with $\dim(R) = d$. Assume that R admits a canonical module ω_R. Then, for every $P \in \operatorname{Spec}(R)$, $(\omega_R)_P$ is a canonical module for R_P. For every $P \in \operatorname{Spec}(R)$, let $\mathrm{E}(R/P)$ be the injective hull of R/P, so that $\mathrm{E}(R/P)$ is automatically a module over $\widehat{R_P}^P$, the P-adic (or P_P-adic) completion of R_P. Let M be a finitely generated R-module. By local duality (and Matlis duality), for all i, we have

(1) $\mathrm{H}^i_{\mathfrak{m}}(M) \cong \operatorname{Hom}_R(\operatorname{Ext}_R^{d-i}(M, \omega_R), \mathrm{E}(R/\mathfrak{m}))$ over $\widehat{R}^{\mathfrak{m}}$;

(2) $\operatorname{Hom}_R(\mathrm{H}^i_{\mathfrak{m}}(M), \mathrm{E}(R/\mathfrak{m})) \cong \widehat{\operatorname{Ext}_R^{d-i}(M, \omega_R)}^{\mathfrak{m}}$ over $\widehat{R}^{\mathfrak{m}}$.

We say that a finitely generated R-module M is *equidimensional* is $\dim(R/P) = \dim(M)$ for all $P \in \mathrm{Min}(M)$. The non-Cohen-Macaulay locus of M consists of $P \in \mathrm{Supp}(M)$ such that M_P is not Cohen-Macaulay over R_P.

Lemma 5.13. *Let $(R, \mathfrak{m}, k)$ be a local ring (of any characteristic) that is excellent or is a homomorphic image of a Gorenstein ring, M be a finitely generated equidimensional R-module with $\dim(M) = d$, and I be an ideal defining the non-Cohen-Macaulay locus of M. Let r, t, $u \in \mathbb{N}$ and $0 \leqslant m \leqslant d-1$. Then*

(1) $\dim(R/I) < r \iff \dim_{\widehat{R}^{\mathfrak{m}}} \left(\mathrm{Hom}_R(\oplus_{i=0}^{d-1} \mathrm{H}^i_{\mathfrak{m}}(M),\, \mathrm{E}(R/\mathfrak{m}))\right) < r.$
(2) *M has $\mathbf{S}_t$* $\iff \dim_{\widehat{R}^{\mathfrak{m}}} \left(\mathrm{Hom}_R(\mathrm{H}^i_{\mathfrak{m}}(M),\, \mathrm{E}(R/\mathfrak{m}))\right) \leqslant i - t$ *for all* $0 \leqslant i < d$.
(3) $\dim_{\widehat{R}^{\mathfrak{m}}}(\mathrm{Hom}_R(\oplus_{i=0}^{m} \mathrm{H}^i_{\mathfrak{m}}(M),\, \mathrm{E}(R/\mathfrak{m}))) \leqslant u \iff \mathrm{depth}(M_P) > \dim(M_P) + m - d$ *for all* $P \in \mathrm{Supp}(M)$ *such that* $\dim(M_P) < d - u$.

Proof. If R is excellent, we may pass to the completion of R. This allows us to assume that R is a homomorphic image of a Gorenstein local ring S. By replacing R by a proper quotient of S (which does not affect *either* side of the claims), we may further assume that R is Cohen-Macaulay with canonical module ω_R (or even that R is Gorenstein) and $\dim(M) = \dim(R) = d$, without loss of generality.

(1) We have (by local duality and Matlis duality, with some of the small details skipped, with P denoting a varying prime ideal)

$$\begin{aligned}
\dim(R/I) < r &\iff M_P \text{ is CM (or 0) for all } P \text{ such that } \dim(R/P) \geqslant r \\
&\iff \oplus_{i\geqslant 1} \mathrm{Ext}^i_{R_P}(M_P,\, \omega_{R_P}) = 0 \text{ for all } P \text{ with } \dim(R/P) \geqslant r \\
&\iff \dim_R \left(\oplus_{i\geqslant 1} \mathrm{Ext}^i_R(M,\, \omega_R)\right) < r \\
&\iff \dim_R \left(\oplus_{i=0}^{d-1} \mathrm{Ext}^{d-i}_R(M,\, \omega_R)\right) < r \\
&\iff \dim_{\widehat{R}^{\mathfrak{m}}} \left(\oplus_{i=0}^{d-1} \widehat{\mathrm{Ext}^{d-i}_R(M,\, \omega_R)}^{\mathfrak{m}}\right) < r \\
&\iff \dim_{\widehat{R}^{\mathfrak{m}}} \left(\mathrm{Hom}_{\widehat{R}^{\mathfrak{m}}}(\oplus_{i=0}^{d-1} \mathrm{H}^i_{\mathfrak{m}}(M),\, \mathrm{E}(R/\mathfrak{m}))\right) < r.
\end{aligned}$$

