INFINITE RINGS WITH PLANAR ZERO-DIVISOR GRAPHS

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Abstract. For any commutative ring R that is not a domain, there is a zero-divisor graph, denoted $\Gamma(R)$, in which the vertices are the nonzero zero-divisors of R and two distinct vertices x and y are joined by an edge exactly when $xy = 0$. In [Sm2], Smith characterized the graph structure of $\Gamma(R)$ provided it is infinite and planar. In this paper, we give a ring-theoretic characterization of R such that $\Gamma(R)$ is infinite and planar.

0. Introduction

Throughout this paper, every ring is assumed commutative with $1 \neq 0$. For any such ring R, let $Z(R)$ be the set of all zero-divisors of R and then let $Z(R)^* = Z(R) \setminus \{0\}$. The zero-divisor graph, denoted $\Gamma(R)$, is defined as follows: Its vertex set is $V = Z(R)^*$ and, for any distinct $x, y \in Z(R)^*$, there is an undirected edge between them exactly when $xy = 0$ (see [AL] and [Beck]). It is clear that $\Gamma(R) = \emptyset$ if and only if R is an integral domain. For this reason, we assume R is *not* an integral domain throughout this paper when studying $\Gamma(R)$.

Notation 0.1. The following notations will be used throughout this paper.

(1) For any set S, we use $|S|$ to denote its cardinality. We usually use m, n to denote finite cardinalities (e.g., $|\mathbb{Z}_n| = n$) and use α, β to denote general cardinalities. Thus, to say $|S| = \infty$ is the same as to say $|S| \geq |\mathbb{Z}|$.
(2) For any positive integer n and any cardinality $\alpha > 0$, we use $K_{n,\alpha}$ to denote a complete bipartite graph with a bipartition into two vertex subsets of cardinalities n and α respectively. In particular, $K_{1,\alpha}$ is often referred to as a star graph.
(3) In part (2) above, if we add edges to $K_{n,\alpha}$ by joining all distinct vertices within the vertex subset of cardinality n, the resulted graph is denoted by $K_{[n],\alpha}$. It is clear that $K_{1,\alpha} = K_{[1],\alpha}$. In fact, $K_{[n],\alpha}$ is simply a complete $(n+1)$-partite graph with the vertex set partitioned into subsets of cardinalities $1, 1, \ldots, 1, \alpha$ respectively.

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(4) The spectrum of a ring R, i.e., the set of all prime ideals of R, is denoted by $\operatorname{Spec}(R)$. The Krull dimension, $\dim(R)$, is the supremum of the lengths of all chains of prime ideals of R.
(5) Let R be a ring and M an R-module. Then $R \ltimes M$ stands for the ring structure defined on the abelian group $\{(r,x) \mid r \in R, x \in M\}$ in which the multiplication is defined by $(r,x)(s,y) = (rs, ry+sx)$ for all $r,s \in R$ and $x,y \in M$. This is often called the *idealization* of M.
(6) The notation p stands for a prime (hence positive) integer. Let $n \in \mathbb{Z}$ be such that $p \nmid n$. We say n is a quadratic residue modulo p if $n \equiv k^2 \mod p$ for some $k \in \mathbb{Z}$. Otherwise we say n is a quadratic non-residue modulo p.

One of the questions concerning $\Gamma(R)$ is when is it planar. In [AFLL], Anderson et al. have studied this question in case $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ (with $n_1, \ldots, n_k$ positive integers) and posed the question as to when $\Gamma(R)$ is planar for a general finite ring R. This question was completely answered in [Sm1] by Smith, and was also investigated in [AMY].

If R is an infinite ring, we see that $\Gamma(R)$ is also infinite unless R is a domain. Naturally, one would be interested in when $\Gamma(R)$ is planar in case $|R| = \infty$. This was studied by Smith in [Sm2], in which he showed that, for an infinite ring R (not a domain), $\Gamma(R)$ is planar if and only if $\Gamma(R)$ is graph-isomorphic to either $K_{1,\alpha}$, $K_{2,\alpha}$ or $K_{[2],\alpha}$ with $|\mathbb{Z}| \le \alpha \le |\mathbb{R}|$.

In spite of the above graph-theoretic characterization of $\Gamma(R)$ being planar, it was still open (as indicated at the end of [Sm2, Section 2]) to find a ring-theoretic characterization of an infinite ring R such that $\Gamma(R)$ is graph-isomorphic to $K_{1,\alpha} = K_{[1],\alpha}$, $K_{2,\alpha}$ or $K_{[2],\alpha}$. We are to study this in Section 1. In fact, we study the ring structure of R such that $\Gamma(R)$ is $K_{p-1,\alpha}$ or $K_{[p-1],\alpha}$ for any prime number p. Then, in Section 2, we characterize all infinite rings R such that $\Gamma(R)$ is planar. For example, we have

Theorem (See Theorem 2.1)**.** *Let R be a ring that is not a domain. The following statements are equivalent.*

(1) R is an infinite *ring such that $\Gamma(R)$ is planar.*
(2) $|\mathbb{Z}| \le |R| \le |\mathbb{R}|$ and, moreover, R falls into exactly one of the two categories.
  (I) $R \cong R_1 \times \mathbb{Z}_p$ with R_1 a domain while $p = 2$ or $p = 3$.
  (II) The nilradical of R is a prime ideal of order 2 or 3.
(3) $|\mathbb{Z}| \le |R| \le |\mathbb{R}|$ and R has a prime ideal of order 2 or 3.

