PRIMARY DECOMPOSITION II: PRIMARY COMPONENTS AND LINEAR GROWTH

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Abstract. We study the following properties about primary decomposition over a Noetherian ring R: (1) For finitely generated modules $N \subseteq M$ and a given subset $X = \{P_1, P_2, \ldots, P_r\} \subseteq \operatorname{Ass}(M/N)$, we define an X-primary component of $N \subsetneq M$ to be an intersection $Q_1 \cap Q_2 \cap \cdots \cap Q_r$ for some P_i-primary components Q_i of $N \subseteq M$ and we study the maximal X-primary components of $N \subseteq M$; (2) We give a proof of the 'linear growth' property of Ext and Tor, which says that for finitely generated modules N and M, any fixed ideals $I_1, I_2, \ldots, I_t$ of R and any fixed integer $i \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that for any $\underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$ there exists a primary decomposition of 0 in $E_{\underline{n}} = \operatorname{Ext}^i_R(N, M/I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M)$ (or 0 in $T_{\underline{n}} = \operatorname{Tor}_i^R(N, M/I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M)$) such that every P-primary component Q of that primary decomposition contains $P^{k|\underline{n}|} E_{\underline{n}}$ (or $P^{k|\underline{n}|} T_{\underline{n}}$), where $|\underline{n}| = n_1 + n_2 + \cdots + n_t$.

0. Introduction

Throughout this paper R is a Noetherian ring and every R-module is assumed to be finitely generated unless stated otherwise explicitly. Let $N \subsetneq M$ be a proper R-submodule of M. By a primary decomposition $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ of N in M, we always mean a minimal (hence irredundant) primary decomposition, where Q_i is a P_i-primary submodule of M, i.e. $\operatorname{Ass}(M/Q_i) = \{P_i\}$, for each $i = 1, 2, \ldots, s$, unless mentioned otherwise explicitly. Then $\operatorname{Ass}(M/N) = \{P_1, P_2, \ldots, P_s\}$ and we say that Q_i is a P_i-primary component of N in M. As a subset of $\operatorname{Spec}(R)$ with the *Zariski* topology, $\operatorname{Ass}(M/N)$ inherits the subspace topology. It is easy to see that if $N = M$, then $\operatorname{Ass}(M/N) = \emptyset$ and everything becomes trivial. For an ideal I in R, we use $(N :_M I^\infty)$ to denote $\cup_{i=1}^\infty (N :_M I^i)$. If $U \subset R$ is a multiplicatively closed subset of R, we use $R[U^{-1}]$ to denote the localized ring at U and use $M[U^{-1}] \cong M \otimes R[U^{-1}]$ to denote and localized $R[U^{-1}]$-module for any R-module M. We also use $N[U^{-1}] \cap M$ to denote the pre-image of $N[U^{-1}]$ under the natural map $M \to M[U^{-1}]$. Since $\operatorname{Ass}(M/N)$ is finite, every subset $X \subseteq \operatorname{Ass}(M/N)$ has a unique minimal open superset in $\operatorname{Ass}(M/N)$, which we denote by $\mathrm{o}(X)$. For any $P \in \operatorname{Ass}(M/N)$, we may simply write $\mathrm{o}(\{P\})$ as $\mathrm{o}(P)$. In fact it is easy to see that $\mathrm{o}(X) = \{P \in \operatorname{Ass}(M/N) \mid P \subseteq \cup_{P' \in X} P'\}$. We use $\mathbb{N}$ to denote the set of all non-negative integers.

Notation 0.1. Let $N \subseteq M$ be finitely generated R-modules and $X \subseteq \operatorname{Ass}(M/N)$ a subset of $\operatorname{Ass}(M/N)$. Say $X = \{P_1, P_2, \ldots, P_r\} \subseteq \{P_1, P_2, \ldots, P_r, P_{r+1}, \ldots, P_s\} = \operatorname{Ass}(M/N)$.

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(1) If $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ is a primary decomposition of N in M with Q_i being P_i-primary, then we say $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_r$ is an X-primary component (or a primary component over X) of $N \subseteq M$. If $X = \emptyset$, then we agree that M is the only X-primary component of $N \subseteq M$.

(2) We call an X-primary component of $N \subseteq M$ maximal if it is not properly contained in any X-primary component of $N \subseteq M$.

(3) We use $\Lambda_X(N \subseteq M)$, or Λ_X if the R-modules $N \subseteq M$ are clear from the context, to denote the set of all possible X-primary components of N in M.

(4) We use $\overset{\circ}{\Lambda}_X(N \subseteq M)$, or $\overset{\circ}{\Lambda}_X$ if the R-modules $N \subseteq M$ are clear from the context, to denote the set of all maximal X-primary components of N in M.

(5) In the above notations, if $X = \{P\} \subseteq \operatorname{Ass}(M/N)$, i.e. $P \in \operatorname{Ass}(M/N)$, we may simply write Λ_P or $\overset{\circ}{\Lambda}_P$ instead of $\Lambda_{\{P\}}$ or $\overset{\circ}{\Lambda}_{\{P\}}$.

Remark 0.2. Let $N \subseteq M$ and X be as in Notation 0.1.

(1) Due to the compatibility property of primary decomposition (c.f. [Yao, Theorem 1.1]), we can equivalently say that Q is an X-primary component of $N \subseteq M$ if $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_r$, where Q_i is a P_i-primary component of $N \subseteq M$. But we shall stick to the definition as in Notation 0.1(1) and avoid using the compatibility property until after we have given a new proof of the compatibility property in Corollary 1.2.

(2) Using the notion of X-primary component, we may equivalently agree to say that the primary decompositions of $N \subseteq M$ are independent over X if $\Lambda_X(N \subseteq M)$ contains a unique X-primary component (c.f. [Yao, Definition 0.2]). It is well-known that $\Lambda_X(N \subseteq M)$ contains a unique X-primary component if X is an open subset of $\operatorname{Ass}(M/N)$ (see, for example, [Ei, page 101, Proposition 3.13]). Conversely, the uniqueness of X-primary component implies X is open in $\operatorname{Ass}(M/N)$ (c.f. [Yao, Theorem 2.2]). A slightly stronger version of this result will be proved in Corollary 1.5.

(3) Recall that $X = \{P_1, P_2, \ldots, P_r\} \subseteq \operatorname{Ass}(M/N)$. It is easy to see that the unique $\mathrm{o}(X)$-primary component in $\Lambda_{\mathrm{o}(X)}(N \subseteq M)$ is equal to $N[U^{-1}] \cap M$ where $U = R \setminus \cup_{i=1}^{r} P_i$ (see, for example, [Ei, page 113, Exercise 3.12]). Recall that $\mathrm{o}(X) = \{P \in \operatorname{Ass}(M/N) \mid P \subseteq \cup_{i=1}^{r} P_i\}$ is the minimal open superset of X.

Let $N \subseteq M$ be finitely generated R-modules and $P \in \operatorname{Ass}(M/N)$. Then the P-primary component of $N \subseteq M$ is unique if and only if $P \in \operatorname{Ass}(M/N)$ is minimal. That is to say that the embedded primary components are not unique, which is an easy consequence of [HRS]. In fact W. Heinzer, L. J. Ratliff, Jr. and K. Shah proved stronger results regarding the (maximal) embedded primary components in [HRS]:

Theorem 0.3 (Heinzer, Ratliff and Shah)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I \subseteq R$ an ideal of R. Assume that $\mathfrak{m} \in \operatorname{Ass}(R/I)$.*

(1) The intersection of all maximal $\mathfrak{m}$-primary components of I in R, which is equal to the intersection of all $\mathfrak{m}$-primary components of I in R by part (2) below, is equal to I, which implies that there are infinitely many maximal $\mathfrak{m}$-primary components of $I \subseteq R$ if $\mathfrak{m}$ is embedded (c.f. [HRS, Theorem 2.8]).

(2) *For every* $\mathfrak{m}$*-primary component* Q *of* $I \subseteq R$*,* Q *is a finite intersection of maximal* $\mathfrak{m}$*-primary components of* $I \subseteq R$*.* (*c.f.* [HRS, Theorem 2.13]).

The above results of [HRS] can be translated to the following statement in a more general situation:

Theorem 0.4 (Heinzer, Ratliff and Shah)**.** *Let* R *be a Noetherian ring and* $N \subseteq M$ *finitely generated* R*-modules. Assume that* $P \in \mathrm{Ass}(M/N)$*.*

(1) *Every* $Q \in \Lambda_P(N \subseteq M)$ *can be written as an intersection of finitely many* $Q' \in \overset{\circ}{\Lambda}_P(N \subseteq M)$ (*c.f.* [HRS, Theorem 2.13]).

(2) *The intersection* $\cap\{Q \,|\, Q \in \overset{\circ}{\Lambda}_P(N \subseteq M)\} = \cap\{Q \,|\, Q \in \Lambda_P(N \subseteq M)\}$ *is equal to* $N_P \cap M$*, which implies that there are infinitely many maximal* P*-primary components of* $N \subseteq M$ *if* P *is embedded* (*c.f.* [HRS, Theorem 2.8]).