(2) Similarly, we have (by local duality and Matlis duality, with some of the small details skipped, with P denoting prime ideals in $\mathrm{Supp}(M)$)

$$\begin{aligned}
M \text{ has } \mathbf{S}_t &\iff \mathrm{depth}(M_P) \geqslant \min\{t, \dim(M_P)\} \text{ for all } P \\
&\iff \mathrm{Ext}^{\dim(M_P)-i}_{R_P}(M_P, \omega_{R_P}) = 0, \forall P \text{ and } \forall i < \min\{t, \dim(M_P)\} \\
&\iff \mathrm{Ext}^{i}_{R_P}(M_P, \omega_{R_P}) = 0, \forall P \text{ and } \forall i > \dim(M_P) - \min\{t, \dim(M_P)\} \\
&\iff \mathrm{Ext}^{i}_{R_P}(M_P, \omega_{R_P}) = 0 \text{ for all } P \text{ and all } i > \dim(M_P) - t \\
&\iff \mathrm{Ext}^{i}_{R_P}(M_P, \omega_{R_P}) = 0 \text{ as long as } \dim(M_P) < i + t \text{ and } i > 0 \\
&\iff \dim_R \left(\mathrm{Ext}^i_R(M,\, \omega_R)\right) \leqslant d - i - t \text{ for all } i \in \{1, \ldots, d\} \\
&\iff \dim_R \left(\mathrm{Ext}^{d-i}_R(M,\, \omega_R)\right) \leqslant i - t \text{ for all } i \in \{0, \ldots, d-1\} \\
&\iff \dim_{\widehat{R}^{\mathfrak{m}}} \left(\widehat{\mathrm{Ext}^{d-i}_R(M,\, \omega_R)}^{\mathfrak{m}}\right) \leqslant i - t \text{ for all } 0 \leqslant i \leqslant d-1 \\
&\iff \dim_{\widehat{R}^{\mathfrak{m}}} \left(\mathrm{Hom}_R(\mathrm{H}^i_{\mathfrak{m}}(M),\, \mathrm{E}(R/\mathfrak{m}))\right) \leqslant i - t \text{ for all } 0 \leqslant i \leqslant d-1.
\end{aligned}$$

(3) Again, we have (by local duality and Matlis duality, with some of the small details skipped, with $P \in \operatorname{Supp}(M)$)

$$\begin{aligned}
&\dim_{\widehat{R}^{\mathfrak{m}}}(\operatorname{Hom}_R(\oplus_{i=0}^m \operatorname{H}^i_{\mathfrak{m}}(M), \operatorname{E}(R/\mathfrak{m}))) \leqslant u \\
&\iff \dim(\oplus_{i \geqslant d-m} \operatorname{Ext}^i_R(M, \omega_R)) \leqslant u \\
&\iff \oplus_{i \geqslant d-m} \operatorname{Ext}^i_{R_P}(M_P, \omega_{R_P}) = 0, \forall P \text{ such that } \dim(M_P) < d-u \\
&\iff \operatorname{depth}(M_P) > \dim(R_P) - (d-m), \forall P \text{ such that } \dim(M_P) < d-u.
\end{aligned}$$

Note that, for all $P \in \operatorname{Supp}(M)$, $\dim(R_P) = \dim(M_P)$. □

Lemma 5.14. *Let $R = \oplus_{i \in \mathbb{N}^s} R_i$ be an F-finite $\mathbb{N}^s$-graded ring with R_0 being a field, and $M = \oplus_{i \in \mathbb{Q}^s} M_i$ a finitely generated $\mathbb{Q}^s$-graded R-module. Then $\Theta({}^eM) \geqslant \theta({}^eM) q^{\alpha(R_{\mathfrak{m}})}$ for all $e \in \mathbb{N}$.*

Proof. Fix any $e \in \mathbb{N}$ and denote $n := q^{\alpha(R_{\mathfrak{m}})} = [R_0 : R_0^{p^e}]$. Let $\{b_1, \ldots, b_n\}$ be a basis of R_0 as a module over its subfield R_0^q, where $q = p^e$.

As in Remark 5.4, we have the following direct sum decomposition over R

$${}^eM = \bigoplus_{i \in [0,\, p^e)^s \cap \mathbb{Q}^s} {}^e(M_{i+p^e\mathbb{N}^s}), \qquad \text{where} \qquad M_{i+p^e\mathbb{N}^s} := \bigoplus_{j \in i+p^e\mathbb{N}^s} M_j.$$

Since R_0 is a field, $M_{i+p^e\mathbb{N}^s}$ and M_j are all free over R_0 for all $i \in [0,\, p^e)^s \cap \mathbb{Q}^s$ and $j \in i + p^e\mathbb{N}^s$. Therefore we have the following direct sum decompositions over R_0^q:

$$\begin{aligned}
M_{i+p^e\mathbb{N}^s} &= \bigoplus_{r=1}^n (R_0^q b_r M_{i+p^e\mathbb{N}^s}) \\
&= \bigoplus_{r=1}^n \left(R_0^q b_r \bigoplus_{j \in i+p^e\mathbb{N}^s} M_j \right) = \bigoplus_{r=1}^n \left(\bigoplus_{j \in i+p^e\mathbb{N}^s} R_0^q b_r M_j \right).
\end{aligned}$$