1. The ring structure of R with $\Gamma(R) \cong K_{p-1,\alpha}$ or $\Gamma(R) \cong K_{[p-1],\alpha}$

Throughout this section, we assume that $\Gamma(R) \cong K_{p-1,\alpha}$ or $\Gamma(R) \cong K_{[p-1],\alpha}$ in which p is a prime number. (Notice that $K_{1,\alpha} = K_{[1],\alpha}$.) In either case, there is a bipartition of the vertex set $V = Z(R)^*$, say $V = U \cup W$, a disjoint union, such that $|U| = p-1$ while $|W| = \alpha$. Let

- $\mathfrak{a} := U \cup \{0\}$, which, *a priori*, is a subset of R; and
- $\mathfrak{m} := (0 :_R \mathfrak{a}) = \{r \in R \mid ru = 0, \forall u \in \mathfrak{a}\}$, an ideal of R.

These notations and assumptions will be kept throughout this section. It is noted that in [AL], the structure of R is studied given that $\Gamma(R) \cong K_{1,\alpha}$. In this paper, we treat the case $\Gamma(R) \cong K_{p-1,\alpha}$ or $\Gamma(R) \cong K_{[p-1],\alpha}$ for all prime p simultaneously.

Before stating the the results, we also remark that any ring R is an algebra over $\mathbb{Z}$ under the unique ring homomorphism $\phi : \mathbb{Z} \to R$ sending 1 to 1. By abuse of notation, for any $n \in \mathbb{Z}$, we use '$n \in R$' to identify the element $\phi(n)$ in R. For example, '$0 \neq n \in R$' simply means '$0 \neq \phi(n) \in R$'.

Lemma 1.1. *Let R be a ring such that $\Gamma(R) \cong K_{p-1,\alpha}$ or $\Gamma(R) \cong K_{[p-1],\alpha}$. Moreover, assume $\alpha \geq 2$.*

1. *We have $W \subset \mathfrak{m}$ and the subset $\mathfrak{a}$ is an ideal of R. In fact, we have $\mathfrak{a} = (0 :_R W) = (0 :_R w)$ and $ww' \neq 0$ for any $w, w' \in W$.*
2. *Thus $\mathfrak{a} \cong \mathbb{Z}_p$ as $\mathbb{Z}$-modules. Hence, $\mathfrak{a}$ is generated by u over $\mathbb{Z}$ (and therefore over R) for any $u \in U$. Consequently, we have $\mathfrak{m} = (0 :_R u)$ for every $u \in U$.*
3. *We have $\mathfrak{a} \cong R/\mathfrak{m}$ as R-modules and $R/\mathfrak{m} \cong \mathbb{Z}_p$ as rings, which implies that $\mathfrak{m}$ is a maximal ideal of R. In particular, $p \in \mathfrak{m}$.*
4. *If $\mathfrak{a} \not\subseteq \mathfrak{m}$, then $\Gamma(R) \cong K_{p-1,\alpha}$. If $\mathfrak{a} \subseteq \mathfrak{m}$, then $\Gamma(R) \cong K_{[p-1],\alpha}$.*
5. *$\mathfrak{a} \subseteq \mathfrak{m}$ if and only if $\mathfrak{a}^2 = 0$ if and only if $uu' = 0$ for some $u, u' \in U$. In this case, we have a disjoint union $\mathfrak{m} = \{0\} \cup U \cup W$, which forces $\mathfrak{a} \subsetneq \mathfrak{m}$.*

Proof. (1). First, it is clear that $W \subset (0 :_R \mathfrak{a}) = \mathfrak{m}$ from the assumption on $\Gamma(R)$. For every $w \in W$, we have $\mathfrak{a} \subseteq (0 :_R w) \subseteq \mathfrak{a} \cup \{w\}$. Thus, as $|W| = \alpha \geq 2$, we see that $\mathfrak{a} = \cap_{w\in W}(0 :_R w) = (0 :_R W)$, which is an ideal of R. Now, suppose $w^2 = 0$ for some $w \in W$. Then we get $(0 :_R w) = \mathfrak{a} \cup \{w\} \supset \mathfrak{a}$, which produces an abelian group of order $p+1$ with a subgroup of order p, a contradiction. This shows $w^2 \neq 0$, which implies $\mathfrak{a} = (0 :_R w)$ and hence $ww' \neq 0$ for any $w, w' \in W \subset R \setminus \mathfrak{a}$.

(2). In particular, $\mathfrak{a}$ is an abelian group under addition. Thus, as $|\mathfrak{a}| = p$ is a prime number, we see that $\mathfrak{a} \cong \mathbb{Z}_p$ and is generated by every $u \in U$ over $\mathbb{Z}$ (and therefore over R). Thus $\mathfrak{m} = (0 :_R \mathfrak{a}) = (0 :_R u)$ for every $u \in U$.

(3). This follows from the isomorphism $R/\mathfrak{m} = R/(0 :_R u) \cong Ru = \mathfrak{a}$ for any $u \in U$ combined with $\mathfrak{a} \cong \mathbb{Z}_p$. It is clear that $p \in \mathfrak{m}$.

(4) and (5) should follow immediately. (If $\mathfrak{a} \subseteq \mathfrak{m}$, then $\mathfrak{m}$ consists of all the zero-divisors and hence $\mathfrak{m} = \{0\} \cup V = \{0\} \cup U \cup W = \mathfrak{a} \cup W$, all as disjoint unions.) □

Now we study the ring structure of R assuming $\Gamma(R)$ is $K_{p-1,\alpha}$ or $K_{[p-1],\alpha}$. We assume $\alpha \geq 2$ most of the time so that $\mathfrak{a}$ is an ideal of R. (See Remark 1.7 for what happens when $\alpha = 1$.)

Proposition 1.2. *Assume $\alpha \geq 2$. If $\mathfrak{a} \not\subseteq \mathfrak{m}$, then $\mathfrak{a} \cap \mathfrak{m} = 0$, $\mathfrak{a} \in \operatorname{Spec}(R)$ and $R \cong R_1 \times \mathbb{Z}_p$ with R_1 an integral domain such that $|R_1| = \alpha + 1$.*

Conversely, if $R \cong R_1 \times \mathbb{Z}_p$ for any integral domain R_1, we have $\Gamma(R) \cong K_{p-1,|R_1|-1}$.

