THE COMPATIBILITY, INDEPENDENCE, AND LINEAR GROWTH PROPERTIES

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Abstract. The first part is about primary decomposition. After reviewing the basic definitions, we survey the compatibility, independence, and linear growth properties that have been known. Then, we prove the linear growth property of primary decomposition for a new family of modules.

In the remaining sections, we study secondary representation, which can be viewed as a dual of primary decomposition. Correspondingly, we study the compatibility, independence, and linear growth properties of secondary representations.

0. Introduction

Throughout this paper, all rings are assumed to be commutative with one; and they are not necessarily Noetherian unless we state so explicitly.

Sections 1–5 are dedicated to the theory of primary decomposition. In its classic form, it states that every ideal in a Noetherian ring can be expressed as an intersection of finitely many primary ideals. Later, the theory of primary decomposition was developed for modules. In particular, if a module is Noetherian, then every submodule is decomposable.

Although the primary decompositions are not unique in general, there are certain uniqueness properties governing the primary decompositions.

In Section 1, basic definitions and properties in the theory of primary decomposition are reviewed. In Section 2, we go over the compatibility property, which says that primary components from different primary decompositions of a fixed submodule can be put together and the resulting intersection is still a primary decomposition of the submodule. Maximal primary components are studied in Section 3. In Section 4, the linear growth property of primary decomposition is reviewed. We establish the linear growth property for a new family of modules in Section 5.

In Sections 6–11, we study the secondary representation theory. This can be viewed as a dual of the primary decomposition theory. In this theory, a module is representable if it can be expressed as a finite sum of secondary submodules. It turns out that every Artinian module has a secondary representation.

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Many of the results in the theory of secondary representation have their dual forms in the theory of primary decomposition. Because of this, one often draws inspiration from one theory and then applies it to the other. In this note, the theory of secondary representation is presented in a way that would make the duality between the two theories evident.

In Section 6, we go over the fundamentals of the theory of secondary representation. In the subsequent sections, we study and prove the compatibility, minimal components, independence, and linear growth properties of secondary representation. In Section 8, we discuss a result of Sharp [Sh2] that makes the classic Matlis duality applicable to Artinian modules even if the ring is not Noetherian. This allows us to establish results on secondary representation by reducing them to the dual results in the theory of primary decomposition.

Many of the results in Sections 7–11 were obtained in [Yao3].

1. Primary decomposition

In this section, we give a brief introduction to the notions of associated prime and primary decomposition. Systematic treatments of primary decomposition can be found in many textbooks, for example, [AM], [Bo], [Ei] or [Mat].

Let R be a ring (not necessarily Noetherian) and M an R-module.

We say that a prime ideal $P \in \operatorname{Spec}(R)$ is *associated* to M if there exists $x \in M$ such that $(0 :_R x) = P$. The set of all primes associated to M is denoted $\operatorname{Ass}_R(M)$, or simply $\operatorname{Ass}(M)$ when R is understood from the context.

Following [AM], we say that a prime ideal $P \in \operatorname{Spec}(R)$ *belongs* to M if there exists $x \in M$ such that $\sqrt{(0 :_R x)} = P$. (In fact, the terminology "*P belongs to 0 in M*" was used in [AM].) The set of all primes belonging to M is denoted $\operatorname{Ass}'_R(M)$, or simply $\operatorname{Ass}'(M)$ when R is understood from the context.

We say that M is *coprimary* (over R) if, for every $r \in R$, either r is M-regular (i.e., $(0 :_M r) = 0$) or $r \in \sqrt{\operatorname{Ann}(M)}$. (Under this definition, 0 is a coprimary module.) It turns out that, if $M \neq 0$ is coprimary and if we let $P = \sqrt{\operatorname{Ann}(M)}$, then $P \in \operatorname{Spec}(R)$; in this case, we say M is P-coprimary. (This definition recovers the definition of primary ideals in that an ideal Q is P-primary (in R) if and only if R/Q is P-coprimary as an R-module.)

We also define $\operatorname{Ass}''_R(M) := \{P \in \operatorname{Spec}(R) \mid \exists K \subseteq M,\, K \text{ is } P\text{-coprimary}\}$; or, equivalently, $\operatorname{Ass}''_R(M) := \{P \in \operatorname{Spec}(R) \mid \exists\, x \in M,\, R/(0 :_R x) \text{ is } P\text{-coprimary}\}$. This notion Ass'' and the notion Att (to be defined in §6) are dual to each other.

Quite generally, if M is P-coprimary, then $\operatorname{Ass}'_R(M) = \{P\} = \operatorname{Ass}''(M)$.

For R-modules $Q \subseteq M$, we say that Q is (P-)primary if M/Q is (P-)coprimary. For R-modules $N \subsetneq M$, we say that N is *decomposable* in M (over R) if there exist R-submodules Q_i that are P_i-primary in M, for $i = 1, \ldots, s$, such that

$$N = Q_1 \cap \cdots \cap Q_s.$$

This intersection is called a *primary decomposition* of N in M (over R). One can always convert a primary decomposition to a *minimal* one in the sense that $P_i \neq P_j$ for all $i \neq j$ and $N \neq \cap_{i\neq k} Q_i$ for every $k = 1, \dots, s$. So from now on, as a general rule, all primary decompositions are assumed to be minimal unless stated otherwise explicitly.

For every R-module M, we agree that M is decomposable in M with $M = M$ being the unique primary decomposition of M in M.

Given R-modules $N \subseteq M$, N is decomposable in M if and only if 0 is decomposable in M/N; and the primary decompositions of N in M are in one-to-one correspondence with the primary decompositions of 0 in M/N.

Similarly, let $N \subseteq M$ be R-modules and let I be an ideal of R such that $I \subseteq \operatorname{Ann}(M)$, so that $N \subseteq M$ can be naturally viewed as modules over R/I. Then N is decomposable in M as R-modules if and only if N is decomposable in M as (R/I)-modules.

Next, we list some properties of primary decomposition. We need to introduce some notation that will be used in the sequel: Given an R-module M, we use $\operatorname{Min}(M)$ to denote the set of all the minimal primes over $\operatorname{Ann}(M)$. For a multiplicative subset $U \subseteq R$, we use $M[U^{-1}]$ to denote the module of fractions after inverting all the elements in U, so that $M[U^{-1}] \cong M \otimes_R R[U^{-1}]$.

Theorem 1.1. *Let $N \subseteq M$ be R-modules and suppose $N = Q_1 \cap \cdots \cap Q_s$ is a (minimal) primary decomposition of N in M in which Q_i is P_i-primary.*

(1) *We have $\{P_1, \dots, P_s\} = \operatorname{Ass}'_R(M/N) = \operatorname{Ass}''_R(M/N)$, which is independent of the particular (minimal) primary decompositions in M.*

(1') *We have $\operatorname{Min}(M/N) \subseteq \{P_1, \dots, P_s\}$. In fact, $\operatorname{Min}(M/N)$ equals the set of the minimal members of $\{P_1, \dots, P_s\}$ (under inclusion).*

(2) *If P_i is minimal in $\operatorname{Ass}'_R(M/N)$, then Q_i is uniquely determined as $Q_i = \operatorname{Ker}\big(M \to (M/N)_{P_i}\big)$. See (4) below.*

(3) *Let $h\colon A \to R$ be a ring homomorphism, so that $N \subseteq M$ may be viewed as A-modules. Let K be an A-submodule of M such that $N \subseteq K$ (e.g., $K = M$). Then N is decomposable in K as A-modules. If $N \subsetneq K$, then*

$$N = \bigcap_{Q_i \not\supseteq K} (Q_i \cap K)$$

is a (not necessarily minimal) primary decomposition of N in K over A, in which $Q_i \cap K$ is $h^{-1}(P_i)$-primary in K provided that $Q_i \not\supseteq K$.

(3') *In particular, $\operatorname{Ass}'_A(M) = h^*(\operatorname{Ass}'_R(M))$, in which $h^*\colon \operatorname{Spec}(R) \to \operatorname{Spec}(A)$ is the continuous map naturally induced by h.*

(4) *Let $U \subseteq R$ be a multiplicative set. Then $N[U^{-1}] = \cap_{U \cap P_i = \varnothing} Q_i[U^{-1}]$ is a primary decomposition in $M[U^{-1}]$, in which $Q_i[U^{-1}]$ is $P_i[U^{-1}]$-primary in $M[U^{-1}]$; and $\operatorname{Ker}(M \to (M/N)[U^{-1}]) = \cap_{U \cap P_i = \varnothing} Q_i$.*

(5) *For any finitely generated ideal I of R, $\cap_{I \not\subseteq P_i} Q_i = (N :_M I^n)$ for $n \gg 0$.*

(5') *For any non-empty subset I of R, $\cap_{I \not\subseteq P_i} Q_i = \cap_{r \in I}(\cup_{n \in \mathbb{N}}(N :_M r^n))$.*

Remark 1.2. In [Bo, Chapter IV], the notion of primary decomposition is generalized to *weak primary decomposition*. (This was simply called *primary decomposition* in [Bo]. We add the word "weak" into the terminology in order to distinguish it from the notion of (ordinary) primary decomposition.) For an R-module M and $P \in \operatorname{Spec}(R)$, we say that P is *weakly associated* to M if P is minimal over the ideal $\operatorname{Ann}_R(x)$ (i.e., $P \in \operatorname{Min}(Rx)$) for some $x \in M$. Denote by $\operatorname{Ass_f}({}_RM)$, or simply $\operatorname{Ass_f}(M)$, the set of all the prime ideals weakly associated to M (cf. [Bo, page 289, Chapter IV, §1, Exercise 17]). It is clear $M = 0 \iff \operatorname{Ass_f}(M) = \varnothing$.

We say that M is *weakly coprimary* if, for all $r \in R$, either $(0 :_M r) = 0$ or $\cup_{n \geqslant 0}(0 :_M r^n) = M$. If $M \neq 0$ is coprimary, it follows that $\{r \in R \mid (0 :_M r) \neq 0\} =: P$ is prime, and we say that M is weakly P-coprimary. It turns out that M is weakly P-coprimary if and only if $\operatorname{Ass_f}(M) = \{P\}$. See [Bo, page 292, Chapter IV, §2, Exercises 11, 12].

Given R-modules $Q \subseteq M$, we say that Q is *weakly P-primary* in M if M/Q is weakly P-coprimary, i.e., $\operatorname{Ass_f}(M/Q) = \{P\}$. Now, for $N \subseteq M$, we say that N is *weakly decomposable* in M if there exist weakly P_i-primary submodules Q_i in M, $i = 1, \ldots, s$, such that $N = Q_1 \cap \cdots \cap Q_s$. If such decompositions exist, we can make them minimal. Weak primary decompositions enjoy many of the properties of primary decompositions; see [Bo, page 294, Chapter IV, §2, Exercise 20] and Theorem 1.3 below. Conversely, if Q is P-primary in M then Q is weakly P-primary in M; thus every primary decomposition is a weak primary decomposition.

In [St], some of the basic properties of $\operatorname{Ass_f}$ and weak primary decomposition were worked out in detail via elementary techniques.

We state the following weak-primary-decomposition analogue of Theorem 1.1.

Theorem 1.3. *Suppose $N = Q_1 \cap \cdots \cap Q_s$ is a minimal weak primary decomposition of N in M, in which Q_i is weakly P_i-primary.*

(1) *We have $\{P_1, \ldots, P_s\} = \operatorname{Ass}'(M/N) = \operatorname{Ass}''(M/N) = \operatorname{Ass_f}(M/N)$, which is independent of the particular (minimal) primary decompositions in M.*

(1') *We have $\operatorname{Min}(K/N) \subseteq \{P_1, \ldots, P_s\}$ for all R-submodule K satisfying $N \subseteq K \subseteq M$ and K/N is finitely generated.*

(2) *If P_i is minimal in $\{P_1, \ldots, P_s\} = \operatorname{Ass_f}(M/N)$ (under inclusion), then Q_i is uniquely determined as $Q_i = \operatorname{Ker}\big(M \to (M/N)_{P_i}\big)$. See (4) below.*

(3) *Let $h\colon A \to R$ be a ring homomorphism, so that $N \subseteq M$ may be viewed as A-modules. Let K be an A-submodule of M such that $N \subseteq K$ (e.g., $K = M$). Then N is weakly decomposable in K as A-modules. If $N \subsetneq K$, then*

$$N = \bigcap_{Q_i \not\supseteq K} (Q_i \cap K)$$

is a (not necessarily minimal) weak primary decomposition of N *in* K *over* A*, in which* $Q_i \cap K$ *is weakly* $h^{-1}(P_i)$*-primary in* K *provided that* $Q_i \not\supseteq K$*.*

(3') *In particular,* $\mathrm{Ass_f}({}_AM) = h^*(\mathrm{Ass_f}({}_RM))$*, in which* $h^*\colon \mathrm{Spec}(R) \to \mathrm{Spec}(A)$ *is the continuous map naturally induced by* h*.*

(4) *Let* $U \subseteq R$ *be a multiplicative set. Then* $N[U^{-1}] = \cap_{U\cap P_i=\varnothing} Q_i[U^{-1}]$ *is a weak primary decomposition in* $M[U^{-1}]$*, in which* $Q_i[U^{-1}]$ *is* $P_i[U^{-1}]$*-primary in* $M[U^{-1}]$*; and* $\mathrm{Ker}(M \to (M/N)[U^{-1}]) = \cap_{U\cap P_i=\varnothing} Q_i$*.*

(5) *For any finitely generated ideal* I *of* R*,* $\cap_{I\not\subseteq P_i} Q_i = \cup_{n\in\mathbb{N}}(N :_M I^n)$*.*

(5') *For any non-empty subset* I *of* R*,* $\cap_{I\not\subseteq P_i} Q_i = \cap_{r\in I}(\cup_{n\in\mathbb{N}}(N :_M r^n))$*.*

It is well-known that if $0 \to M_1 \to M_2 \to M_3$ is an exact sequence of R-modules then $\mathrm{Ass}(M_1) \subseteq \mathrm{Ass}(M_2) \subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_3)$; and $\mathrm{Ass}(\oplus_{i\in\Delta}K_i) = \cup_{i\in\Delta}\mathrm{Ass}(K_i)$ for any family $\{K_i\}_{i\in\Delta}$ of R-modules. The analogue also holds if we replace Ass with Ass', Ass'' or $\mathrm{Ass_f}$. (See [Bo, page 289, Ch IV, §1, Ex 17(c)] for the $\mathrm{Ass_f}$-analogue.) Here we present the Ass'-analogue, as it will be referred to in the proof of Lemma 2.2.

Lemma 1.4. *Let* $0 \to M_1 \to M_2 \to M_3$ *be an exact sequence of modules over a ring* R*. Then* $\mathrm{Ass}'(M_1) \subseteq \mathrm{Ass}'(M_2) \subseteq \mathrm{Ass}'(M_1) \cup \mathrm{Ass}'(M_3)$*.*

Moreover, $\mathrm{Ass}'(\oplus_{i\in\Delta}K_i) = \cup_{i\in\Delta}\mathrm{Ass}'(K_i)$ *for any family* $\{K_i\}_{i\in\Delta}$ *of* R*-modules.*

Proof. We sketch a proof of the first claim. Without loss of generality, assume $M_1 \subseteq M_2$ and $M_2/M_1 \subseteq M_3$. As $\mathrm{Ass}'(M_1) \subseteq \mathrm{Ass}'(M_2)$ is clear, it remains to show $\mathrm{Ass}'(M_2) \subseteq \mathrm{Ass}'(M_1) \cup \mathrm{Ass}'(M_2/M_1)$. Let $P \in \mathrm{Ass}'(M_2)$, so that $P = \sqrt{(0 :_R x)}$ for some $x \in M_2$. If there exists $r \in R \setminus P$ such that $rx \in M_1$, then it is straightforward to see that $P = \sqrt{(0 :_R rx)}$ and hence $P \in \mathrm{Ass}'(M_1)$. If $rx \notin M_1$ for all $r \in R \setminus P$, then it follows that $P = \sqrt{(0 :_R \overline{x})}$, where $\overline{x} = x + M_1 \in M_2/M_1$, and hence $P \in \mathrm{Ass}'(M_2/M_1$.

The second claim follows from the first when Δ is finite. In the general case, it is easy to see $\mathrm{Ass}'(\oplus_{i\in\Delta}K_i) \supseteq \cup_{i\in\Delta}\mathrm{Ass}'(K_i)$. Conversely, if $P \in \mathrm{Ass}'(\oplus_{i\in\Delta}K_i)$, then there exists a finite subset $\Delta' \subseteq \Delta$ such that $P \in \mathrm{Ass}'(\oplus_{i\in\Delta'}K_i)$. It then follows that $P \in \cup_{i\in\Delta'}\mathrm{Ass}'(K_i) \subseteq \cup_{i\in\Delta}\mathrm{Ass}'(K_i)$. □

We end this section with some basic facts concerning various kinds of associated prime ideals as well as decomposability. Let M be an R-module. It is clear that $\mathrm{Ass}(M) \subseteq \mathrm{Ass}''(M) \subseteq \mathrm{Ass}'(M) \subseteq \mathrm{Ass_f}(M) \subseteq \mathrm{Spec}(R)$. Consequently, as there is the Zariski topology on $\mathrm{Spec}(R)$, all the others are topological (sub)spaces. Quite generally, for any subset X of $\mathrm{Spec}(R)$, the Zariski topology on $\mathrm{Spec}(R)$ induces a topological structure on X in such a way that the closed sets of X are of the form $V_X(I) := \{P \in X \mid P \supseteq I\}$ with $I \subseteq R$.

If R is Noetherian or M is Noetherian over R, then $\mathrm{Ass}_R(M) = \mathrm{Ass}'_R(M) = \mathrm{Ass}''(M) = \mathrm{Ass_f}(M)$, and $\mathrm{Ass}_R(M) = \varnothing \iff M = 0$.

If M is Noetherian, then M is P-coprimary $\iff \mathrm{Ass}_R(M) = \{P\}$.

If $N \subseteq M$ are R-modules such that the quotient M/N is Noetherian over R, then N is decomposable in M. This is a classic result due to E. Noether.

There are more definitions of associated primes in the literature. See a list of these definitions in [Sw2, Remark 3.11].

2. Compatibility of primary components

Throughout this section, let R be a (not necessarily Noetherian) ring and let $N \subseteq M$ be R-modules such that N is decomposable in M.

Notation 2.1. Let $X \subseteq \operatorname{Ass}'(M/N)$. Say $X = \{P_1, \dots, P_r\} \subseteq \{P_1, \dots, P_r, \dots, P_s\} = \operatorname{Ass}'(M/N)$.