Remark 0.5. (1) For example, we can translate the results in Theorem 0.3 from the $I \subseteq R$ situation to the R-modules $N \subseteq M$ by using Nagata's Idealization technique.

(2) The claims in the case where $P \in \mathrm{Ass}(M/N)$ is minimal over $\mathrm{Ann}(M/N)$ are trivially true.

In section 1, inspired by the results of [HRS], we are going to prove the following results regarding the X-primary component of $N \subseteq M$ for a subset X of $\mathrm{Ass}(M/N)$. Namely,

Theorem 1.3. *Let* R *be a Noetherian ring and* $N \subseteq M$ *finitely generated* R*-modules. Assume that* $X \subset \mathrm{Ass}(M/N)$*. Say* $X = \{P_1, P_2, \ldots, P_r\}$*. Set* $U = R \setminus \cup\{P \,|\, P \in X\}$*. Then,*

(1) $\overset{\circ}{\Lambda}_X(N \subseteq M) = \{\cap_{i=1}^r Q_i \,|\, Q_i \in \overset{\circ}{\Lambda}_{P_i}(N \subseteq M), 1 \le i \le r\}$.

Consequently, we also have the following.

(2) *For every* $Q' \in \Lambda_X(N \subseteq M)$*,* $Q' = \cap\{Q \,|\, Q \in \overset{\circ}{\Lambda}_X(N \subseteq M), Q' \subseteq Q\}$*. Actually every* $Q' \in \Lambda_X(N \subseteq M)$ *is an intersection of finite members of* $Q \in \overset{\circ}{\Lambda}_X(N \subseteq M)$*.*

(3) *The intersection* $\cap\{Q \,|\, Q \in \overset{\circ}{\Lambda}_X(N \subseteq M)\} = \cap\{Q \,|\, Q \in \Lambda_X(N \subseteq M)\}$ *is equal to* $N[U^{-1}] \cap M$*. This result implies that there are infinitely many maximal* X*-primary components of* $N \subseteq M$ *if* X *is not open in* $\mathrm{Ass}(M/N)$*.*

From section 2 onwards, we study the 'linear growth' property of the primary decompositions of a family of R-modules. The linear growth property measures the 'sizes' of the primary components. Roughly speaking, it says that the primary components are big enough in some specific primary decompositions (see Definition 0.6 below for its precise meaning). We give a tentative definition of the linear growth property as we are going to study the linear growth property abstractly.

Definition 0.6. Given a family $\mathcal{F} = \{M_{\underline{n}} \,|\, \underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t\}$ of finitely generated R-modules, we say $\mathcal{F}$ satisfies the linear growth property (of primary decomposition) if there exist $k, b \in \mathbb{N}$ such that, for any $\underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$, there exists a primary decomposition of 0 in $M_{\underline{n}}$

$$0 = Q_{\underline{n}_1} \cap Q_{\underline{n}_2} \cap \cdots \cap Q_{\underline{n}_{s_{\underline{n}}}},$$

where the $Q_{\underline{n}_i}$'s are $P_{\underline{n}_i}$-primary components of the primary decomposition such that $P_{\underline{n}_i}^{k|\underline{n}|+b} M_{\underline{n}} \subseteq Q_{\underline{n}_i}$ for all $i = 1, 2, \ldots, s_{\underline{n}}$, where $|\underline{n}| = n_1 + n_2 + \cdots + n_t$.

Notice that if $M_{(0,0,\ldots,0)} = 0$, then we can always additionally require $b = 0$.

The first family of R-modules proved to satisfy the linear growth property is $\{M_{\underline{n}} = \frac{M}{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M} \,|\, \underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t\}$. We state the result as follows,

Theorem 0.7 (Linear growth; [Sw, Theorem 3.4] and [Sh2, Theorem 2.1])**.** *Let R be a Noetherian ring, M a finitely generated R-module and $I_1, I_2, \ldots, I_t$ ideals of R. Then there exists an integer $k \in \mathbb{N}$ such that for any $\underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$, there exists a primary decomposition of* 0 *in* $M_{\underline{n}} := \frac{M}{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M}$

$$0 = Q_{\underline{n}_1} \cap Q_{\underline{n}_2} \cap \cdots \cap Q_{\underline{n}_{s_{\underline{n}}}},$$

where the $Q_{\underline{n}_i}$'s are $P_{\underline{n}_i}$-primary components of the primary decomposition such that $P_{\underline{n}_i}^{k|\underline{n}|} M_{\underline{n}} \subseteq Q_{\underline{n}_i}$ for all $i = 1, 2, \ldots, s_{\underline{n}}$, where $|\underline{n}| = n_1 + n_2 + \cdots + n_t$.

The essential case (i.e. $M = R$) of the above linear growth property was first proved in [Sw] by I. Swanson. Then R. Y. Sharp, by using injective modules, proved the linear growth property in the general situation as stated in the above theorem (see [Sh2]). Recently the author gave another (short) proof of the above linear growth property by using Artin-Rees numbers (c.f. [Yao]).

The second family of R-modules that satisfies the linear growth property is the family $\{R_{\underline{n}} = R/\overline{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t}} \,|\, \underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t\}$, which was proved by R. Y. Sharp, as stated below:

Theorem 0.8 (Linear growth; [Sh1, Theorem 4.1])**.** *Let R be a Noetherian ring and $I_1, I_2, \ldots, I_t$ ideals of R. Then there exists an integer $k \in \mathbb{N}$ such that for any $\underline{n} \in \mathbb{N}^t$, there exists a primary decomposition of* 0 *in* $R_{\underline{n}} := R/\overline{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t}}$

$$0 = Q_{\underline{n}_1} \cap Q_{\underline{n}_2} \cap \cdots \cap Q_{\underline{n}_{s_{\underline{n}}}},$$

where the $Q_{\underline{n}_i}$'s are $P_{\underline{n}_i}$-primary components of the primary decomposition such that $P_{\underline{n}_i}^{k|\underline{n}|} R_{\underline{n}} \subseteq Q_{\underline{n}_i}$ for all $i = 1, 2, \ldots, s_{\underline{n}}$, where $|\underline{n}| = n_1 + n_2 + \cdots + n_t$.

Remark 0.9. In [Sh1], the linear growth property was only proved on the family $\{R/\overline{I^n} \,|\, n \in \mathbb{N}\}$. It seems that the same technique and method can be readily used to proved the linear growth property of the family $\{M/\overline{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t}} M \,|\, \underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t\}$ where M is any faithful R-module, provided that the set $\cup_{\underline{n} \in \mathbb{N}^t} \operatorname{Ass}(M/\overline{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t}} M)$ is finite.

It seems that the linear growth property in the above two cases is related to the fact that certain graded modules are Noetherian. See [Yao, Section 3].

In Section 2, we study the linear growth property theoretically and show how certain kinds of Artin-Rees numbers can be used to prove the linear growth property.

Then we prove the linear growth property of a specific family consisting of (co)homology modules in Section 3, which is then used in Section 4 to prove that the family $\{E_{\underline{n}} = \operatorname{Ext}_R^c(N, \frac{M}{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M}) \,|\, \underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t\}$ and the family $\{T_{\underline{n}} = \operatorname{Tor}_c^R(N, \frac{M}{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M}) \,|\, \underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t\}$ satisfy the linear growth property for any finitely generated R-modules N, M and any fixed $c \in \mathbb{N}$. In fact, the general form of the result in Section 4 is the following:

Theorem 4.1. *Let A be a Noetherian ring, N a finitely generated A-module, R a Noetherian A-algebra, M a finitely generated R-module, $I_1, I_2, \ldots, I_t$ fixed ideals of R and $c \in \mathbb{N}$. Then there exists a $k \in \mathbb{N}$ such that for any $\underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$ there exists a primary decomposition of 0 in $E_{\underline{n}} = \operatorname{Ext}^c_A(N, \frac{M}{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M})$, (0 in $T_{\underline{n}} = \operatorname{Tor}^A_c(N, \frac{M}{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M})$, respectively,) all regarded as R-modules,*

$$0 = Q_{\underline{n}_1} \cap Q_{\underline{n}_2} \cap \cdots \cap Q_{\underline{n}_{s_{\underline{n}}}},$$

where the $Q_{\underline{n}_i}$ are $P_{\underline{n}_i}$-primary components of the primary decomposition such that $P^{k|\underline{n}|}_{\underline{n}_i} E_{\underline{n}} \subseteq Q_{\underline{n}_i}$ ($P^{k|\underline{n}|}_{\underline{n}_i} T_{\underline{n}} \subseteq Q_{\underline{n}_i}$, respectively) for all $i = 1, 2, \ldots, s_{\underline{n}}$, where $|\underline{n}| = n_1 + n_2 + \cdots + n_t$.

1. Primary components over subsets

Lemma 1.1. *Let $N \subseteq M$ be finitely generated R-modules and $X \subseteq \operatorname{Ass}(M/N)$ a subset of $\operatorname{Ass}(M/N)$. For an R-module Q such that $N \subseteq Q \subseteq M$, the following are equivalent:*

(1) Q is an X-primary component of $N \subseteq M$, i.e. $Q \in \Lambda_X(N \subseteq M)$.
(2) $\operatorname{Ass}(M/Q) \subseteq X$ and $\operatorname{Ass}(Q/N) \subseteq \operatorname{Ass}(M/N) \setminus X$.
(3) $\operatorname{Ass}(M/Q) = X$ and $\operatorname{Ass}(Q/N) = \operatorname{Ass}(M/N) \setminus X$.

