[Preliminary Version]

AN EMBEDDING THEOREM FOR MODULES OF FINITE (G-)PROJECTIVE DIMENSION

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ABSTRACT. Let M be any finitely generated module of finite projective dimension (respectively, finite G-dimension) over a commutative Noetherian ring R. Then M embeds into a finite direct sum Z of cyclic R-modules each of which is the quotient of R by an ideal generated by an R-regular sequence. This can be done so that both Z/M and hence Z have projective dimension (respectively, G-dimension) no more than the projective dimension (respectively, G-dimension) of M. Consequently, we also get a similar embedding theorem for finitely generated modules of finite injective dimension over any Cohen-Macaulay ring that has a global canonical module.

0. INTRODUCTION

Throughout this paper R is a commutative Noetherian ring with 1. It is well-known that any quotient module of R modulo an ideal generated by an R-regular sequence has finite projective dimension.

The main theorem of the paper is to embed any finitely generated R-module with finite projective dimension (or finite G-dimension) into a module that *obviously* has finite projective dimension.

Theorem 2.3. *Let R be a Noetherian ring and M a finitely generated R-module with projective dimension (respectively, G-dimension) $= r < \infty$. Then there exist a proper R-regular sequence $\underline{z} = z_1, \dots, z_r$, non-negative integers $n_0, n_1, \dots, n_r$, and a short exact sequence*

$$0 \longrightarrow M \longrightarrow Z \longrightarrow N \longrightarrow 0$$

with $Z = \oplus_{i=0}^{r} \big(R/\sum_{j=1}^{i} z_j R\big)^{n_i}$ and N having projective dimension (respectively, G-dimension) $\leq r$.

As an immediate corollary, we generalize a result by I. Aberbach that, for any F-rational ring R (of prime characteristic p) and any finitely generated R-module M of finite projective dimension (more generally, of finite G-dimension), 0 is tightly closed in M. See Theorem 3.2.

We also observe that Theorem 2.3 can be applied to show the existence of uniform test exponents for tight closure and Frobenius closure for finitely generated R-modules of finite projective dimension. See Theorem 3.3 and Theorem 3.6.

Date: October 8, 2009.

2000 *Mathematics Subject Classification.* Primary 13C05; Secondary 13A35, 13C13, 13H10.

Both authors were partially supported by the National Science Foundation and the second author was also partially supported by the Research Initiation Grant of Georgia State University.

Then, using a theorem of R. Y. Sharp ([Sh1], cf. Theorem 4.1), we obtain a similar result for embedding modules with locally finite injective dimension over any Cohen-Macaulay ring with a global canonical module ω. See Theorem 4.2.

We set up notations that will be used throughout this paper.

Notation 0.1. Let R be a Noetherian ring, M an R-module, I an ideal of R and $\underline{x} = x_1, \dots, x_l$ a sequence in R.

(1) For any integer i with $0 \le i \le l$, we denote by $\underline{x}_{[i]}$ the subsequence $x_1, \dots, x_i$ of $\underline{x}$. Then we write $(\underline{x}_{[i]}) = (x_1, \dots, x_i)R = \sum_{j=1}^{i} x_i R$, the ideal generated by $\underline{x}_{[i]}$. If $i = 0$, we agree that $\underline{x}_{[0]}$ is empty and $(\underline{x}_{[0]}) = 0$, the zero ideal.

(2) We say that an element $x \in R$ is regular on M (or a non-zero-divisor on M, or M-regular) if $x\theta \neq 0$ for any $0 \neq \theta \in M$. We agree that any $x \in R$ is regular on 0, the zero module.

(3) We say $\underline{x}$ is a *(possibly improper)* M-regular sequence if x_i is regular on $M/(\underline{x}_{[i-1]})M$ for all $i = 1, \dots, l$. If, furthermore, $(\underline{x})M \subsetneq M$, then we call $\underline{x}$ a *proper* regular sequence on M. We agree that any empty sequence (that is, any sequence of length 0, for example $\underline{x}_{[0]}$) is a proper regular sequence on any R-module.

(4) The *depth* of I on M, denoted by $\operatorname{depth}(I, M)$, is defined as follows

$$\operatorname{depth}(I, M) = \sup\{n \mid \exists \text{ a proper } M\text{-regular sequence } y_1, \dots, y_n \in I\}.$$

(5) Furthermore, the depth of M, denoted by $\operatorname{depth}(M)$, is defined as

$$\operatorname{depth}(M) = \sup\{n \mid \exists \text{ a proper } M\text{-regular sequence } y_1, \dots, y_n \in R\}.$$

When $(R, \mathfrak{m})$ is local, it is obvious that $\operatorname{depth}(M) = \operatorname{depth}(\mathfrak{m}, M)$. In general, if M is finitely generated over R, then

$$\operatorname{depth}(M) = \sup\{\operatorname{depth}(\mathfrak{m}R_{\mathfrak{m}}, M_{\mathfrak{m}}) \mid \mathfrak{m} \text{ is a maximal ideal in } R\}.$$

(6) We denote by $\operatorname{pd}_R(M)$ and $\operatorname{id}_R(M)$ the projective dimension and injective dimension of M respectively. When R is clearly understood, we simply denote them by $\operatorname{pd}(M)$ and $\operatorname{id}(M)$ respectively.

(7) We say that M is of *locally* finite injective dimension if, for any maximal ideal (or, equivalently, for any prime ideal) $\mathfrak{m}$ of R, $\operatorname{id}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) < \infty$. Then denote by $\mathcal{I}(R)$ the category of finitely generated R-modules with locally finite injective dimension. (In case $\operatorname{depth}(R) < \infty$ and M is finitely generated over R, we have $M \in \mathcal{I}(R) \iff \operatorname{id}_R(M) < \infty$.)

(8) Let $\mathcal{P}(R)$ denote the category of finitely generated R-modules with finite projective dimension. (One may similarly define the notion of locally finite projective dimension. However, a finitely generated R-module has locally finite projective dimension if and only if it has finite projective dimension.)

1. Preliminaries

We first state a (refined) version of Prime Avoidance. See [Ka, Theorem 124] or [Ei, Exercise 3.18].

Theorem 1.1 (Prime Avoidance)**.** *Let R be a commutative ring, $g, f_1, \ldots, f_l \in R$ and $P_1, \ldots, P_k \in \operatorname{Spec}(R)$. If $(g, f_1, \ldots, f_l) \not\subseteq P_i$ for all $1 \leq i \leq k$, then there exists $g' \in g+(f_1, \ldots, f_l)$ (that is, $g' = g+f$ for some $f \in (f_1, \ldots, f_l)$) such that $g' \notin \cup_{i=1}^k P_i$.*

Proof. This is a classic result. See [Ka, Theorem 124] for details. □

Theorem 1.2. *Let R be a Noetherian ring, $I \subseteq R$ an ideal and M an R-module. Suppose $\underline{x} = x_1, \ldots, x_d$ is any M-regular sequence in I.*

(1) Suppose $\underline{x}$ is also R-regular. Then, for any R-module N such that $\operatorname{Ann}_R(N) \supseteq I$, there is a natural isomorphism

$$\phi_N^i : \operatorname{Ext}_R^i(N, M) \cong \operatorname{Ext}_{R/(\underline{x})}^{i-d}(N, M/(\underline{x})M) \quad \text{for every } i \in \mathbb{Z}.$$

In particular, this shows $\operatorname{Ext}_R^i(N, M) = 0$ for all $i \leq d-1$ and there is a natural R-isomorphism $\phi_N^d : \operatorname{Ext}_R^d(N, M) \cong \operatorname{Hom}_{R/(\underline{x})}(N, M/(\underline{x})M)$, the latter of which may be naturally identified with $\operatorname{Hom}_R(N, M/(\underline{x})M)$.

(1') More generally, for any R-module N such that $\operatorname{Ann}_R(N) \supseteq I$, there is a natural isomorphism

$$\phi_N^i : \operatorname{Ext}_R^i(N, M) \cong \operatorname{Ext}_R^{i-d}(N, M/(\underline{x})M) \quad \text{for every } i \leq d.$$

In particular, this shows $\operatorname{Ext}_R^i(N, M) = 0$ for all $i \leq d-1$ and there is a natural R-isomorphism $\phi_N^d : \operatorname{Ext}_R^d(N, M) \cong \operatorname{Hom}_R(N, M/(\underline{x})M)$.

(2) Suppose M is finitely generated over R, $\operatorname{Ann}_R(M) + I \neq R$ and $\underline{x}$ is a maximal M-regular sequence in I. Then, for any finitely generated R-module N with $\operatorname{Ann}_R(N) = I$, we have $\operatorname{Ext}_R^d(N, M) \cong \operatorname{Hom}_R(N, M/(\underline{x})M) \neq 0$, which implies

$$d = \min\{i \mid \operatorname{Ext}_R^i(N, M) \neq 0\}.$$

Thus every maximal M-regular sequence in I has length d, which implies that $\operatorname{depth}(I, M) = d$.

Proof. See [Rees] for details. A proof for part (2) may also be found in [Ei, Proposition 18.4]. □

Next, we state some results concerning regular sequences. We make repeated use of these facts in the sequel. Lacking convenient references, we have included proofs in the lemma that follows. For part (4), which is more subtle, we have given complete details.

Lemma 1.3. *Let $\underline{x} = x_1, \ldots, x_r$ and $\underline{y} = y_1, \ldots, y_s$ be R-regular sequences.*

(1) Suppose $r = s$ and there is an element $u \in R$ that is regular on $R/(\underline{x})$. Then there exists $u' \in u + (\underline{x})$ such that u' is regular on both $R/(\underline{y})$ and $R/(\underline{x})$.

(2) If $r > s$, then any R-homomorphism $h : R/(\underline{x}) \to R/(\underline{y})$ vanishes (in other words, $\operatorname{Hom}_R(R/(\underline{x}), R/(\underline{y})) = 0$).