1. If $N = Q_1 \cap \cdots \cap Q_r \cap \cdots \cap Q_s$ is a primary decomposition of N in M with Q_i being P_i-primary, then we say $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_r$ is an *X-primary component* (or a *primary component* over X) of $N \subseteq M$. If $X = \varnothing$, then we agree that M is the only X-primary component of $N \subseteq M$.
2. We call an X-primary component of $N \subseteq M$ maximal if it is not properly contained in any X-primary component of $N \subseteq M$.
3. We use $\Lambda_X(N \subseteq M)$ to denote the set of all possible X-primary components of N in M.
4. We use $\mathring{\Lambda}_X(N \subseteq M)$ to denote the set of all maximal X-primary components of N in M. (Note that $\mathring{\Lambda}_X(N \subseteq M) \neq \varnothing$ if M/N is Noetherian.)
5. In case $X = \{P\} \subseteq \operatorname{Ass}(M/N)$, we may simply write Λ_P and $\mathring{\Lambda}_P$ instead of $\Lambda_{\{P\}}$ and $\mathring{\Lambda}_{\{P\}}$ respectively.

Note that, for $P \in \operatorname{Ass}'(M/N)$, the P-primary components are not necessarily unique in general (cf. Corollary 3.4). The compatibility property (see Theorem 2.3) says that if one takes a P-primary component of $N \subseteq M$ for each $P \in \operatorname{Ass}'(M/N)$ (from possibly different decompositions), then they are "compatible" in the sense that their intersection is exactly N, thus producing a primary decomposition of N in M. This was proved in [Yao1] and [Yao2] under the Noetherian assumption (but see [Yao1, Remark 1.2]). Here we state the results more generally.

Lemma 2.2 (Compare with [Yao2, Lemma 1.1]). *Let $N \subseteq M$ be R-modules such that N is decomposable in M, and $X \subseteq \operatorname{Ass}'(M/N)$. For an R-module Q such that $N \subseteq Q \subseteq M$, the following are equivalent:*

1. *Q is an X-primary component of $N \subseteq M$, i.e., $Q \in \Lambda_X(N \subseteq M)$.*
2. *Q is decomposable in M, $\operatorname{Ass}'\left(\frac{M}{Q}\right) \subseteq X$ and $\operatorname{Ass}'\left(\frac{Q}{N}\right) \subseteq \operatorname{Ass}'\left(\frac{M}{N}\right) \setminus X$.*
3. *Q is decomposable in M, $\operatorname{Ass}'\left(\frac{M}{Q}\right) = X$ and $\operatorname{Ass}'\left(\frac{Q}{N}\right) = \operatorname{Ass}'\left(\frac{M}{N}\right) \setminus X$.*

Proof. The proof of [Yao2, Lemma 1.1], with Ass' instead of Ass, should work here, in light of Lemma 1.4 and the fact that N is automatically decomposable in Q. $\square$

Theorem 2.3 (Compatibility)**.** *Let $N \subseteq M$ be R-modules such that N is decomposable in M. Let $X_i \subseteq \operatorname{Ass}'(M/N)$ and $Q_{X_i} \in \Lambda_{X_i}(N \subseteq M)$ for $1 \leqslant i \leqslant n$.*

1. *Then $\cap_{i=1}^n Q_{X_i} \in \Lambda_X(N \subseteq M)$, where $X = \cup_{i=1}^n X_i$.*
2. *In particular, suppose $\operatorname{Ass}'(M/N) = \{P_1, \ldots, P_s\}$ and $Q_i \in \Lambda_{P_i}(N \subseteq M)$ for each $i = 1, 2, \ldots, s$. Then $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$, which is necessarily a minimal primary decomposition of $N \subseteq M$.*

Proof. The proof of [Yao2, Corollary 1.2], with Ass' instead of Ass, should work here. Note that, by construction, $\cap_{i=1}^n Q_{X_i}$ is decomposable in M. □

Remark 2.4. As noted in [Yao1, Remark 1.2], the compatibility property is also shared by weak primary decompositions (cf. Remark 1.2). In fact, the analogues of Lemmas 1.4, 2.2 and Theorem 2.3 hold after every Ass' is replaced with $\operatorname{Ass}_{\mathrm{f}}$, "decomposable" with "weakly decomposable", and after $\Lambda_X(N \subseteq M)$ is interpreted accordingly. In [St], Stalvey presented a detailed proof of the compatibility for weak primary decomposition, following the proof given in [Yao1, Theorem 1.1].

3. Maximal primary components, independence

In this section, let $N \subseteq M$ be R-modules such that N is decomposable in M and $X \subseteq \operatorname{Ass}'(M/N)$. Note that $\operatorname{Ass}'(M/N)$ is a topological space in Zariski topology.

Notation 3.1. Let $N \subseteq M$ be as above. Since $\operatorname{Ass}'(M/N)$ is finite, every subset $X \subseteq \operatorname{Ass}'(M/N)$ has a unique minimal open superset in $\operatorname{Ass}'(M/N)$, which we denote by $\mathrm{o}(X)$. (Although this notation does not reflect the ambient space $\operatorname{Ass}'(M/N)$, there should be no danger of ambiguity.) For any $P \in \operatorname{Ass}'(M/N)$, we may simply write $\mathrm{o}(\{P\})$ as $\mathrm{o}(P)$. In fact, $\mathrm{o}(X) = \{P \in \operatorname{Ass}'(M/N) \mid P \subseteq \cup_{P' \in X} P'\}$, by prime avoidance.

Note that, if X is open in $\operatorname{Ass}'(M/N)$ (i.e., $X = \mathrm{o}(X)$), then there is a unique X-primary component of $N \subseteq M$, which is determined as $\operatorname{Ker}\big(M \to (M/N)[U^{-1}]\big)$ with $U = R \setminus \big(\cup_{P \in X} P\big)$ (cf. Theorem 1.1 (4)). This inspires the following definition.

Definition 3.2. Let R be a ring, $N \subseteq M$ be R-modules such that N is decomposable in M, and $X \subseteq \operatorname{Ass}'_R(M/N)$. We say that the primary decompositions of N in M are *independent* over X, or X-independent, if $\Lambda_X(N \subseteq M)$ consists of exactly one component, i.e., $|\Lambda^*_X(M)| = 1$.

Now assume that M/N is Noetherian over R. Thus, for any $X \subseteq \operatorname{Ass}(M/N) = \operatorname{Ass}'(M/N)$, maximal X-primary components exist. (When studying primary decompositions of N in M, we may simply study the primary decompositions of 0 in M/N as modules over $R/\operatorname{Ann}(M/N)$. Note that $R/\operatorname{Ann}(M/N)$ is Noetherian under the current assumption.)

In case $(R, \mathfrak{m})$ is local, maximal $\mathfrak{m}$-primary components were studied in [HRS]. In [Yao2, Theorem 1.3], maximal X-primary components of $N \subseteq M$ were studied for general $X \subseteq \operatorname{Ass}(M/N)$. This is stated below.

Theorem 3.3. *Let $N \subseteq M$ be R-modules such that M/N is Noetherian over R, and $X \subseteq \operatorname{Ass}(M/N)$. Say $X = \{P_1, P_2, \dots, P_r\}$ and set $U = R \setminus \big(\cup_{i=1}^r P_i\big)$. Then*

(1) $\mathring{\Lambda}_X(N \subseteq M) = \{\cap_{i=1}^r Q_i \,|\, Q_i \in \mathring{\Lambda}_{P_i}(N \subseteq M),\, 1 \leqslant i \leqslant r\}$.

Consequently, we also have the following:

(2) *For every $Q \in \Lambda_X(N \subseteq M)$, $Q = \cap\{Q' \,|\, Q' \in \mathring{\Lambda}_X(N \subseteq M),\, Q \subseteq Q'\}$. In fact, every $Q \in \Lambda_X(N \subseteq M)$ is an intersection of finitely many $Q' \in \mathring{\Lambda}_X(N \subseteq M)$.*

(3) *The intersection $\cap\{Q \,|\, Q \in \mathring{\Lambda}_X(N \subseteq M)\} = \cap\{Q \,|\, Q \in \Lambda_X(N \subseteq M)\}$ is equal to $\operatorname{Ker}\big(M \to (M/N)[U^{-1}]\big)$, which is the unique $\operatorname{o}(X)$-primary component in $\Lambda_{\operatorname{o}(X)}(N \subseteq M)$.*

Proof. We may assume $N = 0$. Then M/N can be viewed as a finitely generated module over the Noetherian ring $R/\operatorname{Ann}(M/N)$; and the same proof of [Yao2, Theorem 1.3] works here. □

Now we study the property of X-independence. Quite generally, X-independence holds when X is open in $\operatorname{Ass}'(M/N)$ by Theorem 3.3 (3) (also see [AM, Theorem 4.10]). In fact, Theorem 3.3 implies that the primary decompositions of $N \subseteq M$ are independent over X if and only if X is open in $\operatorname{Ass}'(M/N) = \operatorname{Ass}(M/N)$ under the assumption that M/N) is Noetherian.

Theorem 3.4 ([Yao2, Corollary 1.5])**.** *Let $N \subseteq M$ be R-modules such that M/N is Noetherian over R, and $X \subseteq \operatorname{Ass}(M/N)$. The following are equivalent:*

(1) *X is open in $\operatorname{Ass}(M/N)$.*
(2) *$\Lambda_X(N \subseteq M)$ consists of only one X-primary component.*
(3) *$\Lambda_X(N \subseteq M)$ is finite.*
(4) *$\mathring{\Lambda}_X(N \subseteq M)$ is finite.*

Proof. This follows from Theorem 3.3; or see the proof of [Yao2, Corollary 1.5]. □

Remark 3.5. As above, assume that M/N is Noetherian over R. By Theorem 3.4, there are infinitely many P-primary components of N in M if $P \in \operatorname{Ass}(M/N)$ is an embedded prime.

4. Linear growth of primary components

Swanson showed the following *linear growth* property concerning the primary decompositions of I^n in R:

Theorem 4.1 ([Sw1])**.** *Let R be a Noetherian ring and I an ideal of R. Then there exists $k \in \mathbb{N} := \{0, 1, 2, \dots\}$ such that, for every $n \in \mathbb{N}$, there exists a primary decomposition (of I^n in R)*

$$I^n = Q_{n,1} \cap Q_{n,2} \cap \cdots \cap Q_{n,s(n)} \quad \textit{(with } Q_{n,i} \textit{ being } P_{n,i}\textit{-primary in } R)$$

such that $(P_{n,i})^{kn} \subseteq Q_{n,i}$ *for all* $i = 1, 2, \ldots, s(n)$.

This was later generalized to any Noetherian R-module M together with several ideals in [Sh5] via a study of injective modules. The same result was also later obtained in [Yao1, Yao2] via different methods. In [Yao2], this kind of property was also proved for families of Tor and Ext modules. (See Theorem 4.4 for the precise statements.)

Inspired by the above, we formulate the following definition of the linear growth property of primary decomposition.

Definition 4.2. Given a family $\mathcal{F} = \{M_a \,|\, a = (a_1, a_2, \ldots, a_r) \in \mathbb{N}^r\}$ consisting of R-modules, we say $\mathcal{F}$ satisfies the *linear growth* property of primary decomposition (over R) if there exists $k \in \mathbb{N}$ such that, for every $a = (a_1, a_2, \ldots, a_r) \in \mathbb{N}^r$ such that $M_a \neq 0$, there exists a primary decomposition of 0 in M_a,

$$0 = Q_{a,1} \cap Q_{a,2} \cap \cdots \cap Q_{a,s(a)} \quad (\text{with } Q_{a,i} \text{ being } P_{a,i}\text{-primary in } M_a)$$

such that $(P_{a,i})^{k|a|}M_a \subseteq Q_{a,i}$ for all $i = 1, 2, \ldots, s(a)$, where $|a| = a_1 + \cdots + a_r$.

When the above occurs, we refer to k as a *slope* of $\mathcal{F}$. (Clearly, if k is a slope of $\mathcal{F}$, then all the integers greater than k are also slopes of $\mathcal{F}$.)

The linear growth property is a measure of the 'sizes' of the primary components as $a \in \mathbb{N}^r$ varies. Roughly speaking, it says that there are primary decompositions in which the primary components are "not too small".

Next, we set up some notation, which will also be used in §5 and §11.

Notation 4.3. Let R be a ring, I_i, J_j ideals of R and X_i, Y_j indeterminates, for $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$ with s and t positive integers.

(1) By $m \in \mathbb{Z}^s$, we mean $m := (m_1, \ldots, m_s) \in \mathbb{Z}^s$; similarly for $n \in \mathbb{Z}^t$.
(2) For $m \in \mathbb{Z}^s$ and $n \in \mathbb{Z}^t$, denote $(m, n) := (m_1, \ldots, m_s, n_1, \ldots, n_t) \in \mathbb{Z}^{s+t}$.
(3) For any ideal I of R, $I^e = R$ if $e \leqslant 0$.
(4) For $m \in \mathbb{Z}^s$ and $n \in \mathbb{Z}^t$, denote $I^m := I_1^{m_1} \cdots I_s^{m_s}$ and $J^n := J_1^{n_1} \cdots J_t^{n_t}$.
(5) For $m \in \mathbb{Z}^s$ and $n \in \mathbb{Z}^t$, denote $X^m := X_1^{m_1} \cdots X_s^{m_s}$ and $Y^n := Y_1^{n_1} \cdots Y_t^{n_t}$.
(6) Denote $\mathbb{N} = \{i \,|\, i \in \mathbb{Z},\, i \geqslant 0\} = \{0, 1, 2, \ldots\}$.
(7) For all $m \in \mathbb{N}^s$ and $n \in \mathbb{N}^t$ (so that $(m, n) \in \mathbb{N}^{s+t}$), denote $|m| = \sum_{i=1}^s m_i$, $|n| = \sum_{j=1}^t n_j$ and $|(m, n)| = |m| + |n|$.
(8) By $0 \in \mathbb{Z}^s$, we mean $0 := (0, \ldots, 0) \in \mathbb{Z}^s$; similarly for $0 \in \mathbb{Z}^t$.
(9) Denote $e_i := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^s$, with the i-th component 1.
(10) Denote $f_j := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^t$, with the j-th component 1.

We list some results on the linear growth property, including [Sw1], as follows:

Theorem 4.4. *Let A be a Noetherian ring, M a finitely generated A-module, R an A-algebra, N a Noetherian R-module, and $J_1, \ldots, J_t$ ideals of R. Then each of the following families of R-modules has the linear growth property for primary decomposition (over R):*

(1) *The family* $\{N/J^nN \mid n \in \mathbb{N}^t\}$*; see* [Sw1, Sh5, Yao1, Yao2].
(2) *The family* $\{R/\overline{J^n} \mid n \in \mathbb{N}^t\}$ *if* R *is Noetherian; see* [Sh4].
(3) *The family* $\{\operatorname{Tor}_c^A(M, N/J^nN) \mid n \in \mathbb{N}^t\}$*; see* [Yao2].
(4) *The family* $\{\operatorname{Ext}_A^c(M, N/J^nN) \mid n \in \mathbb{N}^t\}$*; see* [Yao2].

Note that, in Theorem 4.4, N is a finitely generated module over $R/\operatorname{Ann}_R(N)$, which is a Noetherian A-algebra. Also note that each of (3) and (4) recovers (1) as a special case. In fact, both (3) and (4) are direct consequences of the following:

Theorem 4.5 ([Yao2, Theorem 3.2])**.** *Let* A *be a ring and* R *an* A*-algebra. Let* N *be any Noetherian* R*-module,* $J_1, \ldots, J_t$ *fixed ideals of* R*, and* $c \in \mathbb{Z}$*. Fix any complex*

$$F_\bullet : \qquad \cdots \longrightarrow F_{c+1} \longrightarrow F_c \longrightarrow F_{c-1} \longrightarrow \cdots$$

of finitely generated flat A*-modules. For any* $n \in \mathbb{N}^t$*, denote*

$$E_n = \mathrm{H}^c\left(\operatorname{Hom}_A(F_\bullet, \tfrac{N}{J^nN})\right) \qquad \text{and} \qquad T_n = \mathrm{H}_c\left(F_\bullet \otimes_A \tfrac{N}{J^nN}\right),$$

the c*-th cohomology and homology of the respective complexes. Then the family* $\{E_n \mid n \in \mathbb{N}^t\}$ *and the family* $\{T_n \mid n \in \mathbb{N}^t\}$*, both consisting of finitely generated* R*-modules, satisfy the linear growth property of primary decomposition over* R*.*

Proof. This was essentially proved in [Yao2, Theorem 3.2]: By replacing R with $R/\operatorname{Ann}_R(N)$, we may assume R is Noetherian. Then, for each i, $F_i \otimes_A R$ is flat and finitely presented over R. Hence $F_\bullet \otimes_A R$ is a complex of finitely generated projective modules over R. By Hom-$\otimes$ adjointness and associativity of tensor,

$$E_n \cong \mathrm{H}^c\left(\operatorname{Hom}_R(F_\bullet \otimes_A R, \tfrac{N}{J^nN})\right) \qquad \text{and} \qquad T_n \cong \mathrm{H}_c\left((F_\bullet \otimes_A R) \otimes_R \tfrac{N}{J^nN}\right).$$

Now [Yao2, Theorem 3.2] applies, which completes the proof. $\square$

Theorem 4.5 will be used to prove the linear growth property of primary decompositions for $\left\{\operatorname{Tor}_c^R\left(\frac{M}{I^mM}, \frac{N}{J^nN}\right) \,\middle|\, (m,n) \in \mathbb{N}^s \times \mathbb{N}^t\right\}$ in the next section §5.

We end this section with an easy fact concerning the linear growth property of primary decomposition.

Lemma 4.6. *Let* $h\colon A \to R$ *be a ring homomorphism,* $\{M_n \mid n \in \mathbb{N}^t\}$ *a family of* R*-modules,* $\{K_n \mid n \in \mathbb{N}^t\}$ *a family of* A*-modules such that* $K_n \subseteq M_n$ *as* A*-modules for all* $n \in \mathbb{N}^t$*, and* U *a multiplicative subset of* R*.*

If $\{M_n \mid n \in \mathbb{N}^t\}$ *satisfies the linear growth property of primary decomposition over* R *with a slope* k*, then* $\{K_n \mid n \in \mathbb{N}^t\}$ *and* $\{M_n[U^{-1}] \mid n \in \mathbb{N}^t\}$ *satisfy the linear growth property of primary decomposition over* A *and* $R[U^{-1}]$ *respectively with the same slope* k*.*

Proof. This follows (almost immediately) from Theorem 1.1 (3)(4). $\square$

5. Linear growth of $\left\{\operatorname{Tor}_c^R\left(\frac{M}{I^m M}, \frac{N}{J^n N}\right)\right\}$

Assume that R is a Noetherian ring, $I_1, \ldots, I_s$, $J_1, \ldots, J_t$ are ideals of R, M and N are finitely generated R-modules, and $c \in \mathbb{Z}$. For all $m \in \mathbb{N}^s$ and all $n \in \mathbb{N}^t$, denote (cf. Notation 4.3)

$$T_{(m,n)} := \operatorname{Tor}_c^R\left(\frac{M}{I^m M}, \frac{N}{J^n N}\right) \quad \text{and} \quad E_{(m,n)} := \operatorname{Ext}_R^c\left(\frac{M}{I^m M}, \frac{N}{J^n N}\right).$$

The families $\left\{T_{(m,n)} \,|\, (m,n) \in \mathbb{N}^s \times \mathbb{N}^t\right\}$ and $\left\{E_{(m,n)} \,|\, (m,n) \in \mathbb{N}^s \times \mathbb{N}^t\right\}$ consist of finitely generated R-modules indexed by $\mathbb{N}^s \times \mathbb{N}^t = \mathbb{N}^{s+t}$.