Proof. Without loss of generality, we assume $N = 0$. Say $X = \{P_1, P_2, \ldots, P_r\} \subseteq \{P_1, P_2, \ldots, P_r, P_{r+1}, \ldots, P_s\} = \operatorname{Ass}(M/N)$.

(1) $\Rightarrow$ (2): Condition (1) means there is a primary decomposition $0 = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ of 0 in M with Q_i being P_i-primary such that $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_r$. Then there is an injective R-homomorphism

$$\frac{M}{Q} = \frac{M}{\cap_{i=1}^r Q_i} \to \bigoplus_{i=1}^r \frac{M}{Q_i},$$

which implies that $\operatorname{Ass}(M/Q) \subseteq \operatorname{Ass}(\oplus_{i=1}^r M/Q_i) = X$. Also we have an injective R-homomorphism

$$Q = \frac{Q}{Q \cap (\cap_{i=r+1}^s Q_i)} \cong \frac{Q + (\cap_{i=r+1}^s Q_i)}{\cap_{i=r+1}^s Q_i} \subseteq \frac{M}{\cap_{i=r+1}^s Q_i},$$

which implies that $\operatorname{Ass}(Q) \subseteq \{P_{r+1}, \ldots, P_s\} = \operatorname{Ass}(M) \setminus X$.

(2) $\Rightarrow$ (3): This is evident since $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(M/Q) \cup \operatorname{Ass}(Q)$.

(3) $\Rightarrow$ (1): As $\operatorname{Ass}(M/Q) = \{P_1, P_2, \ldots, P_r\}$, we choose an arbitrary primary decomposition $Q = Q_1' \cap Q_2' \cap \cdots \cap Q_r'$ of $Q \subset M$ where Q_i' is the P_i-primary component for $i = 1, 2, \ldots, r$. Next we choose an arbitrary primary decomposition $0 = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ of 0 in M with Q_i being P_i-primary and let $Q' = \cap_{i=r+1}^s Q_i$, i.e. $Q' \in \Lambda_{\operatorname{Ass}(M) \setminus X}(0 \subseteq M)$. Therefore, by the argument (1) $\Rightarrow$ (2), $\operatorname{Ass}(Q') \subseteq X$. Finally we know that $Q \cap Q' = 0$ since $\operatorname{Ass}(Q \cap Q') \subseteq \operatorname{Ass}(Q) \cap \operatorname{Ass}(Q') = \emptyset$. Hence we know that

$$0 = Q \cap Q' = Q_1' \cap Q_2' \cap \cdots \cap Q_r' \cap Q_{r+1} \cap \cdots Q_s$$

is a primary decomposition of $0 \subseteq M$, which implies that $Q = Q_1' \cap Q_2' \cap \cdots \cap Q_r'$ is an X-primary component of $N \subseteq M$, i.e. $Q \in \Lambda_X(N \subseteq M)$. $\square$

As a corollary, we give an alternative proof of the compatibility property of primary decomposition (c.f. [Yao, Theorem 1.1]).

Corollary 1.2 (Compatibility). *Let $N \subseteq M$ be finitely generated R-modules. Then*

(1) Let $X_1, X_2, \dots, X_n$ be subsets of $\operatorname{Ass}(M/N)$ and $Q_{X_i} \in \Lambda_{X_i}(N \subseteq M)$ for $1 \le i \le n$. Then $\cap_{i=1}^n Q_{X_i} \in \Lambda_X(N \subseteq M)$, where $X = \cup_{i=1}^n X_i$.

(2) In particular, suppose $\operatorname{Ass}(M/N) = \{P_1, P_2, \dots, P_s\}$ and Q_i is a P_i-primary component of N in M, i.e. $Q_i \in \Lambda_{P_i}$ for each $i = 1, 2, \dots, s$. Then $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$, which is necessarily a minimal primary decomposition of $N \subseteq M$.

Proof. (1) : By the above lemma, we know $\operatorname{Ass}(M/Q_{X_i}) = X_i$ and $\operatorname{Ass}(Q_{X_i}/N) = \operatorname{Ass}(M/N) \backslash X_i$ for $1 \le i \le n$. Therefore $\operatorname{Ass}(M/(\cap_{i=1}^n Q_{X_i})) \subseteq \cup_{i=1}^n \operatorname{Ass}(M/Q_{X_i}) = \cup_{i=1}^n X_i = X$ as there is an embedding of $M/(\cap_{i=1}^n Q_{X_i})$ into $\oplus_{i=1}^n M/Q_{X_i}$. Also $\operatorname{Ass}((\cap_{i=1}^n Q_{X_i})/N) \subseteq \cap_{i=1}^n \operatorname{Ass}(Q_{X_i}/N) = \operatorname{Ass}(M/N) \setminus X$. The result follows from Lemma 1.1.

(2) : This is just a special case of (1) as N is the only $\operatorname{Ass}(M/N)$-primary component of $N \subseteq M$. $\square$

In [HRS], quoted as Theorem 0.4, it was shown that, for any R-modules $N \subseteq M$ and any $P \in \operatorname{Ass}(M/N)$, the intersection of all maximal P-primary components of $N \subseteq M$ is equal to $M \cap N_P$, the pre-image of N_P under the natural map $M \to M_P$. Notice that $M \cap N_P$ is exactly the unique $\mathrm{o}(P)$-primary component in $\Lambda_{\mathrm{o}(P)}(M/N)$ (see Remark 0.2(3)). It was also shown that every P-primary component is a finite intersection of maximal P-primary components of $N \subseteq M$. We are going to show similar results for maximal X-primary components:

Theorem 1.3. *Let R be a Noetherian ring and $N \subseteq M$ finitely generated R-modules. Let $X \subset \operatorname{Ass}(M/N)$. Say $X = \{P_1, P_2, \dots, P_r\}$ and set $U = R \backslash \cup \{P \,|\, P \in X\}$. Recall that $\mathrm{o}(X) = \{P \in \operatorname{Ass}(M/N) \,|\, P \subseteq \cup_{i=1}^r P_i\}$.*

(1) $\mathring{\Lambda}_X(N \subseteq M) = \{\cap_{i=1}^r Q_i \,|\, Q_i \in \mathring{\Lambda}_{P_i}(N \subseteq M), 1 \le i \le r\}$.

Consequently, we also have the following.

(2) For every $Q \in \Lambda_X(N \subseteq M)$, $Q = \cap\{Q' \,|\, Q' \in \mathring{\Lambda}_X(N \subseteq M), Q \subseteq Q'\}$. Actually every $Q \in \Lambda_X(N \subseteq M)$ is an intersection of finitely many $Q' \in \mathring{\Lambda}_X(N \subseteq M)$.

(3) The intersection $\cap\{Q \,|\, Q \in \mathring{\Lambda}_X(N \subseteq M)\} = \cap\{Q \,|\, Q \in \Lambda_X(N \subseteq M)\}$ is equal to $N[U^{-1}] \cap M$, i.e. the only $\mathrm{o}(X)$-primary component in $\Lambda_{\mathrm{o}(X)}(N \subseteq M)$.

Proof. (1): It is easy to show $\mathring{\Lambda}_X(N \subseteq M) \subseteq \{\cap_{i=1}^r Q_i \,|\, Q_i \in \mathring{\Lambda}_{P_i}(N \subseteq M), 1 \le i \le r\}$: For any $Q \in \mathring{\Lambda}_X(N \subseteq M)$, write $Q = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_r$, where $Q'_i \in \Lambda_{P_i}$ for each $1 \le i \le r$. Then we choose $Q_i \in \mathring{\Lambda}_{P_i}$ such that $Q'_i \subseteq Q_i$ for each $1 \le i \le r$ so that $Q = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_r \subseteq Q_1 \cap Q_2 \cap \cdots \cap Q_r$. But $Q_1 \cap Q_2 \cap \cdots \cap Q_r \in \Lambda_X$ by compatibility property (Corollary 1.2), which forces $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_r$.