(3) There exists an R-regular sequence $\underline{z} = z_1, \ldots, z_l$ with $l = \max\{r, s\}$ such that $(\underline{z}_{[i]}) \subseteq (\underline{x}_{[i]})$ and $(\underline{z}_{[j]}) \subseteq (\underline{y}_{[j]})$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$.

(4) Assume that $r = s$ and $(\underline{x}) \supseteq (\underline{y})$. Say $y_j = \sum_{i=1}^r a_{ij} x_i$ for $j = 1, \ldots, r$. Let $A = (a_{ij})$ be the resulted $r \times r$ matrix and denote by $\delta = \det(A)$ the determinant of A. Then

(a) $(\underline{y}) :_R (\underline{x}) = (\underline{y}, \delta)$.

(b) $(\underline{y}) :_R \delta = (\underline{x})$, *which implies the following short exact sequence*

$$0 \longrightarrow R/(\underline{x}) \xrightarrow{\overline{\delta}} R/(\underline{y}) \longrightarrow R/(\underline{y}, \delta) \longrightarrow 0,$$

in which $\overline{\delta}$ *denotes the (well-defined)* R*-linear map sending* $1 + (\underline{x}) \in R/(\underline{x})$ *to* $\delta + (\underline{y}) \in R/(\underline{y})$. *Consequently, any element that is regular on* $R/(\underline{y})$ *is automatically regular on* $R/(\underline{x})$.

(c) Moreover, $\operatorname{pd}(R/(\underline{y}, \delta)) \le r$.

(d) Suppose, for some $u \in R$*, that* u *is regular on* $R/(\underline{y})$ *(and hence on* $R/(\underline{x})$*). Then* $(\underline{y}, u\delta) = (\underline{y}, u) \cap (\underline{y}, \delta)$*, which gives the following short exact sequence*

$$0 \longrightarrow R/(\underline{y}, u\delta) \longrightarrow R/(\underline{y}, u) \oplus R/(\underline{y}, \delta) \longrightarrow R/(\underline{y}, u, \delta) \longrightarrow 0,$$

with $\operatorname{pd}(R/(\underline{y}, \delta)) \le r$ *and* $\operatorname{pd}(R/(\underline{y}, u, \delta)) \le r + 1$.

Proof. (1). If $\underline{y}$ is improper, then this is trivially true. Assume $\underline{y}$ is proper. Then $(\underline{x}, u) \nsubseteq P$ for any $P \in \operatorname{Ass}(R/(\underline{y}))$. (Otherwise, suppose $(\underline{x}, u) \subseteq P$ for some $P \in \operatorname{Ass}(R/(\underline{y}))$. Then $y_1/1, \ldots, y_s/1$ form a maximal proper R_P-regular sequence (of length s). But, on the other hand, $x_1/1, \ldots, x_r/1, u/1$ would form a proper R_P-regular sequence of length $r+1 = s+1$, a contradiction to Theorem 1.2(2).) By prime avoidance (cf. Theorem 1.1), there exists $u' \in u + (\underline{x})$ such that $u' \notin \cup_{P \in \operatorname{Ass}(R/(\underline{y}))} P$. In other words, u' is regular on $R/(\underline{y})$. Obviously, u' remains regular on $R/(\underline{x})$.

(2). By part (1), there exists $x'_{s+1} \in x_{s+1} + (\underline{x}_{[s]}) \subseteq (\underline{x})$ such that x'_{s+1} is regular on $R/(\underline{y})$. Then, for any R-homomorphism $h : R/(\underline{x}) \to R/(\underline{y})$ and any $w \in R/(\underline{x})$, we have $x'_{s+1} h(w) = h(x'_{s+1} w) = h(0) = 0$, which forces $h(w) = 0$, which shows that h is a zero map.

(3). First observe that if $r \ne s$, say $r > s$, then we may extend $y_1, \ldots, y_s$ to an R-regular sequence $y_1, \ldots, y_s, y_{s+1}, \ldots, y_r$ with $y_{s+1} = \cdots = y_r = 1$. Therefore, without loss of generality, we assume $r = s$. We inductively construct the desired $z_1, \ldots, z_r$ as follows: Let $z_1 = x_1 y_1$, which is evidently R-regular and satisfying $(z_1) \subseteq (x_1) \cap (y_1)$. Suppose that, for some integer k with $1 \le k < r$, we have constructed an R-regular sequence $z_1, \ldots, z_k$ such that $(z_1, \ldots, z_i) \subseteq (x_1, \ldots, x_i) \cap (y_1, \ldots, y_i)$ for all $i = 1, \ldots, k$ as desired. Then, by Part (1) above, there exist $x'_{k+1} \in x_{k+1} + (\underline{x}_{[k]})$ and $y'_{k+1} \in y_{k+1} + (\underline{y}_{[k]})$ such that both x'_{k+1} and y'_{k+1} are regular on $R/(z_1, \ldots, z_k)$. Let $z_{k+1} = x'_{k+1} y'_{k+1}$. Then, clearly, $z_1, \ldots, z_k, z_{k+1}$ form an R-regular sequence such that $(z_1, \ldots, z_i) \subseteq (x_1, \ldots, x_i) \cap (y_1, \ldots, y_i)$ for all $i = 1, \ldots, k, k+1$ as desired. In particular, when $k = r = s$, we are able to construct an R-regular sequence $z_1, \ldots, z_r$ such that $(z_1, \ldots, z_i) \subseteq (x_1, \ldots, x_i) \cap (y_1, \ldots, y_i)$ for all $i = 1, \ldots, r = s$ as desired.

It remains to prove (4). First of all, the fact that $(x_1, \ldots, x_r)A = (y_1, \ldots, y_r)$ implies that $(\underline{x})\delta \subseteq (\underline{y})$. Thus, if $\underline{x}$ is improper (i.e. $(\underline{x}) = R$), then $\delta \in (\underline{y})$ and all the claims in (4) are trivial. So we assume $R \supsetneq (\underline{x}) \supseteq (\underline{y})$ from now on.

Let $K_\bullet(\underline{x}, R)$ and $K_\bullet(\underline{y}, R)$ be the Koszul complexes of $\underline{x}$ and $\underline{y}$ respectively. As $\underline{x}$ and $\underline{y}$ are R-regular sequences, we see that $K_\bullet(\underline{x}, R)$ and $K_\bullet(\underline{y}, R)$ are free resolutions of $R/(\underline{x})$ and $R/(\underline{y})$ respectively. This implies that $\operatorname{pd}(R/(\underline{x})) = \operatorname{pd}(R/(\underline{y})) = r$, which implies that $\operatorname{pd}((\underline{x})/(\underline{y})) \leq r$ because of the short exact sequence

$$0 \longrightarrow (\underline{x})/(\underline{y}) \xrightarrow{\iota} R/(\underline{y}) \xrightarrow{\pi} R/(\underline{x}) \longrightarrow 0,$$

in which ι and π denote the natural inclusion and surjection homomorphisms respectively. The matrix A induces an R-linear map $h_\bullet : K_\bullet(\underline{y}, R) \to K_\bullet(\underline{x}, R)$ lifting $\pi : R/(\underline{y}) \to R/(\underline{x})$ so that $h_0 : R \to R$ and $h_r : R \to R$ are multiplications by 1 and $\delta = \det(A)$ respectively. As $\operatorname{Ann}((\underline{x})/(\underline{y})) \supseteq (\underline{y})$, Theorem 1.2 implies that $\operatorname{Ext}^i_R((\underline{x})/(\underline{y}), R) = 0$ for all $i = 0, \ldots, r-1$. Therefore there is a short exact sequence

$$0 \longrightarrow \operatorname{Ext}^r_R(R/(\underline{x}), R) \xrightarrow{\pi^*} \operatorname{Ext}^r_R(R/(\underline{y}), R) \xrightarrow{\iota^*} \operatorname{Ext}^r_R((\underline{x})/(\underline{y}), R) \longrightarrow 0,$$

in which $\iota^* = \operatorname{Ext}^r_R(\iota, R)$ and $\pi^* = \operatorname{Ext}^r_R(\pi, R)$ are the R-linear homomorphisms naturally induced by $(\underline{x})/(\underline{y}) \xrightarrow{\iota} R/(\underline{y})$ and $R/(\underline{y}) \xrightarrow{\pi} R/(\underline{x})$.

Consider the following diagram (with all the unlabeled maps being natural)

$$(\star)\qquad \begin{tikzcd}
0 \arrow[r] & ((\underline{y}):(\underline{x}))/(\underline{y}) \arrow[r] & R/(\underline{y}) \arrow[r] & R/((\underline{y}):(\underline{x})) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Hom}_R\left(\frac{R}{(\underline{x})}, \frac{R}{(\underline{y})}\right) \arrow[r] \arrow[u, "\cong"'] & \operatorname{Hom}_R\left(\frac{R}{(\underline{y})}, \frac{R}{(\underline{y})}\right) \arrow[r] \arrow[u, "\cong"'] & \operatorname{Hom}_R\left(\frac{(\underline{x})}{(\underline{y})}, \frac{R}{(\underline{y})}\right) \arrow[r, dotted] \arrow[u, dotted, "\cong"'] & 0 \\
0 \arrow[r] & \operatorname{Ext}^r_R\left(\frac{R}{(\underline{x})}, R\right) \arrow[r, "\pi^*"] \arrow[u, "\phi^r_{R/(\underline{x})}", "\cong"'] \arrow[d, "\psi_1"', "\cong"] & \operatorname{Ext}^r_R\left(\frac{R}{(\underline{y})}, R\right) \arrow[r, "\iota^*"] \arrow[u, "\phi^r_{R/(\underline{y})}", "\cong"'] \arrow[d, "\psi_2"', "\cong"] & \operatorname{Ext}^r_R\left(\frac{(\underline{x})}{(\underline{y})}, R\right) \arrow[r] \arrow[u, "\phi^r_{(\underline{x})/(\underline{y})}", "\cong"'] \arrow[d, dotted, "\cong"] & 0 \\
0 \arrow[r, dotted] & R/(\underline{x}) \arrow[r, "\overline{\delta}"] & R/(\underline{y}) \arrow[r] & R/(\underline{y}, \delta) \arrow[r] & 0
\end{tikzcd},$$

in which $\phi^r_{R/(\underline{x})}$, $\phi^r_{R/(\underline{y})}$ and $\phi^r_{(\underline{x})/(\underline{y})}$ are the natural isomorphisms as in Theorem 1.2(1) while the isomorphisms ψ_i (for $i = 1, 2$) are simply realizations of $\operatorname{Ext}^r_R(R/(\underline{x}), R)$ and $\operatorname{Ext}^r_R(R/(\underline{y}), R)$ as $H^r(\operatorname{Hom}(K_\bullet(\underline{x}, R), R))$ and $H^r(\operatorname{Hom}(K_\bullet(\underline{y}, R), R))$ respectively. As all the maps are natural, we see that $(\star)$ is a commutative diagram. As a result, all the rows in $(\star)$ are short exact sequences and all the dotted vertical arrows in $(\star)$ are isomorphisms.