In [Yao2], the author asked whether the family $\left\{T_{(m,n)}\right\}$ or $\left\{E_{(m,n)}\right\}$ could satisfy the linear growth property of primary decomposition. Although this is still open for $\left\{E_{(m,n)} \mid (m,n) \in \mathbb{N}^s \times \mathbb{N}^t\right\}$ (see Question 5.6), we are going to establish this for $\left\{T_{(m,n)} \mid (m,n) \in \mathbb{N}^s \times \mathbb{N}^t\right\}$ in this section. In fact, it is a corollary of the following theorem.

Theorem 5.1. *Let R be a ring, A and B flat R-algebras such that A, B and $A \otimes_R B$ are all Noetherian rings. Let A' and B' be homomorphic images (i.e., quotient rings) of A and B respectively, M a finitely generated A'-module, $I_1, \ldots, I_s$ ideals of A', N a finitely generated B'-module, and $J_1, \ldots, J_t$ ideals of B'. Fix any $c \in \mathbb{Z}$.*

Then the family $\left\{\operatorname{Tor}_c^R\left(\frac{M}{I^m M}, \frac{N}{J^n N}\right) \,\middle|\, (m,n) \in \mathbb{N}^s \times \mathbb{N}^t = \mathbb{N}^{s+t}\right\}$ satisfies the linear growth property of primary decomposition over (the Noetherian ring) $A' \otimes_R B'$.

Proof. It suffices to prove the linear growth property over $A \otimes_R B$, which maps onto $A' \otimes_R B'$. Thus, without loss of generality, we may assume $A = A'$ and $B = B'$.

There exists $g \in \mathbb{N}$, large enough, such that

$$I_i = (x_{i1}, \ldots, x_{ig})A \qquad \text{and} \qquad J_j = (y_{j1}, \ldots, y_{jg})B$$

in which $x_{ik} \in A$ and $y_{jk} \in B$ for all $i \in \{1, \ldots, s\}$, all $j \in \{1, \ldots, t\}$ and all $k \in \{1, \ldots, g\}$. (We pick a uniform g only to make the notation simpler.)

Define the following ($\mathbb{Z}^s$-graded) rings and module (cf. Notation 4.3):

$$\begin{aligned}
\mathscr{A} &:= A[X_{ik}, X_i \,|\, 1 \leqslant k \leqslant g,\, 1 \leqslant i \leqslant s], \\
\mathcal{A} &:= \bigoplus_{m \in \mathbb{Z}^s} I^m X^{-m} = A[I_i X_i^{-1}, X_i \,|\, 1 \leqslant i \leqslant s] \subseteq A[X_i^{-1}, X_i \,|\, 1 \leqslant i \leqslant s], \\
\mathcal{M} &:= \bigoplus_{m \in \mathbb{Z}^s} I^m M X^{-m},
\end{aligned}$$

in which X_{ik} and X_i are (independent) variables. Both $\mathscr{A}$ and $\mathcal{A}$ are naturally rings via the polynomial operations, and $\mathcal{M}$ is naturally an $\mathcal{A}$-module, which is finitely generated. Moreover, we make all of them $\mathbb{Z}^s$-graded by assigning degrees as follows (cf. Notation 4.3):

$$\begin{aligned}
&\deg(A) = \deg(M) = 0 := (0, \dots, 0) \in \mathbb{Z}^s,\\
&\deg(X_{ik}) = \deg(X_i^{-1}) = e_i := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^s,\\
&\deg(X_i) = -e_i = (0, \dots, 0, -1, 0, \dots, 0) \in \mathbb{Z}^s.
\end{aligned}$$

Under the gradings, $\mathcal{M}$ is a graded $\mathcal{A}$-module. There is a surjective homogeneous A-algebra homomorphism $\phi\colon \mathscr{A} \twoheadrightarrow \mathcal{A}$ determined by

$$X_{ik} \longmapsto x_{ik}X_i^{-1}, \qquad X_i \longmapsto X_i.$$

This makes $\mathcal{M}$ a finitely generated graded module over $\mathscr{A}$. (Clearly, both $\mathscr{A}$ and $\mathcal{A}$ are finitely generated A-algebras and hence Noetherian.)

Similarly, we define the following $\mathbb{Z}^t$-graded rings and module (cf. Notation 4.3):

$$\begin{aligned}
\mathscr{B} &:= B[Y_{jk}, Y_j \mid 1 \leqslant k \leqslant g,\ 1 \leqslant j \leqslant t],\\
\mathcal{B} &:= \bigoplus_{n \in \mathbb{Z}^t} J^n Y^{-n} = B[J_jY_j^{-1}, Y_j \mid 1 \leqslant j \leqslant t] \subseteq B[Y_j^{-1}, Y_j \mid 1 \leqslant j \leqslant t],\\
\mathcal{N} &:= \bigoplus_{n \in \mathbb{Z}^t} J^n N Y^{-n},
\end{aligned}$$

with Y_{jk} and Y_j variables and with the gradings given by (cf. Notation 4.3)

$$\begin{aligned}
&\deg(B) = \deg(N) = 0 := (0, \dots, 0) \in \mathbb{Z}^t,\\
&\deg(Y_{jk}) = \deg(Y_j^{-1}) = f_j := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^t,\\
&\deg(Y_j) = -f_j = (0, \dots, 0, -1, 0, \dots, 0) \in \mathbb{Z}^t.
\end{aligned}$$

There is a surjective homogeneous B-algebra homomorphism $\psi\colon \mathscr{B} \twoheadrightarrow \mathcal{B}$ given by

$$Y_{jk} \longmapsto y_{jk}Y_j^{-1}, \qquad Y_j \longmapsto Y_j.$$

This makes $\mathcal{N}$ a finitely generated graded module over $\mathscr{B}$, since $\mathcal{N}$ is (naturally) a finitely generated graded module over $\mathcal{B}$. (Clearly, both $\mathscr{B}$ and $\mathcal{B}$ are finitely generated B-algebras and hence Noetherian.)

We now consider $\mathscr{C} := \mathscr{A} \otimes_R \mathscr{B}$ and $\mathcal{C} := \mathcal{A} \otimes_R \mathcal{B}$, which are clearly Noetherian (since they are finitely generated algebras over $A \otimes_R B$). In the sequel, we use $[-]_h$ to denote the h-th homogeneous component of a graded module. (For example, $[\mathscr{A}]_\alpha$ stands for the homogeneous component of $\mathscr{A}$ of degree α with the understanding that $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$, since $\mathscr{A}$ is $\mathbb{Z}^s$-graded.) Keeping this in mind, we observe that both $\mathscr{C}$ and $\mathcal{C}$ are naturally $\mathbb{Z}^{s+t}$-graded rings with

$$\begin{aligned}
&[\mathscr{C}]_{(\alpha,\beta)} = [\mathscr{A} \otimes_R \mathscr{B}]_{(\alpha,\beta)} = [\mathscr{A}]_\alpha \otimes_R [\mathscr{B}]_\beta \quad \text{and}\\
&[\mathcal{C}]_{(\alpha,\beta)} = [\mathcal{A} \otimes_R \mathcal{B}]_{(\alpha,\beta)} = [\mathcal{A}]_\alpha \otimes_R [\mathcal{B}]_\beta
\end{aligned}$$

for all $(\alpha,\beta)\in\mathbb{Z}^s\times\mathbb{Z}^t=\mathbb{Z}^{s+t}$. In particular, for $(0,0)\in\mathbb{Z}^s\times\mathbb{N}^t$, we have

$$[\mathscr{C}]_{(0,0)}=[\mathscr{A}\otimes_R\mathscr{B}]_{(0,0)}=[\mathscr{A}]_0\otimes_R[\mathscr{B}]_0=A\otimes_R B \quad \text{and}$$
$$[\mathcal{C}]_{(0,0)}=[\mathcal{A}\otimes_R\mathcal{B}]_{(0,0)}=[\mathcal{A}]_0\otimes_R[\mathcal{B}]_0=A\otimes_R B.$$

Moreover, the surjective homogeneous R-algebra homomorphisms $\phi\colon\mathscr{A}\twoheadrightarrow\mathcal{A}$ and $\psi\colon\mathscr{B}\twoheadrightarrow\mathcal{B}$ induce an surjective homogeneous R-algebra homomorphism

$$\phi\otimes\psi\colon\mathscr{C}\twoheadrightarrow\mathcal{C}.$$

Write down graded free resolutions of $\mathcal{M}$ over $\mathscr{A}$ and of $\mathcal{N}$ over $\mathscr{B}$ respectively by (free) modules of finite ranks (over $\mathscr{A}$ and over $\mathscr{B}$ respectively)

$$\begin{aligned}\mathscr{F}_\bullet:&\qquad\cdots\longrightarrow\mathscr{F}_i\longrightarrow\mathscr{F}_{i-1}\longrightarrow\cdots\longrightarrow\mathscr{F}_1\longrightarrow\mathscr{F}_0\ (\longrightarrow\mathcal{M})\longrightarrow 0,\\ \mathscr{G}_\bullet:&\qquad\cdots\longrightarrow\mathscr{G}_j\longrightarrow\mathscr{G}_{j-1}\longrightarrow\cdots\longrightarrow\mathscr{G}_1\longrightarrow\mathscr{G}_0\ (\longrightarrow\mathcal{N})\longrightarrow 0.\end{aligned}$$

Then $\mathscr{F}_\bullet\otimes_R\mathscr{G}_\bullet$ is (naturally) a $\mathbb{Z}^{s+t}$-graded complex composed of finitely generated free $\mathscr{C}$-modules over the $\mathbb{Z}^{s+t}$-graded ring $\mathscr{A}\otimes_R\mathscr{B}=:\mathscr{C}$.

By abuse of notation, we use X^mY^n to denote $(X^m\otimes 1)(1\otimes Y^n)=X^m\otimes Y^n\in\mathscr{C}$. By Theorem 4.5, the following family (of $\mathscr{C}$-modules)

$$\left\{\mathrm{H}_c\left((\mathscr{F}_\bullet\otimes_R\mathscr{G}_\bullet)\otimes_{\mathscr{C}}\frac{\mathscr{C}}{(X^mY^n)}\right)\;\middle|\;(m,n)\in\mathbb{N}^s\times\mathbb{N}^t=\mathbb{N}^{s+t}\right\}$$

has the linear growth property of primary decomposition over $\mathscr{C}=\mathscr{A}\otimes_R\mathscr{B}$.

We are going to show that the above linear growth property implies the linear growth property of $\left\{\mathrm{Tor}_c^R\left(\frac{M}{I^mM},\frac{N}{J^nN}\right)\mid(m,n)\in\mathbb{N}^s\times\mathbb{N}^t=\mathbb{N}^{s+t}\right\}$ over $A\otimes_R B$.

Firstly, for all $(m,n)\in\mathbb{N}^s\times\mathbb{N}^t$, the modules $\mathrm{H}_c\left((\mathscr{F}_\bullet\otimes_R\mathscr{G}_\bullet)\otimes_{\mathscr{C}}\frac{\mathscr{C}}{(X^mY^n)}\right)$ are annihilated by $\mathrm{Ker}(\phi\otimes\psi)$; so, naturally, they are all graded modules over $\mathcal{C}=\mathcal{A}\otimes_R\mathcal{B}$. (This follows directly from how $\mathcal{M}$, $\mathcal{N}$, $\mathscr{F}_\bullet$ and $\mathscr{G}_\bullet$ are constructed: Multiplication by every element in $\mathrm{Ker}(\phi)$ (resp. $\mathrm{Ker}(\psi)$) is homotopic to 0 on $\mathscr{F}_\bullet$ (resp. $\mathscr{G}_\bullet$); and $\mathrm{Ker}(\phi\otimes\psi)$ is generated by $\mathrm{Ker}(\phi)$ and $\mathrm{Ker}(\psi)$ since both ϕ and ψ are surjective.) Hence $\left\{\mathrm{H}_c\left((\mathscr{F}_\bullet\otimes_R\mathscr{G}_\bullet)\otimes_{\mathscr{C}}\frac{\mathscr{C}}{(X^mY^n)}\right)\;\middle|\;(m,n)\in\mathbb{N}^s\times\mathbb{N}^t=\mathbb{N}^{s+t}\right\}$ has the linear growth property of primary decomposition over $\mathcal{C}$.

Secondly, for every $m\in\mathbb{N}^s$ and $n\in\mathbb{N}^t$, there is a canonical homogeneous isomorphism of $(\mathscr{A}\otimes_R\mathscr{B})$-complexes

$$(\mathscr{F}_\bullet\otimes_R\mathscr{G}_\bullet)\otimes_{\mathscr{C}}\frac{\mathscr{C}}{(X^mY^n)}\cong\left(\mathscr{F}_\bullet\otimes_{\mathscr{A}}\frac{\mathscr{A}}{(X^m)}\right)\otimes_R\left(\mathscr{G}_\bullet\otimes_{\mathscr{B}}\frac{\mathscr{B}}{(Y^n)}\right).$$

Therefore, for each $(\alpha,\beta)\in\mathbb{Z}^s\times\mathbb{Z}^t$, there is an isomorphism between the following $(\mathscr{A}_0\otimes_R\mathscr{B}_0)$-complexes

$$\left[(\mathscr{F}_\bullet\otimes_R\mathscr{G}_\bullet)\otimes_{\mathscr{C}}\frac{\mathscr{C}}{(X^mY^n)}\right]_{(\alpha,\beta)}\cong\left[\left(\mathscr{F}_\bullet\otimes_{\mathscr{A}}\frac{\mathscr{A}}{(X^m)}\right)\right]_\alpha\otimes_R\left[\left(\mathscr{G}_\bullet\otimes_{\mathscr{B}}\frac{\mathscr{B}}{(Y^n)}\right)\right]_\beta.$$

Thirdly, observe that X_i is regular on both $\mathcal{M}$ and $\mathscr{A}$ for every $i \in \{1, \dots, s\}$ while Y_j is regular on both $\mathcal{N}$ and $\mathscr{B}$ for every $j \in \{1, \dots, t\}$. Thus X^m is regular on both $\mathcal{M}$ and $\mathscr{A}$ while Y^n is regular on both $\mathcal{N}$ and $\mathscr{B}$ for every $m \in \mathbb{N}^s$ and $n \in \mathbb{N}^t$. Consequently,

(a) $\mathscr{F}_\bullet \otimes_{\mathscr{A}} \frac{\mathscr{A}}{(X^m)}$ is a graded free resolution of $\frac{\mathcal{M}}{X^m\mathcal{M}}$ over graded ring $\frac{\mathscr{A}}{(X^m)}$;
(b) $\mathscr{G}_\bullet \otimes_{\mathscr{B}} \frac{\mathscr{B}}{(Y^n)}$ is a graded free resolution of $\frac{\mathcal{N}}{Y^n\mathcal{N}}$ over graded ring $\frac{\mathscr{B}}{(Y^n)}$.

Moreover, by the construction of $\mathscr{A}$ and $\mathscr{B}$, all of their homogeneous components are free A-modules and free B-modules respectively; so they are all flat R-modules. It follows that all of the homogeneous components of $\frac{\mathscr{A}}{(X^m)}$ and $\frac{\mathscr{B}}{(Y^n)}$ are free A-modules and free B-modules respectively and hence flat over R, for all $m \in \mathbb{N}^s$ and $n \in \mathbb{N}^t$. In light of this, statements (a) and (b) above imply the following (for all $m \in \mathbb{N}^s$, $n \in \mathbb{N}^t$, $\alpha \in \mathbb{Z}^s$ and $\beta \in \mathbb{Z}^t$):

($[\mathrm{a}]_\alpha$) $\left[\mathscr{F}_\bullet \otimes_{\mathscr{A}} \frac{\mathscr{A}}{(X^m)}\right]_\alpha$ is a flat resolution of $\left[\frac{\mathcal{M}}{X^m\mathcal{M}}\right]_\alpha = \frac{I^\alpha M}{I^{\alpha+m}M}$ over over R;
($[\mathrm{b}]_\beta$) $\left[\mathscr{G}_\bullet \otimes_{\mathscr{B}} \frac{\mathscr{B}}{(Y^n)}\right]_\beta$ is a flat resolution of $\left[\frac{\mathcal{N}}{Y^n\mathcal{N}}\right]_\beta = \frac{J^\beta N}{J^{\beta+n}N}$ over R.

In particular, for $\alpha = 0 \in \mathbb{Z}^s$ and $\beta = 0 \in \mathbb{Z}^t$, we have (for all $m \in \mathbb{N}^s$ and $n \in \mathbb{N}^t$)

($[\mathrm{a}]_0$) $\left[\mathscr{F}_\bullet \otimes_{\mathscr{A}} \frac{\mathscr{A}}{(X^m)}\right]_0$ is a flat resolution of $\left[\frac{\mathcal{M}}{X^m\mathcal{M}}\right]_0 = \frac{M}{I^mM}$ over R;
($[\mathrm{b}]_0$) $\left[\mathscr{G}_\bullet \otimes_{\mathscr{B}} \frac{\mathscr{B}}{(Y^n)}\right]_0$ is a flat resolution of $\left[\frac{\mathcal{N}}{Y^n\mathcal{N}}\right]_0 = \frac{N}{J^nN}$ over R.