To show $\mathring{\Lambda}_X(N \subseteq M) \supseteq \{\cap_{i=1}^r Q_i \,|\, Q_i \in \mathring{\Lambda}_{P_i}(N \subseteq M), 1 \le i \le r\}$ we use induction on $|X|$, the cardinality of X. If $|X| = 1$, there is nothing to prove. Assuming the containment is true for $|X| = r-1$, we show the containment for $X = \{P_1, P_2, \dots, P_r\}$. After rearrangement if necessary, we may assume that $P_r \not\subseteq P_i$ for $1 \le i \le r-1$. Set $U = R \setminus \cup_{i=1}^{r-1} P_i$. Let $Q = \cap_{i=1}^r Q_i$ such that $Q_i \in \mathring{\Lambda}_{P_i}(N \subseteq M)$

for $1 \le i \le r$. For any $Q' \in \Lambda_X$ such that $Q \subseteq Q'$, we need to show $Q = Q'$. Write $Q' = \cap_{i=1}^r Q'_i$ such that $Q'_i \in \Lambda_{P_i}$ for $1 \le i \le r$. Then we have

$$\cap_{i=1}^{r-1} Q_i = M \cap Q[U^{-1}] \subseteq M \cap Q'[U^{-1}] = \cap_{i=1}^{r-1} Q'_i,$$

which forces $\cap_{i=1}^{r-1} Q_i = \cap_{i=1}^{r-1} Q'_i$ by the induction hypothesis. Therefore

$$\begin{aligned}&\big(\cap_{i=1}^{r-1} Q_i\big) \cap (Q' + Q_r)\\ &\qquad = Q' + \cap_{i=1}^r Q_i = Q' \qquad (\text{since } \cap_{i=1}^{r-1} Q_i = \cap_{i=1}^{r-1} Q'_i \supset Q').\end{aligned}$$

Hence we can derive a primary decomposition $Q' = \cap_{i=1}^r Q''_i$ of $Q' \subseteq M$ from any particular primary decompositions of $\cap_{i=1}^{r-1} Q_i \subseteq M$ and of $(Q' + Q_r) \subseteq M$. In this particular primary decomposition $Q' = \cap_{i=1}^r Q''_i$ of $Q' \subseteq M$, the P_r-primary component, Q''_r, must come from the P_r-primary component of $(Q' + Q_r) \subseteq M$, hence must contain $Q' + Q_r$. But $Q''_r \in \Lambda_{P_r}(Q' \subseteq M)$ and $Q' \in \Lambda_X(N \subseteq M)$, in light of Corollary 1.2, imply that $Q''_r \in \Lambda_{P_r}(N \subseteq M)$, which forces $Q''_r = Q_r$. Therefore $Q' \subseteq Q''_r = Q_r$, which implies

$$Q = \cap_{i=1}^r Q_i = \big(\cap_{i=1}^{r-1} Q_i\big) \cap Q_r = \big(\cap_{i=1}^{r-1} Q'_i\big) \cap Q_r \supseteq Q'.$$

Consequently, we conclude that $Q = Q'$.

(2): For any $Q \in \Lambda_X(N \subseteq M)$, write

$$Q = Q_1 \cap Q_2 \cap \cdots \cap Q_r.$$

By Theorem 0.4(1), each Q_i is a finite intersection of maximal P_i-primary components, i.e. there is an $n \in \mathbb{N}$ such that

$$Q_i = Q_{i1} \cap Q_{i2} \cap \cdots \cap Q_{in},$$

where $Q_{ij} \in \mathring{\Lambda}_{P_i}(N \subseteq M)$ for every $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, n$. Let $Q'_j = Q_{1j} \cap Q_{2j} \cap \cdots \cap Q_{rj}$ for each $j = 1, 2, \ldots, n$. Then $Q'_j \in \mathring{\Lambda}_X(N \subseteq M)$ by part (1) and

$$Q = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_n.$$

(3): The equality $\cap\{Q \,|\, Q \in \mathring{\Lambda}_X(N \subseteq M)\} = \cap\{Q \,|\, Q \in \Lambda_X(N \subseteq M)\}$ follows directly from part (2). By part (1), we have

$$\cap\{Q \,|\, Q \in \mathring{\Lambda}_X(N \subseteq M)\} = \cap_{i=1}^r Q''_i,$$

where $Q''_i = \cap\{Q \,|\, Q \in \mathring{\Lambda}_{P_i}(N \subset M)\}$ for each $i = 1, 2, \ldots, r$. But then, by Theorem 0.4(2), Q''_i is equal to the only $\mathrm{o}(P_i)$-primary component in $\Lambda_{\mathrm{o}(P_i)}(N \subset M)$. Therefore $\cap\{Q \,|\, Q \in \mathring{\Lambda}_X\} = \cap_{i=1}^r Q''_i$ is the unique $\mathrm{o}(X)$-primary component in $\Lambda_{\mathrm{o}(X)}(N \subseteq M)$ by Corollary 1.2 and the fact that $\cup_{i=1}^r \mathrm{o}(P_i) = \mathrm{o}(\cup_{i=1}^r P_i) = \mathrm{o}(X)$. $\square$

Remark 1.4. If X is open in $\mathrm{Ass}(M/N)$, then $\Lambda_X(N \subseteq M)$ contains a unique X-component and the above theorem becomes trivial.

As promised in Remark 0.2(2), here is a result recovering and generalizing [Yao, Theorem 2.2].

Corollary 1.5. *The following are equivalent:*

(1) X is open in $\mathrm{Ass}(M/N)$.

(2) $\Lambda_X(N \subseteq M)$ consists of only one X-primary component.

(3) $\Lambda_X(N \subseteq M)$ *is finite.*

(4) $\mathring{\Lambda}_X(N \subseteq M)$ *is finite.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are evident.

(4) $\Rightarrow$ (1): Say $\mathring{\Lambda}_X(N \subseteq M) = \{Q'_1, Q'_2 \ldots, Q'_t\}$ and let $Q = \cap_{i=1}^t Q'_i$. By Corollary 1.2, $Q \in \Lambda_X(N \subseteq M)$. On the other hand, by Theorem 1.3(3), $Q \in \Lambda_{\mathrm{o}(X)}(N \subseteq M)$. Therefore, by Lemma 1.1, $X = \mathrm{o}(X)$ is open in $\mathrm{Ass}(M/N)$. $\square$

2. The linear growth property and Artin-Rees numbers

In this section, we are going to study the linear growth property of the primary decompositions of families of R-modules (see definition 0.6). Let $\mathcal{F} = \{M_{\underline{n}} \mid \underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t\}$ be a family of finitely generated R-modules. By the compatibility property (see Corollary 1.2), we may equivalently say that the family $\mathcal{F}$ satisfies the linear growth property if there exist $k, b \in \mathbb{N}$ such that for any $\underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$ and any $P \in \mathrm{Ass}(M_{\underline{n}})$, there exists a P-primary component, say Q, of $0 \subseteq M_{\underline{n}}$ such that $P^{k|\underline{n}|+b} M_{\underline{n}} \subseteq Q$. Notice that if $M_{(0,0,\ldots,0)} = 0$, then we can always additionally require $b = 0$.

Notation 2.1. Let R be a Noetherian ring, M a finitely generated module over R and J an ideal of R. We write $\mathrm{G}(J, M) = \min\{n \in \mathbb{N} \mid J^n M \cap (0 :_M J^\infty) = 0\}$.

Lemma 2.2. *Let $\mathcal{F} = \{M_{\underline{n}} \mid \underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t\}$ be a family of finitely generated R-modules. Then the following are equivalent:*

(1) *The family $\mathcal{F}$ satisfies the linear growth property.*

(2) *There exist integers $k, b \in \mathbb{N}$ such that $J^{k|\underline{n}|+b} M_{\underline{n}} \cap (0 :_{M_{\underline{n}}} J^\infty) = 0$, that is $\mathrm{G}(J, M_{\underline{n}}) \leq k|\underline{n}| + b$ for all $\underline{n} \in \mathbb{N}^t$ and all ideals J of R.*

(3) *There exist integers $k, b \in \mathbb{N}$ such that $P^{k|\underline{n}|+b} M_{\underline{n}} \cap (0 :_{M_{\underline{n}}} P^\infty) = 0$, that is $\mathrm{G}(P, M_{\underline{n}}) \leq k|\underline{n}| + b$ for all $\underline{n} \in \mathbb{N}^t$ and all prime ideals $P \in \mathrm{Ass}(M_{\underline{n}})$.*

If $\mathrm{Ass}(\mathcal{F}) := \cup_{\underline{n} \in \mathbb{N}^t} \mathrm{Ass}(M_{\underline{n}})$ is finite, then the above conditions (1), (2) and (3) are also equivalent to the following.

(4) *For any prime ideal $P \in \mathrm{Ass}(\mathcal{F})$, there exist integers $k, b \in \mathbb{N}$, which may depend on P, such that $P^{k|\underline{n}|+b} M_{\underline{n}} \cap (0 :_{M_{\underline{n}}} P^\infty) = 0$, i.e. $\mathrm{G}(P, M_{\underline{n}}) \leq k|\underline{n}| + b$ for all $\underline{n} \in \mathbb{N}^t$ such that $P \in \mathrm{Ass}(M_{\underline{n}})$.*

Proof. (1) $\Rightarrow$ (2): By the meaning of the linear growth property, there exist integers $k, b \in \mathbb{N}$ such that, for any $\underline{n} \in \mathbb{N}^t$, there exists a primary decomposition of 0 in $M_{\underline{n}}$

$$0 = Q_{\underline{n}_1} \cap Q_{\underline{n}_2} \cap \cdots \cap Q_{\underline{n}_{s_{\underline{n}}}},$$

where the $Q_{\underline{n}_i}$ are $P_{\underline{n}_i}$-primary components of the primary decomposition with $P_{\underline{n}_i}^{k|\underline{n}|+b} M_{\underline{n}} \subseteq Q_{\underline{n}_i}$ for all $i = 1, 2, \ldots, s_{\underline{n}}$. Let J be an arbitrary ideal of R and $\underline{n}$ an arbitrary vector in $\mathbb{N}^t$. By possibly rearranging the primary components, we may assume that $J \subseteq P_{\underline{n}_i}$ for $1 \leq i \leq r$ and $J \nsubseteq P_{\underline{n}_i}$ for $r+1 \leq i \leq s_{\underline{n}}$. Then

$$\begin{aligned} J^{k|\underline{n}|+b} M_{\underline{n}} &\subseteq Q_{\underline{n}_1} \cap Q_{\underline{n}_2} \cap \cdots \cap Q_{\underline{n}_r} \qquad \text{and} \\ (0 :_{M_{\underline{n}}} J^\infty) &= Q_{\underline{n}_{r+1}} \cap \cdots \cap Q_{\underline{n}_{s_{\underline{n}}}}. \end{aligned}$$

Therefore $J^{k|\underline{n}|+b} M_{\underline{n}} \cap (0 :_{M_{\underline{n}}} J^\infty) = 0$.