(4)(a). This follows from the isomorphism $R/((\underline{y}) : (\underline{x})) \xrightarrow{\cong} R/(\underline{y}, \delta)$.

(4)(b). This is forced by the fact that $\overline{\delta} : R/(\underline{x}) \to R/(\underline{y})$ is injective.

(4)(c). Fix a resolution $F_\bullet$ of $(\underline{x})/(\underline{y})$ by finitely generated projective R-modules and assume the length of $F_\bullet$ is $\operatorname{pd}((\underline{x})/(\underline{y})) \leq r$. Then, making use of the fact that $\operatorname{Ext}^i_R((\underline{x})/(\underline{y}), R) = 0$ for all $i = 0, \ldots, r-1$ (cf. Theorem 1.2), we conclude that $\operatorname{Hom}_R(F_\bullet, R)$ constitutes a projective resolution of $\operatorname{Ext}^r_R((\underline{x})/(\underline{y}), R) \cong R/(\underline{y}, \delta)$ of length $\leq r$.

(4)(d). Let $A' = \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\det(A') = \det(A) = \delta$ and $(x_1, \ldots, x_k, u)A' = (y_1, \ldots, y_k, u)$. For any $v \in (\underline{y}, u) \cap (\underline{y}, \delta)$, write $v = y + c_1 u = y' + c_2\delta$ for $y, y' \in (\underline{y})$ and $c_1, c_2 \in R$. Thus $c_2\delta = y - y' + c_1 u \in (\underline{y}, u)$, which implies that $c_2 \in (\underline{y}, u) : \delta = (\underline{y}, u) : \det(A') = (\underline{x}, u)$ by the above part (b) applied to the regular sequences $\underline{x}, u$ and $\underline{y}, u$. Therefore $c_2\delta \in (\underline{x}, u)\delta \subseteq (\underline{y}, u\delta)$ by part (a) above and, hence, $v = y' + c_2 \in (\underline{y}, u\delta)$.

This proves that $(\underline{y}, u) \cap (\underline{y}, \delta) \subseteq (\underline{y}, u\delta)$. As the reverse inclusion is obvious, we conclude that $(\underline{y}, u\delta) = (\underline{y}, u) \cap (\underline{y}, \delta)$. Then the claimed exact sequence simply follows from the more general exact sequence

$$0 \longrightarrow R/(I \cap J) \longrightarrow R/I \oplus R/J \longrightarrow R/(I + J) \longrightarrow 0$$

for any ideals $I, J \subseteq R$. Finally, the claim that $\mathrm{pd}(R/(\underline{y}, u, \delta)) \leq r + 1$ follows from the part (c) above applied to the regular sequences $\underline{x}, u$ and $\underline{y}, u$, both of length $r+1$, while the claim $\mathrm{pd}(R/(\underline{y}, \delta)) \leq r$ has been proved in part (c) already. □

Lemma 1.4. *Let X, Y_1 and Y_2 be R-modules and $\phi_i : X \to Y_i$ be R-homomorphisms for $i = 1, 2$. Let $Z' := \{(\phi_1(x), \phi_2(x)) \mid x \in X\} \subseteq Y_1 \oplus Y_2$ and let $Z := (Y_1 \oplus Y_2)/Z'$. Then there are naturally induced R-linear maps $\psi_i : Y_i \to Z$ for $i = 1, 2$. The R-module Z is called the* push-out *of the maps $X \xrightarrow{\phi_i} Y$ for $i = 1, 2$. There is a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & Y_1 \arrow[r] \arrow[d, "\psi_1"] & N_1 \arrow[r] \arrow[d, equal, "\mathrm{id}"] & 0 \\
Y_2 \arrow[r, "\psi_2"'] \arrow[d] & Z \arrow[r] \arrow[d] & N_1 \arrow[r] & 0 \\
N_2 \arrow[r, equal, "\mathrm{id}"'] \arrow[d] & N_2 \arrow[d] & & \\
0 & 0 & &
\end{tikzcd}$$

in which $N_i := \mathrm{Coker}(\phi_i)$ for $i = 1, 2$, all the unlabeled maps are naturally induced R-homomorphisms and the following properties hold:

(1) All the rows and columns are exact.
(2) Furthermore, $\phi_1(\mathrm{Ker}(\phi_2)) = \mathrm{Ker}(\psi_1)$ and $\phi_2(\mathrm{Ker}(\phi_1)) = \mathrm{Ker}(\psi_2)$.

Proof. This is well-known and straightforward. □

Lemma 1.5. *Let M_1, M_2 and M_3 be R-modules, $\phi \in \mathrm{Hom}_R(M_1, M_2)$ and $\psi \in \mathrm{Hom}_R(M_2, M_3)$ such that $\mathrm{Ker}(\psi) = 0$. Then there exists a short exact sequence*

$$0 \longrightarrow \mathrm{Coker}(\phi) \longrightarrow \mathrm{Coker}(\psi \circ \phi) \longrightarrow \mathrm{Coker}(\psi) \longrightarrow 0.$$

Proof. This is simply the short exact sequence

$$0 \longrightarrow M_2/\phi(M_1) \longrightarrow M_3/\psi(\phi(M_1)) \longrightarrow M_3/\psi(M_2) \longrightarrow 0,$$

which should follow immediately from the injectivity of ψ. □

Next we review the notion of G-dimension, which was first introduced by Auslander and Bridger in [AB]. For a modern treatment of this subject, see [Ch]. For any R-module M, denote $M^* := \mathrm{Hom}_R(M, R)$.

Definition 1.6 ([AB])**.** Let R be a Noetherian ring and M a finitely generated R-module.

(1) We say M is totally reflexive if $M \cong M^{**}$ under the natural map and $\mathrm{Ext}^i_R(M \oplus M^*, R) = 0$ for all $i \geq 1$.
(2) The G-dimension of M, denoted $\text{G-dim}_R(M)$ (or $\text{G-dim}(M)$ if R is understood), is the minimal n such that there is an exact sequence

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

in which M_i is totally reflexive for all $i = 0, 1, \ldots, n$.

Remark 1.7. (1) It is clear that $\text{G-dim}(M) = 0$ if and only if M is totally reflexive. If this is the case, then from the definition we see that M^* is also totally reflexive (as $M^{***} \cong M^*$).
(2) The G-dimension behaves in a very similar way as the projective dimension. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of finitely generated R-modules. Then
  (a) $\text{G-dim}(M_1) \leq \max\{\text{G-dim}(M_2), \text{G-dim}(M_3) - 1\}$,
  (b) $\text{G-dim}(M_2) \leq \max\{\text{G-dim}(M_1), \text{G-dim}(M_3)\}$ and
  (c) $\text{G-dim}(M_3) \leq \max\{\text{G-dim}(M_1) + 1, \text{G-dim}(M_2)\}$.
(3) For any finitely generated R-module M with $\mathrm{pd}(M) < \infty$, it is forced that $\text{G-dim}(M) = \mathrm{pd}(M)$. (This may follow from the fact that, if $M \neq 0$ and $\text{G-dim}(M) < \infty$, then $\text{G-dim}(M) = \max\{n \mid \mathrm{Ext}^n(R, M) \neq 0\}$. See [Ch, 1.2.7].)

2. Embedding modules of finite projective dimension or finite G-dimension

Notation 2.1. Let R be a Noetherian ring and M a finitely generated R-module with $\text{G-dim}(M) < \infty$. We say that a finitely generated R-module M_1 is a *permissible* extension of M if there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow M_1 \longrightarrow N \longrightarrow 0$$

such that $\text{G-dim}(N) \leq \text{G-dim}(M)$.

Lemma 2.2. *Let R be a Noetherian ring and M a finitely generated R-module with* $\text{G-dim}(M) < \infty$*.*

(1) If M_1 is a permissible extension of M, then $\text{G-dim}(M_1) = \text{G-dim}(M)$*.*
(2) Therefore, a permissible extension of a permissible extension of M is again a permissible extension of M.
(3) Suppose we have an exact sequence $X \xrightarrow{\phi} Y \to M \to 0$ (i.e. $M = \mathrm{Coker}(\phi)$) and $\psi : Y \hookrightarrow Y'$ is an injective map such that $\text{G-dim}(\mathrm{Coker}(\psi)) \leq \text{G-dim}(M)$*. Then* $\mathrm{Coker}(X \xrightarrow{\psi\circ\phi} Y')$ *is a permissible extension of M.*

Proof. All the claims are straightforward. See Remark 1.7(2). In particular, (3) follows immediately from Lemma 1.5. □

Now we are ready to state and prove the embedding theorem for finitely generated modules of finite projective dimension (or finite G-dimension).