Now we study $\mathrm{H}_c\left((\mathscr{F}_\bullet \otimes_R \mathscr{G}_\bullet) \otimes_{\mathscr{C}} \frac{\mathscr{C}}{(X^mY^n)}\right)$, which is a $\mathbb{Z}^{s+t}$-graded module, in terms of its homogeneous components. Recall that $[\mathscr{C}]_{(0,0)} = A \otimes_R B = [\mathcal{C}]_{(0,0)}$. Combining the three paragraphs above, we obtain the following isomorphisms over $A \otimes_R B$:

$$
\begin{aligned}
&\mathrm{H}_c\left((\mathscr{F}_\bullet \otimes_R \mathscr{G}_\bullet) \otimes_{\mathscr{C}} \frac{\mathscr{C}}{(X^mY^n)}\right) \\
&\qquad = \bigoplus_{(\alpha,\beta)\in\mathbb{Z}^s\times\mathbb{Z}^t} \left[\mathrm{H}_c\left((\mathscr{F}_\bullet \otimes_R \mathscr{G}_\bullet) \otimes_{\mathscr{C}} \frac{\mathscr{C}}{(X^mY^n)}\right)\right]_{(\alpha,\beta)} \\
&\qquad = \bigoplus_{(\alpha,\beta)\in\mathbb{Z}^s\times\mathbb{Z}^t} \mathrm{H}_c\left(\left[(\mathscr{F}_\bullet \otimes_R \mathscr{G}_\bullet) \otimes_{\mathscr{C}} \frac{\mathscr{C}}{(X^mY^n)}\right]_{(\alpha,\beta)}\right) \\
&\qquad \cong \bigoplus_{(\alpha,\beta)\in\mathbb{Z}^s\times\mathbb{Z}^t} \mathrm{H}_c\left(\left[\mathscr{F}_\bullet \otimes_{\mathscr{A}} \frac{\mathscr{A}}{(X^m)}\right]_\alpha \otimes_R \left[\mathscr{G}_\bullet \otimes_{\mathscr{B}} \frac{\mathscr{B}}{(Y^n)}\right]_\beta\right) \\
&\qquad = \bigoplus_{(\alpha,\beta)\in\mathbb{Z}^s\times\mathbb{Z}^t} \mathrm{Tor}_c^R\left(\frac{I^\alpha M}{I^{\alpha+m}M}, \frac{J^\beta N}{J^{\beta+n}N}\right)
\end{aligned}
$$

for all $(m,n) \in \mathbb{N}^s \times \mathbb{N}^t$. In particular, for all $(m,n) \in \mathbb{N}^s \times \mathbb{N}^t$,

$$\begin{aligned}\operatorname{Tor}_c^R\left(\frac{M}{I^m M}, \frac{N}{J^n N}\right) &\cong \left[\mathrm{H}_c\left((\mathscr{F}_\bullet \otimes_R \mathscr{G}_\bullet) \otimes_{\mathscr{C}} \frac{\mathscr{C}}{(X^m Y^n)}\right)\right]_{(0,0)} \\ &\subseteq \mathrm{H}_c\left((\mathscr{F}_\bullet \otimes_R \mathscr{G}_\bullet) \otimes_{\mathscr{C}} \frac{\mathscr{C}}{(X^m Y^n)}\right)\end{aligned}$$

as $(A \otimes_R B)$-modules.

In summary, the family $\left\{\mathrm{H}_c\left((\mathscr{F}_\bullet \otimes_R \mathscr{G}_\bullet) \otimes_{\mathscr{C}} \frac{\mathscr{C}}{(X^m Y^n)}\right) \;\middle|\; (m,n) \in \mathbb{N}^s \times \mathbb{N}^t\right\}$ satisfies the linear growth property of primary decomposition over the graded ring $\mathcal{C}$ with $[\mathcal{C}]_{(0,0)]} = A \otimes_R B$; and for each $(m,n) \in \mathbb{N}^s \times \mathbb{N}^t$, $\operatorname{Tor}_c^R\left(\frac{M}{I^m M}, \frac{N}{J^n N}\right)$ is an $(A \otimes_R B)$-submodule of $\mathrm{H}_c\left((\mathscr{F}_\bullet \otimes_R \mathscr{G}_\bullet) \otimes_{\mathscr{C}} \frac{\mathscr{C}}{(X^m Y^n)}\right)$ up to isomorphism.

Finally, by Lemma 4.6, the family

$$\left\{\operatorname{Tor}_c^R\left(\frac{M}{I^m M}, \frac{N}{J^n N}\right) \;\middle|\; (m,n) \in \mathbb{N}^s \times \mathbb{N}^t = \mathbb{N}^{s+t}\right\}$$

satisfies the linear growth of primary decomposition over $A \otimes_R B$. □

In fact, the proof of Theorem 5.1 implies the following (apparently) stronger result concerning infinitely many families and a *uniform* slope (cf. Definition 4.2).

Theorem 5.2. *Keep the notation and the assumptions in Theorem 5.1.*

Then there exists k such that for all $(\alpha,\beta) \in \mathbb{Z}^s \times \mathbb{Z}^t$, the family

$$\boldsymbol{\mathcal{T}}^{(\alpha,\beta)} := \left\{\operatorname{Tor}_c^R\left(\frac{I^\alpha M}{I^{\alpha+m} M}, \frac{J^\beta N}{J^{\beta+n} N}\right) \;\middle|\; (m,n) \in \mathbb{N}^s \times \mathbb{N}^t = \mathbb{N}^{s+t}\right\}$$

satisfies the linear growth property of primary decomposition over $A' \otimes_R B'$ with the uniform slope k. More explicitly, for every $(\alpha,\beta) \in \mathbb{Z}^{s+t}$ and $(m,n) \in \mathbb{N}^{s+t}$ such that $\operatorname{Tor}_c^R\left(\frac{I^\alpha M}{I^{\alpha+m} M}, \frac{J^\beta N}{J^{\beta+n} N}\right) \neq 0$, there exists a primary decomposition of 0 in $\operatorname{Tor}_c^R\left(\frac{I^\alpha M}{I^{\alpha+m} M}, \frac{J^\beta N}{J^{\beta+n} N}\right)$ over $A' \otimes_R B'$,

$$0 = Q_{\alpha,\beta,m,n,1} \cap Q_{\alpha,\beta,m,n,2} \cap \cdots \cap Q_{\alpha,\beta,m,n,s(\alpha,\beta,m,n)},$$

with $Q_{\alpha,\beta,m,n,i}$ being $P_{\alpha,\beta,m,n,i}$-primary in $\operatorname{Tor}_c^R\left(\frac{I^\alpha M}{I^{\alpha+m} M}, \frac{J^\beta N}{J^{\beta+n} N}\right)$, such that

$$(P_{\alpha,\beta,m,n,i})^{k|(m,n)|} \operatorname{Tor}_c^R\left(\tfrac{I^\alpha M}{I^{\alpha+m} M}, \tfrac{J^\beta N}{J^{\beta+n} N}\right) \subseteq Q_{\alpha,\beta,m,n,i}$$

for all $i = 1, 2, \ldots, s(\alpha,\beta,m,n)$.

Proof. As seen in the proof of Theorem 5.1, for all $(m,n) \in \mathbb{N}^s \times \mathbb{N}^t$ and for all $(\alpha,\beta) \in \mathbb{Z}^s \times \mathbb{Z}^t$, we have

$$\begin{aligned}\operatorname{Tor}_c^R\left(\frac{I^\alpha M}{I^{\alpha+m}M}, \frac{J^\beta N}{J^{\beta+n}N}\right) &\cong \left[\mathrm{H}_c\left((\mathscr{F}_\bullet \otimes_R \mathscr{G}_\bullet) \otimes_{\mathscr{C}} \frac{\mathscr{C}}{(X^m Y^n)}\right)\right]_{(\alpha,\beta)} \\ &\subseteq \mathrm{H}_c\left((\mathscr{F}_\bullet \otimes_R \mathscr{G}_\bullet) \otimes_{\mathscr{C}} \frac{\mathscr{C}}{(X^m Y^n)}\right)\end{aligned}$$

as $(A \otimes_R B)$-modules.

Say k is a slope for $\left\{\mathrm{H}_c\left((\mathscr{F}_\bullet \otimes_R \mathscr{G}_\bullet) \otimes_{\mathscr{C}} \frac{\mathscr{C}}{(X^m Y^n)}\right) \;\middle|\; (m,n) \in \mathbb{N}^s \times \mathbb{N}^t = \mathbb{N}^{s+t}\right\}$ over $\mathcal{C}$. By Lemma 4.6, all the families $\boldsymbol{\mathcal{T}}^{(\alpha,\beta)}$, $(\alpha,\beta) \in \mathbb{Z}^s \times \mathbb{Z}^t$, satisfy the linear growth property of primary decomposition over $A \otimes_R B$ with the same slope k. □

Remark 5.3. Recall that an R-algebra S is said to be *essentially of finite type over R* if $S \cong T[U^{-1}]$ with T a finitely generated R-algebra and U a multiplicative subset of T. We remark that Theorems 5.1 and 5.2 apply when A' and B' are essentially of finite type over R. This is because one can then let A and B be of the form $T[U^{-1}]$ with T being a polynomial ring over R (hence flat over R) with finitely many variables.

Remark 5.4. Note that Theorems 5.1 and 5.2 include the case of $s = 0$ or $t = 0$. For example, when $s = 0$, Theorem 5.1 states that the family $\left\{\operatorname{Tor}_c^R\left(M, \frac{N}{J^n N}\right) \;\middle|\; n \in \mathbb{N}^t\right\}$ satisfies the linear growth property of primary decomposition over $A' \otimes_R B'$, which is slightly different from Theorem 4.4 (3); and Theorem 5.2 says that, for all $\beta \in \mathbb{Z}^t$, the families $\left\{\operatorname{Tor}_c^R\left(M, \frac{J^\beta N}{J^{\beta+n}N}\right) \;\middle|\; n \in \mathbb{N}^t\right\}$ satisfy the linear growth property of primary decomposition over $A' \otimes_R B'$ with a uniform slope.

In fact, if $s = 0$, we can relax the condition on A and M by assuming that A is any R-algebra such that $A \otimes_R B$ is Noetherian and M is any finitely generated A-module, while the other assumptions remain the same. The proof is similar, but we construct $\mathscr{G}_\bullet$ only. By Theorem 4.5, the family $\left\{\mathrm{H}_c\left((A \otimes_R \mathscr{G}_\bullet) \otimes_{A \otimes_R \mathscr{B}} \frac{M \otimes_R \mathscr{B}}{Y^n(M \otimes_R \mathscr{B})}\right) \;\middle|\; n \in \mathbb{N}^t\right\}$ has linear growth property of primary decomposition over $A \otimes_R \mathscr{B}$. The rest follows in a similar way, by considering the homogeneous components graded by $\mathbb{Z}^t$. It might be helpful to note the natural homogeneous $(A \otimes_R \mathscr{B})$-isomorphisms

$$(A \otimes_R \mathscr{G}_\bullet) \otimes_{A \otimes_R \mathscr{B}} \frac{M \otimes_R \mathscr{B}}{Y^n(M \otimes_R \mathscr{B})} \cong M \otimes_R \left(\mathscr{G}_\bullet \otimes_{\mathscr{B}} \frac{\mathscr{B}}{(Y^n)}\right) \quad \text{and}$$

$$\begin{aligned}\left[\mathrm{H}_c\left(M \otimes_R \left(\mathscr{G}_\bullet \otimes_{\mathscr{B}} \frac{\mathscr{B}}{(Y^n)}\right)\right)\right]_\beta &= \mathrm{H}_c\left(M \otimes_R \left[\mathscr{G}_\bullet \otimes_{\mathscr{B}} \frac{\mathscr{B}}{(Y^n)}\right]_\beta\right) \\ &\cong \operatorname{Tor}_c^R\left(M, \frac{J^\beta N}{J^{\beta+n}N}\right)\end{aligned}$$

over $A \otimes_R B = [A \otimes_R \mathscr{B}]_0$.

As promised, we state the following corollary (when $A = R = B$).

Corollary 5.5. *Let R be a Noetherian ring, M and N finitely generated R-modules, $I_1, \ldots, I_s$, $J_1, \ldots, J_t$ ideals of R, and $c \in \mathbb{Z}$.*

Then the family $\left\{\operatorname{Tor}_c^R\left(\frac{M}{I^m M}, \frac{N}{J^n N}\right) \mid (m,n) \in \mathbb{N}^s \times \mathbb{N}^t = \mathbb{N}^{s+t}\right\}$ satisfies the linear growth property of primary decomposition over R.

More generally, the families $\left\{\operatorname{Tor}_c^R\left(\frac{I^\alpha M}{I^{\alpha+m} M}, \frac{J^\beta N}{J^{\beta+n} N}\right) \;\middle|\; (m,n) \in \mathbb{N}^s \times \mathbb{N}^t\right\}$, for all $(\alpha, \beta) \in \mathbb{Z}^s \times \mathbb{Z}^t$, satisfy the linear growth property of primary decomposition over R with a uniform slope.

Question 5.6. Keep the notation and the assumptions in Corollary 5.5. Does the family $\left\{\operatorname{Ext}_R^c\left(\frac{M}{I^m M}, \frac{N}{J^n N}\right) \mid (m,n) \in \mathbb{N}^s \times \mathbb{N}^t\right\}$ satisfy the linear growth property?

When $c \leqslant 0$, the linear growth property of primary decomposition can be easily established for $\left\{\operatorname{Ext}_R^c\left(\frac{M}{I^m M}, \frac{N}{J^n N}\right) \mid (m,n) \in \mathbb{N}^s \times \mathbb{N}^t\right\}$. For general c, the question is open even for the family $\left\{\operatorname{Ext}_R^c\left(\frac{M}{I^m M}, N\right) \mid m \in \mathbb{N}^s\right\}$.

6. Secondary representation

Secondary representations were first studied by I. G. Macdonald [Mac] and D. Kirby [Ki]. The theory can be viewed as a dual of the theory of primary decomposition. (See Theorem 8.2 and Observation 8.4 for example, where this duality is demonstrated explicitly.) For this reason, it was called *coprimary decomposition* in [Ki]. Systematic treatment of secondary representation can be found in many sources, for example, see [Ki], [Mac], [Mat], and [Sh1] as well as many other papers authored or co-authored by R. Y. Sharp.

Assume that R is a ring (not necessarily Noetherian) and M is an R-module. In this section, we briefly review some of the basic definitions and properties.

We say that M is *secondary* if, for all $r \in R$, either $rM = M$ or $r \in \sqrt{\operatorname{Ann}(M)}$. (Note that, under this definition, 0 is a secondary module.) If $M \neq 0$ is secondary, then $P := \sqrt{\operatorname{Ann}(M)}$ is a prime ideal; and we say M is P-secondary in this case.

It is easy to see that if M is P-secondary then, for any multiplicatively closed subset U of R and any finitely generated ideal I of R, we have

$$[U]M = \begin{cases} M & \text{if } P \cap U = \varnothing \\ 0 & \text{if } P \cap U \neq \varnothing \end{cases} \qquad \text{and} \qquad \cap_{i \in \mathbb{N}} \left(I^i M\right) = \begin{cases} M & \text{if } I \not\subseteq P \\ 0 & \text{if } I \subseteq P \end{cases}$$

in which $[U]M := \cap_{u \in U}(uM)$.

For a general R-module M and a prime ideal P, we say that P is *attached* to M if there is an R-submodule N of M such that M/N is P-secondary, that is, a homomorphic image of M is P-secondary. The set of all the primes attached to M is denoted $\operatorname{Att}_R(M)$, or simply $\operatorname{Att}(M)$ if R is understood. (Note that Att and Ass$''$ are dual to each other.)

As $\mathrm{Att}(M) \subseteq \mathrm{Spec}(R)$, there is a topology on $\mathrm{Att}(M)$ that is induced by the Zariski topology on $\mathrm{Spec}(R)$.

If M is P-secondary, then $\mathrm{Att}(M) = \{P\}$. If M is an Artinian R-module, then M is P-secondary $\iff$ $\mathrm{Att}(M) = \{P\}$, and $M = 0 \iff \mathrm{Att}(M) = \varnothing$.

Example 6.1. Let $(R, \mathfrak{m})$ be any Noetherian local domain, not necessarily complete. Then $\mathrm{E}_R(R/\mathfrak{m})$, the injective hull of the residue field $R/\mathfrak{m}$, is 0-secondary; so that $\mathrm{Att}_R(\mathrm{E}_R(R/\mathfrak{m})) = \{0\}$. Note that $\mathrm{E}_R(R/\mathfrak{m})$ is Artinian, and the zero ideal 0 is not the maximal ideal $\mathfrak{m}$ if $\dim(R) > 0$. (However, $\mathrm{Ass}_R(\mathrm{E}_R(R/\mathfrak{m})) = \{\mathfrak{m}\}$.)

We also note that $R/\mathfrak{m}$ is both $\mathfrak{m}$-secondary and $\mathfrak{m}$-coprimary as an R-module with $\mathrm{Att}_R(R/\mathfrak{m}) = \{\mathfrak{m}\} = \mathrm{Ass}_R(R/\mathfrak{m})$.

For an R-module $M \neq 0$, we say M is *representable* (over R) if there exist submodules Q_i that are P_i-secondary, for $i = 1, \dots, s$, such that

$$M = Q_1 + \cdots + Q_s.$$

This summation is called a *secondary representation* of M. One can always convert a secondary representation to a *minimal* one in the sense that $P_i \neq P_j$ for all $i \neq j$ and $M \neq \sum_{i \neq k} Q_i$ for every $k = 1, \dots, s$. So from now on and as a general rule, all secondary representations are assumed to be minimal unless stated otherwise explicitly.

By convention, the zero R-module 0 is representable with $0 = 0$ being the unique secondary representation.

For concrete examples of secondary representation, see Examples 9.4 and 9.5.

Here is a theorem on the existence of secondary representation, cf. [Mac].

Theorem 6.2. *Every Artinian R-module is representable (over R).*

For any R-module M and any ideal $I \subseteq \mathrm{Ann}(M)$, the following is clear: M is representable over R if and only if M is representable over R/I.

Next, we state some useful results about secondary representations; compare with Theorem 1.1. We do not need to assume M is Artinian in Theorem 6.3, as long as M is representable. In case $U = R \setminus P$ with $P \in \mathrm{Spec}(R)$, we write $M^P := [U]M$.

Theorem 6.3 (Cf. [Mac]; compare with Theorem 1.1)**.** *Let $M = Q_1 + \cdots + Q_s$ be a (minimal) secondary representation of an R-module M in which Q_i is P_i-secondary for each $i = 1, \dots, s$. Then the following hold*

(1) $\{P_1, \dots, P_s\} = \mathrm{Att}(M)$, *which is independent of the particular (minimal) secondary representation (cf.* [Mac, Theorem 2.2]*).*

(1') *We have* $\mathrm{Min}(M) \subseteq \{P_1, \dots, P_s\}$. *In fact,* $\mathrm{Min}(M)$ *consists of the minimal members of* $\{P_1, \dots, P_s\}$ *(under inclusion) precisely.*

(2) *If P_i is minimal in* $\mathrm{Att}(M)$ *(i.e., $P_i \in \mathrm{Min}(M)$), then $Q_i = M^{P_i}$. See (4).*

(3) *Let $h\colon A \to R$ be a ring homomorphism, so that M is naturally an A-module. Let K be an A-submodule of M (e.g., $K = 0$). Then M/K is representable*

over A. In fact, if $M/K \neq 0$, then

$$M/K = \sum_{Q_i \nsubseteq K} (Q_i + K)/K$$

is a (not necessarily minimal) secondary representation of M/K over A, in which $(Q_i + K)/K$ is $h^{-1}(P_i)$-secondary provided that $Q_i \nsubseteq K$.