(2) $\Rightarrow$ (3): This is evident.

(3) $\Rightarrow$ (1): Because of the compatibility property of primary components, it suffices to prove that, for an arbitrary $\underline{n}$ in $\mathbb{N}^t$ and an arbitrary $P \in \operatorname{Ass}(M_{\underline{n}})$, there exists a P-primary component, say Q, of $0 \subseteq M_{\underline{n}}$ such that $P^{k|\underline{n}|+b}M_{\underline{n}} \subseteq Q$. Since $P^{k|\underline{n}|+b}M_{\underline{n}} \cap (0 :_{M_{\underline{n}}} P^\infty) = 0$, we can derive a primary decomposition of $0 \subset M_{\underline{n}}$ from any primary decompositions of $P^{k|\underline{n}|+b}M_{\underline{n}} \subset M_{\underline{n}}$ and of $(0 :_{M_{\underline{n}}} P^\infty) \subset M_{\underline{n}}$. As $P \notin \operatorname{Ass}(M_{\underline{n}}/(0 :_{M_{\underline{n}}} P^\infty))$ it is easy to see that the P-primary component Q in this derived primary decomposition has to be a P-primary component of $P^{k|\underline{n}|+b}M_{\underline{n}} \subset M_{\underline{n}}$, which forces $P^{k|\underline{n}|+b}M_{\underline{n}} \subseteq Q$.

(3) $\Leftrightarrow$ (4): This is evident under the assumption that $\operatorname{Ass}(\mathcal{F}) = \cup_{\underline{n}\in\mathbb{N}^t} \operatorname{Ass}(M_{\underline{n}})$ is finite. $\square$

Lemma 2.3. *Let R, S be Noetherian rings and $\phi : R \to S$ be a ring homomorphism.*

(1) Suppose that M' is a finitely generated R-module, M'' is a finitely generated S-module such that there is an injective R-homomorphism $\psi : M' \to M''$. Then $\mathrm{G}(J, M') \leq \mathrm{G}(JS, M'')$ for any ideal J of R.

(2) Suppose there are two families, $\mathcal{F}_1 = \{M'_{\underline{n}} \,|\, \underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t\}$ of finitely generated R-modules and $\mathcal{F}_2 = \{M''_{\underline{n}} \,|\, \underline{n} \in \mathbb{N}^t\}$ of finitely generated S-modules, such that there is an injective R-homomorphism $\psi_{\underline{n}} : M'_{\underline{n}} \to M''_{\underline{n}}$ for every $\underline{n} \in \mathbb{N}^t$. If the family $\mathcal{F}_2$ satisfies the linear growth property, then so does the family $\mathcal{F}_1$.

Proof. It is enough to prove (1) as (2) follows from (1) immediately by Lemma 2.2. Without loss of generality, we assume that $M' \subseteq M''$. Say $\mathrm{G}(JS, M'') = n$, which implies that $(JS)^n M'' \cap (0 :_{M''} (JS)^\infty = 0$. Then

$$J^n M' \cap (0 :_{M'} J^\infty) \subseteq (JS)^n M'' \cap (0 :_{M''} (JS)^\infty) = 0,$$

which proves that $\mathrm{G}(J, M') \leq n = \mathrm{G}(JS, M'')$. $\square$

Various kinds of Artin-Rees numbers play important roles in studying the linear growth property. These numbers have been studied by Huneke in [Hu].

Definition 2.4. Let $M_0 \subseteq M_1$ be finitely generated R-modules over a Noetherian ring R and J an ideal of R. We denote $\mathrm{AR}(J, M_0 \subseteq M_1) := \min\{\, k \,|\, J^n M_1 \cap M_0 \subseteq J^{n-k} M_0 \text{ for all } n \geq k \,\}$. For a set Γ of ideals of R, we denote $\mathrm{AR}(\Gamma, M_0 \subseteq M_1) := \sup\{\mathrm{AR}(J, M_0 \subseteq M_1) \,|\, J \in \Gamma\}$, which could be infinity.

Lemma 2.5 ([Hu, Proposition 2.2])**.** *Let $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r$ be finitely generated R-modules over a Noetherian ring R and J an ideal of R. Then*

(1) $\mathrm{AR}(J, M_0 \subseteq M_r) \leq \sum_{i=1}^r \mathrm{AR}(J, M_{i-1} \subseteq M_i)$.

(2) If furthermore $M_i/M_{i-1} \cong R/I_i$ are cyclic modules for all $i = 1, 2, \ldots, r$, where I_i are ideals of R, then $\mathrm{AR}(J, M_0 \subseteq M_r) \leq \sum_{i=1}^r \mathrm{AR}(J, I_i \subseteq R)$.

Proof. This can be proved by use of arguments in the proof of [Hu, Proposition 2.2]. $\square$

Definition 2.6. Let M be a finitely generated R-module over a Noetherian ring R and J an ideal of R. We define $\mathrm{AR}(J, M)$ to be the maximum of $\mathrm{AR}(J, M' \subseteq M'')$ over all finitely generated R-modules $M' \subseteq M''$ such that $M''/M' \cong M$, which is well-defined and finite by the above Lemma 2.5(2). For a set Γ of ideals of R, we denote $\mathrm{AR}(\Gamma, M) := \sup\{\mathrm{AR}(J, M) \,|\, J \in \Gamma\}$, which could be infinity. Also $\mathrm{AR}(M) := \sup\{\mathrm{AR}(J, M) \,|\, J \text{ is an ideal of } R\}$.

Remark 2.7.
(1) For $M_0 \subseteq M_1$ as in Definition 2.4, if $J^n M_1 \subseteq M_0$ for some n, then $\mathrm{AR}(J, M_0 \subseteq M_1) \le n$.
(2) If $0 \to K \to M \to L \to 0$ is exact, then $\mathrm{AR}(J,M) \le \mathrm{AR}(J,K) + \mathrm{AR}(J,L)$. Or equivalently, we have $\mathrm{AR}(J, M_2/M_0) \le \mathrm{AR}(J, M_1/M_0) + \mathrm{AR}(J, M_2/M_1)$ if $M_0 \subseteq M_1 \subseteq M_2$.
(3) If the ring R has the uniform Artin-Rees property (see [Hu]), then $\mathrm{AR}(M)$ is finite for every finitely generated R-module M.
(4) Actually, it is not hard to see that $\mathrm{AR}(J,M) = \mathrm{AR}(J, N \subseteq R^n)$ whenever $M \cong R^n/N$.

Lemma 2.8. *Let R be a Noetherian ring and $0 \to M' \xrightarrow{\psi} M'' \to M \to 0$ an exact sequence of finitely generated R-modules. Then, for any ideal J of R,*

$$\begin{aligned} \mathrm{G}(J, M'') &\le \max\{\mathrm{G}(J,M), \mathrm{G}(J,M') + \mathrm{AR}(J, \psi(M') \subseteq M'')\} \\ &\le \max\{\mathrm{G}(J,M), \mathrm{G}(J,M') + \mathrm{AR}(J,M)\}. \end{aligned}$$

Proof. Without loss of generality, we may assume $M' = \psi(M') \subseteq M''$ so that $M''/M' \cong M$. Say $\mathrm{G}(J,M') = m', \mathrm{G}(J,M) = m$ and $\mathrm{AR}(J, M' \subseteq M'') = k$. That is to say that $J^{m'}M' \cap (0 :_{M'} J^\infty) = 0$, $J^m M \cap (0 :_M J^\infty) = 0$ and $M' \cap J^n M'' \subseteq J^{n-k}M'$ for all $n \ge k$. Then we have

$$\begin{aligned} &J^{\max\{m, m'+k\}} M'' \cap (0 :_{M''} J^\infty) \\ &= (J^{\max\{m,m'+k\}} M'' \cap (0 :_{M''} J^\infty)) \cap M' && \text{(by the meaning of } m) \\ &\subseteq (J^{m'+k} M'' \cap M') \cap (0 :_{M'} J^\infty) \\ &\subseteq J^{m'} M' \cap (0 :_{M'} J^\infty) && \text{(by the meaning of } k) \\ &= 0 && \text{(by the meaning of } m'), \end{aligned}$$

which gives the desired result. □

As an immediate consequence, we have the following lemma concerning the linear growth property and short exact sequences, which is used in the proof of the linear growth property on the two families of Ext and Tor R-modules in Section 4.