It is immediate to verify that ${}^e\left(\bigoplus_{j \in i+p^e\mathbb{N}^s} R_0^q b_r M_j\right)$ is actually a graded R-module for every $r \in \{1, \ldots, n\}$ and every $i \in [0,\, p^e)^s \cap \mathbb{Q}^s$. This establishes the following decomposition of ${}^eM$ as a direct sum of graded R-modules:

$$(\dagger) \qquad {}^eM = \bigoplus_{i \in [0,\, p^e)^s \cap \mathbb{Q}^s} \bigoplus_{r=1}^n {}^e\left(\bigoplus_{j \in i+p^e\mathbb{N}^s} R_0^q b_r M_j \right).$$

Moreover, ${}^e\left(\bigoplus_{j \in i+p^e\mathbb{N}^s} R_0^q b_r M_j\right) \neq 0$ for some (equivalently, for all) $r \in \{1, \ldots, n\}$ if and only if ${}^e\left(\bigoplus_{j \in i+p^e\mathbb{N}^s} M_j\right) \neq 0$.

Thus there are exactly $\theta({}^eM)n$ non-zero graded direct summands in the above decomposition (†) of ${}^eM$. This proves $\Theta({}^eM) \geqslant \theta({}^eM)n = \theta({}^eM)q^{\alpha(R_{\mathfrak{m}})}$. □

Theorem 5.15. *Let $R = \oplus_{i \in \mathbb{N}^s} R_i$ be an F-finite $\mathbb{N}^s$-graded local ring and M a finitely generated equidimensional $\mathbb{Q}^s$-graded R-module. Assume that, for some $n \in \mathbb{N}$, one of the following conditions hold:*

(a) $\limsup_{e \to \infty} \frac{\Theta({}^eM)}{q^{n+\alpha(R_{\mathfrak{m}})}} = \infty$; *or*

(b) *R_0 is a field and* $\limsup_{e \to \infty} \frac{\theta({}^eM)}{q^n} = \infty$.

Then (with $\mathfrak{m}$ denoting the maximal graded ideal of R)

(1) *If the defining ideal of the non-Cohen-Macaulay locus of M, denoted I, satisfies* $\dim(R/I) \leqslant n$, *then there exists an $\mathbb{N}^s$-graded small Cohen-Macaulay R-module N of dimension* $\dim(M)$.
(2) *If M satisfies $\mathbf{S}_t$ for some $t \in \mathbb{N}$, then there exists a finitely generated $\mathbb{N}^s$-graded R-module N such that* $\operatorname{depth}(\mathfrak{m}, N) \geqslant \min\{t+n+1, \dim(M)\}$.
(3) *If* $\operatorname{depth}(M_P) > \dim(M_P) + m - d$, *for some $m \in \{0, \ldots, \dim(M)-1\}$ and for all $P \in \operatorname{Supp}(M)$ such that* $\dim(M_P) < d - n$, *then there exists a finitely generated $\mathbb{N}^s$-graded R-module N such that* $\operatorname{depth}(\mathfrak{m}, N) \geqslant m+1$.

In each case, N can be chosen as a graded direct summand of ${}^eM$, for some $e \in \mathbb{N}$, up to a shift.

Proof. Since condition (b) implies condition (a), it suffices to assume condition (a). Then the claim follows from Corollary 5.10 and Lemma 5.13. □

To find a small Cohen-Macaulay module over an $\mathbb{N}^s$-graded local ring R, it suffices to find one such module over R/P for some $P \in \operatorname{Min}(R)$ such that $\dim(R/P) = \dim(R)$. One advantage of working over an $\mathbb{N}^s$-graded local domain is that we can use group theory to estimate the growth of $\theta({}^eR)$ as $e \to \infty$.

Remark 5.16. Let $R = \oplus_{i \in \mathbb{N}^s} R_i$ be an $\mathbb{N}^s$-graded domain (of any characteristic). Then

$$G := \{i \in \mathbb{N}^s \mid R_i \neq 0\} \text{ is a subsemigroup of } \mathbb{N}^s.$$

Fix homogeneous elements $g_j \in R_{i_j} \setminus \{0\}$, with $j \in \{1, \ldots, n\}$ and $i_j \in \mathbb{N}^s$, such that $\{g_1, \ldots, g_n\}$ generates R as an algebra over R_0. It is clear that G is precisely the subsemigroup of $\mathbb{N}^s$ generated by $\{i_1, \ldots, i_n\}$. Let (G) denote the subgroup of $\mathbb{Z}^s$ generated by G; so that $(G) \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$. Clearly, r is the maximum cardinality of all linearly independent subsets of $\{i_1, \ldots, i_n\}$ over $\mathbb{Z}$ (or over $\mathbb{Q}$). By proper permutation, we may assume that $\{i_1, \ldots, i_r\}$ is linearly independent over $\mathbb{Z}$; so that the subsemigroup (H) generated by $H := \{i_1, \ldots, i_r\}$ is free, i.e., $(H) \cong \mathbb{N}^r$. This further implies the following (see Remark 5.3 and Remark 5.6 for relevant notations):