(3') *In particular,* $\mathrm{Att}_A(M) = h^*(\mathrm{Att}_R(M))$, *in which* $h^*\colon \mathrm{Spec}(R) \to \mathrm{Spec}(A)$ *is the continuous map naturally induced by h.*

(4) *For any multiplicative subset U of R, $[U]M = \sum_{P_i \cap U = \varnothing} Q_i$ is a secondary representation over R (cf.* [Mac, Theorem 3.1]*).*

(5) *For any finitely generated ideal I of R, $\cap_{j\in\mathbb{N}}(I^j M) = \sum_{I \nsubseteq P_i} Q_i = I^n M$ for all $n \gg 0$ (cf.* [Mac, Theorem 3.3]*).*

(5') *For any non-empty subset I of R, $\cup_{r\in I}(\cap_{n\in\mathbb{N}}(r^n M)) = \sum_{I \nsubseteq P_i} Q_i$.*

Very much like $\mathrm{Ass}(-)$ (as well as Ass', Ass'' and $\mathrm{Ass}_{\mathrm{f}}$), the sets of attached primes are relatively well-behaved with exact sequences, as stated in the following well-known lemma. This will be referred to in the proof of Lemma 7.2.

Lemma 6.4 (Compare with Lemma 1.4)**.** *Let $M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules. Then* $\mathrm{Att}(M_3) \subseteq \mathrm{Att}(M_2) \subseteq \mathrm{Att}(M_1) \cup \mathrm{Att}(M_3)$.

Moreover, $\mathrm{Att}(\oplus_{i=1}^n K_i) = \cup_{i=1}^n \mathrm{Att}(K_i)$ *for R-modules $K_1, \ldots, K_n$ with $n \in \mathbb{N}$.*

7. Compatibility of secondary components

Throughout this section, we assume that R is a (not necessarily Noetherian) ring and M a (not necessarily Artinian) R-module. The reader should observe the similarity (or rather, "duality") between this section and §2.

The results in this section were obtained in [Yao3].

Notation 7.1. Let M be a representable R-module and $X \subseteq \mathrm{Att}(M)$. Say $X = \{P_1, \ldots, P_r\} \subseteq \{P_1, \ldots, P_r, \ldots, P_s\} = \mathrm{Att}(M)$.

(1) If $M = Q_1 + \cdots + Q_r + \cdots + Q_s$ is a secondary representation of M with Q_i being P_i-secondary, then we say $Q = Q_1 + \cdots + Q_r$ is an X-secondary component (or a secondary component over X) of M. If $X = \varnothing$, then we agree that 0 is the only $\varnothing$-secondary component.

(2) We call an X-secondary component of M minimal if it does not properly contain any X-secondary component of M.

(3) Denote by $\Lambda^*_X(M)$ the set of all possible X-secondary components of M.

(4) We use $\underset{\circ}{\Lambda}{}^*_X(M)$ to denote the set of all minimal X-secondary components of M. (Note that $\underset{\circ}{\Lambda}{}^*_X(M) \neq \varnothing$ if M is Artinian.)

(5) In case $X = \{P\} \subseteq \mathrm{Att}(M)$, we may simply write Λ^*_P and $\underset{\circ}{\Lambda}{}^*_P$ instead of $\Lambda^*_{\{P\}}$ and $\underset{\circ}{\Lambda}{}^*_{\{P\}}$ respectively.

Lemma 7.2 (Compare with Lemma 2.2)**.** *Let M be a representable R-module and $X \subseteq \mathrm{Att}(M)$. For an R-module Q such that $Q \subseteq M$, the following are equivalent:*

(1) *Q is an X-secondary component of M, i.e., $Q \in \Lambda^*_X(M)$.*
(2) *Q is representable, $\mathrm{Att}(Q) \subseteq X$ and $\mathrm{Att}(M/Q) \subseteq \mathrm{Att}(M) \setminus X$.*
(3) *Q is representable, $\mathrm{Att}(Q) = X$ and $\mathrm{Att}(M/Q) = \mathrm{Att}(M) \setminus X$.*

Proof. Say $X = \{P_1, \dots, P_r\} \subseteq \{P_1, \dots, P_r, P_{r+1}, \dots, P_s\} = \mathrm{Att}(M)$.

(1) $\Rightarrow$ (2): Condition (1) means that there is a secondary representation $M = Q_1 + \cdots + Q_r + \cdots + Q_s$ with Q_i being P_i-secondary such that $Q = Q_1 + \cdots + Q_r$. Then evidently $\mathrm{Att}(Q) \subseteq X$ (since they are equal, see Theorem 6.3 (1)). Also, we have an R-linear isomorphism

$$\frac{M}{Q} = \frac{Q + \sum_{i=r+1}^{s} Q_i}{Q} \cong \frac{\sum_{i=r+1}^{s} Q_i}{Q \cap \sum_{i=r+1}^{s} Q_i},$$

which implies that $\mathrm{Att}(M/Q) \subseteq \mathrm{Att}(\sum_{i=r+1}^{s} Q_i) = \{P_{r+1}, \dots, P_s\} = \mathrm{Att}(M) \setminus X$.

(2) $\Rightarrow$ (3): This is evident, since $\mathrm{Att}(M) \subseteq \mathrm{Att}(M/Q) \cup \mathrm{Att}(Q)$ by Lemma 6.4.

(3) $\Rightarrow$ (1): As Q is representable and $\mathrm{Att}(Q) = \{P_1, \dots, P_r\}$, we fix a secondary representation $Q = Q_1 + \cdots + Q_r$ in which Q_i is the P_i-secondary component for $i = 1, \dots, r$. Next, we fix a secondary representation $M = Q'_1 + \cdots + Q'_r + \cdots + Q'_s$ of M with Q'_i being P_i-secondary and let $Q' = \cap_{i=r+1}^{s} Q'_i$, so that $Q' \in \Lambda^*_{\mathrm{Att}(M) \setminus X}(M)$. By the argument (1) $\Rightarrow$ (2), $\mathrm{Att}(M/Q') \subseteq X$. Since $\frac{M}{Q+Q'}$ is a homomorphic image of both M/Q and M/Q', we know that $\mathrm{Att}(\frac{M}{Q+Q'}) \subseteq \mathrm{Att}(M/Q) \cap \mathrm{Att}(M/Q') = \varnothing$. Note that $\frac{M}{Q+Q'}$ is representable since M is so (cf. Theorem 6.3 (3)). In light of this, the fact that $\mathrm{Att}(\frac{M}{Q+Q'}) = \varnothing$ necessarily implies $\frac{M}{Q+Q'} = 0$ (cf. Theorem 6.3 (1)), and hence $M = Q + Q'$. That is,

$$M = Q + Q' = Q_1 + \cdots + Q_r + Q'_{r+1} + \cdots Q'_s,$$

which is necessarily a (minimal) secondary representation of M. This implies that $Q = Q_1 + \cdots + Q_r$ is an X-secondary component of M, i.e., $Q \in \Lambda^*_X(M)$. $\square$

As a consequence, we establish the following 'compatibility' property of secondary representation, as follows.

Theorem 7.3 (Compatibility)**.** *Let M be a representable R-module. Then*

(1) *If $X_i \subseteq \mathrm{Att}(M)$ and $Q_{X_i} \in \Lambda^*_{X_i}(M)$ for $1 \leqslant i \leqslant n$. Then $\sum_{i=1}^{n} Q_{X_i} \in \Lambda^*_X(M)$, where $X = \cup_{i=1}^{n} X_i$.*
(2) *In particular, suppose $\mathrm{Att}(M) = \{P_1, \dots, P_s\}$ and $Q_i \in \Lambda^*_{P_i}(M)$ for each $i = 1, \dots, s$. Then $M = Q_1 + \cdots + Q_s$, which is necessarily a minimal secondary representation of M.*

Proof. (1) By Lemma 7.2, we see $\mathrm{Att}(Q_{X_i}) = X_i$ and $\mathrm{Att}(M/Q_{X_i}) = \mathrm{Att}(M) \setminus X_i$ for $1 \leqslant i \leqslant n$. Therefore

$$\mathrm{Att}(M/(\textstyle\sum_{i=1}^{n} Q_{X_i}) \subseteq \cap_{i=1}^{n} \mathrm{Att}(M/Q_{X_i}) = \mathrm{Att}(M) \setminus X$$

because of the natural surjections from M/Q_{X_i} onto $M/(\sum_{i=1}^n Q_{X_i})$. Also observe that $\operatorname{Att}(\sum_{i=1}^n Q_{X_i}) \subseteq \cup_{i=1}^n \operatorname{Att}(Q_{X_i}) = X$ (since there is an obvious surjection from $\oplus_{i=1}^n Q_{X_i}$ to $\sum_{i=1}^n Q_{X_i}$). Now Lemma 7.2 gives the desired result.

(2) This is a special case of (1). By definition, M is the only $\operatorname{Att}(M)$-secondary component of M. (This can also be proved by "dualizing" the proof in [Yao1, Theorem 1.1]; see [Yao3, Theorem 4.1.2] for details.) $\square$

8. Applying a result of Sharp on Artinian modules

Throughout this section, R is a ring and M is an Artinian R-module. Although R is not necessarily Noetherian, we are going to see that M can be naturally realized as an Artinian module over a Noetherian complete semi-local ring, thanks to a theorem of R. Y. Sharp in [Sh2] (cf. Theorem 8.3). This would make the classic Matlis duality applicable, which then allows us to transform secondary representations to primary decompositions, as we are going to see in Observation 8.4. (Also see [Sh3] for another result on Artinian modules.)

Notation 8.1. We will use the following notation in the sequel.

(1) Let $\operatorname{MSpec}(R) := \{\mathfrak{m} \in \operatorname{Spec}(R) \mid \mathfrak{m} \text{ is maximal in } R\}$.
(2) For every $\mathfrak{m} \in \operatorname{MSpec}(R)$, denote $\Gamma_{\mathfrak{m}}(M) := \cup_{n\geqslant 0}(0 :_M \mathfrak{m}^n)$, which is isomorphic to $M_{\mathfrak{m}}$ since M is Artinian.
(3) For every $\mathfrak{m} \in \operatorname{MSpec}(R)$, let $\widehat{R_{\mathfrak{m}}}$ be the $\mathfrak{m}$-adic completion of $R_{\mathfrak{m}}$ (or R), which is a quasi-local ring (i.e., a ring, not necessarily Noetherian, with a unique maximal ideal).
(4) Let $\widehat{R} := \prod_{\mathfrak{m}\in\operatorname{MSpec}(R)} \widehat{R_{\mathfrak{m}}}$, which is a ring (not necessarily Noetherian).
(5) Let $\phi\colon R \to \widehat{R}$ be the natural ring homomorphism.
(6) Let $\phi^*\colon \operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$ denote the induced continuous map, that is, $\phi^*(P) = \phi^{-1}(P)$ for all $P \in \operatorname{Spec}(\widehat{R})$.
(7) Let ϕ^*_M denote the resulting map if we restrict ϕ^* to $\operatorname{Att}_{\widehat{R}}(M) \to \operatorname{Att}_R(M)$. Thus, for $X \subseteq \operatorname{Att}_R(M)$, ${\phi^*_M}^{-1}(X) = \{P \in \operatorname{Att}_{\widehat{R}}(M) \mid \phi^{-1}(P) \in X\}$.
(8) For each $\mathfrak{m} \in \operatorname{MSpec}(R)$, let $\operatorname{E}_{\widehat{R}}(R/\mathfrak{m})$ denote the injective hull of $R/\mathfrak{m}$ over $\widehat{R}$ (which is canonically isomorphic to its injective hull over $\widehat{R_{\mathfrak{m}}}$).
(9) Let $E := \prod_{\mathfrak{m}} \operatorname{E}_{\widehat{R}}(R/\mathfrak{m})$, which is injective over $\widehat{R}$.
(10) Define the Matlis dualizing functor, denoted $\operatorname{D}(-)$, as follows: for every $\widehat{R}$-module N (e.g., N is an Artinian R-module), let $\operatorname{D}(N) := \operatorname{Hom}_{\widehat{R}}(N, E)$.

Let us recall the classic Matlis duality (over a Noetherian complete semi-local ring) and some consequences.

Theorem 8.2 (Matlis duality)**.** *Let R be a Noetherian semi-local ring that is complete (with respect to its Jacobson radical) and M be an R-module that is Artinian or*

Noetherian. Say $\operatorname{MSpec}(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$, *so that* $R = \widehat{R} = \prod_{i=1}^n \widehat{R_{\mathfrak{m}_i}}$ *and* $E = \oplus_{i=1}^n \mathrm{E}_{R_{\mathfrak{m}_i}}(R/\mathfrak{m}_i)$. *Then*

(1) *If* M *is Artinian (resp. Noetherian), then* $\mathrm{D}(M)$ *is Noetherian (resp. Artinian).*
(2) $\mathrm{D}(\mathrm{D}(M)) = M$ *and (hence)* $\mathrm{D}(\mathrm{D}(\mathrm{D}(M))) = \mathrm{D}(M)$.
(3) *If* $\{N_i\}_{i\in\Delta}$ *is a family of (possibly infinitely many)* R*-submodules of* M, *then*
$$\begin{aligned}
&\mathrm{D}\left(M/\textstyle\sum_{i\in\Delta} N_i\right) = \cap_{i\in\Delta} \mathrm{D}(M/N_i),\\
&\textstyle\sum_{i\in\Delta} N_i = \mathrm{D}\left(\mathrm{D}(M)/\cap_{i\in\Delta} \mathrm{D}(M/N_i)\right),\\
&\mathrm{D}\left(M/\cap_{i\in\Delta} N_i\right) = \textstyle\sum_{i\in\Delta} \mathrm{D}(M/N_i),\\
&\cap_{i\in\Delta} N_i = \mathrm{D}\left(\mathrm{D}(M)/\textstyle\sum_{i\in\Delta} \mathrm{D}(M/N_i)\right),\\
&N_i \subseteq N_j \iff \mathrm{D}(M/N_i) \supseteq \mathrm{D}(M/N_j).
\end{aligned}$$
(4) *For any* R*-submodule* Q *of* M *and* $P \in \operatorname{Spec}(R)$, Q *is* P*-secondary if and only if* $\mathrm{D}(Q)$ *is* P*-coprimary if and only if* $\mathrm{D}(M/Q)$ *is* P*-primary in* $\mathrm{D}(M)$.
(4') *For any submodule* Q' *of* M *and* $P \in \operatorname{Spec}(R)$, Q' *is* P*-primary in* M *if and only if* $\mathrm{D}(M/Q')$ *is* P*-secondary.*
(5) $M = \sum_{i=1}^s Q_i$ *is a (minimal) secondary representation of* M *if and only if* $0 = \cap_{i=1}^s \mathrm{D}(M/Q_i)$ *is a (minimal) primary decomposition of* 0 *in* $\mathrm{D}(M)$.
(5') $0 = \cap_{i=1}^s Q'_i$ *is a (minimal) primary decomposition of* 0 *in* M *if and only if* $\mathrm{D}(M) = \sum_{i=1}^s \mathrm{D}(M/Q'_i)$ *is a (minimal) secondary representation of* $\mathrm{D}(M)$.
(6) $\operatorname{Att}_R(M) = \operatorname{Ass}_R(\mathrm{D}(M))$ *and* $\operatorname{Ass}_R(M) = \operatorname{Att}_R(\mathrm{D}(M))$.

(By abuse of notation, we use "=" to denote natural isomorphisms, and regard $\mathrm{D}(M/N_i)$, $\mathrm{D}(M/Q)$ *and* $\mathrm{D}(M/Q_i)$ *as* R*-submodules of* $\mathrm{D}(M)$ *via the natural injections.)*

Proof. Statements (1), (2) and (3) are standard results of the classic Matlis duality.

(4) It is clear that $Q \neq 0 \iff \mathrm{D}(Q) \neq 0 \iff \mathrm{D}(M/Q) \subsetneq \mathrm{D}(M)$. So we assume $Q \neq 0$. Then we have that
$$\begin{aligned}
Q \text{ is } P\text{-secondary} &\iff f\colon Q \xrightarrow{r\cdot} Q \text{ is } \begin{cases}\text{surjective} & \text{if } r \in R\setminus P\\ \text{nilpotent} & \text{if } r \in P\end{cases}\\
&\iff g\colon\ \mathrm{D}(Q) \xrightarrow{r\cdot} \mathrm{D}(Q) \text{ is } \begin{cases}\text{injective} & \text{if } r \in R\setminus P\\ \text{nilpotent} & \text{if } r \in P\end{cases}\\
&\iff \mathrm{D}(Q) \text{ is } P\text{-coprimary}\\
&\iff \mathrm{D}(M)/\mathrm{D}(M/Q) \text{ is } P\text{-coprimary}\\
&\iff \mathrm{D}(M/Q) \text{ is } P\text{-primary in } \mathrm{D}(M).
\end{aligned}$$

(4') This can be proved in a similar way. (This also follows from (4) in light of the duality results (1) and (2).)

Finally, (5), (5') and (6) all follow from (1), (2), (3), (4) and (4') directly. □

Let R be a general commutative ring (not necessarily Noetherian). Since M is Artinian, we see that $M = \oplus_{\mathfrak{m}\in \mathrm{MSpec}(R)}\Gamma_{\mathfrak{m}}(M)$ and $\Gamma_{\mathfrak{m}}(M) = 0$ for all but finitely many $\mathfrak{m}$. For each $\mathfrak{m} \in \mathrm{MSpec}(R)$, $\Gamma_{\mathfrak{m}}(M) = M_{\mathfrak{m}}$ is naturally a module over $\widehat{R_{\mathfrak{m}}}$. Thus M can be naturally viewed as a module over $\widehat{R}$ (via component-wise scalar multiplications). If we compose this derived $\widehat{R}$-module structure of M with ϕ, we recover the original R-module structure of M. Moreover, for a subset N of M, it is straightforward to see that

$$N \text{ is an } R\text{-submodule of } M \iff N \text{ is an } \widehat{R}\text{-submodule of } M.$$

So M must be Artinian over $\widehat{R}$, since M is Artinian over R. To study the R-module structure of M, one approach would be to study its $\widehat{R}$-module structure.