Theorem 2.3. *Let R be a Noetherian ring and M a finitely generated R-module. Then*

(1) *If* $\text{G-dim}(M) = r < \infty$*, then there exist an R-regular sequence $\underline{z} = z_1, \ldots, z_r$, integers $n_0, n_1, \ldots, n_r \geq 0$, and a short exact sequence*
$$0 \longrightarrow M \longrightarrow Z \longrightarrow N \longrightarrow 0$$
with $Z = \oplus_{i=0}^r (R/(\underline{z}_{[i]}))^{n_i}$ and $\text{G-dim}(N) \leq r$*.*

(2) *If* $\text{pd}(M) = r < \infty$*, then there exist an R-regular sequence $\underline{z} = z_1, \ldots, z_r$, integers $n_0, n_1, \ldots, n_r \geq 0$, and a short exact sequence*
$$0 \longrightarrow M \longrightarrow Z \longrightarrow N \longrightarrow 0$$
with $Z = \oplus_{i=0}^r (R/(\underline{z}_{[i]}))^{n_i}$ and $\text{pd}(N) \leq r$*.*

(3) *In (1) and (2) above, it is forced that $\underline{z}$ is a proper R-regular sequence and, in case $M \neq 0$, $n_r > 0$.*

Proof. First, we notice that (2) is an immediate consequence of (1). Indeed, if $\text{pd}(M) = r$, then $\text{G-dim}(M) = r$ by Remark 1.7(3) and the claim in (1) would give an embedding as stated in (1) with $\text{G-dim}(N) \leq r$. Then, as $\text{pd}(M) + \text{pd}(Z) < \infty$, we see that $\text{pd}(N) < \infty$, which forces $\text{pd}(N) = \text{G-dim}(N) \leq r$.

Once (1) and (2) are proved, we observe that (3) always holds. (To show this, we assume that (1) holds without loss of generality (cf. Remark 1.7(3)). Evidently, claim (3) holds when $\text{G-dim}(M) = r = 0$. Assume that $\text{G-dim}(M) = r > 0$ and, on the contrary, suppose that claim (1) occurs with $(\underline{z}) = R$ or $n_r = 0$. Then we have the following short exact sequence
$$0 \longrightarrow M \longrightarrow Z \longrightarrow N \longrightarrow 0$$
with $Z = \oplus_{i=0}^{r-1} (R/(\underline{z}_{[i]}))^{n_i}$ and $\text{G-dim}(N) \leq r$. But, then, the above short exact sequence would force $\text{G-dim}(M) < r$ (cf. Remark 1.7(2)), a contradiction.)

Therefore, it remains to prove (1). Before starting the proof, we make some easy observations.

Observation 2.3.1. In order to prove claim (1) for M with $\text{G-dim}(M) = r$, it suffices to prove the same claim for a single permissible extension of M. Indeed, suppose that claim (1) is proved for a permissible extension M_1 of M. That is, we have short exact sequences
$$0 \longrightarrow M \xrightarrow{f} M_1 \longrightarrow N \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow M_1 \xrightarrow{g} Z \longrightarrow N_1 \longrightarrow 0,$$
for some R-regular sequence $\underline{z} = z_1, \ldots, z_r$, non-negative integers $n_0, n_1, \ldots, n_r$ such that $Z = \oplus_{i=0}^r (R/(\underline{z}_{[i]}))^{n_i}$, $\text{G-dim}(N) \leq r$ and $\text{G-dim}(N_1) \leq r$. Then, denoting $N' = \text{Coker}(g \circ f)$, we have short exact sequences
$$0 \longrightarrow M \xrightarrow{g \circ f} Z \longrightarrow N' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N \longrightarrow N' \longrightarrow N_1 \longrightarrow 0,$$
the latter of which follows from Lemma 1.5 and implies that $\text{G-dim}(N') \leq r$.

Observation 2.3.2. Suppose $M \cong M' \oplus M''$. In order to prove claim (1) for M, it suffices to prove the same claim for both M' and M''. Say $\text{G-dim}(M') = \text{G-dim}(M) = r$ and $\text{G-dim}(M'') = s \leq r$. Suppose there exist R-regular sequences $\underline{z}' = z'_1, \ldots, z'_r$

and $\underline{z}'' = z''_1, \dots, z''_s$, non-negative integers $m_0, m_1, \dots, m_r, n_0, n_1, \dots, n_s$, and short exact sequences (with $Z' = \oplus_{i=0}^r (R/(\underline{z}'_{[i]}))^{m_i}$ and $Z'' = \oplus_{i=0}^s (R/(\underline{z}''_{[i]}))^{n_i}$)

$$0 \longrightarrow M' \xrightarrow{f'} Z' \longrightarrow N' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow M'' \xrightarrow{f''} Z'' \longrightarrow N'' \longrightarrow 0$$

such that G-dim$(N') \leq r$ and G-dim$(N'') \leq s \leq r$. Then, by Lemma 1.3(3), there exists an R-regular sequence $\underline{z} = z_1, \dots, z_r$ such that $(\underline{z}_{[i]}) \subseteq (\underline{z}'_{[i]})$ and $(\underline{z}_{[j]}) \subseteq (\underline{z}''_{[j]})$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$. Applying Lemma 1.3(4)(b), we get exact sequences

$$0 \longrightarrow \oplus_{i=0}^r (R/(\underline{z}'_{[i]}))^{m_i} \xrightarrow{g'} \oplus_{i=0}^r (R/(\underline{z}_{[i]}))^{m_i} \longrightarrow L' \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow \oplus_{i=0}^s (R/(\underline{z}''_{[i]}))^{n_i} \xrightarrow{g''} \oplus_{i=0}^s (R/(\underline{z}_{[i]}))^{n_i} \longrightarrow L'' \longrightarrow 0$$

with pd$(L') \leq r$ and pd$(L'') \leq s$. Thus there are short exact sequences

$$0 \longrightarrow M' \xrightarrow{g' \circ f'} \oplus_{i=0}^r (R/(\underline{z}_{[i]}))^{m_i} \longrightarrow K' \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow M'' \xrightarrow{g'' \circ f''} \oplus_{i=0}^s (R/(\underline{z}_{[i]}))^{n_i} \longrightarrow K'' \longrightarrow 0$$

with $K' := \operatorname{Coker}(g' \circ f')$ and $K'' := \operatorname{Coker}(g'' \circ f'')$. From Lemma 1.5, we see that G-dim$(K') \leq r$ and G-dim$(K'') \leq s$. Consequently, we get a short exact sequence

$$0 \longrightarrow M' \oplus M'' \xrightarrow{(g' \circ f') \oplus (g'' \circ f'')} \oplus_{i=0}^r (R/(\underline{z}_{[i]}))^{m_i + n_i} \longrightarrow K' \oplus K'' \longrightarrow 0,$$

in which we agree that $n_i = 0$ for all $i > s$. This verifies the desired claim (1) for $M \cong M' \oplus M''$ as G-dim$(K' \oplus K'') \leq r =$ G-dim(M).

Now we proceed to prove (1) by induction on G-dim(M). If G-dim$(M) = 0$, then both M and M^* are totally reflexive (cf. Remark 1.7(1). Fix a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow F \longrightarrow M^* \longrightarrow 0,$$

in which F is a free R-module of finite rank. Then G-dim$(M_1) = 0$ and hence G-dim$(M_1^*) = 0$. Applying $\operatorname{Hom}(\_, R)$ to the above short exact sequence and noting the fact that $\operatorname{Ext}_R^1(M^*, R) = 0$, we get a short exact sequence

$$0 \longrightarrow M \longrightarrow F^* \longrightarrow M_1^* \longrightarrow 0,$$

which proves claim (1) (and hence claim (2)). (In fact, when pd$(M) = 0$, claim (2) is clear.)

Now, we assume that, for some $r \geq 0$, the claim (1) holds when the G-dimension is $\leq r$. For any finitely generated R-module M with G-dim$(M) = r + 1$, it suffices to verify claim (1) on M in order to complete the induction step.

There is a short exact sequence

$$0 \longrightarrow V_1 \xrightarrow{\eta_1} F \longrightarrow M \longrightarrow 0,$$

in which F is a finitely generated projective (e.g., free) R-module. Thus G-dim$(V_1) = r, V_1 \neq 0$ and, by the induction hypothesis, there exists a proper R-regular sequence $\underline{x} = x_1, \dots, x_r$, non-negative integers $m_0, \dots, m_r$ with $m_r > 0$, and a short exact sequence

$$0 \longrightarrow V_1 \xrightarrow{\eta_2} X \longrightarrow W_1 \longrightarrow 0$$

such that $X = \oplus_{i=0}^r (R/(\underline{x}_{[i]}))^{m_i}$ and G-dim$(W_1) \leq r$.

Let V_2 be the push-out of $V_1 \overset{\eta_1}{\hookrightarrow} F$ and $V_1 \overset{\eta_2}{\hookrightarrow} X$. By Lemma 1.4, there are short exact sequences

$$0 \longrightarrow X \xrightarrow{\phi} V_2 \longrightarrow M \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F \longrightarrow V_2 \longrightarrow W_1 \longrightarrow 0,$$

the latter of which implies that $\text{G-dim}(V_2) \leq r$. Denote $X' = (R/(\underline{x}))^{m_r}$ and $X'' = \oplus_{i=0}^{r-1}(R/(\underline{x}_{[i]}))^{m_i}$ so that $X = X' \oplus X''$. Also denote $V = \text{Coker}(\phi|_{X''})$ where $\phi|_{X''} : X'' \to V_2$ is the restriction of ϕ to X''. Then there are short exact sequences

$$0 \longrightarrow X'' \xrightarrow{\phi|_{X''}} V_2 \xrightarrow{\pi} V \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow X' \xrightarrow{\pi\circ\phi|_{X'}} V \longrightarrow M \longrightarrow 0.$$