(1) Let M be a finitely generated $\mathbb{Q}^s$-graded faithful R-module. Then there exists $x \in M_{i_0}$ such that $\operatorname{Ann}_R(x) = 0$, which implies (for any fixed $p \in \mathbb{N}_+$ and $e \in \mathbb{N}$)

$$\begin{aligned}\operatorname{Card}\{i \in [0,\, p^e)^s \cap \mathbb{Q}^s \mid M_{i+p^e\mathbb{N}^s} \neq 0\} &\geqslant \operatorname{Card}([0,\, p^e)^s \cap (i_0 + G)) \\ &\geqslant \operatorname{Card}([0,\, p^e)^s \cap (i_0 + (H))).\end{aligned}$$

It is routine to see that

$$\limsup_{e \to \infty} \frac{\operatorname{Card}([0,\, p^e)^s \cap (i_0 + (H)))}{p^{er}} > 0.$$

(2) Consequently, when R is further assumed to have prime characteristic $p > 0$, we see (cf. Remark 5.6)

$$\limsup_{e \to \infty} \frac{\theta({}^eR)}{p^{er}} > 0.$$

Definition 5.17. In the situation of Remark 5.16, we call r the *grading dimension* of the $\mathbb{N}^s$-graded domain R.

Theorem 5.18. *Let $R = \oplus_{i \in \mathbb{N}^s} R_i$ be an $\mathbb{N}^s$-graded domain with R_0 being an F-finite field. Assume that the grading dimension of R is r. Let M be a finitely generated $\mathbb{Q}^s$-graded faithful R-module (e.g., $M = R$). Denote by $\mathfrak{m}$ the unique homogeneous maximal ideal of R.*

1. *If the non-Cohen-Macaulay locus of M is defined by an ideal I that satisfies $\dim(R/I) < r$, then there exists an $\mathbb{N}^s$-graded small Cohen-Macaulay R-module (of maximal dimension).*
2. *If M satisfies the $\mathbf{S}_t$ condition with $t \in \mathbb{N}$, then there exists a finitely generated $\mathbb{N}^s$-graded R-module N such that $\operatorname{depth}(\mathfrak{m}, N) \geqslant \min\{t + r, \dim(R)\}$.*
3. *If $\operatorname{depth}(M_P) > \dim(M_P) + m - d$, for some $m \in \{0, \dots, \dim(M) - 1\}$ and for all $P \in \operatorname{Supp}(M)$ such that $\dim(M_P) \leqslant d - r$, then there exists a finitely generated $\mathbb{N}^s$-graded R-module N such that $\operatorname{depth}(\mathfrak{m}, N) \geqslant m + 1$.*

Proof. This follows from Theorem 5.15 and Remark 5.16. □

Theorem 5.19. *Let $R = \oplus_{i \in \mathbb{N}^s} R_i$ be an $\mathbb{N}^s$-graded domain with R_0 being an F-finite field. Assume that the grading dimension of R is r. Then there exists a finitely generated $\mathbb{N}^s$-graded R-module N such that $\operatorname{depth}(N) \geqslant \min\{2 + r, \dim(R)\}$.*

In particular, if $\dim(R) \leqslant r + 2$, then there exists an $\mathbb{N}^s$-graded small Cohen-Macaulay R-module.

Proof. Let $\overline{R}$ be the integral closure of R in its fraction field. Then $\overline{R}$ is module-finite over R and is $\mathbb{N}^s$-graded. Moreover, $\overline{R}$ satisfies the $\mathbf{S}_2$ condition as a ring, hence it has the $\mathbf{S}_2$ condition as an R-module. Now both claims follow from Theorem 5.18. □

6. Examples of regular Noetherian domains that are not very strongly F-regular

In this section we generalize a construction in [EHo, Remarks (2), p. 159] to give a large family of regular rings that are not very strongly F-regular. We shall construct many regular Noetherian domains R with a nonzero element $c \in R$ such that the map $R \to {}^eR$ with $1 \mapsto {}^ec$ is not pure over R for any choice of $e \in \mathbb{N}_+$. Of course, all regular rings R are strongly F-regular in the sense of Definition 2.2. Moreover, we can arrange that R/cR is a strongly F-regular domain such that every local ring at a maximal ideal has an isolated singularity, and, in fact, is the local ring at a maximal ideal of an affine hypersurface over a field. However, the rings we construct are not excellent. In each of these domains there is an element c such that c is in $M_s^{h_s}$ for a family of maximal ideals $\{M_s : s \in \mathbb{N}_+\}$ such that the positive integers $h_s \to \infty$ as $s \to \infty$. Note:

Proposition 6.1. *If the map $R \to {}^eR$ such that $1 \mapsto {}^ec$ is pure, then for every maximal ideal M of R we have that $c \notin M^{[p^e]}$.*

Proof. The purity implies that the induced map $R/M \to {}^eR/(M({}^eR)) \cong R/M^{[p^e]}$, sending $\overline{1}$ to $\overline{c}$, is injective, and so $c \notin M^{[p^e]}$. □

Discussion 6.2. The construction is based on a refinement of the method in [Ho1], which utilizes an idea of Nagata [Na, p. 203] for constructing Noetherian rings of infinite Krull dimension. In [Ho1], the rings constructed have the property that every nonzero element is in only finitely many maximal ideals, and then the Noetherian property for the local rings at maximal ideals implies that the ring is Noetherian.