Let us study $\mathrm{Ann}_{\widehat{R}}(M)$, the annihilator of M over $\widehat{R}$. By the above, we see that

$$\mathrm{Ann}_{\widehat{R}}(M) = \prod_{\mathfrak{m}\in \mathrm{MSpec}(R)} \mathrm{Ann}_{\widehat{R_{\mathfrak{m}}}}(\Gamma_{\mathfrak{m}}(M)).$$

Thus M is naturally an Artinian module over the following quotient ring

$$\frac{\widehat{R}}{\mathrm{Ann}_{\widehat{R}}(M)} = \frac{\prod_{\mathfrak{m}\in \mathrm{MSpec}(R)} \widehat{R_{\mathfrak{m}}}}{\prod_{\mathfrak{m}\in \mathrm{MSpec}(R)} \mathrm{Ann}_{\widehat{R_{\mathfrak{m}}}}(\Gamma_{\mathfrak{m}}(M))} \cong \prod_{\mathfrak{m}\in \mathrm{MSpec}(R)} \frac{\widehat{R_{\mathfrak{m}}}}{\mathrm{Ann}_{\widehat{R_{\mathfrak{m}}}}(\Gamma_{\mathfrak{m}}(M))}.$$

As $\Gamma_{\mathfrak{m}}(M) = 0$ for all but finitely many $\mathfrak{m}$, say $\{\mathfrak{m} \in \mathrm{MSpec}(R) \mid \Gamma_{\mathfrak{m}}(M) \neq 0\} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. Then $M = \oplus_{i=1}^{n}\Gamma_{\mathfrak{m}_i}(M)$ is naturally an Artinian module over

$$\prod_{\mathfrak{m}\in \mathrm{MSpec}(R)} \frac{\widehat{R_{\mathfrak{m}}}}{\mathrm{Ann}_{\widehat{R_{\mathfrak{m}}}}(\Gamma_{\mathfrak{m}}(M))} = \prod_{i=1}^{n} \frac{\widehat{R_{\mathfrak{m}_i}}}{\mathrm{Ann}_{\widehat{R_{\mathfrak{m}_i}}}(\Gamma_{\mathfrak{m}_i}(M))}.$$

So we study $\Gamma_{\mathfrak{m}_i}(M)$ over $\frac{\widehat{R_{\mathfrak{m}_i}}}{\mathrm{Ann}_{\widehat{R_{\mathfrak{m}_i}}}(\Gamma_{\mathfrak{m}_i}(M))}$ for $i = 1, \ldots, n$. By construction, $\Gamma_{\mathfrak{m}_i}(M)$ is Artinian over the quasi-local ring $\frac{\widehat{R_{\mathfrak{m}_i}}}{\mathrm{Ann}_{\widehat{R_{\mathfrak{m}_i}}}(\Gamma_{\mathfrak{m}_i}(M))}$; moreover, if we compose this module structure with the natural map $R \to \widehat{R_{\mathfrak{m}_i}}$, we recover the original R-module structure of $\Gamma_{\mathfrak{m}_i}(M)$.

Therefore, to study the secondary representations of an Artinian R-module M, it (usually) suffices to study them over $\widehat{R}$ (as secondary representations behave well under scalar restriction, see Theorem 6.3 (3)). Then it suffices to regard M as an (Artinian) module over $\frac{\widehat{R}}{\mathrm{Ann}_{\widehat{R}}(M)} = \prod_{i=1}^{n} \frac{\widehat{R_{\mathfrak{m}_i}}}{\mathrm{Ann}_{\widehat{R_{\mathfrak{m}_i}}}(\Gamma_{\mathfrak{m}_i}(M))}$.

The following theorem of R. Y. Sharp verifies that each of the rings $\frac{\widehat{R_{\mathfrak{m}_i}}}{\mathrm{Ann}_{\widehat{R_{\mathfrak{m}_i}}}(\Gamma_{\mathfrak{m}_i}(M))}$ is actually Noetherian. In the sequel, we say a ring is local if it is Noetherian with a unique maximal ideal. We say a ring is semi-local if it is Noetherian with finitely many maximal ideals.

Theorem 8.3 ([Sh2]). *Let M be an Artinian R-module as above. Then for each $\mathfrak{m} \in \operatorname{MSpec}(R)$, $\frac{\widehat{R_{\mathfrak{m}}}}{\left(0:_{\widehat{R_{\mathfrak{m}}}} \Gamma_{\mathfrak{m}}(M)\right)}$ is (either the zero ring or) a local (Noetherian) ring that is complete with respect to its maximal ideal. Therefore $\frac{\widehat{R}}{\operatorname{Ann}_{\widehat{R}}(M)}$ is a complete semi-local (Noetherian) ring (i.e., a direct product of finitely many complete local rings).*

Since each $\frac{\widehat{R_{\mathfrak{m}_i}}}{\operatorname{Ann}_{\widehat{R_{\mathfrak{m}_i}}}(\Gamma_{\mathfrak{m}_i}(M))}$ is complete local (Noetherian), the classic Matlis duality (Theorem 8.2) applies. It then follows that the functor $\mathrm{D}(-)$, which is defined over $\widehat{R}$, enjoys many of the properties of the classic Matlis duality, even though $\widehat{R}$ may not be Noetherian. Consequently, secondary representations of Artinian R-modules are in one-to-one correspondence with primary decompositions of Noetherian $\widehat{R}$-modules. (This is demonstrated in Observation 8.4 next.)

The following observations would show how the classic Matlis duality is applied, thanks to Theorem 8.3. This duality allows us to make a connection between the theory of secondary representation and the theory of primary decomposition.

Observation 8.4. Let R be a ring and M be an Artinian R-module. Keep all the above notation in this section. By abuse of notation, we may use "=" to denote natural isomorphisms. To further simplify the notation, let

$$T_{\mathfrak{m}} := \frac{\widehat{R_{\mathfrak{m}}}}{\operatorname{Ann}_{\widehat{R_{\mathfrak{m}}}}(\Gamma_{\mathfrak{m}}(M))} = \frac{\widehat{R_{\mathfrak{m}}}}{I_{\mathfrak{m}}},\ \forall\, \mathfrak{m} \quad \text{and} \quad T := \prod_{\mathfrak{m} \in \operatorname{MSpec}(R)} T_{\mathfrak{m}} = \prod_{i=1}^{n} T_{\mathfrak{m}_i} = \frac{\widehat{R}}{I}$$

with $I := \operatorname{Ann}_{\widehat{R}}(M)$ and $I_{\mathfrak{m}} := \operatorname{Ann}_{\widehat{R_{\mathfrak{m}}}}(\Gamma_{\mathfrak{m}}(M))$. Then M is an Artinian T-module; and Theorem 8.3 says that T is a complete semi-local (Noetherian) ring. We make the following observations (many of them obvious):

(1) Although $\mathrm{D}(M)$ is defined as $\operatorname{Hom}_{\widehat{R}}(M, E)$ over $\widehat{R}$, $\mathrm{D}(M)$ is the same as taking the Matlis dual over the complete semi-local ring T, and it is also the same as taking the Matlis dual of each $\Gamma_{\mathfrak{m}_i}(M)$ individually over the complete local ring $T_{\mathfrak{m}_i}$ and then taking their direct sum. This is because, by Hom-$\otimes$ adjointness,

$$\begin{aligned}
\operatorname{Hom}_{\widehat{R}}(M, E) &= \bigoplus_{i=1}^{n} \operatorname{Hom}_{\widehat{R_{\mathfrak{m}_i}}}\left(\Gamma_{\mathfrak{m}_i}(M), \mathrm{E}_{\widehat{R_{\mathfrak{m}_i}}}(R/\mathfrak{m}_i)\right) \\
&= \bigoplus_{i=1}^{n} \operatorname{Hom}_{T_{\mathfrak{m}_i}}\left(\Gamma_{\mathfrak{m}_i}(M), \operatorname{Hom}_{\widehat{R_{\mathfrak{m}_i}}}\left(T_{\mathfrak{m}_i}, \mathrm{E}_{\widehat{R_{\mathfrak{m}_i}}}(R/\mathfrak{m}_i)\right)\right) \\
&= \bigoplus_{i=1}^{n} \operatorname{Hom}_{T_{\mathfrak{m}_i}}\left(\Gamma_{\mathfrak{m}_i}(M), \mathrm{E}_{T_{\mathfrak{m}_i}}(R/\mathfrak{m}_i)\right) \\
&= \operatorname{Hom}_{T}\left(M, \bigoplus_{i=1}^{n} \mathrm{E}_{T_{\mathfrak{m}_i}}(R/\mathfrak{m}_i)\right),
\end{aligned}$$

in which $\mathrm{E}_{T_{\mathfrak{m}_i}}(R/\mathfrak{m}_i)$ denotes the injective hull of $R/\mathfrak{m}_i$ over the ring $T_{\mathfrak{m}_i}$.

(2) Thus $\mathrm{D}(M)$ is a Noetherian T-module, and hence a Noetherian $\widehat{R}$-module.

(3) Therefore, $\mathrm{D}(\mathrm{D}(M)) = M$ and $\mathrm{D}(\mathrm{D}(\mathrm{D}(M))) = \mathrm{D}(M)$ (up to the canonical isomorphisms) as T-modules and hence as $\widehat{R}$-modules. This follows from the classic Matlis duality (cf. Theorem 8.2) over T.

(4) If $\{N_i\}_{i\in\Delta}$ is a family of (possibly infinitely many) $\widehat{R}$-submodules of M and $\{K_i\}_{i\in\Delta}$ is a family of $\widehat{R}$-submodules of $\mathrm{D}(M)$, then

$$\begin{aligned}
&\mathrm{D}\left(M/\textstyle\sum_{i\in\Delta} N_i\right) = \cap_{i\in\Delta}\mathrm{D}(M/N_i),\\
&\textstyle\sum_{i\in\Delta} N_i = \mathrm{D}\left(\mathrm{D}(M)/\cap_{i\in\Delta}\mathrm{D}(M/N_i)\right),\\
&N_i \subseteq N_j \iff \mathrm{D}(M/N_i) \supseteq \mathrm{D}(M/N_j),\\
&N_i = N_j \iff \mathrm{D}(M/N_i) = \mathrm{D}(M/N_j),\\
&\mathrm{D}\left(\mathrm{D}(M)/\cap_{i\in\Delta} K_i\right) = \textstyle\sum_{i\in\Delta}\mathrm{D}(\mathrm{D}(M)/K_i),\\
&\cap_{i\in\Delta} K_i = \mathrm{D}\left(M/\textstyle\sum_{i\in\Delta}\mathrm{D}(\mathrm{D}(M)/K_i)\right),\\
&K_i \subseteq K_j \iff \mathrm{D}(\mathrm{D}(M)/K_i) \supseteq \mathrm{D}(\mathrm{D}(M)/K_j),\\
&K_i = K_j \iff \mathrm{D}(\mathrm{D}(M)/K_i) = \mathrm{D}(\mathrm{D}(M)/K_j).
\end{aligned}$$

(Indeed, the above equations and equivalences hold over T (cf. Theorem 8.2); hence they also hold over $\widehat{R}$.)

(5) For $Q \subseteq M$, Q is P-secondary if and only if $\mathrm{D}(Q)$ is P-coprimary if and only if $\mathrm{D}(M/Q)$ is P-primary in $\mathrm{D}(M)$ (over T or over $\widehat{R}$, no difference). This follows immediately from Theorem 8.2 over T (and hence over $\widehat{R}$).

(6) $M = \sum_{i=1}^{s} Q_i$ is a (minimal) secondary representation of M over T (hence over $\widehat{R}$) if and only if $0 = \cap_{i=1}^{s}\mathrm{D}(M/Q_i)$ is a (minimal) primary decomposition of 0 in $\mathrm{D}(M)$ over T (hence over $\widehat{R}$). Thus $\mathrm{Att}_{\widehat{R}}(M) = \mathrm{Ass}_{\widehat{R}}(\mathrm{D}(M))$. Since every $\widehat{R}$-submodule of $\mathrm{D}(M)$ is of the form $\mathrm{D}(M/Q)$, the above is also a criterion of primary decompositions of 0 in $\mathrm{D}(M)$. Put directly, over T (and $\widehat{R}$), $0 = \cap_{i=1}^{s} Q'_i$ is a primary decomposition of 0 in $\mathrm{D}(M)$ if and only if $M = \sum_{i=1}^{s}\mathrm{D}(\mathrm{D}(M)/Q'_i)$ is a secondary representation of M. All these follow from Theorem 8.2 over T.

(7) Thus, for any $Y \subseteq \mathrm{Att}_{\widehat{R}}(M)$ and $\widehat{R}$-submodules $Q \subseteq M$ and $Q' \subseteq \mathrm{D}(M)$,

$$\begin{aligned}
Q \in \Lambda^*_Y(M) &\iff \mathrm{D}(M/Q) \in \Lambda_Y(0 \subseteq \mathrm{D}(M)),\\
Q \in \underset{\circ}{\Lambda}{}^*_Y(M) &\iff \mathrm{D}(M/Q) \in \overset{\circ}{\Lambda}_Y(0 \subseteq \mathrm{D}(M)),\\
\mathrm{D}(\mathrm{D}(M)/Q') \in \Lambda^*_Y(M) &\iff Q' \in \Lambda_Y(0 \subseteq \mathrm{D}(M)),\\
\mathrm{D}(\mathrm{D}(M)/Q') \in \underset{\circ}{\Lambda}{}^*_Y(M) &\iff Q' \in \overset{\circ}{\Lambda}_Y(0 \subseteq \mathrm{D}(M)).
\end{aligned}$$

Note that an R-submodule of M is the same as an $\widehat{R}$-submodule of M.

(8) $\mathrm{D}(0 :_M J) = \mathrm{D}(M)/J\,\mathrm{D}(M)$ for every ideal J of T. This remains true if J is an ideal of R or $\widehat{R}$ (because of the natural maps $R \to \widehat{R} \to T$).

(9) For convenience, we usually state the above results over $\widehat{R}$ rather than T, even though T (being Noetherian complete semi-local) is the reason why the results hold. This is because $\widehat{R}$ does not depend on the Artinian module M while T does, and sometimes we study several Artinian R-modules.

(10) Lastly, we make a summary as follows: For any Artinian R-module M, applying $\mathrm{D}(-)$ to M (over $\widehat{R}$ by construction) is the same as taking the Matlis dual of M over the complete semi-local (Noetherian) ring $T = \frac{\widehat{R}}{\mathrm{Ann}_{\widehat{R}}(M)}$. As a result, $\mathrm{D}(M)$ is a Noetherian module, $\mathrm{D}(\mathrm{D}(M)) = M$ and $\mathrm{D}(\mathrm{D}(\mathrm{D}(M))) = \mathrm{D}(M)$ over $\widehat{R}$; and studying the secondary representations of M over $\widehat{R}$ is equivalent to studying the primary decompositions of 0 in $\mathrm{D}(M)$ over $\widehat{R}$. All the above hold for every Artinian R-module M over $\widehat{R}$. In this sense, we (essentially) have the classic Matlis duality over $\widehat{R}$ for Artinian R-modules even though $\widehat{R}$ may not be Noetherian. For this reason, we also refer to $\mathrm{D}(-)$ (over $\widehat{R}$) as the Matlis functor.

(Again, the reader please be reminded that, by abuse of notation, we used "=" to denote natural isomorphisms in the above statements.)

In light of the above, we will frequently employ the following strategy in the remaining sections: To study the secondary representations of a given Artinian R-module M, we instead study the secondary representations of M over $\widehat{R}$ or, equivalently, over the complete semi-local ring $T = \frac{\widehat{R}}{\mathrm{Ann}_{\widehat{R}}(M)}$. Applying Matlis duality $\mathrm{D}(-)$, we obtain a Noetherian module $\mathrm{D}(M)$ (over the complete semi-local ring T). If we can show (or if we already know) certain properties of the primary decompositions of $\mathrm{D}(M)$, then, after applying Matlis duality $\mathrm{D}(-)$ again, we get corresponding properties of the secondary representations for $\mathrm{D}(\mathrm{D}(M)) = M$ (over the complete semi-local ring T). This in turn should reveal properties of secondary representation of the *original* Artinian R-module M that we intend to study (via the map $R \to \widehat{R} \to T$).

Next, we state a lemma concerning relations between the secondary representations of M as an R-module and the secondary representations of M as an $\widehat{R}$-module. To avoid confusion, we may use ${}_RM$ to indicate that the R-module structure of M is being considered; similarly, ${}_{\widehat{R}}M$ indicates the $\widehat{R}$-module structure.

Lemma 8.5. *Let R be a ring and M an Artinian R-module. Then the following hold:*

(1) $\Lambda^*_{{\phi^*_M}^{-1}(X)}({}_{\widehat{R}}M) \subseteq \Lambda^*_X({}_RM)$ *for all* $X \subseteq \mathrm{Att}_R(M)$.

(2) *For every* $X \subseteq \mathrm{Att}_R(M)$ *and every* $Q_X \in \Lambda^*_X({}_RM)$, *there exists* $Q_{{\phi^*_M}^{-1}(X)} \in \Lambda^*_{{\phi^*_M}^{-1}(X)}({}_{\widehat{R}}M)$ *such that* $Q_{{\phi^*_M}^{-1}(X)} \subseteq Q_X$.

(3) $\underset{\circ}{\Lambda}{}^*_X({}_RM) = \underset{\circ}{\Lambda}{}^*_{{\phi^*_M}^{-1}(X)}({}_{\widehat{R}}M)$ *for all* $X \subseteq \mathrm{Att}_R(M)$.

Proof. Say $\mathrm{Att}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. By Theorem 6.3 (3'), we may write $\mathrm{Att}_{\widehat{R}}(M) = \{P_{i,j} \,|\, 1 \leqslant i \leqslant s;\ 1 \leqslant j \leqslant n(i)\}$ such that ${\phi_M^*}^{-1}(\mathfrak{p}_i) = \{P_{i,j} \,|\, 1 \leqslant j \leqslant n(i)\}$.

(1) Let $Q_i \in \Lambda^*_{{\phi_M^*}^{-1}(\mathfrak{p}_i)}({}_{\widehat{R}}M)$. Then there is a secondary representation $M = \sum_{i,j} Q_{i,j}$ of M with $Q_{i,j}$ being $P_{i,j}$-secondary such that $Q_i = \sum_{j=1}^{n(i)} Q_{i,j}$. Note that $M = \sum_i(\sum_{j=1}^{n(i)} Q_{i,j})$ is a secondary representation of M over R with $\sum_{j=1}^{n(i)} Q_{i,j}$ being the $\mathfrak{p}_i$-secondary component. Hence $Q_i = \sum_{j=1}^{n(i)} Q_{i,j} \in \Lambda^*_{\mathfrak{p}_i}({}_RM)$. This verifies the claim for $X = \{\mathfrak{p}_i\}$. The general claim follows, cf. Theorem 7.3 (1).