Proof. As $\mathfrak{a} \not\subseteq \mathfrak{m}$, we have $\mathfrak{a} \cap \mathfrak{m} \subsetneq \mathfrak{a}$, which forces $\mathfrak{a} \cap \mathfrak{m} = 0$ as $|\mathfrak{a}| = p$ is a prime number.

Next, we prove $\mathfrak{a}$ is a prime ideal of R by contradiction. Suppose there exist $r, s \in R \setminus \mathfrak{a}$ such that $rs \in \mathfrak{a}$. Choose any $w \in W$. As $(0 :_R w) = \mathfrak{a}$ by Lemma 1.1(1), we get $rsw = 0$ while $sw \neq 0$. Thus r is a nonzero zero-divisor, which then forces

$r \in W$ as $r \notin U$. Now, by Lemma 1.1(1) again, we see that $sw \in (0 :_R r) = \mathfrak{a}$ and $sw \in \mathfrak{m}$ as $w \in W \subset \mathfrak{m}$. This shows $0 \neq sw \in \mathfrak{a} \cap \mathfrak{m}$, which is a contradiction to $\mathfrak{a} \cap \mathfrak{m} = 0$. Thus $\mathfrak{a} \in \operatorname{Spec}(R)$.

As $\mathfrak{m}$ is a maximal ideal of R (cf. Lemma 1.1(3)) and $\mathfrak{a} \not\subseteq \mathfrak{m}$, we see that $\mathfrak{a}+\mathfrak{m} = R$. By Chinese Remainder Theorem, we see that

$$R = R/(\mathfrak{a} \cap \mathfrak{m}) \cong (R/\mathfrak{a}) \times (R/\mathfrak{m}) \cong R_1 \times \mathbb{Z}_p,$$

in which $R_1 = R/\mathfrak{a}$ is a domain such that $|R_1| = \alpha + 1$.

It is clear that $\Gamma(R_1 \times \mathbb{Z}_p) \cong K_{p-1,|R_1|-1}$ for any integral domain R_1. □

Now that we are done with the case where $\mathfrak{a} \not\subseteq \mathfrak{m}$, we next study the case where $\mathfrak{a} \subseteq \mathfrak{m}$. In light of Lemma 1.1(5), we see that it can not actually happen that $\mathfrak{a} = \mathfrak{m}$ under our assumption on $\Gamma(R)$ and that $\alpha \geq 2$. Thus it remains to study the case where $\mathfrak{a} \subsetneq \mathfrak{m}$ only.

Proposition 1.3. *Assume $\alpha \geq 2$. If $\mathfrak{a} \subsetneq \mathfrak{m}$ and $\mathfrak{a} \in \operatorname{Spec}(R)$, then R can be characterized as an infinite ring with a prime ideal $\mathfrak{a}$ such that $\mathfrak{a}^2 = 0$ and $|\mathfrak{a}| = p$. In this case, we have $\dim(R) \geq 1$ and $\mathfrak{a} = \sqrt{0}$, in which $\sqrt{0}$ denotes the nilradical of R.*

Conversely, if R is an infinite ring such that the nilradical $\sqrt{0}$ is a prime ideal and $|\sqrt{0}| = p$, then $\Gamma(R) \cong K_{[p-1],|R|}$.

Proof. All the claims in the first paragraph follow immediately from the assumption and Lemma 1.1. Concerning $\dim(R)$, we see that $\dim(R) \geq 1$ as there is a chain of primes $\mathfrak{a} \subsetneq \mathfrak{m}$. Thus R is an infinite ring as any finite ring has Krull dimension 0.

Conversely, suppose R is an infinite ring such that $\mathfrak{b} := \sqrt{0}$ is a prime ideal and $|\mathfrak{b}| = p$. Let $\mathfrak{m} = (0 :_R \mathfrak{b})$. Then, as $\mathfrak{b}$ is a simple R-module, we see that $\mathfrak{b} \cong R/\mathfrak{m}$ and $\mathfrak{m}$ is maximal. As $\mathfrak{b} = \sqrt{0}$ is finitely generated, we see that $\mathfrak{b}^n = 0$ for some integer $n \geq 1$. If $\mathfrak{b}^2 \neq 0$, then $\mathfrak{b}^2 = \mathfrak{b}$ and hence $\mathfrak{b}^n = \mathfrak{b} \neq 0$ for all $n \geq 1$, a contradiction. So $\mathfrak{b}^2 = 0$ and thus $\mathfrak{b} \subseteq \mathfrak{m}$. If $\mathfrak{b} = \mathfrak{m}$, then the filtration $0 \subsetneq \mathfrak{b} = \mathfrak{m} \subsetneq R$ would imply that $|R| = p^2$, a contradiction. So $\mathfrak{b} \subsetneq \mathfrak{m}$. Suppose $rs = 0$ for $r, s \in R \setminus \{0\}$. Then, as $\mathfrak{b} \in \operatorname{Spec}(R)$, we see that $r \in \mathfrak{b}$ or $s \in \mathfrak{b}$. Say $r \in \mathfrak{b}$, which implies $\mathfrak{b} = (r)$ and $s \in (0 :_R r) = (0 :_R \mathfrak{b}) = \mathfrak{m}$. This shows that $\mathfrak{m}$ consists of all zero-divisors, in which $uu' = 0$, $uw = 0$ and $ww' \notin \mathfrak{b}$ for all $u, u' \in \mathfrak{b} \setminus \{0\}$ and all $w, w' \in \mathfrak{m} \setminus \mathfrak{b}$. This shows $\Gamma(R) \cong K_{[p-1],|\mathfrak{m}|-p}$. As $|R| = \infty$ and $|R/\mathfrak{m}| = |\mathfrak{b}| = p$, we see that $|R| = |\mathfrak{m}| = |\mathfrak{m}| - p$, so that $\Gamma(R) \cong K_{[p-1],|R|}$. □