We summarize some results of [Ho1] in the simplest case, which is all that we will need here. Let $B_1, \dots, B_n, \dots$ be a sequence of domains finitely generated over an algebraically closed field k, and for every $s \in \mathbb{N}_+$ let $\mathfrak{M}_s$ be a maximal ideal of B_s. In this situation, the conditions that certain rings be absolutely Noetherian or absolutely domains imposed in [Ho1] are automatic.

Let $C = \bigotimes_{s\in\mathbb{N}_+} B_s$. Then each $\mathfrak{M}_n C$ is prime in C, and if V is the complement of the union of these prime ideals in C, $V^{-1}C$ has as its maximal ideals precisely the ideals $\mathfrak{m}_s = V^{-1}\mathfrak{M}_s C$. The local rings $(V^{-1}C)_{\mathfrak{m}_s}$ are all essentially of finite type over a field, although the field depends on s. In fact the residue class field is the fraction field $\mathcal{L}_s$ of the domain $D_s = \bigotimes_{t\in\mathbb{N}_+, t\neq s} B_t$, and D_s embeds in $C \cong D_s \otimes_k B_s$ with image disjoint from $\mathfrak{M}_s C$. It follows that $\mathcal{L}_s$ embeds in

$$(V^{-1}C)_{\mathfrak{m}_s} \cong C_{\mathfrak{M}_s C} \cong (\mathcal{L}_s \otimes B_s)_{\mathcal{Q}_s},$$

where $\mathcal{Q}_s$ is the extension $\mathfrak{M}_s(\mathcal{L}_s \otimes_k B_s) = \mathcal{L}_s \otimes_k \mathfrak{M}_s$. Thus, the local rings of $V^{-1}C$ are Noetherian, and even excellent, since they are essentially of finite type over a field. In [Ho1] it is shown that $V^{-1}C$ is Noetherian, by proving that every nonzero element belongs to only finitely many maximal ideals.

The situation we deal with here has an extra difficulty, because we want there to be an element c that is in every maximal ideal. The following result is used to prove the Noetherian property.

Theorem 6.3. *Let $R = \varinjlim_\lambda R_\lambda$, where the rings R_λ are Noetherian, the maps in the direct limit system are faithfully flat (and so injective) and let $c \in R_{\lambda_0}$ be an element that is in the Jacobson radical of R_μ for all $\mu \geq \lambda_0$. Suppose that R/cR is Noetherian. Then R is Noetherian.*

Proof. If not, let P be an ideal of R that is maximal with respect to not being finitely generated. As in the proof of Cohen's theorem [Na, Theorem (3.4), p. 8] P is prime. Since R/cR is Noetherian, every ideal of R containing c is finitely generated. Thus, we may assume $c \notin P$ but that we have finitely many elements $r_1, \dots, r_h \in P$ that generate $P(R/cR)$. Choose λ_1 so that these elements and c are in $R_1 := R_{\lambda_1}$. Let $\mathfrak{p}$ denote the contraction of P to R_1. Then $P \subseteq \mathfrak{p}R + cR$. But when we write $u \in P$ as $v + cr$ with $v \in \mathfrak{p}R$ and $r \in R$, we have that $cr = u - v \in P$, and so $r \in P$. It follows that $P = \mathfrak{p}R + cP$. But if $P = \mathfrak{p}R + c^j P$ for some integer $j \geq 1$, then we have $P = \mathfrak{p}R + c^j(\mathfrak{p}R + cP) = \mathfrak{p}R + c^{j+1}P$. It follows by induction on j that $P = \mathfrak{p}R + c^j P \subseteq \mathfrak{p}R + c^j R$ for all $j \in \mathbb{N}_+$.

To complete the proof, it suffices to show that $P = \mathfrak{p}R$. Suppose, to the contrary, that $r \in P \setminus \mathfrak{p}R$ and that $r \in R_\mu$ for some $\mu \geq \lambda_1$. Then, for all $j \in \mathbb{N}_+$, $r \in (\mathfrak{p}R + c^j R) \cap R_\mu$. Since $R_\mu \to R$ is faithfully flat, we have that $r \in \mathfrak{p}R_\mu + c^j R_\mu$ for all $j \in \mathbb{N}_+$. If r is not 0 in $R_\mu/\mathfrak{p}R_\mu$, choose a maximal ideal $\mathfrak{M}$ of R_μ that contains $\mathrm{Ann}_{R_\mu}(r \mod \mathfrak{p}R_\mu)$. This maximal ideal also contains c. After we localize at $\mathfrak{M}$, we have that the image of r is not 0, but is in every power of the maximal ideal, a contradiction. □