(2) Let $M = \sum_i Q_i$ be any secondary representation of M over R with Q_i being the $\mathfrak{p}_i$-secondary component, so that $Q_i \in \Lambda^*_{\mathfrak{p}_i}({}_RM)$. Since each Q_i is Artinian, it is representable over $\widehat{R}$. Say $Q_i = \sum_{j=1}^{m(i)} Q_{i,j}$ is a secondary representation of Q_i over $\widehat{R}$ with $Q_{i,j}$ being $P'_{i,j}$-secondary (over $\widehat{R}$). After rearrangement, there is $k(i)$, $0 \leqslant k(i) \leqslant \min\{m(i), n(i)\}$, such that $P'_{i,j} = P_{i,j}$ for $1 \leqslant j \leqslant k(i)$ but $P'_{i,j} \notin \{P_{i,1}, \ldots, P_{i,n(i)}\}$ for all $j > k(i)$. Then we have

$$(\dagger) \qquad M = \sum_{i=1}^{s} \sum_{j=1}^{m(i)} Q_{i,j} = Q_{1,1} + \cdots + Q_{1,m(1)} + \cdots + Q_{s,1} + \cdots + Q_{s,m(s)},$$

which is a not necessarily minimal secondary representation of M over $\widehat{R}$. We claim that, if we make (†) minimal, then $Q_{i,j}$ must be redundant for all $j > k(i)$. (Here is why: Suppose, for some $j > k(i)$, $Q_{i,j}$ remains in the minimized form of the above summation (†). Then we must have $P'_{i,j} \in \mathrm{Att}_{\widehat{R}}(M)$, cf. Theorem 6.3 (1). Because $j > k(i)$, we must have $P'_{i,j} = P_{a,b} \in \mathrm{Att}_{\widehat{R}}(M)$ for some $a \neq i$. But $\phi^*(P'_{i,j}) = \mathfrak{p}_i \neq \mathfrak{p}_a = \phi^*(P_{a,b})$, which is a contradiction.) Thus, we can throw out all the components $Q_{i,j}$ with $j > k(i)$, so that we get

$$(\ddagger) \qquad M = \sum_{i=1}^{s} \sum_{j=1}^{k(i)} Q_{i,j} = Q_{1,1} + \cdots + Q_{1,k(1)} + \cdots + Q_{s,1} + \cdots + Q_{s,k(s)}.$$

But this implies $\mathrm{Att}_{\widehat{R}}(M) \subseteq \{P_{i,j} \,|\, 1 \leqslant i \leqslant s;\ 1 \leqslant j \leqslant k(i)\}$, which forces $k(i) = n(i)$ for all i in light of Theorem 6.3 (1). Consequently, (‡) must be a minimal secondary representation of M over $\widehat{R}$. Therefore, for each $i = 1, \ldots, s$, we see

$$\Lambda^*_{\mathfrak{p}_i}({}_RM) \ni Q_i \supseteq \sum_{j=1}^{k(i)} Q_{i,j} \in \Lambda^*_{{\phi_M^*}^{-1}(\mathfrak{p}_i)}({}_{\widehat{R}}M).$$

This verifies the claim for $X = \{\mathfrak{p}_i\}$. The general claim follows, cf. Theorem 7.3 (1).

(3) This follows from (1) and (2). □

Thus, when we study the minimal secondary components of an Artinian R-module, it suffices to do so over $\widehat{R}$, where Matlis duality applies.

We will frequently use Matlis duality to go between secondary representations of Artinian R-modules and primary decompositions of Noetherian $\widehat{R}$-modules. Most of the results in the following sections were obtained in [Yao3].

9. Independence

Let R be a ring and M be a representable R-module. Note that $\operatorname{Att}(M)$ is finite, and $\operatorname{Att}(M)$ is a topological space because of the Zariski topology on $\operatorname{Spec}(R)$. As in Notation 3.1, for every $Y \subseteq \operatorname{Att}(M)$, we use $\mathrm{o}(Y)$ to denote the smallest superset of Y that is open in $\operatorname{Att}(M)$. The notation $\mathrm{o}(Y)$ depends on the ambient space, which should be made clear in the context.

If Y is an open subset of $\operatorname{Att}(M)$, then there is only one Y-secondary component in $\Lambda^*_Y(M)$, and it is $[U]M$ where $U = R \setminus \cup_{P \in Y} P$; see Theorem 6.3 (4).

Definition 9.1. Let M be an R-module and $X \subseteq \operatorname{Att}_R(M)$. We say that the secondary representations of M are *independent* over X, or X-independent, if $\Lambda^*_X(M)$ consists of exactly one component, i.e., $|\Lambda^*_X(M)| = 1$.

Obviously, this definition is parallel to the definition of independence of primary decompositions (cf. Definition 3.2). In Theorem 3.4, it was shown that if K is a Noetherian R-module and $X \subseteq \operatorname{Ass}(K)$, then the primary decompositions of 0 in K are independent over X if and only if X is open in $\operatorname{Ass}(K)$.

Naturally, we ask the following question.

Question 9.2. Let R be a ring, M an Artinian R-module, and $X \subseteq \operatorname{Att}_R(M)$ such that the secondary representations of M are independent over X. Then is X an open subset of $\operatorname{Att}_R(M)$?

The next theorem indicates an answer of 'almost yes'. (The answer to the question is actually 'no', as explained in Example 9.4.)

Theorem 9.3. *Let M be an Artinian R-module and $X \subseteq \operatorname{Att}_R(M)$. Denote by ϕ^*_M the natural map from $\operatorname{Att}_{\widehat{R}}(M)$ to $\operatorname{Att}_R(M)$. If the secondary representations of M are independent over X, then ${\phi^*_M}^{-1}(X)$ is open in $\operatorname{Att}_{\widehat{R}}(M)$.*

Proof. As $\Lambda^*_{{\phi^*_M}^{-1}(X)}({}_{\widehat{R}}M) \subseteq \Lambda^*_X({}_RM)$ (by Lemma 8.5 (1)) and $|\Lambda^*_X({}_RM)| = 1$, we see $|\Lambda^*_{{\phi^*_M}^{-1}(X)}({}_{\widehat{R}}M)| = 1$. Now let us apply Matlis duality to M (over $\widehat{R}$). We see $|\Lambda_{{\phi^*_M}^{-1}(X)}(0 \subseteq \mathrm{D}(M))| = 1$ in light of the one-to-one correspondence in Observation 8.4 (7). That is, the primary decompositions of 0 in $\mathrm{D}(M)$ over $\widehat{R}$ are independent over $\Lambda_{{\phi^*_M}^{-1}}(X)$. Since $\mathrm{D}(M)$ is Noetherian over $\widehat{R}$, we conclude that $\Lambda_{{\phi^*_M}^{-1}}(X)$ is open in $\operatorname{Ass}_{\widehat{R}}(\mathrm{D}(R))$ by Theorem 3.4. Now the proof is complete since $\operatorname{Ass}_{\widehat{R}}(\mathrm{D}(R)) = \operatorname{Att}_{\widehat{R}}(M)$, by Observation 8.4 (6). $\square$

The following example provides a negative answer to Question 9.2. (The ring in the example, i.e., $\mathbb{Z}$, is actually Noetherian.)

Example 9.4. Let $R = \mathbb{Z}$ and let $p \neq q$ be primes. Let $M := \Gamma_p(\frac{\mathbb{Q}}{\mathbb{Z}}) \oplus \frac{\mathbb{Z}}{(q)}$, which is Artinian over $\mathbb{Z}$. (Note that $\Gamma_p(\frac{\mathbb{Q}}{\mathbb{Z}})$ is the injective hull of $\mathbb{Z}/(p)$.) It is not hard to verify that the above direct sum is actually the unique secondary representation of M and $\mathrm{Att}_R(M) = \{(0), (q)\}$. In particular, the secondary representation of M is independent over $\{(q)\}$, but $\{(q)\}$ is not open in $\mathrm{Att}_R(M)$.

One might wonder whether the converse of Theorem 9.3 holds. It turns out that it fails, as shown in the following example.

Example 9.5. Let $(R, \mathfrak{m})$ be a Noetherian local ring that satisfies all the following conditions in relation with its completion $\widehat{R}$:

- There exist incomparable prime ideals P_1, $P_2 \in \mathrm{Spec}(\widehat{R})$ such that
$$P_1 \cap R =: \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 := P_2 \cap R.$$
It follows that $\{P_2\}$ is an open subset of $\{P_1, P_2\}$.
- There are infinitely many (distinct) $\widehat{R}$-submodules $\frac{K_n}{P_1}$ of $\frac{\widehat{R}}{P_1}$, $n \geqslant 1$, such that $\frac{\widehat{R}}{K_n}$ are $\mathfrak{p}_2$-coprimary. (Thus, K_n are ideals of $\widehat{R}$, containing P_1.)

(Such a ring exists. For example, let $R := \mathbb{Q}[X, Y, Z]_{(X,Y,Z)}$, so that $\widehat{R} = \mathbb{Q}[\![X, Y, Z]\!]$. Let $e^Y := \sum_{k=0}^{\infty} \frac{Y^k}{k!}$, $P_1 := (X - e^Y + 1)\widehat{R}$ and $P_2 := Z\widehat{R}$. Then $\mathfrak{p}_1 = 0 \subsetneq ZR = \mathfrak{p}_2$; and $\left(\frac{\widehat{R}}{P_1 + Z\widehat{R}}\right)_{\mathfrak{p}_2} \neq 0$. Let $K_n := \mathrm{Ker}\left(\widehat{R} \to \left(\frac{\widehat{R}}{P_1 + Z^n\widehat{R}}\right)_{\mathfrak{p}_2}\right)$, so that $\frac{\widehat{R}}{K_n}$ are $\mathfrak{p}_2$-coprimary as R-modules for all $n \geqslant 1$. Note that $K_n \supsetneq K_{n+1}$, since $\left(\frac{K_n}{K_{n+1}}\right)_{\mathfrak{p}_2} \cong \left(\frac{P_1 + Z^n\widehat{R}}{P_1 + Z^{n+1}\widehat{R}}\right)_{\mathfrak{p}_2} \cong \left(\frac{\widehat{R}}{P_1 + Z\widehat{R}}\right)_{\mathfrak{p}_2} \neq 0$, for all $n \geqslant 0$.)

It is straightforward to see that both
$$0 = \left(0 \oplus \frac{\widehat{R}}{P_2}\right) \cap \left(\frac{\widehat{R}}{P_1} \oplus 0\right) \qquad \text{and} \qquad 0 = \left(0 \oplus \frac{\widehat{R}}{P_2}\right) \cap \left(\frac{K_n}{P_1} \oplus 0\right),\ n \geqslant 1,$$
are (minimal) primary decompositions of 0 in $\frac{\widehat{R}}{P_1} \oplus \frac{\widehat{R}}{P_2}$ over R. Let E be the injective hull of residue field $R/\mathfrak{m}$, and let $M := (0 :_E P_1) \oplus (0 :_E P_2)$.

Applying Matlis duality $\mathrm{Hom}_{\widehat{R}}(-, E)$ to the above primary decompositions, we see that both
$$M = \big((0 :_E P_1) \oplus 0\big) + \big(0 \oplus (0 :_E P_2)\big) \qquad \text{and}$$
$$M = \big((0 :_E P_1) \oplus 0\big) + \big((0 :_E K_n) \oplus (0 :_E P_2)\big),\ n \geqslant 1,$$
are (minimal) secondary representations of M over R. In the above, $0 \oplus (0 :_E P_2)$ and $(0 :_E K_n) \oplus (0 :_E P_2)$, $n \geqslant 1$, give rise to infinitely many (distinct) $\mathfrak{p}_2$-secondary components of ${}_RM$. Note that M is Artinian over R and over $\widehat{R}$, and the above secondary representations (over R) show that $\mathrm{Att}_R(M) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. It is also easy to

see that

$$(*) \qquad M = \big((0 :_E P_1) \oplus 0\big) + \big(0 \oplus (0 :_E P_2)\big)$$

is a (minimal) secondary representation of M over $\widehat{R}$ and $\operatorname{Att}_{\widehat{R}}(M) = \{P_1, P_2\}$. (Thus $(*)$ is the unique secondary representation of M over $\widehat{R}$ by the choice of P_i.)

In summary, $(\phi_M^*)^{-1}(\{\mathfrak{p}_2\}) = \{P_2\}$ is open in $\operatorname{Att}_{\widehat{R}}(M)$, but the secondary representations of ${}_RM$ are not independent over $\{\mathfrak{p}_2\}$. In fact, $|\Lambda^*_{\mathfrak{p}_2}({}_RM)| = \infty$.

10. Minimal secondary components

Let M be an Artinian R-module. Using the notation introduced in Section 8, we present the following theorem about minimal secondary components. (The result was first obtained in [Yao3].)

Theorem 10.1. *Let M be an Artinian R-module and $X \subseteq \operatorname{Att}_R(M)$. Say $X = \{P_1, \dots, P_r\}$. Then the following hold*

(1) $\underset{\circ}{\Lambda}{}^*_X(M) = \Big\{\sum_{i=1}^r Q_i \,\Big|\, Q_i \in \underset{\circ}{\Lambda}{}^*_{P_i}(M),\ 1 \leqslant i \leqslant r\Big\}$.

(1') $\underset{\circ}{\Lambda}{}^*_X({}_RM) = \underset{\circ}{\Lambda}{}^*_{{\phi_M^*}^{-1}(X)}({}_{\widehat{R}}M) = \Big\{\sum_{P \in {\phi_M^*}^{-1}(X)} Q_P \,\Big|\, Q_P \in \underset{\circ}{\Lambda}{}^*_P({}_{\widehat{R}}M)\Big\}$.

(2) *For all $Q \in \Lambda^*_{{\phi_M^*}^{-1}(X)}({}_{\widehat{R}}M)$, $Q = \sum\Big\{Q' \,\Big|\, Q' \in \underset{\circ}{\Lambda}{}^*_{{\phi_M^*}^{-1}(X)}({}_{\widehat{R}}M),\ Q' \subseteq Q\Big\}$. In fact, every such Q is a sum of finitely many $Q' \in \underset{\circ}{\Lambda}{}^*_{{\phi_M^*}^{-1}(X)}({}_{\widehat{R}}M)$.*

(3) $\sum\Big\{Q \mid Q \in \underset{\circ}{\Lambda}{}^*_X({}_RM)\Big\} = \sum\Big\{Q \mid Q \in \underset{\circ}{\Lambda}{}^*_{{\phi_M^*}^{-1}(X)}({}_{\widehat{R}}M)\Big\}$ *equals the unique* $\mathrm{o}({\phi_M^*}^{-1}(X))$*-secondary component of M over $\widehat{R}$, in which* $\mathrm{o}({\phi_M^*}^{-1}(X))$ *is the smallest open superset of ${\phi_M^*}^{-1}(X)$ in $\operatorname{Att}_{\widehat{R}}(M)$.*

Proof. (1) and (1'): A direct proof will be given in Remark 10.2. But here we present a proof by the duality method described in §8. For (1'), the first equality follows from Lemma 8.5 (3). To show the second equality, we regard M as an $\widehat{R}$-module. Then by Observation 8.4, it suffice to show

$$\mathring{\Lambda}_{{\phi_M^*}^{-1}(X)}(0 \subseteq \mathrm{D}(M)) = \left\{ \bigcap_{P \in {\phi_M^*}^{-1}(X)} \mathrm{D}\left(\tfrac{M}{Q_P}\right) \,\middle|\, \mathrm{D}\left(\tfrac{M}{Q_P}\right) \in \mathring{\Lambda}_P(0 \subseteq \mathrm{D}(M)) \right\}.$$

But this holds by the virtue of Theorem 3.3 (1). Then (1) follows from (1').

(2) By Observation 8.4, it suffices to show that $\mathrm{D}(M/Q)$ equals the following

$$\bigcap \left\{ \mathrm{D}(M/Q') \,\middle|\, \mathrm{D}(M/Q') \in \mathring{\Lambda}_{{\phi_M^*}^{-1}(X)}(0 \subseteq \mathrm{D}(M)),\ \mathrm{D}(M/Q') \supseteq \mathrm{D}(M/Q) \right\}.$$

But this follows from Theorem 3.3 (2). The finiteness claim follows similarly.

(3) The first equality follows from Lemma 8.5 (3). To show the remaining claim, we use Matlis duality $\mathrm{D}(-)$. By Observation 8.4, it suffices to show that

$$\bigcap \left\{ \mathrm{D}(M/Q) \,\middle|\, \mathrm{D}(M/Q) \in \mathring{\Lambda}_{\phi_M^{*}{}^{-1}(X)}(0 \subseteq \mathrm{D}(M)) \right\}$$

is the unique $\mathrm{o}(\phi_M^{*}{}^{-1}(X))$-primary component of 0 in $\mathrm{D}(M)$ as an $\widehat{R}$-module. But this follows from Theorem 3.3 (3) as $\mathrm{D}(M)$ is a Noetherian $\widehat{R}$-module. $\square$

Remark 10.2. We would like to present the following direct proofs of (1) and (1') of Theorem 10.1 without using Matlis duality:

For (1), it is easy to show $\underset{\circ}{\Lambda}{}^*_X(M) \subseteq \left\{ \sum_{i=1}^r Q_i \,\middle|\, Q_i \in \underset{\circ}{\Lambda}{}^*_{P_i}(M),\, 1 \leqslant i \leqslant r \right\}$: For any $Q \in \underset{\circ}{\Lambda}{}^*_X(M)$, write $Q = Q'_1 + \cdots + Q'_r$, where $Q'_i \in \Lambda^*_{P_i}$ for each $1 \leqslant i \leqslant r$. There exists $Q_i \in \underset{\circ}{\Lambda}{}^*_{P_i}$ such that $Q'_i \supseteq Q_i$ for each $i = 1, \ldots, r$, so that $Q = Q'_1 + \cdots + Q'_r \supseteq Q_1 + \cdots + Q_r$. But $Q_1 + \cdots + Q_r \in \Lambda^*_X$ by the compatibility property (Theorem 7.3), which shows $Q = Q_1 + \cdots + Q_r$.