Example 1.4. Rings as in Proposition 1.3 include idealizations of the form $R_1 \ltimes \mathbb{Z}_p$ for any domain R_1 that maps onto $\mathbb{Z}_p$ (so that $\mathbb{Z}_p$ is an R_1-module), for example,

- $\mathbb{Z}[X_\lambda, Y \mid \lambda \in \Lambda]/(pY, X_\lambda Y, Y^2 \mid \lambda \in \Lambda) \cong \mathbb{Z}[X_\lambda \mid \lambda \in \Lambda] \ltimes \mathbb{Z}_p$ and
- $\mathbb{Z}[X_\lambda, Y \mid \lambda \in \Lambda]/(p, X_\lambda Y, Y^2 \mid \lambda \in \Lambda) \cong \mathbb{Z}_p[X_\lambda \mid \lambda \in \Lambda] \ltimes \mathbb{Z}_p$,

in which Λ is an index set. However, if we consider the ring

$$R = \mathbb{Z}[X_\lambda \mid \lambda \in \Lambda]/(p^2, pX_\lambda \mid \lambda \in \Lambda) \quad \text{with } \Lambda \neq \emptyset,$$

we see that it has prime ideals $\mathfrak{b} = (\overline{p}) = \sqrt{0}$ with $|\mathfrak{b}| = p$ and $\mathfrak{m} = (\overline{p}, \overline{X}_\lambda \mid \lambda \in \Lambda)$ such that $R/\mathfrak{m} \cong \mathbb{Z}_p$ as in Proposition 1.3. But R can not be written as $R_1 \ltimes \mathbb{Z}_p$ (nor can it be written as $R_1 \times \mathbb{Z}_p$) for any ring R_1.

Proposition 1.5. *Assume $\alpha \geq 2$, $\mathfrak{a} \subsetneq \mathfrak{m}$ and $\mathfrak{a} \notin \operatorname{Spec}(R)$. Then the following statements hold.*

(1) *We have $w_1 w_2 \in U$ for all $w_1, w_2 \in W$. Hence we have $\mathfrak{m}^2 \subseteq \mathfrak{a}$ and $\mathfrak{m}^3 = 0$.*
(2) *Fix any $x \in W$. Then $px^2 = 0$ and every element $r \in \mathfrak{m}$ can be written as $r = n_2 x^2 + n_1 x$ with unique $n_i \in \mathbb{Z}$ such that $0 \leq n_i \leq p-1$. Thus, $|\mathfrak{m}| = p^2$.*
(3) *Consequently, any element $s \in R$ can be written as $s = n_2 x^2 + n_1 x + n_0$ with unique $n_i \in \mathbb{Z}$ such that $0 \leq n_i \leq p-1$. Thus R is finite with $|R| = p^3$ and, as an algebra, R is generated by x over $\mathbb{Z}$.*

Proof. (1). The set of associated primes of R is

$$\operatorname{Ass}(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) \,|\, \mathfrak{p} = (0 :_R r) \text{ for some } r \in R\}.$$

For any $r \in R$, the annihilator ideal $(0 :_R r)$ could only possibly be

$$(0 :_R r) = \begin{cases} 0 \notin \operatorname{Spec}(R) & \text{in case } r \text{ is not a zero-divisor} \\ \mathfrak{a} \notin \operatorname{Spec}(R) & \text{in case } r \in W \\ \mathfrak{m} \in \operatorname{Spec}(R) & \text{in case } r \in U \\ R \notin \operatorname{Spec}(R) & \text{in case } r = 0. \end{cases}$$

This shows $\operatorname{Ass}(R) = \{\mathfrak{m}\}$. Moreover, by [Beck, Theorem 4.3], we have $\min(R) \subseteq \operatorname{Ass}(R)$. Thus $\mathfrak{m}$ is the only minimal prime and hence the nilradical of R. Then, by the argument in the proof of [AMY, Theorem 3.2(c)(d)], we see that $w_1 w_2 \in U$ for all $w_1, w_2 \in W$. Indeed, for any $w \in W$, we have $\min\{k \,|\, w^k = 0\} = 3$ and hence $w^2 \in (0 :_R w) \setminus \{0\} = U$. Consequently, we have $w_1 (w_2)^2 = 0$, which implies $w_1 w_2 \in (0 :_R w_2) \setminus \{0\} = U$. Now, as $\mathfrak{m} = \{0\} \cup U \cup W$, it should be clear that $\mathfrak{m}^2 \subseteq \mathfrak{a}$, which then implies $\mathfrak{m}^3 = 0$.

(2). Fix any $x \in W$. By part (1) above, we have $x^2 \in U = \mathfrak{a} \setminus \{0\}$. Then, as $|\mathfrak{a}| = p$, we see that

$$\mathfrak{a} = \{x^2, 2x^2, \ldots, (p-1)x^2, px^2 = 0x^2 = 0\} = \{nx^2 \,|\, 0 \leq n \leq p-1\}.$$

Let r be any element in $\mathfrak{m}$. As $rx \in \mathfrak{m}^2 \subseteq \mathfrak{a}$, we have $rx = n_1 x^2$ for an integer n_1 with $0 \leq n_1 \leq p-1$. Thus $0 = rx - n_1 x^2 = (r - n_1 x)x$, which implies $r - n_1 x \in (0 :_R x) = \mathfrak{a}$. Hence $r - n_1 x = n_2 x^2$ for some integer n_2 with $0 \leq n_2 \leq p-1$, showing that $r = n_2 x^2 + n_1 x$ with $0 \leq n_i \leq p-1$ as desired. Also, it is straightforward to show that such n_1, n_2 are unique to r. Consequently, we have $|\mathfrak{m}| = p^2$.