Construction 6.4. We now give the construction for a family of choices for the ring R. Let k be an algebraically field. Let $\mathcal{X}_1, \dots, \mathcal{X}_s, \dots$ be a sequence of mutually disjoint finite sets of indeterminates over k. For every $s \in \mathbb{N}_+$, let G_s denote an irreducible polynomial of positive degree without constant term in $k[\mathcal{X}_s]$. Let

$\mathcal{X} = \bigcup_{s\in\mathbb{N}_+} \mathcal{X}_s$, and let $T = k[\mathcal{X}]$. Let $S = T/(G_s - G_{s'} : s, s' \in \mathbb{N}_+)$. Let c denote the common image of the various G_s in S, and also, by slight abuse of notation, in localizations of S. Let $\mathfrak{P}_s = (\mathcal{X}_s)T$, and let Q_s be its image $\mathfrak{P}_s S$ in S. We shall show that the ideals $\mathfrak{P}_s S$ are prime. Let $W = S \setminus \bigcup_{s\in\mathbb{N}_+} Q_s$. We shall show that $R = W^{-1}S$ is a regular Noetherian ring whose maximal ideals are simply the ideals of the form $W^{-1}Q_s =: M_s$, $s \in \mathbb{N}_+$, that c is a nonzero element in the Jacobson radical of R, and R/cR is a domain, each of whose local rings $(R/cR)_{M_s}$ has the form $\mathcal{L}_s[\mathcal{X}_s]/(G_s)$ for a suitable field extension $\mathcal{L}_s$ of k. Specific examples are discussed in 6.8.

In proving that R has the right properties it will be convenient to introduce additional rings as follows. For $t \geq 1$, let $T_t := k[\mathcal{X}_s : 1 \leq s \leq t]$ and let $S_t := T_t/(G_s - G_{s'} : 1 \leq s, s' \leq t)$. Note that S_1 may be identified with T_1, since the set of differences $G_s - G_{s'}$ contains only 0 in this case. We let c_t denote the common value of the G_s for $1 \leq s \leq t$ in S_t. Let $Q_{s,t} = \mathfrak{P}_s S_t$. Let $W_t = S_t \setminus \bigcup_{s=1}^{t} Q_{s,t}$. We have an obvious map $S_t \to S_{t'}$ for $t \leq t'$ induced by $T_t \subseteq T_{t'}$. Let $R_t := (W_t)^{-1}S_t$. The following result gives a catalogue of the properties we need to know about this construction. Note that once we have established this result, we shall simply write c for the image of any of the elements c_t in R.

Before giving the result we note the following fact: Let B be an arbitrary ring and $c \in B$ a nonzerodivisor. Suppose that $\bigcap_{n\in\mathbb{N}_+} c^n B = 0$, which holds if either (1) B is an $\mathbb{N}$-graded ring and c is a form of positive degree or (2) B is any Noetherian ring and c is in the Jacobson radical of B. If B/cB is a domain, then B is a domain. (Otherwise there are nonzero elements in B whose product is zero, and we can factor a highest power of c from each to produce an example where neither element is divisible by c. But then, taking images modulo cB, we get a contradiction, since B/cB is a domain. Some hypothesis like (1) or (2) is necessary. If x, c are indeterminates and k is a field, the ring $B := k[x, c]/(x - xc)$ is not a domain and the image $\overline{c}$ of c is a nonzerodivisor in B, but $B/\overline{c}B \cong k$ is a domain.)

Proposition 6.5. *Let all notation be as in 6.4.*

(1) *For all $t \geq 1$, $S_{t+1} \cong S_t[\mathcal{X}_{t+1}]/(G_{t+1} - c_t)$ and is nonzero free module over S_t. Hence, $S_{t'}$ is free over S_t for all $t \leq t'$. Since c_1 is a nonzerodivisor in S_1, c_t is a nonzerodivisor in S_t for all $t \geq 1$.*
(2) *$S = \varinjlim_t S_t$ with injective maps, and we think of the maps as inclusions.*
(3) *For all t, $c_t S_t$ is a prime ideal in S_t. In fact, $S_t/c_t S_t$ is the tensor product over k of the domains $T_s/G_s T_s$, $1 \leq s \leq t$.*
(4) *Each of the ideals $Q_{s,t}$ is a prime ideal in S_t containing c_t. The complement of $\bigcup_{s=1}^{t} Q_{s,t}/c_t S_t$ in $S_t/c_t S_t$ is the image of $W_t \mod c_t S_t$.*
(5) *For all $t \geq 1$ and for all s, $1 \leq s \leq t$, we have $Q_{s,t+1} \cap S_t = Q_{s,t}$, while $Q_{t+1,t+1} \cap S_t = c_t S_t$.*
(6) *For all $t \geq 1$, $W_{t+1} \cap S_t = W_t$, and $\bigcup_{t\in\mathbb{N}_+} W_t = W$.*
(7) *The element c_t is in the Jacobson radical of R_t for all $t \geq 1$.*
(8) *All of the rings R_t are regular domains, and $R = \varinjlim_t R_t$.*

Proof. (1) The isomorphism stated is clear. Freeness follows because, after a change of variables over k, G_{t+1} may be thought of as monic in one of the variables in $\mathcal{X}_{t+1}$, and this is also true for $G_{t+1} - c_t$ viewed as a polynomial in $S_t[\mathcal{X}_{t+1}]$.