Lemma 2.9. *Let R be a Noetherian ring. Suppose there are three families, $\mathcal{F}_1 = \{M'_{\underline{n}} \,|\, \underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t\}$, $\mathcal{F}_2 = \{M''_{\underline{n}} \,|\, \underline{n} \in \mathbb{N}^t\}$ and $\mathcal{F} = \{M_{\underline{n}} \,|\, \underline{n} \in \mathbb{N}^t\}$, of finitely generated R-modules such that there is an exact sequence*

$$0 \longrightarrow M'_{\underline{n}} \xrightarrow{\psi_{\underline{n}}} M''_{\underline{n}} \longrightarrow M_{\underline{n}} \longrightarrow 0$$

for every $\underline{n} \in \mathbb{N}^t$. Assume that the families $\mathcal{F}_1$ and $\mathcal{F}$ both satisfy the linear growth property. Then the family $\mathcal{F}_2$ satisfies the linear growth property if one of the following conditions holds:

(1) The numbers $\mathrm{AR}(\mathrm{Ass}(M''_{\underline{n}}), \psi_{\underline{n}}(M'_{\underline{n}}) \subseteq M''_{\underline{n}})$ are finite for all $\underline{n} \in \mathbb{N}^t$ and the function defined by $\underline{n} \mapsto \mathrm{AR}(\mathrm{Ass}(M''_{\underline{n}}), \psi_{\underline{n}}(M'_{\underline{n}}) \subseteq M''_{\underline{n}})$ is bounded above by a linear function of $|\underline{n}|$.
(2) The numbers $\mathrm{AR}(\mathrm{Ass}(M''_{\underline{n}}), M_{\underline{n}})$ are finite for all $\underline{n} \in \mathbb{N}^t$ and the function defined by $\underline{n} \mapsto \mathrm{AR}(\mathrm{Ass}(M''_{\underline{n}}), M_{\underline{n}})$ is bounded above by a linear function of $|\underline{n}|$.
(3) The set $\mathrm{Ass}(\mathcal{F}_2) := \cup_{\underline{n} \in \mathbb{N}^t} \mathrm{Ass}(M''_{\underline{n}})$ is finite and for any prime ideal $P \in \mathrm{Ass}(\mathcal{F}_2)$, the function defined by $\underline{n} \mapsto \mathrm{AR}(P, M_{\underline{n}})$ is bounded above by a linear function of $|\underline{n}|$.

Proof. The result follows immediately from Lemma 2.8 and Lemma 2.2. $\square$

Remark 2.10. We would like to apply Lemma 2.8 and sketch a proof of Theorem 0.7: Without loss of generality, we assume $I_i = (x_i)$ and each $x_i \in R$ is M-regular (c.f. the proof of [Sw, Theorem 3.4] or [Yao, Theorem 3.3]). So $\mathcal{F} = \{M_{\underline{n}} := \frac{M}{x^{\underline{n}}M} \,|\, \underline{n} \in \mathbb{N}^t\}$, in which $\underline{n} := (n_1, n_2, \ldots, n_t)$ and $x^{\underline{n}} := x_1^{n_1} \cdots, x_t^{n_t}$. Then $\mathrm{Ass}(\mathcal{F})$ is a finite set. Hence both $k_1 := \max\{\mathrm{G}(P, \frac{M}{x_i M}) \,|\, P \in \mathrm{Ass}(\mathcal{F}), 1 \le i \le t\}$ and $k_2 := \max\{\mathrm{AR}(P, \frac{M}{x_i M}) \,|\, P \in \mathrm{Ass}(\mathcal{F}), 1 \le i \le t\}$ are finite. Set $k := \max\{k_1, k_2\} < \infty$. For any $\underline{n} \ne (0, \ldots, 0)$, say $n_1 \ne 0$, there is an exact sequence $0 \to M_{(n_1-1, n_2, \ldots, n_t)} \to M_{\underline{n}} \to \frac{M}{x_1 M} \to 0$. Then, by induction on $|\underline{n}|$ and Lemma 2.8, we get $\mathrm{G}(P, M_{\underline{n}}) \le k|\underline{n}|$ for all $P \in \mathrm{Ass}(\mathcal{F})$ and all $\underline{n} \in \mathbb{N}^t$ (the case $\underline{n} = (0, \ldots, 0)$ is trivial), which proves the linear growth property of family $\mathcal{F}$ by Lemma 2.2. Actually this sketch is very similar to the proof of [Yao, Theorem 3.3].

3. The linear growth property of families of (co)homology modules

In this section, we are going to study the linear growth property of the primary decompositions of certain families of (co)homology modules. The main result, Theorem 3.2, will be used in the next section to prove the linear growth property of the primary decompositions of certain families of Ext and Tor R-modules.

The next lemma, which is needed in our proof of Theorem 3.2, is probably well-known. It is about a property of Noetherian partially ordered sets and we include a proof nonetheless for completeness. Recall that a partially ordered set D with partial order "$\le$" is called Noetherian if every ascending chain $x_1 \le x_2 \le \cdots \le x_n \le \cdots$ eventually stabilizes. For example, if M is a Noetherian R module, then the set of all R-submodules of M, partially ordered by containment, is a Noetherian partially ordered set. Also recall that $\mathbb{N}^t$ is a partially ordered set in which $(n_1, n_2, \ldots, n_t) \le (m_1, m_2, \ldots, m_t)$ if and only if $n_i \le m_i$ for all $i = 1, 2, \ldots, t$.

Lemma 3.1. *Let $(D, \le)$ be a Noetherian partially ordered set and $f : \mathbb{N}^t \to D$ an order-preserving map. Then there exists $\underline{m} \in \mathbb{N}^t$ such that $f(\underline{n}) = f(\underline{m})$ for all $\underline{n} \in \mathbb{N}^t$ satisfying $\underline{m} \le \underline{n}$. Moreover, the image $f(\mathbb{N}^t)$ is a finite subset of D.*

Proof. We prove the lemma by induction on t, the number of components in $\underline{n} \in \mathbb{N}^t$. We interpret $\mathbb{N}^0$ as a set consisting of one element. Therefore the lemma is trivially true in the case where $t = 0$. Assuming the lemma is true for $t-1$, we prove the lemma for t.

For any $r \in \mathbb{N}$, write $\underline{(r)} = (r, r, \ldots, r) \in \mathbb{N}^t$. Since D is Noetherian, there is a $c \in \mathbb{N}$ such that $f(\underline{(c)}) = f(\underline{(c')})$ for all $c \le c' \in \mathbb{N}$. It is easy to see that $f(\underline{(c)}) = f(\underline{n})$ for all $\underline{n} \in \mathbb{N}^t$ satisfying $\underline{(c)} \le \underline{n}$.

It remains to prove the set $\{f(n_1, n_2, \ldots, n_t) \,|\, n_i \le c-1 \text{ for some } 1 \le i \le t\}$ is finite. For any integers i and b such that $1 \le i \le t$ and $0 \le b \le c-1$, set $\mathbb{N}^t_{n_i=b} = \{(n_1, n_2, \ldots, n_t) \,|\, n_i = b\}$. It is easy to see that $\mathbb{N}^t_{n_i=b}$ is isomorphic to $\mathbb{N}^{t-1}$ as partially ordered sets and therefore $f(\mathbb{N}^t_{n_i=b})$ is finite for every $1 \le i \le t$ and $0 \le b \le c-1$ by the induction hypothesis. So $\{f(n_1, n_2, \ldots, n_t) \,|\, n_i \le c-1 \text{ for some } 1 \le i \le t\} = \cup_{\substack{1 \le i \le t \\ 0 \le b \le c-1}} f(\mathbb{N}^t_{n_i=b}))$ is finite, which completes the proof. $\square$

Theorem 3.2. *Let A be a Noetherian ring and R a Noetherian A-algebra. Fix a complex*

$$F_\bullet : \qquad \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$$

of finitely generated projective A-modules. Let M be a finitely generated R-module, $I_1, I_2, \ldots, I_t$ fixed ideals of R and $c \in \mathbb{Z}$. For any $\underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$, set $E_{\underline{n}} = \mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \frac{M}{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M}))$ and $T_{\underline{n}} = \mathrm{H}_c(F_\bullet \otimes_A \frac{M}{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M})$, the c-th cohomology and homology R-modules of the respective complexes. Then the family $\{E_{\underline{n}} \mid \underline{n} \in \mathbb{N}^t\}$ and the family $\{T_{\underline{n}} \mid \underline{n} \in \mathbb{N}^t\}$, both of which consist of finitely generated R-modules, satisfy the linear growth property.

Proof. First let us construct

$$\begin{aligned} \mathcal{R}_I &= R[I_1 X_1^{-1}, I_2 X_2^{-1}, \ldots, I_t X_t^{-1}, X_1, X_2, \ldots, X_t] \qquad \text{and} \\ \mathcal{M} &= \bigoplus_{n_1, n_2, \ldots, n_t \in \mathbb{Z}} I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M X_1^{-n_1} X_2^{-n_2} \cdots X_t^{-n_t}. \end{aligned}$$

Here, by convention, we agree that $I_i^{n_i} = R$ whenever $n_i \le 0$. To shorten our notations, we write $X^{\underline{n}} = X_1^{n_1} X_2^{n_2} \cdots X_t^{n_t}$ and $I^{\underline{n}} = I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t}$ for every $\underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$.