(3). As $R/\mathfrak{m} \cong \mathbb{Z}_p$, there are exactly p distinct (right) cosets of $\mathfrak{m}$, namely, $\mathfrak{m} + n$ with $n = 0, 1, \ldots, p-1$. Thus, for any $s \in R$, there is an integer n_0 with $0 \leq n_0 \leq p-1$ and some $r \in \mathfrak{m}$ such that $s = r + n_0$. By part (2) above, we see that $s = n_2 x^2 + n_1 x + n_0$. And it is routine to verify that such n_0, n_1, n_2 are unique to s. Consequently, we see that R is a finite ring with $|R| = p^3$. Evidently, R is generated by x as a $\mathbb{Z}$-algebra. □

The next proposition was essentially covered in [AMY, Theorem 3.4] in case $p \geq 3$. However, our result is stated in a slightly different way and it holds for all prime number p under the assumption $\mathfrak{a} \subsetneq \mathfrak{m}$. Following the suggestion of the referee, we provide a complete proof.

Proposition 1.6 (cf. [AMY, Theorem 3.4]). *Assume $\alpha \geq 2$, $\mathfrak{a} \subsetneq \mathfrak{m}$ and $\mathfrak{a} \notin \operatorname{Spec}(R)$ as in Proposition 1.5 above. Then $\Gamma(R) \cong K_{[p-1],p(p-1)}$ and* exactly one *of the following will happen.*

(1) If $p \notin \mathfrak{a}$, then $R \cong \mathbb{Z}_{p^3}$.
(2) If $0 = p \in R$, then $R \cong \mathbb{Z}[X]/(p, X^3) \cong \mathbb{Z}_p[X]/(X^3)$.
(3) If $0 \neq p \in \mathfrak{a}$ and $x^2 = p \in R$ for some $x \in W$, then $R \cong \mathbb{Z}[X]/(pX, X^2 - p) \cong \mathbb{Z}_{p^2}[X]/(pX, X^2 - p)$.
(4) If $0 \neq p \in \mathfrak{a}$ and $w^2 \neq p \in R$ for any $w \in W$, then $R \cong \mathbb{Z}_{p^2}[X]/(pX, X^2 - np)$, in which n is the least positive integer that is a quadratic non-residue modulo p. This case will never happen when $p = 2$.

Proof. (1). Assume $p \notin \mathfrak{a}$. Then $p \in \mathfrak{m} \setminus \mathfrak{a} = W$ by Lemma 1.1(3)(5). From Proposition 1.5(3) above, we see that R is generated by $p \in R$ as a $\mathbb{Z}$-algebra and $|R| = p^3$. As $p \in \mathbb{Z}$, the (unique) ring homomorphism $\phi : \mathbb{Z} \to R$ is surjective, which forces $R \cong \mathbb{Z}/(p^3) = \mathbb{Z}_{p^3}$.

(2). Assume $0 = p \in R$. Fix any $x \in W$. Then there is a (unique) surjective ring homomorphism $\phi : \mathbb{Z}[X] \to R$ such that $\phi(X) = x$ and, moreover, we see that $\{p, X^3\} \subset \ker(\phi)$ (cf. Proposition 1.5(1) above). Thus there is an induced surjective ring homomorphism $\overline{\phi} : \mathbb{Z}[X]/(p, X^3) \to R$. Then, as $|\mathbb{Z}[X]/(p, X^3)| = p^3 = |R|$, we see $\overline{\phi}$ is an isomorphism and, hence, $R \cong \mathbb{Z}[X]/(p, X^3) \cong \mathbb{Z}_p[X]/(X^3)$.

(3). Assume $0 \neq p \in \mathfrak{a}$ and $x^2 = p \in R$ for some $x \in W$. Again, there is a surjective ring homomorphism $\phi : \mathbb{Z}[X] \to R$ such that $\phi(X) = x$. And it is routine to verify that $\{pX, X^2 - p\} \subset \ker(\phi)$ by assumption. Thus there is an induced surjective ring homomorphism $\overline{\phi} : \mathbb{Z}[X]/(pX, X^2 - p) \to R$. As $p^2 = X(pX) - p(X^2 - p)$, we see that $\mathbb{Z}[X]/(pX, X^2 - p) = \mathbb{Z}[X]/(p^2, pX, X^2 - p)$, in which every element can be represented by $n_1 X + n_0$ with $0 \leq n_1 \leq p - 1$ and $0 \leq n_0 \leq p^2 - 1$. This shows that $|\mathbb{Z}[X]/(pX, X^2 - p)| \leq p^3 = |R|$. Thus $\overline{\phi}$ is forced to be an isomorphism. In conclusion, we have $R \cong \mathbb{Z}[X]/(pX, X^2 - p) \cong \mathbb{Z}_{p^2}[X]/(pX, X^2 - p)$.