(2) This is clear: the maps may be thought of as inclusions because they are nonzero free and so faithfully flat.

(3) It is also clear that S_t/c_tS_t is a tensor product as stated, since this is equivalent to dividing by the ideal generated by all the G_s, $1 \le s \le t$. The ring $k[\mathcal{X}_s]/(G_s)$ is a domain because G_s is irreducible. Moreover, a tensor product of domains over an algebraically closed field is a domain.

(4) From the definitions, it is immediate that $S_t/Q_{s,t}$ is the tensor product over k of the rings $k[\mathcal{X}_j]/(G_j)$ as j runs through all values with $j \neq s$ and $1 \le j \le t$. It is also evident that c_t, which may be identified with c_s, is in this prime. The second statement is clear, since if $u \in S_t$, we have that $u \notin Q_{s,t}$ if and only if its image in $S_t/Q_{s,t}$ is not 0, and $S_t/Q_{s,t} \cong (S_t/c_tS_t)/(Q_{s,t}/c_tS_t)$.

(5) Note that $Q_{s,t+1} \cap S_t$ is the kernel of the composition map

$$(*) \quad \gamma : S_t \to S_{t+1}/Q_{s,t+1} \cong \bigotimes_{1\le j\le t+1,\, j\neq s} k[\mathcal{X}_j]/G_jk[\mathcal{X}_j] =: C,$$

while $Q_{s,t}$ is the kernel of the map

$$(**) \quad \beta : S_t \to S_t/Q_{s,t} \cong \bigotimes_{1\le j\le t,\, j\neq s} k[\mathcal{X}_j]/G_jk[\mathcal{X}_j] =: B.$$

When $s < t+1$, the result now follows because we have an injection $\iota : B \to C$ (coming from the fact that $C \cong B \otimes_k (k[\mathcal{X}_{t+1}]/G_{t+1}k[\mathcal{X}_{t+1}])$ such that $\gamma = \iota \circ \beta$. When $s = t+1$, we have a similar situation, but C in $(*)$ becomes $\bigotimes_{1\le j\le t} k[\mathcal{X}_j]/G_jk[\mathcal{X}_j]$ and B in $(**)$ becomes the same ring if we replace $Q_{s,t}$ by c_tS_t.

(6) We have that $u \in S_t$ is in W_{t+1} if and only if it is not in any $Q_{s,t+1}$, $1 \le s \le t$, and from part (6) this means that is not in any $Q_{s,t}$, $1 \le s \le t$ and not in c_tS_t. Since c_t is in all of the $Q_{s,t}$, this simply says that u is not in any $Q_{s,t}$, $1 \le s \le t$. It follows that an element of W is in S_t if and only if it is in W_t, and since S is the union of the S_t, the result follows.

(7) By construction, R_t is a semilocal ring whose maximal ideals are the $W_t^{-1}Q_{s,t}$, $1 \le s \le t$, and these all contain c_t.

(8) For every $t \ge 1$ the ring R_t is a domain, because c_t is a nonzerodivisor in the Jacobson radical of R_t and R_t/c_tR_t is a domain. Note that $A := (R_t)_{W_t^{-1}Q_{s,t}}$ is the localization of S_t at the prime ideal generated by $\mathcal{X}_s$. Hence, A/c_tA may be identified with the local ring of $L_{s,t}[\mathcal{X}_s]/(G_s)$ at $(\mathcal{X}_s)$, where $L_{s,t}$ is the fraction field of $\bigotimes_{1\le j\le t,\, j\neq s} k[\mathcal{X}]/(G_j)$. This is the local ring at the origin of a hypersurface domain of dimension $n_s - 1$, where n_s is the cardinality of $\mathcal{X}_s$. Therefore, A has dimension n_s. Since its maximal ideal is generated by the image of $\mathcal{X}_s$, A is regular. □

Remark 6.6. Suppose that we also assume that for every s, the indeterminates in $\mathcal{X}_s$ have been assigned positive integer degrees, so that the polynomial ring $k[\mathcal{X}_s]$ has a (possibly non-standard) $\mathbb{N}$-grading, and that we also assume that every G_s is weighted homogeneous. Then, for $t \ge 1$, all of the rings S_t are domains. We can see this as follows. We may give S_t the structure of an $\mathbb{N}$-graded algebra over k as follows: let $\deg_j$ denote degree with respect to the original weighted grading on the variables in $\mathcal{X}_j$, and for $1 \le s \le t$, let $d_s := \mathrm{lcm}\{\deg_j(G_j) \mid 1 \leqslant j \leqslant t\}/\deg_s(G_s)$. Scale the degrees of all of the variables in $\mathcal{X}_s$ by giving $x \in \mathcal{X}_s$ the new degree $d_s \deg_s(x)$. This gives all of the finitely many variables in $\bigcup_{j=1}^t \mathcal{X}_j$ positive integer degrees in such a way that all of the G_j have the same degree, namely $\mathrm{lcm}\{\deg_j(G_j) \mid 1 \leqslant j \leqslant t\}$. This makes S_t an $\mathbb{N}$-graded k-algebra in which

c_t is a form of degree $\operatorname{lcm}\{\deg_j(G_j) \mid 1 \leqslant j \leqslant t\}$. The element c_t is a nonzerodivisor in S_t by Proposition 6.5(1). Since S_t/c_tS_t is a domain, we see that S_t is a domain.