As $\text{G-dim}(X'') \leq r-1$ and $\text{G-dim}(V_2) \leq r$, we see that $\text{G-dim}(V) \leq r$. Say $\text{G-dim}(V) = s$. (We will see shortly that $s = r$.) By the induction hypothesis, there exists a proper R-regular sequence $\underline{y} = y_1, \ldots, y_s$, non-negative integers $n_0, \ldots, n_s$, and a short exact sequence

$$0 \longrightarrow V \xrightarrow{\psi} Y \longrightarrow W_2 \longrightarrow 0$$

such that $Y = \oplus_{i=0}^{s}(R/(\underline{y}_{[i]}))^{n_i}$ and $\text{G-dim}(W_2) \leq s < \text{G-dim}(M)$. Hence, by Lemma 2.2(3), $\text{Coker}(\psi \circ \pi \circ \phi|_{X'})$ is a permissible extension of M. Therefore, by Observation 2.3.1, we may, without loss of generality, assume that M appears in a short exact sequence of the following form

$$0 \longrightarrow (R/(\underline{x}))^{m_r} \xrightarrow{\varphi} \oplus_{i=0}^{s}(R/(\underline{y}_{[i]}))^{n_i} \longrightarrow M \longrightarrow 0,$$

in which $\varphi = \psi \circ \pi \circ \phi|_{X'}$ while $\underline{x}$, $\underline{y}$, $m_r > 0$ and n_i $(0 \leq i \leq s)$ are as described above. Notice that, by Lemma 1.3(2), $\text{Hom}((R/(\underline{x}))^{m_r}, (R/(\underline{y}_{[i]}))^{n_i}) = 0$ for all $i = 0, 1, \ldots, \min\{r-1, s\}$. At this point, it is easy to see that $r = s$. (Otherwise, we would have $r > s$, which would mean $\varphi = 0$, a contradiction to the fact that φ is injective and $(R/(\underline{x}))^{m_r} \neq 0$.) Moreover, we may think of φ as an injective R-linear map in $\text{Hom}((R/(\underline{x}))^{m_r}, (R/(\underline{y}))^{n_r})$ and, consequently, we have the following isomorphism

$$M \cong \oplus_{i=0}^{r-1}(R/(\underline{y}_{[i]}))^{n_i} \bigoplus \text{Coker}\left((R/(\underline{x}))^{m_r} \xrightarrow{\varphi} (R/(\underline{y}))^{n_r}\right).$$

As the desired claim obviously holds for $\oplus_{i=0}^{r-1}(R/(\underline{y}_{[i]}))^{n_i}$, it suffices to prove it for $\text{Coker}\left((R/(\underline{x}))^{m_r} \xrightarrow{\varphi} (R/(\underline{y}))^{n_r}\right)$. Moreover, by replacing $\underline{y}$ with a suitable R-regular sequence of length r, we may further assume that $(\underline{y}) \subseteq (\underline{x})$ without loss of generality (cf. Lemma 1.3(3), Lemma 1.3(4)(b,c), Lemma 2.2(3) and Observation 2.3.1, etcetera; details omitted). Therefore, without loss of generality, we may simply assume that M occurs in a short exact sequence of the following form

$$(*) \qquad 0 \longrightarrow (R/(\underline{x}))^{m} \xrightarrow{\varphi} (R/(\underline{y}))^{n} \longrightarrow M \longrightarrow 0,$$

in which $m = m_r > 0$, $n = n_r > 0$ while both $\underline{x} = x_1, \ldots, x_r$ and $\underline{y} = y_1, \ldots, y_r$ are proper R-regular sequences of length r such that $(\underline{y}) \subseteq (\underline{x})$.

Next, we will set up some notations that we will use throughout the remainder of the proof.

Notation 2.3.3. From now on, we let $\underline{x}$ and $\underline{y}$ be as in $(*)$. In particular, we assume $(\underline{y}) \subseteq (\underline{x})$. Say $y_j = \sum_{i=1}^r a_{ij}x_i$ for $j = 1, \ldots, r$ and let $\delta = \det(a_{ij})$ be the determinant of the $r \times r$ matrix (a_{ij}). By Lemma 1.3(4), $(\underline{y}) : (\underline{x}) = (\underline{y}, \delta)$ and $(\underline{y}) : \delta = (\underline{x})$. Also, we refresh our notations and denote $(R/(\underline{x}))^m = X$ and $(R/(\underline{y}))^n = Y$. Thus, for any $\varphi \in \operatorname{Hom}(X, Y)$, $\operatorname{Image}(\varphi) \subseteq \delta Y$. In fact, it is easy to see that $\delta Y \cong (R/(\underline{x}))^n$ in light of Lemma 1.3(4)(b). Choose $\underline{b} = b_1, \ldots, b_m \in X$ and $\underline{e} = e_1, \ldots, e_n \in Y$ such that they form bases for X and Y over $R/(\underline{x})$ and $R/(\underline{y})$ respectively. Given $\varphi \in \operatorname{Hom}(X, Y)$, we may write $\varphi(b_j) = \sum_{i=1}^n \delta f_{i,j} e_i$ in which $f_{i,j} \in R$ for all $i = 1, \ldots, n$ and all $j = 1, \ldots, m$. This produces a $n \times m$ matrix

$$\delta \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,m-1} & f_{1,m} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,m-1} & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n-1,1} & f_{n-1,2} & \cdots & f_{n-1,m-1} & f_{n-1,m} \\ f_{n,1} & f_{n,2} & \cdots & f_{n,,m-1} & f_{n,m} \end{pmatrix} = \delta(f_{i,j}) = (\delta f_{i,j}),$$

which we will use to represent φ. Evidently, the entries $f_{i,j}$ depend on the choices of $\underline{b} = b_1, \ldots, b_m \in X$ and $\underline{e} = e_1, \ldots, e_n \in Y$. However, once $\underline{b} = b_1, \ldots, b_m$ and $\underline{e} = e_1, \ldots, e_n$ are given, then the entries $f_{i,j}$, modulo $(\underline{x})$, are uniquely determined by φ and vice versa.

Here we make another observation. We remark that this kind of observation had been made and utilized in, for example, [Du] and [Sm].

Observation 2.3.4. Let $M = \operatorname{Coker}\left((R/(\underline{x}))^m \xrightarrow{\varphi} (R/(\underline{y}))^n\right)$ be as in $(*)$ and keep the notations as in Notation 2.3.3. Suppose that $u_1, \ldots, u_n$ are non-zero-divisors over $R/(\underline{y}))$. Let $\xi : Y \to Y$ be an R-linear map defined by $e_i \mapsto u_i e_i$ for $i = 1, \ldots, n$. Then ξ is injective and $\operatorname{pd}(\operatorname{Coker}(\xi)) = \operatorname{pd}(\oplus_{i=1}^n R/(\underline{y}, u_i)) \leq r+1$. Hence $\operatorname{Coker}(\xi \circ \varphi)$ is a permissible extension of M by Lemma 2.2(3). Therefore, to prove the claim (1) for M, it suffices to prove the same claim for $\operatorname{Coker}(\xi \circ \varphi)$. Finally, we observe that $\xi \circ \varphi \in \operatorname{Hom}(X, Y)$ is represented by the following matrix

$$\delta \begin{pmatrix} u_1 f_{1,1} & u_1 f_{1,2} & \cdots & u_1 f_{1,m-1} & u_1 f_{1,m} \\ u_2 f_{2,1} & u_2 f_{2,2} & \cdots & u_2 f_{2,m-1} & u_2 f_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{n-1} f_{n-1,1} & u_{n-1} f_{n-1,2} & \cdots & u_{n-1} f_{n-1,m-1} & u_{n-1} f_{n-1,m} \\ u_n f_{n,1} & u_n f_{n,2} & \cdots & u_n f_{n,,m-1} & u_n f_{n,m} \end{pmatrix} = (\delta u_i f_{i,j}).$$

We continue to prove Theorem 2.3 for M by induction. For our module $M = \operatorname{Coker}(X \xrightarrow{\varphi} Y)$ as in $(*)$ and Notation 2.3.3, our next goal is to prove the following

Claim 2.3.5. *For each $k = 1, \dots, m$, there exists an injective R-linear map $\varphi_k \in \operatorname{Hom}(X, Y)$ such that $M_k := \operatorname{Coker}(\varphi_k)$ is a permissible extension of M and, moreover, φ_k can be represented by a $n \times m$ matrix of the following form*

$$(\dagger) \qquad \delta \begin{pmatrix} u_k & 0 & \dots & 0 & 0 & f_{1,k+1} & f_{1,k+2} & \cdots & f_{1,m-1} & f_{1,m} \\ 0 & u_k & \dots & 0 & 0 & f_{2,k+1} & f_{2,k+2} & \cdots & f_{2,m-1} & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & u_k & 0 & f_{k-1,k+1} & f_{k-1,k+2} & \cdots & f_{k-1,m-1} & f_{k-1,m} \\ 0 & 0 & \dots & 0 & u_k & f_{k,k+1} & f_{k,k+2} & \cdots & f_{k,m-1} & f_{k,m} \\ 0 & 0 & \dots & 0 & 0 & f_{k+1,k+1} & f_{k+1,k+2} & \cdots & f_{k+1,m-1} & f_{k+1,m} \\ 0 & 0 & \dots & 0 & 0 & f_{k+2,k+1} & f_{k+2,k+2} & \cdots & f_{k+2,m-1} & f_{k+2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & f_{n-1,k+1} & f_{n-1,k+2} & \cdots & f_{n-1,m-1} & f_{n-1,m} \\ 0 & 0 & \dots & 0 & 0 & f_{n,k+1} & f_{n,k+2} & \cdots & f_{n,m-1} & f_{n,m} \end{pmatrix},$$

in which $f_{i,j} = 0$ whenever $i \neq j \leq k$ and, moreover, $f_{1,1} = \cdots = f_{k,k} = u_k$ is regular on both $R/(\underline{x})$ and $R/(\underline{y})$. (When $k = m$, the result also implies $m \leq n$.)

We are going to prove Claim 2.3.5 by induction on k. Here we think of $k = 0$ as the initial case, which is deemed trivially true. Now assume Claim 2.3.5 is proved for certain $k \geq 0$. We need to prove the claim for $k + 1$. (The induction step from $k = 0$ to $k = 1$ indeed produces a proof of Claim 2.3.5 in the case of $k = 1$. So what we are doing is a valid induction after all.)