(4). Assume $0 \neq p \in \mathfrak{a}$ and $w^2 \neq p \in R$ for any $w \in W$. This situation never happens when $p = 2$. Indeed, if $p = 2$, then $p \in R$ is the only element in $\mathfrak{a} \setminus \{0\} = U$ as $|U| = p - 1 = 1$. Thus $w^2 = p \in R$ for all $w \in W$ by Proposition 1.5(1) above. So we assume $p \geq 3$. Fix any $y \in W$ and let $u = y^2 \in U$. Then, by Proposition 1.5(2) above, every element $w \in W \subset \mathfrak{m}$ can be written as $w = n_2 y^2 + n_1 y$ with $0 \leq n_i \leq p - 1$. Then $w^2 = n_1^2 y^2 = n_1^2 u$ (as $y^3 = 0$ by Proposition 1.5(1)). Since u generates $\mathfrak{a}$ as a $\mathbb{Z}$-module, we see that $mu = p \in \mathfrak{a}$ for some integer m with $1 \leq m \leq p - 1$. We claim that m is a quadratic non-residue modulo p. Indeed, if $\overline{m} = \overline{l^2} \in \mathbb{Z}_p$ for some integer l, then we would have $(ly)^2 = l^2 u = mu = p \in \mathfrak{a}$, a contradiction. Let n be the least positive integer that is a quadratic non-residue modulo p, which exists since $p \geq 3$. Then $2 \leq n \leq p - 1$ and nm, the product of two quadratic non-residues, is a quadratic residue modulo p, i.e., $nm \equiv k^2 \mod p$ for some $k \in \mathbb{Z}$. Let $x = ky$. Then $x^2 = k^2 y^2 = k^2 u = nmu = np \in \mathfrak{a} \setminus \{0\} = U$ as $\gcd(n, p) = 1$. Also notice that $x = ky \in \mathfrak{m}$ (since $y \in W \subset \mathfrak{m}$) and $x \notin \mathfrak{a}$ (because $x^2 = np \neq 0 \in R$ but $\mathfrak{a}^2 = 0$). Thus $x \in \mathfrak{m} \setminus \mathfrak{a} = W$ by Lemma 1.1(5). As in part (3), there is a surjective ring homomorphism $\phi : \mathbb{Z}[X] \to R$ such that $\phi(X) = x$ and we see that $\{p^2, pX, X^2 - np\} \subset \ker(\phi)$. (We have $p^2 \in \ker(\phi)$

because $p^2 \in \mathfrak{a}^2 = 0$ by Lemma 1.1(5).) Thus there is an induced surjective ring homomorphism $\overline{\phi} : \mathbb{Z}[X]/(p^2, pX, X^2 - np) \to R$. Still as in part (3) above, we see that $|\mathbb{Z}[X]/(p^2, pX, X^2 - np)| \leq p^3 = |R|$ and hence $\overline{\phi}$ is an isomorphism. Thus $R \cong \mathbb{Z}[X]/(p^2, pX, X^2 - np) \cong \mathbb{Z}_{p^2}[X]/(pX, X^2 - np)$.

Finally, in each of the four cases above, it is straightforward to verify that $\Gamma(R) \cong K_{[p-1],p(p-1)}$ as the ring structure of R is known. □

We conclude this section by commenting on what happens when $\alpha = 1$, i.e., $\Gamma(R) \cong K_{p-1,1}$ or $\Gamma(R) \cong K_{[p-1],1}$. In fact, we consider cases where $\Gamma(R) \cong K_{n,1}$ and $\Gamma(R) \cong K_{[n],1}$ for any integer $n \geq 1$.

Remark 1.7. Let R be a ring such that $\Gamma(R) \cong K_{n,1}$ or $\Gamma(R) \cong K_{[n],1}$ for any integer $n \geq 1$. As $\Gamma(R)$ is finite, we see that R is a finite ring.

(1) Assume $\Gamma(R) \cong K_{n,1}$ and $n \geq 2$, that is, $\Gamma(R) \cong K_{1,n}$ and $n \geq 2$. By the propositions above, we see that either $R = \mathbb{F}_{n+1} \times \mathbb{Z}_2$ in which $\mathbb{F}_{n+1}$ is the field of cardinality $n+1$ (in case Proposition 1.2 applies and $n+1$ is a prime power), or else $R \cong \mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$ (in case Proposition 1.6 applies and $n = 2$). We observe that Proposition 1.3 never applies to R as R is finite. This was also studied in [AL].

(2) Assume $\Gamma(R) \cong K_{1,1}$ or $\Gamma(R) \cong K_{[n],1}$ for any $n \geq 1$. Then $\Gamma(R)$ is no other than a complete graph on 2 or more vertices. This case was studied in [AL, Theorem 2.10].

This finishes our remark on the cases where $\Gamma(R) \cong K_{n,1}$ or $\Gamma(R) \cong K_{[n],1}$.

As the focus of this paper is infinite rings such that $\Gamma(R)$ is $K_{p-1,\alpha}$ or $K_{[p-1],\alpha}$ (with $p = 2, 3$), the interesting case is when $\alpha = \infty$ where either Proposition 1.2 or Proposition 1.3 (but never Proposition 1.5 or 1.6) applies.

2. The structure of an infinite ring R such that $\Gamma(R)$ is planar

Based on the work in [Sm2] and our investigation in Section 1, we are ready to characterize the structure of an infinite ring R provided that $\Gamma(R)$ is planar.

Theorem 2.1. *Let R be a ring that is not a domain. The following statements are equivalent.*

(1) R is an infinite *ring such that $\Gamma(R)$ is planar.*

(2) $|\mathbb{Z}| \leq |R| \leq |\mathbb{R}|$ and, moreover, R falls into exactly one of the two categories.

(I) $R \cong R_1 \times \mathbb{Z}_p$ with R_1 a domain while $p = 2$ or $p = 3$.

(II) The nilradical of R, denoted $\sqrt{0}$, satisfies $\sqrt{0} \in \operatorname{Spec}(R)$ and $|\sqrt{0}| = p$ with $p = 2$ or $p = 3$. If this is the case, then $\sqrt{0}^2 = 0$.

(3) $|\mathbb{Z}| \leq |R| \leq |\mathbb{R}|$ and there exists $\mathfrak{a} \in \operatorname{Spec}(R)$ such that $|\mathfrak{a}| = p$ with $p = 2$ or $p = 3$.

(4) $R \cong T/I$ in which $T = \mathbb{Z}[X_\lambda \mid \lambda \in \mathbb{R}]$, I is an ideal of T such that $I \subsetneq \mathfrak{p}$ and $\mathfrak{p}/I \cong T/\mathfrak{n}$ as T-modules for some $\mathfrak{p} \in \operatorname{Spec}(T)$ satisfying $|T/\mathfrak{p}| = \infty$ and $\mathfrak{n} = (p, X_\lambda \mid \lambda \in \mathbb{R})$ with $p = 2$ or $p = 3$.