We know that $\mathcal{R}_I$ is a naturally $\mathbb{Z}^t$-graded Noetherian ring and $\mathcal{M}$ is a finitely generated $\mathbb{Z}^t$-graded $\mathcal{R}_I$-module. For any given $\underline{n} \in \mathbb{N}^t$, we denote by $\mathcal{M}[-\underline{n}]$ the 'shift' of $\mathcal{M}$ such that the $\underline{m}$-th degree component of $\mathcal{M}[-\underline{n}]$ is the $(\underline{m}-\underline{n})$-th degree component of $\mathcal{M}$ for every $\underline{m}$. Notice that X_i is $\mathcal{M}$-regular for every $i = 1, 2, \ldots, t$ so that there is an exact sequence of graded $\mathcal{R}_I$-modules (with homogeneous $\mathcal{R}_I$-homomorphisms of degree $(0, 0, \ldots, 0)$)

$$0 \longrightarrow \mathcal{M}[-\underline{n}] \xrightarrow{X^{\underline{n}}} \mathcal{M} \longrightarrow \frac{\mathcal{M}}{X^{\underline{n}} \mathcal{M}} \longrightarrow 0 \tag{3.2.1}$$

for every $\underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$. In particular $\frac{\mathcal{M}}{X^{\underline{n}} \mathcal{M}}$ is $\mathbb{Z}^t$-graded and, evidently, its $(0, 0, \ldots, 0)$-degree component is exactly $\frac{M}{I^{\underline{n}} M}$ for every $\underline{n} \in \mathbb{N}^t$. Moreover, it is easy to see that the cohomology modules of $\mathrm{Hom}_A(F_\bullet, \mathcal{M}), \mathrm{Hom}_A(F_\bullet, \frac{\mathcal{M}}{X^{\underline{n}} \mathcal{M}})$ and the homology modules of $F_\bullet \otimes_A \mathcal{M}, F_\bullet \otimes_A \frac{\mathcal{M}}{X^{\underline{n}} \mathcal{M}}$ are all finitely generated $\mathbb{Z}^t$-graded $\mathcal{R}_I$-modules and the $(0, 0, \ldots, 0)$-degree components of $\mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \frac{\mathcal{M}}{X^{\underline{n}} \mathcal{M}}))$ and $\mathrm{H}_c(F_\bullet \otimes_A \frac{\mathcal{M}}{X^{\underline{n}} \mathcal{M}})$ are exactly $E_{\underline{n}} = \mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \frac{M}{I^{\underline{n}} M}))$ and $T_{\underline{n}} = \mathrm{H}_c(F_\bullet \otimes_A \frac{M}{X^{\underline{n}} M})$ respectively for every $\underline{n} \in \mathbb{N}^t$.

Because the modules in (3.2.1) are finitely generated over $\mathcal{R}_I$, we have long exact sequences of finitely generated $\mathcal{R}_I$-modules

$$\begin{aligned} \mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \mathcal{M}[-\underline{n}])) &\xrightarrow{X^{\underline{n}}} \mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \mathcal{M})) \longrightarrow \mathcal{E}_{\underline{n}} \\ &\longrightarrow \mathrm{H}^{c+1}(\mathrm{Hom}_A(F_\bullet, \mathcal{M})) \xrightarrow{X^{\underline{n}}} \mathrm{H}^{c+1}(\mathrm{Hom}_A(F_\bullet, \mathcal{M})) \\ \text{and} \qquad \mathrm{H}_c(F_\bullet \otimes_A \mathcal{M}[-\underline{n}]) &\xrightarrow{X^{\underline{n}}} \mathrm{H}_c(F_\bullet \otimes_A \mathcal{M}) \longrightarrow \mathcal{T}_{\underline{n}} \\ &\longrightarrow \mathrm{H}_{c-1}(F_\bullet \otimes_A \mathcal{M}) \xrightarrow{X^{\underline{n}}} \mathrm{H}_{c-1}(F_\bullet \otimes_A \mathcal{M}), \end{aligned}$$

where $\mathcal{E}_{\underline{n}} = \mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \frac{\mathcal{M}}{X^{\underline{n}}\mathcal{M}}))$ and $\mathcal{T}_{\underline{n}} = \mathrm{H}_c(F_\bullet \otimes_A \frac{\mathcal{M}}{X^{\underline{n}}\mathcal{M}})$, which are graded $\mathcal{R}_I$-modules. That is to say that there are short exact sequences of $\mathcal{R}_I$-modules

$$0 \longrightarrow \frac{\mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \mathcal{M}))}{X^{\underline{n}}\,\mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \mathcal{M}))} \longrightarrow \mathcal{E}_{\underline{n}} \longrightarrow (0 :_{\mathrm{H}^{c+1}(\mathrm{Hom}_A(F_\bullet,\mathcal{M}))} X^{\underline{n}}) \longrightarrow 0$$

$$\text{and} \quad 0 \longrightarrow \frac{\mathrm{H}_c(F_\bullet \otimes_A \mathcal{M})}{X^{\underline{n}}\,\mathrm{H}_c(F_\bullet \otimes_A \mathcal{M})} \longrightarrow \mathcal{T}_{\underline{n}} \longrightarrow (0 :_{\mathrm{H}_{c-1}(F_\bullet \otimes_A \mathcal{M})} X^{\underline{n}}) \longrightarrow 0.$$

The families $\{\frac{\mathrm{H}^c(\mathrm{Hom}_A(F_\bullet,\mathcal{M}))}{X^{\underline{n}}\,\mathrm{H}^c(\mathrm{Hom}_A(F_\bullet,\mathcal{M}))} \,|\, \underline{n} \in \mathbb{N}^t\}$ and $\{\frac{\mathrm{H}_c(F_\bullet\otimes_A\mathcal{M})}{X^{\underline{n}}\,\mathrm{H}_c(F_\bullet\otimes_A\mathcal{M})} \,|\, \underline{n} \in \mathbb{N}^t\}$ satisfy the linear growth property by Theorem 0.7 (also see Remark 2.10). The sets $\cup_{\underline{n}\in\mathbb{N}^t} \mathrm{Ass}_{\mathcal{R}_I}(\frac{\mathrm{H}^c(\mathrm{Hom}_A(F_\bullet,\mathcal{M}))}{X^{\underline{n}}\,\mathrm{H}^c(\mathrm{Hom}_A(F_\bullet,\mathcal{M}))})$ and $\cup_{\underline{n}\in\mathbb{N}^t} \mathrm{Ass}_{\mathcal{R}_I}(\frac{\mathrm{H}_c(F_\bullet\otimes_A\mathcal{M})}{X^{\underline{n}}\,\mathrm{H}_c(F_\bullet\otimes_A\mathcal{M})})$ are both finite by [Mc] (see also [Br, Ra]). As a result, $\cup_{\underline{n}\in\mathbb{N}^t} \mathrm{Ass}_{\mathcal{R}_I}(\mathcal{E}_{\underline{n}})$ and $\cup_{\underline{n}\in\mathbb{N}^t} \mathrm{Ass}_{\mathcal{R}_I}(\mathcal{T}_{\underline{n}})$ are both finite by the above exact sequences. We see that $\underline{n} \mapsto (0 :_{\mathrm{H}^{c+1}(\mathrm{Hom}_A(F_\bullet,\mathcal{M}))} X^{\underline{n}})$ and $\underline{n} \mapsto (0 :_{\mathrm{H}_{c-1}(F_\bullet\otimes_A\mathcal{M})} X^{\underline{n}})$ define order-preserving maps from $\mathbb{N}^t$ to the Noetherian partially ordered sets that consist of $\mathcal{R}_I$-submodules of the finitely generated $\mathcal{R}_I$-modules $\mathrm{H}^{c+1}(\mathrm{Hom}_A(F_\bullet, \mathcal{M}))$ and $\mathrm{H}_{c-1}(F_\bullet \otimes_A \mathcal{M})$ respectively. Hence both $\{(0 :_{\mathrm{H}^{c+1}(\mathrm{Hom}_A(F_\bullet,\mathcal{M}))} X^{\underline{n}}) \,|\, \underline{n} \in \mathbb{N}^t\}$ and $\{(0 :_{\mathrm{H}_{c-1}(F_\bullet\otimes_A\mathcal{M})} X^{\underline{n}}) \,|\, \underline{n} \in \mathbb{N}^t\}$ are finite sets by Lemma 3.1. As a result, both of them, considered as families of $\mathcal{R}_I$-modules, satisfy the linear growth property and, for any fixed ideal $\mathcal{J} \subseteq \mathcal{R}_I$, the functions defined by $\underline{n} \mapsto \mathrm{AR}(\mathcal{J}, (0 :_{\mathrm{H}^{c+1}(\mathrm{Hom}_A(F_\bullet,\mathcal{M}))} X^{\underline{n}}))$ and by $\underline{n} \mapsto \mathrm{AR}(\mathcal{J}, (0 :_{\mathrm{H}_{c-1}(F_\bullet\otimes_A\mathcal{M})} X^{\underline{n}}))$ are both bounded above by linear (actually constant) functions of $|\underline{n}|$.