The induction hypothesis gives an injective R-linear map φ_k represented by a matrix as in $(\dagger)$ with all the desired properties. Let $I := (f_{k+1,k+1}, f_{k+2,k+1}, \dots, f_{n,k+1})$. First, we show that $I \not\subseteq P$ for any $P \in \operatorname{Ass}(R/(\underline{x}))$. Suppose, on the contrary, that $I \subseteq P$ for some $P \in \operatorname{Ass}(R/(\underline{x}))$. Then there exists $0 \neq \theta \in Rb_{k+1} \cong R/(\underline{x})$ such that $\operatorname{Ann}_R(\theta) = P$. This forces $\varphi_k(\theta) \in \oplus_{i=1}^k \delta Re_i$ and hence $\varphi_k(u_k\theta) \in \oplus_{i=1}^k u_k \delta Re_i$. Say $\varphi_k(u_k\theta) = \sum_{i=1}^k c_i u_k \delta e_i$ where $c_i \in R$ for $1 \leq i \leq k$. Then it is straightforward to see that $\varphi_k(u_k\theta - \sum_{i=1}^k c_i b_i) = 0$. This forces $u_k\theta - \sum_{i=1}^k c_i b_i = 0 \in \oplus_{i=1}^n Rb_i = X$ since φ_k is injective. This in turn forces $u_k\theta = 0$ since $u_k\theta = \sum_{i=1}^k c_i b_i \in Rb_{k+1} \cap \oplus_{i=1}^k Rb_i = 0$. As u_k is regular on $R/(\underline{x}) \cong Rb_{k+1}$, we conclude that $\theta = 0 \in Rb_{k+1}$, a contradiction.

Therefore $I = (f_{k+1,k+1}, f_{k+2,k+1}, \dots, f_{n,k+1}) \not\subseteq P$ for any $P \in \operatorname{Ass}(R/(\underline{x})$. By Prime Avoidance (cf. Theorem 1.1), there exist $c_{k+2}, \dots, c_n \in R$ such that

$$u = f_{k+1,k+1} + c_{k+2} f_{k+2,k+1} \dots, c_n f_{n,k+1} \notin \cup_{P \in \operatorname{Ass}(R/(\underline{x}))} P.$$

That is, u is regular on $R/(\underline{x})$. Then, after a suitable change of basis for $Y \cong (R/(\underline{y}))^n$ over $R/(\underline{y})$ and refreshing the matrix entries accordingly, $\varphi_k : X \to Y$ can be represented by a matrix of the following form

$$\delta\begin{pmatrix} u_k & 0 & \dots & 0 & 0 & f_{1,k+1} & f_{1,k+2} & \dots & f_{1,m-1} & f_{1,m} \\ 0 & u_k & \dots & 0 & 0 & f_{2,k+1} & f_{2,k+2} & \dots & f_{2,m-1} & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & u_k & 0 & f_{k-1,k+1} & f_{k-1,k+2} & \dots & f_{k-1,m-1} & f_{k-1,m} \\ 0 & 0 & \dots & 0 & u_k & f_{k,k+1} & f_{k,k+2} & \dots & f_{k,m-1} & f_{k,m} \\ 0 & 0 & \dots & 0 & 0 & u & f_{k+1,k+2} & \dots & f_{k+1,m-1} & f_{k+1,m} \\ 0 & 0 & \dots & 0 & 0 & f_{k+2,k+1} & f_{k+2,k+2} & \dots & f_{k+2,m-1} & f_{k+2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & f_{n-1,k+1} & f_{n-1,k+2} & \dots & f_{n-1,m-1} & f_{n-1,m} \\ 0 & 0 & \dots & 0 & 0 & f_{n,k+1} & f_{n,k+2} & \dots & f_{n,m-1} & f_{n,m} \end{pmatrix}.$$

Moreover, by Notation 2.3.3, we may replace u by any element in the coset $u + (\underline{x})$ without affecting φ_k. Therefore, we may further assume that u is regular on both $R/(\underline{x})$ and $R/(\underline{y})$ (cf. Lemma 1.3(1)).

Now, let $\xi_1 : Y \to Y$ be an R-linear (injective) map defined by $\xi_1(e_{k+1}) = e_{k+1}$ and $\xi_1(e_i) = ue_i$ for all $i \neq k+1$. Then, according to Observation 2.3.4, $\xi_1 \circ \varphi_k : X \to Y$ is an injective R-homomorphism such that $M'_k := \operatorname{Coker}(\xi_1 \circ \varphi_k)$ is a permissible extension of M_k and, furthermore, $\xi_1 \circ \varphi_k$ is represented by the following matrix

$$\delta\begin{pmatrix} uu_k & 0 & \dots & 0 & 0 & uf_{1,k+1} & uf_{1,k+2} & \dots & uf_{1,m-1} & uf_{1,m} \\ 0 & uu_k & \dots & 0 & 0 & uf_{2,k+1} & uf_{2,k+2} & \dots & uf_{2,m-1} & uf_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & uu_k & 0 & uf_{k-1,k+1} & uf_{k-1,k+2} & \dots & uf_{k-1,m-1} & uf_{k-1,m} \\ 0 & 0 & \dots & 0 & uu_k & uf_{k,k+1} & uf_{k,k+2} & \dots & uf_{k,m-1} & uf_{k,m} \\ 0 & 0 & \dots & 0 & 0 & u & f_{k+1,k+2} & \dots & f_{k+1,m-1} & f_{k+1,m} \\ 0 & 0 & \dots & 0 & 0 & uf_{k+2,k+1} & uf_{k+2,k+2} & \dots & uf_{k+2,m-1} & uf_{k+2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & uf_{n-1,k+1} & uf_{n-1,k+2} & \dots & uf_{n-1,m-1} & uf_{n-1,m} \\ 0 & 0 & \dots & 0 & 0 & uf_{n,k+1} & uf_{n,k+2} & \dots & uf_{n,m-1} & uf_{n,m} \end{pmatrix}.$$

Then, after a suitable change of its basis for $Y \cong (R/(\underline{y}))^n$ over $R/(\underline{y})$ and after refreshing the matrix entries, $\xi_1 \circ \varphi_k : X \to Y$ can be represented by a matrix of the form

$$\delta\begin{pmatrix} uu_k & 0 & \dots & 0 & 0 & 0 & f_{1,k+2} & \dots & f_{1,m-1} & f_{1,m} \\ 0 & uu_k & \dots & 0 & 0 & 0 & f_{2,k+2} & \dots & f_{2,m-1} & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & uu_k & 0 & 0 & f_{k-1,k+2} & \dots & f_{k-1,m-1} & f_{k-1,m} \\ 0 & 0 & \dots & 0 & uu_k & 0 & f_{k,k+2} & \dots & f_{k,m-1} & f_{k,m} \\ 0 & 0 & \dots & 0 & 0 & u & f_{k+1,k+2} & \dots & f_{k+1,m-1} & f_{k+1,m} \\ 0 & 0 & \dots & 0 & 0 & 0 & f_{k+2,k+2} & \dots & f_{k+2,m-1} & f_{k+2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & f_{n-1,k+2} & \dots & f_{n-1,m-1} & f_{n-1,m} \\ 0 & 0 & \dots & 0 & 0 & 0 & f_{n,k+2} & \dots & f_{n,m-1} & f_{n,m} \end{pmatrix}.$$

Finally, let $\xi_2 : Y \to Y$ be an R-linear (injective) map defined by $\xi_2(e_{k+1}) = u_k e_{k+1}$ and $\xi_2(e_i) = e_i$ for all $i \neq k+1$. Denote $\varphi_{k+1} := \xi_2 \circ \xi_1 \circ \varphi_k \in \operatorname{Hom}(X, Y)$ and $M_{k+1} := \operatorname{Coker}(\varphi_{k+1})$. Then, according to Observation 2.3.4 again, $\varphi_{k+1} : X \to Y$ is an injective R-homomorphism such that $M_{k+1} = \operatorname{Coker}(\varphi_{k+1})$ is a permissible extension of M'_k and, furthermore, φ_{k+1} is represented by a matrix (whose entries have been refreshed) of the form

$$\delta \begin{pmatrix} u_{k+1} & 0 & \dots & 0 & 0 & 0 & f_{1,k+2} & \cdots & f_{1,m-1} & f_{1,m} \\ 0 & u_{k+1} & \dots & 0 & 0 & 0 & f_{2,k+2} & \cdots & f_{2,m-1} & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & u_{k+1} & 0 & 0 & f_{k-1,k+2} & \cdots & f_{k-1,m-1} & f_{k-1,m} \\ 0 & 0 & \dots & 0 & u_{k+1} & 0 & f_{k,k+2} & \cdots & f_{k,m-1} & f_{k,m} \\ 0 & 0 & \dots & 0 & 0 & u_{k+1} & f_{k+1,k+2} & \cdots & f_{k+1,m-1} & f_{k+1,m} \\ 0 & 0 & \dots & 0 & 0 & 0 & f_{k+2,k+2} & \cdots & f_{k+2,m-1} & f_{k+2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & f_{n-1,k+2} & \cdots & f_{n-1,m-1} & f_{n-1,m} \\ 0 & 0 & \dots & 0 & 0 & 0 & f_{n,k+2} & \cdots & f_{n,m-1} & f_{n,m} \end{pmatrix},$$

in which $f_{1,1} = \cdots = f_{k+1,k+1} = u_{k+1} := uu_k$ is regular on both $R/(\underline{x})$ and $R/(\underline{y})$ while the (i,j)-th entries are 0 whenever $i \neq j \leq k+1$. By Lemma 2.2(2), M_{k+1} is a permissible extension of M. This concludes the induction step and Claim 2.3.5 is proved.