(5) $R \cong T/I$ *in which* $T = \mathbb{Z}[X_\lambda \,|\, \lambda \in \Lambda]$ *(for some index set* Λ*) and* I *is an ideal of* T *such that* $I \subsetneq \mathfrak{p}$ *for some* $\mathfrak{p} \in \operatorname{Spec}(T)$ *satisfying* $|\mathbb{Z}| \le |T/\mathfrak{p}| \le |\mathbb{R}|$ *and* $|\mathfrak{p}/I| = p$ *with* $p = 2$ *or* $p = 3$.

Proof. (1) $\Rightarrow$ (2). Assume $\Gamma(R)$ is planar. Then, by [Sm2, Theorem 2.19], we have

$$\Gamma(R) \cong K_{p-1,\alpha} \text{ or } \Gamma(R) \cong K_{[p-1],\alpha} \text{ with } |\mathbb{Z}| \le \alpha \le |\mathbb{R}| \text{ while } p = 2 \text{ or } p = 3.$$

Adopt the notations (e.g., $\mathfrak{a}$ and $\mathfrak{m}$) as in Section 1. Then exactly one of the following will occur.

- If $\mathfrak{a} \not\subseteq \mathfrak{m}$, then R falls in category (I) with $|R_1| - 1 = \alpha$ by Proposition 1.2. Thus $|R| = |R_1| = \alpha$.
- If $\mathfrak{a} \subsetneq \mathfrak{m}$ and $\mathfrak{a} \in \operatorname{Spec}(R)$, then R falls in category (II) with $|\mathfrak{m}| - |\mathfrak{a}| = \alpha$ by Proposition 1.3. Also, as $|R/\mathfrak{m}| = |\mathfrak{a}| = p < \infty$ (cf. Lemma 1.1(3)), we have $|R| = |\mathfrak{m}| = |\mathfrak{m}| - |\mathfrak{a}| = \alpha$.

Note that Proposition 1.5 never applies since $|R| = \infty$. Also, the above two cases never overlap because the ring is reduced in (I) but not reduced in (II). Notice that $|\mathbb{Z}| \le |R| \le |\mathbb{R}|$ in both cases.

(2) $\Rightarrow$ (3). This is evident.

(3) $\Rightarrow$ (4). Let $\mathfrak{m} = (0 :_R \mathfrak{a})$. As $|\mathfrak{a}| = p$ with $p = 2$ or $p = 3$, we see that $\mathfrak{a}$ is principal and thus $\mathfrak{a} \cong R/\mathfrak{m}$, which implies $\mathfrak{m}$ is a maximal ideal of R such that $R/\mathfrak{m} \cong \mathbb{Z}_p$ as rings with $p = 2$ or $p = 3$. In particular, we see that R can be generated by elements in $\mathfrak{m}$ as an algebra over $\mathbb{Z}$. As $|\mathbb{Z}| \le |R| \le |\mathbb{R}|$, it is then clear that R can be generated by at most $|\mathbb{R}|$ many elements in $\mathfrak{m}$ as an algebra over $\mathbb{Z}$. Thus there is a surjective ring homomorphism $h : T \to R$, where $T = \mathbb{Z}[X_\lambda \,|\, \lambda \in \mathbb{R}]$ such that $h(X_\lambda) \in \mathfrak{m}$ for all $\lambda \in \mathbb{R}$.

Letting $\mathfrak{n} = h^{-1}(\mathfrak{m})$, we see that $\mathfrak{n} = (p, X_\lambda \,|\, \lambda \in \mathbb{R})$ with $p = 2$ or $p = 3$. Let $I = \ker(h)$ and $\mathfrak{p} = h^{-1}(\mathfrak{a})$. Then it should be clear that $R \cong T/I$, $I \subsetneq \mathfrak{p}$, $\mathfrak{p} \in \operatorname{Spec}(T)$ and $\mathfrak{p}/I \cong T/\mathfrak{n}$ as T-modules. Also, as $|R| = \infty$ and $|\mathfrak{a}| = p$, we see that $|R/\mathfrak{a}| = \infty$, which implies $|T/\mathfrak{p}| = |R/\mathfrak{a}| = \infty$.

(4) $\Rightarrow$ (5). This is clear.

(5) $\Rightarrow$ (3). This is evident.

(3) $\Rightarrow$ (1). Let $\mathfrak{m} = (0 :_R \mathfrak{a})$. As seen above, $\mathfrak{m}$ is a maximal ideal of R such that $R/\mathfrak{m} \cong \mathbb{Z}_p$ as rings with $p = 2$ or $p = 3$.

- Suppose $\mathfrak{a} \cap \mathfrak{m} = 0$. Then by Chinese Remainder Theorem, we have $R \cong (R/\mathfrak{a}) \times (R/\mathfrak{m}) \cong R_1 \times \mathbb{Z}_p$ with $R_1 = R/\mathfrak{a}$ a domain while $p = 2$ or $p = 3$. So $\Gamma(R) \cong K_{p-1,|R_1|-1}$ with $p = 2$ or $p = 3$, which is planar as $|R_1| - 1 \le |\mathbb{R}|$.
- Otherwise, suppose $\mathfrak{a} \cap \mathfrak{m} \ne 0$. Then $\mathfrak{a} \subseteq \mathfrak{m}$ (as $|\mathfrak{a}| = p$ is prime) and hence $\mathfrak{a}^2 = 0$, which implies $\mathfrak{a} = \sqrt{0}$. Then, by Proposition 1.3, we see that $\mathfrak{a} \subsetneq \mathfrak{m}$ and $\Gamma(R) \cong K_{[p-1],|\mathfrak{m}|-|\mathfrak{a}|}$ with $p = 2$ or $p = 3$, which is planar as $|\mathfrak{m}| - |\mathfrak{a}| \le |\mathbb{R}|$.