Therefore by Lemma 2.9(3), the families $\{\mathcal{E}_{\underline{n}} = \mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \frac{\mathcal{M}}{X^{\underline{n}}\mathcal{M}})) \,|\, \underline{n} \in \mathbb{N}^t\}$ and $\{\mathcal{T}_{\underline{n}} = \mathrm{H}_c(F_\bullet \otimes_A \frac{\mathcal{M}}{X^{\underline{n}}\mathcal{M}}) \,|\, \underline{n} \in \mathbb{N}^t\}$ all satisfy the linear growth property.

Finally, by Lemma 2.3, if we contract the linear growth property back to the $(0, 0, \ldots, 0)$-degree components of the members of $\{\mathcal{E}_{\underline{n}} = \mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \frac{\mathcal{M}}{X^{\underline{n}}\mathcal{M}})) \,|\, \underline{n} \in \mathbb{N}^t\}$ and $\{\mathcal{T}_{\underline{n}} = \mathrm{H}_c(F_\bullet \otimes_A \frac{\mathcal{M}}{X^{\underline{n}}\mathcal{M}}) \,|\, \underline{n} \in \mathbb{N}^t\}$, we get the linear growth property of the family $\{E_{\underline{n}} = \mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \frac{M}{I^{\underline{n}}M})) \,|\, \underline{n} \in \mathbb{N}^t\}$ and of the family $\{T_{\underline{n}} = \mathrm{H}_c(F_\bullet \otimes_A \frac{M}{I^{\underline{n}}M}) \,|\, \underline{n} \in \mathbb{N}^t\}$ of R-modules. $\square$

4. The linear growth property of Tor and Ext

In this section, we are going to study the linear growth property of the primary decompositions of certain families of Ext and Tor R-modules.

We first prove the linear growth property of the two families $\{\mathrm{Ext}^c_A(N, \frac{M}{I^{\underline{n}}M})\}$ and $\{\mathrm{Tor}^A_c(N, \frac{M}{I^{\underline{n}}M})\}$.

Theorem 4.1. *Let A be a Noetherian ring, N a finitely generated A-module, R a Noetherian A-algebra, M a finitely generated R-module, $I_1, I_2, \ldots, I_t$ fixed ideals of R and $c \in \mathbb{N}$. For any $\underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$, set $E_{\underline{n}} = \mathrm{Ext}^c_A(N, \frac{M}{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M})$ and $T_{\underline{n}} = \mathrm{Tor}^A_c(N, \frac{M}{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M})$. Then both the family $\{E_{\underline{n}} \,|\, \underline{n} \in \mathbb{N}^t\}$ and the family $\{T_{\underline{n}} \,|\, \underline{n} \in \mathbb{N}^t\}$ of R-modules satisfy the linear growth property. That is to say that there exists an integer $k \in \mathbb{N}$ such that for any $\underline{n} \in \mathbb{N}^t$ there exists a primary decomposition of 0 in $E_{\underline{n}}$ (0 in $T_{\underline{n}}$, respectively)*

$$0 = Q_{\underline{n}_1} \cap Q_{\underline{n}_2} \cap \cdots \cap Q_{\underline{n}_{s_{\underline{n}}}},$$

where the $Q_{\underline{n}_i}$ are $P_{\underline{n}_i}$-primary components of the primary decomposition such that $P_{\underline{n}_i}^{k|\underline{n}|} E_{\underline{n}} \subseteq Q_{\underline{n}_i}$ ($P_{\underline{n}_i}^{k|\underline{n}|} T_{\underline{n}} \subseteq Q_{\underline{n}_i}$, respectively) for all $i = 1, 2, \ldots, s_{\underline{n}}$, where $|\underline{n}| = n_1 + n_2 + \cdots + n_t$.

Proof. To shorten our notations, we write $I^{\underline{n}} = I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t}$ for every $\underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$.

Choose an arbitrary free resolution

$$F_\bullet : \qquad \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to (N) \to 0$$

of the A-module N by finitely generated free A-modules. Then for every $\underline{n} \in \mathbb{N}^t$, we can compute $\mathrm{Ext}_A^c(N, \frac{M}{I^{\underline{n}}M})$ as $\mathrm{H}^c(\mathrm{Hom}_A(F_\bullet, \frac{M}{I^{\underline{n}}M}))$ and compute $\mathrm{Tor}_c^A(N, \frac{M}{I^{\underline{n}}M})$ as $\mathrm{H}_c(F_\bullet \otimes_A \frac{M}{I^{\underline{n}}M})$. Now the desired linear growth property of $\{\mathrm{Ext}_A^c(N, \frac{M}{I^{\underline{n}}M}) \mid \underline{n} \in \mathbb{N}^t\}$ and $\{\mathrm{Tor}_c^A(N, \frac{M}{I^{\underline{n}}M}) \mid \underline{n} \in \mathbb{N}^t\}$ follows from an easy application of Theorem 3.2. $\square$

In the same spirit as in [Yao], Theorem 4.1 can be stated in a more general situation: The modules $\{\frac{M}{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M} \mid \underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{Z}^t\}$ may be replaced by $\{\frac{M_0}{M_{\underline{n}}} \mid \underline{n} \in \mathbb{Z}^t\}$, where $\{M_{\underline{n}} \mid \underline{n} \in \mathbb{Z}^t\}$ is a '$\mathbb{Z}^t$-graded' filtration of M such that

$$\mathcal{M} = \bigoplus_{(n_1, n_2, \ldots, n_t) \in \mathbb{Z}^t} M_{(n_1, n_2, \ldots, n_t)} X_1^{-n_1} X_2^{-n_2} \cdots X_t^{-n_t}$$

naturally forms a $\mathbb{Z}^t$-graded Noetherian module over a $\mathbb{Z}^t$-graded sub-ring $\mathcal{R}$ of the graded ring $R[X_1, X_2, \ldots, X_t, X_1^{-1}, X_2^{-1}, \ldots, X_t^{-1}]$ such that $X_1, X_2, \ldots, X_t$ are all contained in $\mathcal{R}$ and the $(0, 0, \ldots, 0)$-th component of $\mathcal{R}$ is R. We call such a filtration 'Noetherian'. Then we have the following theorem

Theorem 4.2. *Let A be a Noetherian ring, N an finitely generated A-module, R a Noetherian A-algebra, M a finitely generated R-module and $\{M_{\underline{n}} \mid \underline{n} \in \mathbb{Z}^t\}$ a Noetherian filtration of M. For any $\underline{n} \in \mathbb{N}^t$, set $E_{\underline{n}} = \mathrm{Ext}_A^c(N, \frac{M}{M_{\underline{n}}})$ and $T_{\underline{n}} = \mathrm{Tor}_c^A(N, \frac{M}{M_{\underline{n}}})$. Then both the family $\{E_{\underline{n}} \mid \underline{n} \in \mathbb{N}^t\}$ and the family $\{T_{\underline{n}} \mid \underline{n} \in \mathbb{N}^t\}$ satisfy the linear growth property.*

Proof. The proof goes exactly as the proof of the last theorem. $\square$

Question 4.3. Let R be a Noetherian ring, N and M finitely generated R-modules, $I_1, I_2, \ldots, I_t, J_1, J_2, \ldots, J_s$ fixed ideals of R and $c \in \mathbb{N}$. For any $\underline{n} = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$, $\underline{m} = (m_1, m_2, \ldots, m_s) \in \mathbb{N}^s$, set $E_{(\underline{m}, \underline{n})} = \mathrm{Ext}_R^c(\frac{N}{J^{\underline{m}}N}, \frac{M}{I^{\underline{n}}M})$ and $T_{(\underline{m}, \underline{n})} = \mathrm{Tor}_c^R(\frac{N}{J^{\underline{m}}N}, \frac{M}{I^{\underline{n}}M})$, where $(\underline{m}, \underline{n}) = (m_1, m_2, \ldots, m_s, n_1, n_2, \ldots, n_t) \in \mathbb{N}^{s+t}$. Do $\{E_{(\underline{m}, \underline{n})} \mid (\underline{m}, \underline{n}) \in \mathbb{N}^{s+t}\}$ and $\{T_{(\underline{m}, \underline{n})} \mid (\underline{m}, \underline{n}) \in \mathbb{N}^{s+t}\}$ satisfy the linear growth property?

It seems unlikely that an easy proof will be found that establishes the linear growth property for the family $\{E_{(\underline{m}, \underline{n})} \mid (\underline{m}, \underline{n}) \in \mathbb{N}^{s+t}\}$; it might not even hold. Even the family $\{\mathrm{Ext}_R^c(\frac{N}{J^{\underline{m}}N}, M) \mid \underline{m} \in \mathbb{N}^s\}$ seems difficult to handle. However, in the case where $c = 0$, the linear growth property of $\{E_{(\underline{m}, \underline{n})}\}$ can be proved very easily. For general c, the linear growth property of the family $\{T_{(\underline{m}, \underline{n})} \mid (\underline{m}, \underline{n}) \in \mathbb{N}^{s+t}\}$ seems more likely to be tractable.

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