Now that Claim 2.3.5 is proved, we are going to use it to complete the induction step in the course of our proof of Theorem 2.3. Indeed, Claim 2.3.5 states that, when $k = m$, there is a permissible extension M_m of M such that $M_m = \operatorname{Coker}(\varphi_m)$ for some $\varphi_m \in \operatorname{Hom}(X, Y)$ represented by an $n \times m$ matrix of the following form

$$\delta \begin{pmatrix} u_m & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & u_m \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} \delta u_m & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta u_m \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} u_m\delta & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & u_m\delta \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix},$$

in which the (i,j)-th entries are 0 whenever $i \neq j$ while the (i,i)-th entries are equal to $u_m\delta$ with u_m regular on (both $R/(\underline{x})$ and) $R/(\underline{y})$. Consequently, $M_m \cong (R/(\underline{y}, u_m\delta))^m \oplus (R/(\underline{y}))^{n-m}$. Relabeling u_m with u and applying Observation 2.3.1, we see that it suffices to prove Theorem 2.3 assuming $M = (R/(\underline{y}, u\delta))^m \oplus (R/(\underline{y}))^{n-m}$ with u regular on (both $R/(\underline{x})$ and) $R/(\underline{y})$. As $\operatorname{pd}((R/(\underline{y}))^{n-m}) \leq r$, the induction hypothesis gives the desired result for $(R/(\underline{y}))^{n-m}$. By Observation 2.3.2, we only need to prove it for $R/(\underline{y}, u\delta)$ assuming $\operatorname{pd}(R/(\underline{y}, u\delta)) = r+1$. From Lemma 1.3(4)(d), there is a short exact sequence

$$0 \longrightarrow R/(\underline{y}, u\delta) \longrightarrow R/(\underline{y}, u) \oplus R/(\underline{y}, \delta) \longrightarrow R/(\underline{y}, u, \delta) \longrightarrow 0$$

with $\mathrm{pd}(R/(\underline{y}, u, \delta)) \le r+1$ so that $R/(\underline{y}, u) \oplus R/(\underline{y}, \delta)$ is a permissible extension of $R/(\underline{y}, u\delta)$. Now, by Observation 2.3.1 and Observation 2.3.2 again, it remains to prove the assertion for $R/(\underline{y}, u)$ and $R/(\underline{y}, \delta)$ only. The case for $R/(\underline{y}, u)$ is trivially true (as $\underline{y}, u$ form an R-regular sequence of length $r+1$) while the case for $R/(\underline{y}, \delta)$ follows from the induction hypothesis since $\mathrm{pd}(R/(\underline{y}, \delta)) \le r$ (cf. Lemma 1.3(4)(c)). This finishes the desired induction step.

The proof of Theorem 2.3 is now complete. □

Remark 2.4. In the above proof of Theorem 2.3, the case when $\mathrm{pd}(M) < \infty$ follows as a consequence of the case when $\text{G-dim}(M) < \infty$. However, we remark that the embedding theorem for the case when $\mathrm{pd}(M) < \infty$ (i.e., Theorem 2.3(2)) can be proved by simply replacing G-dim with pd in the above proof of Theorem 2.3(1).

Remark 2.5. Let R be Gorenstein. Then the embedding theorem (i.e., Theorem 2.3) applies to any finitely generated R-module M as $\text{G-dim}(M) < \infty$. (When R is Gorenstein, a similar embedding may be achieved by means of primary decomposition, although this approach does not seem to give a way to control the G-dimension of the cokernel.)

3. Applications of the embedding theorem to rings of prime characteristic p

Throughout this section, we assume that R is a commutative Noetherian ring of prime characteristic p. We always use $q = p^e, Q = p^E, q' = p^{e'}$, etcetera, to denote varying powers of p with $e, E, e' \in \mathbb{N}$.

For any R-module M and $e \in \mathbb{N}$, there is a new R-module $F^e(M)$ otained by scalar extension via the iterated Frobenius map $F^e : R \to R$ defined by $r \mapsto r^q$. For any R-modules M, N and $h \in \mathrm{Hom}_R(M, N)$, we correspondingly have $F^e(h) \in \mathrm{Hom}_R(F^e(M), F^e(N))$.

If $N \subseteq M$ and say $\iota : N \hookrightarrow M$ is the inclusion map, then, for any $e \ge 0$, we denote $N_M^{[q]} := \mathrm{Image}\left(F^e(N) \xrightarrow{F^e(\iota)} F^e(M)\right)$. For any $x \in M$ and any $e \ge 0$, we denote the natural image of x in $F^e(M)$ by $x_M^q \in F^e(M)$. (See [HH1] for details.)

A very important concept in studying rings of characteristic p is tight closure. Tight closure was first studied and developed by Hochster and Huneke in the 1980's. Here we denote $R^\circ := R \setminus \cup_{P \in \min(R)} P$, the complement of the union of all minimal primes of the ring R.

Definition 3.1 ([HH1])**.** Let R be a Noetherian ring of prime characteristic p and $N \subseteq M$ be R-modules. The tight closure of N in M, denoted by N_M^*, is defined as follows: An element $x \in M$ is said to be in N_M^* if there exists an element $c \in R^\circ$ such that $cx^q \in N_M^{[q]} \subseteq F^e(M)$ for all $e \gg 0$.

Now let us apply Theorem 2.3 to rings of prime characteristic p. First, we recover a result about F-rational rings (without any Cohen-Macaulay assumption) by I. Aberbach in [Ab]. Recall that R is said to be F-rational if $I_R^* = I$ for any ideal $I = (f_1, \ldots, f_h) \subseteq R$ such that $\mathrm{height}(I) = h$.

Theorem 3.2 (Compare with [Ab])**.** *Let R be a Noetherian ring of prime characteristic p. Assume that every ideal of R generated by an R-regular sequence is tightly closed (e.g., R is F-rational). Then $0^*_M = 0$ for any finitely generated R-module with* $\text{G-dim}(M) < \infty$.

Proof. There exists a short exact sequence

$$0 \longrightarrow M \xrightarrow{h} Z \longrightarrow N \longrightarrow 0$$

as in Theorem 2.3. By the tight closure theory, $h(0^*_M) \subseteq 0^*_Z$. However, by the assumption of R and the condition on Z imposed by Theorem 2.3, we have $0^*_Z = 0$. Thus $h(0^*_M) = 0$, which implies $0^*_M = 0$ as h is injective. $\square$

Theorem 2.3 allows us to show the existence of a uniform test exponent for all R-modules in $\mathcal{P}(R)$. Recall that $Q = p^E$ is said to be a test exponent for some $c \in R$ and R-modules $N \subseteq M$ if, for any $x \in M$, the occurrence of $cx^q \in N^{[q]}_M$ for any single $q \geq Q$ implies $x \in N^*_M$. See [HH2], [Sh2] and [HY1] for details.

Theorem 3.3. *Let R be a Noetherian ring of prime characteristic p and $c \in R$.*

(1) Assume there exists a (uniform) test exponent Q for c and $0 \subseteq R/(\underline{z})$ for all (proper) R-regular sequences $\underline{z} = z_1, \ldots, z_l$. Then Q is a test exponent for c and $0 \subseteq M$ for all finitely generated R-module with $\text{pd}(M) < \infty$.

(2) In particular, if $(R, \mathfrak{m})$ is an excellent equidimensional local ring and $c \in R^\circ$, then there exists a test exponent Q for c and $0 \subseteq M$ for all finitely generated R-module with $\text{pd}(M) < \infty$.

Proof. (1). For any $M \in \mathcal{P}(R)$, there exists a short exact sequence (we may as well assume $M \subseteq Z$ with the inclusion map denoted by h)

$$0 \longrightarrow M \xrightarrow{h} Z \longrightarrow N \longrightarrow 0$$

as in Theorem 2.3. Then, because $\text{pd}(N) < \infty$, we get a short exact sequence

$$0 \longrightarrow F^e(M) \xrightarrow{F^e(h)} F^e(Z) \longrightarrow F^e(N) \longrightarrow 0 \quad \text{ for every } e \in \mathbb{N}.$$

Suppose $cx^q = 0 \in F^e(M)$ for some $x \in M$ and for some $q \geq Q$. Then $cx^q = 0 \in F^e(Z)$, which implies $x \in 0^*_Z$ by the assumption on Q. Therefore, there exists $c' \in R^\circ$ such that $c'x^{q'} = 0 \in F^{e'}(Z)$ for all $q' \gg 1$. As $F^{e'}(h)$ is injective for all $e' \in \mathbb{N}$, we get $c'x^{q'} = 0 \in F^{e'}(M)$ for all $q' \gg 1$, which implies that $x \in 0^*_M$ by the definition of tight closure.

(2). This follows from part (1) as, under the assumption of $(R, \mathfrak{m})$, there is a uniform test exponent Q for c and $0 \subseteq R/(\underline{x})$ for all partial systems of parameters $\underline{x} = x_1, \ldots, x_l$ of R. (This is a result first proved by R. Y. Sharp in [Sh2]. An alternative proof was then given in [HY1].) Also notice that, evidently, every proper R-regular sequence is part of a system of parameters. $\square$

Remark 3.4. In fact, given any excellent equidimensional local ring $(R, \mathfrak{m})$ and any $c \in R^\circ$, we are able to prove the existence of a uniform text exponent for c and $0 \subseteq M$ for all finitely generated R-modules with finite phantom projective dimension. This will be done in [HY2].

In a somewhat similar manner, Theorem 2.3 also allows us to study the Frobenius closure and show the existence of a uniform Frobenius test exponent for (all) R-modules in $\mathcal{P}(R)$. Recall that, given R-modules $N \subseteq M$, the Frobenius closure of N in M, denoted by N_M^F, is defined as $N_M^F := \{x \in M \mid x_M^q \in N_M^{[q]} \text{ for some } q\}$. We say Q is a Frobenius test exponent for $N \subseteq M$ (or, equivalently, for $0 \subseteq M/N$) if $(N_M^F)_M^{[Q]} = N_M^{[Q]}$. Moreover, a local ring $(R, \mathfrak{m})$ is said to be generalized Cohen-Macaulay if $\mathrm{H}^i_{\mathfrak{m}}(R)$ has finite length for every $i < \dim(R)$.