Now the proof is complete. □

Remark 2.2. Theorem 2.1(4) provides a way to construct all infinite rings R (up to isomorphism) such that $\Gamma(R)$ is planar. Form a polynomial ring $T = \mathbb{Z}[X_\lambda \,|\, \lambda \in \mathbb{R}]$. Let $\mathfrak{n} = (p, X_\lambda \,|\, \lambda \in \Lambda)$. Choose any nonzero prime ideal $\mathfrak{p} \ne 0$ of T such that $|T/\mathfrak{p}| = \infty$. We claim that $\mathfrak{p}\mathfrak{n} \subsetneq \mathfrak{p}$ and, hence, $\mathfrak{p}/(\mathfrak{p}\mathfrak{n})$ is a nonzero vector space over $T/\mathfrak{n}$. To show the claim, suppose on the contrary that $\mathfrak{p}\mathfrak{n} = \mathfrak{p}$. Then we would have $\mathfrak{p} = \mathfrak{p}\mathfrak{n}^i$ for all $i \geq 1$ and hence $\mathfrak{p} \subseteq \cap_{i=1}^{\infty}\mathfrak{n}^i = 0$, a contradiction. To see why $\cap_{i=1}^{\infty}\mathfrak{n}^i = 0$, consider any $0 \neq f \in T$. Choose any nonzero term of f, say of monomial degree d and with coefficient $n = p^m a$ such that $p \nmid a$. Then we see that $f \notin \mathfrak{n}^{m+d+1}$. Thus, there exists an ideal I of T such that $\mathfrak{p}\mathfrak{n} \subseteq I \subsetneq \mathfrak{p}$ and $\mathfrak{p}/I \cong T/\mathfrak{n}$ as T-modules. Let $R = T/I$ and let $\mathfrak{m} = \mathfrak{n}/I$ and $\mathfrak{b} = \mathfrak{p}/I$ which are ideals of R. Then $\mathfrak{b}$ is a prime ideal of R such that $|\mathfrak{b}| = |\mathfrak{p}/I| = p$. By Theorem 2.1(4), we see that R is an infinite ring and $\Gamma(R)$ is planar.

Also, Theorem 2.1(2) implies that the set of all infinite rings with planar zero-divisor graphs contains the set of rings of the form $R_1 \times \mathbb{Z}_p$ or $R_2 \ltimes \mathbb{Z}_p$ where each R_i is an integral domain with $|\mathbb{Z}| \leq |R_i| \leq |\mathbb{R}|$, $p = 2$ or 3, and R_2 maps onto $\mathbb{Z}_p$. However, the latter set is a *proper* subset of the former, as indicated in Example 1.4.

More generally (and quite similarly), we have the following theorem characterizing the structure of infinite rings R with $\Gamma(R)$ graph-isomorphic to $K_{p-1,\alpha}$ or $K_{[p-1],\alpha}$.

Theorem 2.3. *Let R be a ring that is not a domain and p be a prime number. The following statements are equivalent.*

(1) R is infinite *and $\Gamma(R)$ is graph-isomorphic to $K_{p-1,\alpha}$ or $K_{[p-1],\alpha}$.*
(2) R is an infinite ring that falls into exactly one of the two categories.
  (I) $R \cong R_1 \times \mathbb{Z}_p$ with R_1 a domain.
  (II) The nilradical of R, denoted $\sqrt{0}$, satisfies $\sqrt{0} \in \operatorname{Spec}(R)$ and $|\sqrt{0}| = p$.
(3) $|R| = \infty$ and there exists $\mathfrak{a} \in \operatorname{Spec}(R)$ such that $|\mathfrak{a}| = p$.
(4) $R \cong T/I$ in which $T = \mathbb{Z}[X_\lambda \,|\, \lambda \in \Lambda]$ (for some index set Λ), I is an ideal of T such that $I \subsetneq \mathfrak{p}$ and $\mathfrak{p}/I \cong T/\mathfrak{n}$ as T-modules for some $\mathfrak{p} \in \operatorname{Spec}(T)$ satisfying $|T/\mathfrak{p}| = \infty$ and $\mathfrak{n} = (p, X_\lambda \,|\, \lambda \in \Lambda)$.
(5) $R \cong T/I$ in which $T = \mathbb{Z}[X_\lambda \,|\, \lambda \in \Lambda]$ (for some index set Λ) and I is an ideal of T such that $I \subsetneq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec}(T)$ satisfying $|T/\mathfrak{p}| = \infty$ and $|\mathfrak{p}/I| = p$.

Proof. This is very similar to the proof of Theorem 2.1 above. Details omitted. □

Remark 2.4. Suppose R is as in Theorem 2.3(3), i.e., R has a prime ideal $\mathfrak{a}$ such that $|\mathfrak{a}| = p$. Then, as in the proof of Theorem 2.1 for (3) ⇒ (1), we see that

$$\Gamma(R) \cong \begin{cases} K_{p-1,\alpha} & \text{if } \mathfrak{a} \cap (0 :_R \mathfrak{a}) = 0, \text{ i.e., if } \mathfrak{a}^2 \neq 0 \\ K_{[p-1],\alpha} & \text{if } \mathfrak{a} \cap (0 :_R \mathfrak{a}) \neq 0, \text{ i.e., if } \mathfrak{a}^2 = 0. \end{cases}$$

And, in a way similar to Remark 2.2, Theorem 2.3(4) provides a way to construct all infinite rings R (up to isomorphism) such that $\Gamma(R)$ is graph-isomorphic to $K_{p-1,\alpha}$ or $K_{[p-1],\alpha}$. In case R is constructed this way, it is routine to verify that

$$\Gamma(R) \cong \begin{cases} K_{p-1,\alpha} & \text{if } \mathfrak{p} \not\subseteq \mathfrak{n} \\ K_{[p-1],\alpha} & \text{if } \mathfrak{p} \subsetneq \mathfrak{n}, \end{cases}$$

in which $\mathfrak{p}$ and $\mathfrak{n}$ are as in Theorem 2.3(4).

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