We now have the following:

Theorem 6.7. *Let all notation be as in 6.4. Then:*

(1) *The ring R is a regular domain, and its maximal ideals are precisely the ideals M_s for $s \in \mathbb{N}_+$. The height of M_s is n_s, where n_s is the cardinality of $\mathcal{X}_s$.*
(2) *The element c is prime. The ring S/cS is the ascending union over t of the rings*
$$k[\mathcal{X}_1]/(G_1) \otimes_k \cdots \otimes_k k[\mathcal{X}_t]/(G_t),$$
and may also be described as $\bigotimes_{s\in\mathbb{N}_+} k[\mathcal{X}_s]/G_sk[\mathcal{X}_s]$.
(3) *The local ring $(R/cR)_{M_s}$ is the same as the localization of the homogeneous affine hypersurface $(\mathcal{L}_s[\mathcal{X}_s])/(G_s)$ at the ideal generated by $\mathcal{X}_s$, where $\mathcal{L}_s$ is the fraction field of $\bigotimes_{t\in\mathbb{N}_+\setminus\{s\}} k[\mathcal{X}_t]/G_tk[\mathcal{X}_t]$.*
(4) *Let h_s denote the largest positive integer h such that $G_s \in (\mathcal{X}_s)^h$. Then we have that $c \in M_s^{h_s}$ for all $s \in \mathbb{N}_+$.*
(5) *If the set of integers $\{h_s : s \in \mathbb{N}_+\}$ defined in part (4) is not bounded, then R is not very strongly F-regular: the map $R \to {}^eR$ with $1 \mapsto {}^ef$ is not pure for any choice of e.*

We note only that (5) follows from Proposition 6.1.

Examples 6.8. We consider the following specific examples. We use Construction 6.4 and Theorem 6.7. Recall from Construction 6.4 that $M_s = (\mathcal{X}_s)R$. In all of these examples, R is regular Noetherian and is not very strongly F-regular. The rings R are not excellent, but the local rings of R/cR are essentially of finite type over a field. We do not know whether R itself is locally excellent.

(1) For every s let $\mathcal{X}_s := \{X_s, Y_s\}$ and let $G_s := X_s^{a_s} - Y_s^{b_s}$, where a_s, b_s are relatively prime. Give X_s degree b_s and Y_s degree a_s. Choose the pairs so that $a_s \to \infty$ and $b_s \to \infty$ as $s \to \infty$. E.g., one may choose $(a_s, b_s) = (s, s+1)$ for all $s \in \mathbb{N}_+$. Then the ring R constructed is regular of dimension 2.

(2) For every s let $\mathcal{X}_s := \{X_s, Y_s, Z_s\}$ with standard grading and choose $G_s := X_s^{h_s} + Y_s^{h_s} + Z_s^{h_s}$ where h_s is chosen to be an integer not divisible by p such that $h_s \to \infty$ as $s \to \infty$. Then R is regular of dimension 3, and $(R/cR)_{M_s}$ has an isolated singularity for all s.

(3) For every s let $\mathcal{X}_s$ consist of $n_s := h_s^2$ variables with the standard grading, where $h_s \geq 2$, and let G_s be a form of degree h_s with coefficients in k chosen in general position (that is, we avoid a proper closed subset of the affine space of possible coefficients for the form). Suppose also that $h_s \to \infty$ as $s \to \infty$. Then R is regular of infinite Krull dimension, and $(R/cR)_{M_s}$ has an isolated singularity that is strongly F-regular for all s. To see this it suffices to see that $k[\mathcal{X}_s]/G_s$ is F-regular (hence, strongly F-regular). We may use the criterion in [HH5, Corollary 7.13(b)]. Since G_s is in general position, it has an isolated singularity. It therefore suffices to check that $K[\mathcal{X}_s]/(G_s)$ is F-pure with a negative **a**-invariant, but the lattter condition is clear. Finally, since G_s is in general position, it suffices from [HoR2, §5(c), Proposition 4.19, p. 158] (the number should have been 5.19, not 4.19) to show that there exists

a homogeneous choice of G of degree h_s in h_s^2 variables whose quotient ring is F-pure, and we may take G to be the determinant of an $h_s \times h_s$ matrix of indeterminates.

Quite recently, in [HoY], we show that every stronly F-regular excellent ring is very strongly F-regular.

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