Theorem 3.5. *Let R be a Noetherian ring of prime characteristic p. Assume that every ideal of R generated by an R-regular sequence is Frobenius closed (e.g., R is F-injective). Then $0_M^F = 0$ for any finitely generated R-module with* $\operatorname{G-dim}(M) < \infty$.

Proof. This is parallel to the proof of Theorem 3.2. Fix a short exact sequence

$$0 \longrightarrow M \xrightarrow{h} Z \longrightarrow N \longrightarrow 0$$

as in Theorem 2.3. Then, it is clear that $h(0_M^F) \subseteq 0_Z^F$. However, by the assumption of R and the condition on Z imposed by Theorem 2.3, we have $0_Z^F = 0$. Thus $h(0_M^F) = 0$, which implies $0_M^F = 0$ as h is injective. □

Theorem 3.6. *Let R be a Noetherian ring of prime characteristic p and $c \in R$.*

(1) Assume there exists a (uniform) Frobenius test exponent Q for $0 \subseteq R/(\underline{z})$ for all (proper) R-regular sequences $\underline{z} = z_1, \ldots, z_l$. Then Q is a uniform Frobenius test exponent for $0 \subseteq M$ for all finitely generated R-module with $\operatorname{pd}(M) < \infty$.

(2) In particular, if $(R, \mathfrak{m})$ is a generalized Cohen-Macaulay local ring, then there exists a uniform Frobenius test exponent Q for $0 \subseteq M$ for all finitely generated R-module with $\operatorname{pd}(M) < \infty$.

Proof. (1). For any $M \in \mathcal{P}(R)$, we only need to show $(0_M^F))^{[Q]} = 0 \subseteq F^E(M)$. As in the proof of Theorem 3.3, there exist Z and N as in Theorem 2.3 and, consequently, a short exact sequence

$$0 \longrightarrow F^e(M) \xrightarrow{F^e(h)} F^e(Z) \longrightarrow F^e(N) \longrightarrow 0 \quad \text{for every } e \in \mathbb{N}.$$

Evidently, $0_M^F \subseteq 0_Z^F$. However, by the assumption of Q, we have $(0_Z^F))^{[Q]} = 0 \subseteq F^E(Z)$. Thus, by the injectivity of $F^E(h)$, we have $(0_M^F))^{[Q]} = 0 \subseteq F^E(M)$, the desired result.

(2). A result in [HKSY] shows that, when $(R, \mathfrak{m})$ is generalized Cohen-Macaulay, there is a uniform Frobenius test exponent Q for $0 \subseteq R/(\underline{x})$ for all partial systems of parameters $\underline{x} = x_1, \ldots, x_l$ of R. Now the desired claim follows from (1). □

Theorem 3.7 ([KS, Corollary 4.3])**.** *Let R be a Noetherian ring of prime characteristic p and $\underline{z} = z_1, \ldots, z_i$ form an R-regular sequence. Then there exists a Frobenius test exponent Q for $(\underline{z})^{[q]} \subseteq R$ for all q, that is, $(0^F_{R/(\underline{z})^{[p]}})^{[Q]}_{R/(\underline{z})^{[p]}} = 0$ for all q.*

Theorem 3.8. *Let R be a Noetherian ring of prime characteristic p and M a finitely generated R-module with* $\operatorname{pd}(M) < \infty$*. Then there exists a Frobenius test exponent Q for $0 \subseteq F^e(M)$ for all $e \in \mathbb{N}$. In other words, $(0^F_{F^e(M)})^{[Q]}_{F^e(M)} = 0$ for all q.*

Proof. Say $\mathrm{pd}(M) = r$. As in the proof of Theorem 3.3, there exist Z and N as in Theorem 2.3 and, consequently, a short exact sequence

$$0 \longrightarrow F^e(M) \xrightarrow{F^e(h)} F^e(Z) \longrightarrow F^e(N) \longrightarrow 0 \quad \text{for every } e \in \mathbb{N}.$$

For each $i = 0, 1, \dots, r$, Theorem 3.7 implies that there is a Frobenius test exponent Q_i for $0 \subseteq R/(\underline{z}_i)^{[q]}$ for all q. Let $Q = \max\{Q_i \mid 0 \le i \le r\}$. Then Q is a Frobenius test exponent for $0 \subseteq F^e(M)$ for all $e \in \mathbb{N}$. Now the desired claim follows from the injectiveness of $F^e(h)$ for all e as in the above short exact sequence. $\square$

4. Embedding modules of locally finite injective dimension over Cohen-Macaulay rings

In this section, we assume that R is a Cohen-Macaulay ring that has a global canonical module, which is denoted by ω. (We no longer assume R has prime characteristic p in this section.) The purpose is to prove a dual version of Theorem 2.3 for modules of locally finite injective dimension. The key point is a result by R. Y. Sharp connecting $\mathcal{P}(R)$ and $\mathcal{I}(R)$ (see Notation 0.1).

Theorem 4.1 (Sharp, [Sh1])**.** *Let R be a Noetherian Cohen-Macaulay ring with a global canonical module ω. Then*

1. *The functor $\mathcal{P}(R) \xrightarrow{\_\otimes_R \omega} \mathcal{I}(R)$ is well-defined and exact.*
2. *The functor $\mathcal{P}(R) \xleftarrow{\mathrm{Hom}_R(\omega,\_)} \mathcal{I}(R)$ is well-defined and exact.*
3. *The above two functors establish an equivalence between $\mathcal{P}(R)$ and $\mathcal{I}(R)$.*

Now, the dual version of Theorem 2.3 is immediate in light of Theorem 4.1.

Theorem 4.2. *Let R be a Noetherian Cohen-Macaulay ring with a global canonical module ω. Then, for any finitely generated R-module M with locally finite injective dimension (i.e., $M \in \mathcal{I}(R)$), there exist an integer $r = \mathrm{pd}(\mathrm{Hom}_R(\omega, M))$, a proper R-regular sequence $\underline{z} = z_1, \dots, z_r$, non-negative integers $n_0, n_1, \dots, n_r$, and a short exact sequence*

$$0 \longrightarrow M \longrightarrow Z \longrightarrow N \longrightarrow 0$$

with $Z = \oplus_{i=0}^r (\omega/(\underline{z}_{[i]})\omega)^{n_i}$, $N \in \mathcal{I}(R)$ and $\mathrm{pd}(\mathrm{Hom}_R(\omega, N)) \le r$. It is automatically true that $n_r > 0$ unless $M = 0$.

Proof. Let $M' = \mathrm{Hom}_R(\omega, M)$. Then $\mathrm{pd}(M') < \infty$ by Theorem 4.1. Thus, from Theorem 2.3, there exist $r = \mathrm{pd}(M')$, an R-regular sequence $\underline{z} = z_1, \dots, z_r$, integers $n_0, n_1, \dots, n_r \ge 0$, and a short exact sequence

$$0 \longrightarrow M' \longrightarrow Z' \longrightarrow N' \longrightarrow 0 \tag{‡}$$

such that $Z' = \oplus_{i=0}^r (R/(\underline{z}_{[i]}))^{n_i}$ and $\mathrm{pd}(N) \le r$.

Now let $Z = \oplus_{i=0}^r (\omega/(\underline{z}_{[i]})\omega)^{n_i} \cong Z' \otimes_R \omega$ and $N = N' \otimes_R \omega$. Also notice that $M \cong M' \otimes_R \omega$. Thus, applying the exact functor $\_\otimes_R \omega$ to (‡), we have a short exact sequence

$$0 \longrightarrow M \longrightarrow Z \longrightarrow N \longrightarrow 0$$

as desired. All the remaining claims follow immediately. $\square$

If, moreover, $(R, \mathfrak{m})$ is local with maximal ideal $\mathfrak{m}$, then, for any finitely generated R-modules M and M' with $\mathrm{id}(M) < \infty$ and $\mathrm{pd}(M') < \infty$, we have

$$\mathrm{depth}(\mathrm{Hom}_R(\omega, M)) = \mathrm{depth}(M) \quad \text{and} \quad \mathrm{depth}(M' \otimes_R \omega) = \mathrm{depth}(M').$$

(Indeed, by Theorem 4.1(1,2), we have inequalities $\mathrm{depth}(\mathrm{Hom}_R(\omega, M)) \geq \mathrm{depth}(M)$ and $\mathrm{depth}(M' \otimes_R \omega) \geq \mathrm{depth}(M')$. Then, in light of Theorem 4.1(3), we get the desired equalities.) Thus we can say more about the cokernel of the embedding of M in the following corollary.

Corollary 4.3. *Let $(R, \mathfrak{m})$ be a Noetherian Cohen-Macaulay local ring that has a canonical module ω. Then, for any finitely generated R-module $M \neq 0$ with $\mathrm{id}(M) < \infty$ and $\dim(R) - \mathrm{depth}(M) = r$, there exist a proper R-regular sequence $\underline{z} = z_1, \ldots, z_r$, non-negative integers $n_0, n_1, \ldots, n_r$ (with $n_r > 0$ automatically), and a short exact sequence*

$$0 \longrightarrow M \longrightarrow Z \longrightarrow N \longrightarrow 0$$

with $Z = \oplus_{i=0}^{r} (\omega/(\underline{z}_{[i]})\omega)^{n_i}$, $\mathrm{id}(N) < \infty$ and, if $N \neq 0$, $\mathrm{depth}(N) \geq \mathrm{depth}(M)$.

Proof. Let M', Z', N', Z, N be as in the proof of Theorem 4.2. Then, by the Auslander-Buchsbaum formula, we have $\mathrm{pd}(M') = \dim(R) - \mathrm{depth}(M') = \dim(R) - \mathrm{depth}(M)$, which gives $\mathrm{depth}(M) = \dim(R) - \mathrm{pd}(M')$. Thus, if $N \neq 0$, we have $\mathrm{depth}(N) = \mathrm{depth}(N') = \dim(R) - \mathrm{pd}(N') \geq \dim(R) - \mathrm{pd}(M') = \mathrm{depth}(M)$, as desired. $\square$

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