We show $\underset{\circ}{\Lambda}{}^*_X(M) \supseteq \left\{ \sum_{i=1}^r Q_i \,\middle|\, Q_i \in \underset{\circ}{\Lambda}{}^*_{P_i}(M),\, 1 \leqslant i \leqslant r \right\}$ by induction on $|X|$, the cardinality of X. If $|X| = 1$, there is nothing to prove. Assuming the containment holds for $|X| = r - 1$, we show the containment for $X = \{P_1, P_2, \ldots, P_r\}$. After rearrangement if necessary, we may assume that $P_r \not\subseteq P_i$ for $1 \leqslant i \leqslant r-1$. Set $U = R \setminus \cup_{i=1}^{r-1} P_i$. Let $Q = \sum_{i=1}^r Q_i$ such that $Q_i \in \underset{\circ}{\Lambda}{}^*_{P_i}(M)$ for $1 \leqslant i \leqslant r$. For any $Q' \in \Lambda^*_X$ such that $Q \supseteq Q'$, we need to show $Q = Q'$. Write $Q' = \sum_{i=1}^r Q'_i$ such that $Q'_i \in \underset{\circ}{\Lambda}{}^*_{P_i}$ for $1 \leqslant i \leqslant r$. Then we have

$$\sum_{i=1}^{r-1} Q_i = [U]Q \supseteq [U]Q' = \sum_{i=1}^{r-1} Q'_i,$$

which forces $\sum_{i=1}^{r-1} Q_i = \sum_{i=1}^{r-1} Q'_i$ by the induction hypothesis. Therefore

$$\sum_{i=1}^{r-1} Q_i + (Q' \cap Q_r) = Q' \cap \sum_{i=1}^{r} Q_i = Q' \qquad (\text{since } \sum_{i=1}^{r-1} Q_i = \sum_{i=1}^{r-1} Q'_i \subset Q').$$

Hence we can derive a secondary representation $Q' = \sum_{i=1}^r Q''_i$ by putting together $\sum_{i=1}^{r-1} Q_i$ and any secondary representation of $(Q' \cap Q_r)$ (and then make it minimal). In this derived secondary representation $Q' = \sum_{i=1}^r Q''_i$, the P_r-secondary component, Q''_r, must come from the P_r-secondary component of $(Q' \cap Q_r)$, hence is contained in $Q' \cap Q_r$. Since $Q''_r \in \Lambda^*_{P_r}(Q')$ and $Q' \in \Lambda^*_X(M)$, we see $Q''_r \in \Lambda^*_{P_r}(M)$ (by compatibility, for example). This forces $Q''_r = Q_r$ since Q_r is already a minimal P_r-secondary component of M. Hence $Q' \supseteq Q''_r = Q_r$, which gives

$$Q = \sum_{i=1}^{r} Q_i = \sum_{i=1}^{r-1} Q_i + Q_r = \sum_{i=1}^{r-1} Q'_i + Q_r \subseteq \sum_{i=1}^{r-1} Q'_i + Q'_r = Q'.$$

Therefore $Q = Q'$, and the proof is complete.

Finally, the first equality of (1') was done in Lemma 8.5 (3); and the last equality follows from (1) applied to M as an Artinian module over $\widehat{R}$.

Because of Theorem 10.1, we can fine-tune Theorem 9.3 as follows.

Theorem 10.3. *Let M be an Artinian R-module and $X \subseteq \operatorname{Att}_R(M)$. Consider the following statements:*

(1) X *is open in* $\operatorname{Att}_R(M)$.	(1') ${\phi^*_M}^{-1}(X)$ *is open in* $\operatorname{Att}_{\widehat{R}}(M)$.
(2) $\lvert\Lambda^*_X(M)\rvert = 1$.	(2') $\lvert\Lambda^*_{{\phi^*_M}^{-1}(X)}({}_{\widehat{R}}M)\rvert = 1$.
(3) $\Lambda^*_X(M)$ *is finite.*	(3') $\Lambda^*_{{\phi^*_M}^{-1}(X)}({}_{\widehat{R}}M)$ *is finite.*
(4) $\underset{\circ}{\Lambda}{}^*_X(M)$ *is finite.*	(4') $\underset{\circ}{\Lambda}{}^*_{{\phi^*_M}^{-1}(X)}({}_{\widehat{R}}M)$ *is finite.*
(5) $\lvert\underset{\circ}{\Lambda}{}^*_X(M)\rvert = 1$.	(5') $\lvert\underset{\circ}{\Lambda}{}^*_{{\phi^*_M}^{-1}(X)}({}_{\widehat{R}}M)\rvert = 1$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (1') $\Leftrightarrow$ (2') $\Leftrightarrow$ (3') $\Leftrightarrow$ (4') $\Leftrightarrow$ (5').

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear.

The implications (1') $\Rightarrow$ (2') $\Rightarrow$ (3') $\Rightarrow$ (4') are clear.

The implications (2') $\Rightarrow$ (5') $\Rightarrow$ (4') are clear.

(4) $\Leftrightarrow$ (4') follows from Lemma 8.5 (3), so does (5) $\Leftrightarrow$ (5').

(4) $\Rightarrow$ (1'): Say $\underset{\circ}{\Lambda}{}^*_X(M) = \{Q_1, \ldots, Q_t\}$, so $\underset{\circ}{\Lambda}{}^*_{{\phi^*_M}^{-1}(X)}({}_{\widehat{R}}M) = \{Q_1, \ldots, Q_t\}$. Let $Q = \sum_{i=1}^t Q_i$. Then $Q \in \Lambda^*_{{\phi^*_M}^{-1}(X)}({}_{\widehat{R}}M)$ by Theorem 7.3. On the other hand, Theorem 10.1 (3) implies $Q \in \Lambda^*_{\mathrm{o}({\phi^*_M}^{-1}(X))}({}_{\widehat{R}}M)$, in which $\mathrm{o}({\phi^*_M}^{-1}(X))$ denotes the smallest open superset of ${\phi^*_M}^{-1}(X)$ in $\operatorname{Att}_{\widehat{R}}(M)$. By Lemma 7.2, we must have ${\phi^*_M}^{-1}(X) = \mathrm{o}({\phi^*_M}^{-1}(X))$, which is open in $\operatorname{Att}_{\widehat{R}}(M)$. □

By Examples 9.4, 9.5, implications (1) $\Leftarrow$ (2) and (3) $\Leftarrow$ (4) are false in general.

11. Linear growth of secondary components

Inspired by the linear growth of primary decomposition, and taking in account the duality between primary decomposition and secondary representation, we formulate a definition of the linear growth property of secondary representation as follows. We use the notation introduced in Notation 4.3. Let R be a ring.

Definition 11.1. Given a family $\mathcal{F} = \{M_a \mid a = (a_1, \ldots, a_r) \in \mathbb{N}^r\}$ of R-modules, we say that $\mathcal{F}$ satisfies the linear growth property of secondary representation over R if there exists $k \in \mathbb{N}$ such that, for every $a = (a_1, \dots, a_r) \in \mathbb{N}^r$ such that $M_a \neq 0$, there exists a secondary representation of M_a

$$M_a = Q_{a,1} + \cdots + Q_{a,s(a)} \quad \text{(with } Q_{a,i} \text{ being } P_{a,i}\text{-secondary)}$$

such that $Q_{a,i} \subseteq \left(0 :_{M_a} (P_{a,i})^{k|a|}\right)$ for all $i = 1, \dots, s(a)$, where $|a| = a_1 + \cdots + a_r$.

When the above occurs, we refer to k as a *slope* of $\mathcal{F}$.

Lemma 11.2. *Let $h\colon A \to R$ be a ring homomorphism, $\{M_n \mid n \in \mathbb{N}^t\}$ a family of R-modules, $\{K_n \mid n \in \mathbb{N}^t\}$ a family of A-modules such that $K_n \subseteq M_n$ as A-modules for all $n \in \mathbb{N}^t$.*

If $\{M_n \mid n \in \mathbb{N}^t\}$ satisfies the linear growth property of secondary representation over R with a slope k, then $\{M_n/K_n \mid n \in \mathbb{N}^t\}$ satisfies the linear growth property of secondary representation over A with the same slope k.

Proof. This follows (almost immediately) from Theorem 6.3 (3). □

The next result is dual to Theorem 4.4.

Theorem 11.3. *Let A be a Noetherian ring, R an A-algebra, M a finitely generated A-module, N an Artinian R-module, and $J_1, \dots, J_t$ ideals of R. Then each of the following families of R-modules has the linear growth property of secondary representation (over R):*

(1) *The family $\{(0 :_N J^n) \mid n \in \mathbb{N}^t\}$.*
(2) *The family $\{\operatorname{Ext}^c_A(M, (0 :_N J^n)) \mid n \in \mathbb{N}^t\}$.*
(3) *The family $\{\operatorname{Tor}^A_c(M, (0 :_N J^n)) \mid n \in \mathbb{N}^t\}$.*

Proof. Note that all the modules in all the families are Artinian R-modules. Apply the Matlis duality functor $\mathrm{D}(-)$ to the modules. By Observation 8.4 and Lemma 11.2, we only need to prove the linear growth property of primary decomposition for each of the following families over $\widehat{R}$:

(1^*) The family $\{\mathrm{D}(N)/J^n\,\mathrm{D}(N) \mid n \in \mathbb{N}^t\}$.
(2^*) The family $\{\operatorname{Tor}^A_c(M,\, \mathrm{D}(N)/J^n\,\mathrm{D}(N)) \mid n \in \mathbb{N}^t\}$.
(3^*) The family $\{\operatorname{Ext}^c_A(M,\, \mathrm{D}(N)/J^n\,\mathrm{D}(N)) \mid n \in \mathbb{N}^t\}$.

Since $\mathrm{D}(N)$ is a Noetherian $\widehat{R}$-module and $\widehat{R}$ is clearly an A-algebra, the desired linear growth property of primary decomposition of the three families follows immediately from Theorem 4.4. □

Theorem 11.3 is a special case of the following dual statement of Theorem 4.5.

Theorem 11.4. *Let A be a ring and R an A-algebra. Let $J_1, J_2, \dots, J_t$ be fixed ideals of R, N an Artinian R-module and $c \in \mathbb{Z}$. Fix a complex*

$$F_\bullet: \qquad \cdots \to F_{c+1} \to F_c \to F_{c-1} \to \cdots \to F_i \to F_{i-1} \to \cdots$$

of finitely generated flat A-modules. For any $n \in \mathbb{N}^t$, let

$$T_n = \mathrm{H}_c(F_\bullet \otimes_A (0 :_N J^n)) \qquad \textit{and} \qquad E_n = \mathrm{H}^c(\operatorname{Hom}_A(F_\bullet, (0 :_N J^n))).$$

Then the families $\{T_n \mid n \in \mathbb{N}^t\}$ *and* $\{E_n \mid n \in \mathbb{N}^t\}$*, both consisting of Artinian* R*-modules, satisfy the linear growth property of secondary representation (over* R*).*

Proof. By Observation 8.4 and Lemma 11.2, it suffices to show the linear growth property of primary decomposition for $\{\mathrm{D}(T_n) \mid n \in \mathbb{N}^t\}$ and $\{\mathrm{D}(E_n) \mid n \in \mathbb{N}^t\}$ over $\widehat{R}$. By Matlis duality, we have

$$\mathrm{D}(T_n) \cong \mathrm{H}^c\left(\mathrm{Hom}_A\left(F_\bullet, \frac{\mathrm{D}(N)}{J^n\,\mathrm{D}(N)}\right)\right) \quad \text{and} \quad \mathrm{D}(E_n) \cong \mathrm{H}_c\left(F_\bullet \otimes_A \frac{\mathrm{D}(N)}{J^n\,\mathrm{D}(N)}\right).$$

As $\mathrm{D}(N)$ is Noetherian over $\widehat{R}$, both $\{\mathrm{D}(T_n) \mid n \in \mathbb{N}^t\}$ and $\{\mathrm{D}(E_n) \mid n \in \mathbb{N}^t\}$ satisfy the linear growth property of primary decomposition by Theorem 4.5. □

The following may also be viewed as a dual of Theorem 4.5.

Theorem 11.5. *Let* A *be a ring,* $J_1, J_2, \ldots, J_t$ *fixed ideals of* A*, and* M *a finitely generated* A*-module. Let* R *be an* A*-algebra and* $c \in \mathbb{Z}$*. Fix a complex*

$$F^\bullet : \qquad \cdots \to F^i \to F^{i+1} \to \cdots \to F^{c-1} \to F^c \to F^{c+1} \to \cdots$$

of injective Artinian R*-modules. Denote* $E_n = \mathrm{H}^c\left(\mathrm{Hom}_A\left(\frac{M}{J^n M}, F^\bullet\right)\right)$*, the* c*-th cohomology, for all* $n \in \mathbb{N}^t$*. Then the family* $\{E_n \mid n \in \mathbb{N}^t\}$*, consisting of Artinian* R*-modules, satisfies the linear growth property of secondary representation over* R*.*

Proof. Without affecting the claim, we assume $F^i = 0$ if $i \notin \{c-1, c, c+1\}$. Denote $I = \mathrm{Ann}_{\widehat{R}}(F^{c-1} \oplus F^c \oplus F^{c+1})$. Then $F^\bullet$ is naturally a complex over the complete semi-local Noetherian ring $\widehat{R}/I$ (cf. Observation 8.4). Clearly, each F_j remains injective and Artinian over $\widehat{R}/I$. Thus, replacing R with $\widehat{R}/I$, we may simply assume that R is Noetherian semi-local with $R = \widehat{R}$ (cf. Lemma 11.2).

Now the classic Matlis duality applies, which is still denoted $\mathrm{D}(-)$. What we observed in Observation 8.4 still holds (of course). For each $n \in \mathbb{N}^t$,

$$\begin{aligned} E_n &\cong \mathrm{H}^c\left(\mathrm{Hom}_A\left(\frac{M}{J^n M}, \mathrm{D}(\mathrm{D}(F^\bullet))\right)\right) \cong \mathrm{H}^c\left(\mathrm{D}\left(\frac{M}{J^n M} \otimes_A \mathrm{D}(F^\bullet)\right)\right) \\ &\cong \mathrm{D}\left(\mathrm{H}_c\left(\frac{M}{J^n M} \otimes_A \mathrm{D}(F^\bullet)\right)\right) \cong \mathrm{D}\left(\mathrm{H}_c\left(\frac{M \otimes_A R}{J^n(M \otimes_A R)} \otimes_R \mathrm{D}(F^\bullet)\right)\right). \end{aligned}$$

By Observation 8.4, it suffices to show that the family

$$\left\{\mathrm{D}(E_n) \cong \mathrm{H}_c\left(\frac{M \otimes_A R}{J^n(M \otimes_A R)} \otimes_R \mathrm{D}(F^\bullet)\right) \;\middle|\; n \in \mathbb{N}^t\right\}$$

has linear growth property of primary decomposition over $R = \widehat{R}$. Note that $\mathrm{D}(F^\bullet)$ is a complex of finitely generated projective R-modules; and $M \otimes_A R$ is Noetherian over R. By Theorem 4.5, $\{\mathrm{D}(E_n) \mid n \in \mathbb{N}^t\}$ satisfies the linear growth property of primary decomposition over $R = \widehat{R}$, which completes the proof. □

Now we prove the linear growth property of secondary representation for another family of Ext modules; compare with Theorem 11.3 (2).

Theorem 11.6. *Let R be a Noetherian ring, $I_1, I_2, \ldots, I_s$ ideals of R, M a finitely generated R-module, N an Artinian R-module, and $c \in \mathbb{Z}$.*

Then the family $\left\{\mathrm{Ext}^c_R(\frac{M}{I^m M}, N) \,|\, m \in \mathbb{N}^s\right\}$, which consists of Artinian R-modules, satisfies the linear growth property of secondary representation over R.

Proof. Since R is Noetherian and N is Artinian, the minimal injective resolution of N consists of Artinian R-modules. Then the result follows from Theorem 11.5. ☐

Finally, we state a result (partially) dual to Theorem 5.2 and Corollary 5.5. It also contains Theorem 11.6 as a particular case.

Theorem 11.7. *Let R be a Noetherian ring, $I_1, \ldots, I_s$, $J_1, \ldots, J_t$ ideals of R, M a finitely generated R-module, N an Artinian R-module, and $c \in \mathbb{Z}$. For every $(\alpha, \beta) \in \mathbb{Z}^s \times \mathbb{Z}^t$, consider the following family (of Artinian R-modules)*

$$\boldsymbol{\mathcal{E}}^{(\alpha,\beta)} := \left\{ \mathrm{Ext}^c_R\left(\frac{I^\alpha M}{I^{\alpha+m}M}, \frac{(0:_N J^{\beta+n})}{(0:_N J^\beta)}\right) \;\middle|\; (m,n) \in \mathbb{N}^s \times \mathbb{N}^t = \mathbb{N}^{s+t} \right\}.$$

Then there exists k such that for all $(\alpha, \beta) \in \mathbb{Z}^s \times \mathbb{Z}^t$, the family $\boldsymbol{\mathcal{E}}^{(\alpha,\beta)}$ satisfies the linear growth property for secondary representation over R with the uniform slope k. That is, for every $(\alpha, \beta) \in \mathbb{Z}^{s+t}$ and for every $(m, n) \in \mathbb{N}^{s+t}$ such that $\mathrm{Ext}^c_R\left(\frac{I^\alpha M}{I^{\alpha+m}M}, \frac{(0:_N J^{\beta+n})}{(0:_N J^\beta)}\right) \neq 0$, there exists a secondary representation

$$\mathrm{Ext}^c_R\left(\tfrac{I^\alpha M}{I^{\alpha+m}M}, \tfrac{(0:_N J^{\beta+n})}{(0:_N J^\beta)}\right) = Q_{\alpha,\beta,m,n,1} + Q_{\alpha,\beta,m,n,2} + \cdots + Q_{\alpha,\beta,m,n,s(\alpha,\beta,m,n)},$$

with $Q_{\alpha,\beta,m,n,i}$ being $P_{\alpha,\beta,m,n,i}$-secondary, such that

$$Q_{\alpha,\beta,m,n,i} \subseteq \left(0 :_{\mathrm{Ext}^c_R\left(\frac{I^\alpha M}{I^{\alpha+m}M}, \frac{(0:_N J^{\beta+n})}{(0:_N J^\beta)}\right)} (P_{\alpha,\beta,m,n,i})^{k|(m,n)|}\right)$$

for all $i = 1, 2, \ldots, s(\alpha, \beta, m, n)$.

In particular, $\left\{\mathrm{Ext}^c_R\left(\frac{M}{I^m M}, (0:_N J^n)\right) \;\middle|\; (m, n) \in \mathbb{N}^s \times \mathbb{N}^t = \mathbb{N}^{s+t}\right\}$ satisfies the linear growth property of secondary representation over R.

Proof. As R is Noetherian, the minimal injective resolution of an Artinian R-module consists of Artinian R-modules. For the Artinian R-module N, there are only finitely many maximal ideals $\mathfrak{m}$ such that $\Gamma_{\mathfrak{m}}(N) \neq 0$; say $\mathfrak{m}_1, \ldots, \mathfrak{m}_r \in \mathrm{MSpec}(R)$ are all such maximal ideals. Let $B = \prod_{i-1}^r \widehat{R_{\mathfrak{m}_i}}$, which is a Noetherian flat R-algebra. Note that N, naturally a B-module, is Artinian over B.

Moreover, $\frac{(0:_N J^{\beta+n})}{(0:_N J^\beta)}$ are all naturally Artinian B-modules for all $\beta \in \mathbb{Z}^t$ and all $n \in \mathbb{N}^t$. Therefore, $\mathrm{Ext}^c_R\left(\frac{I^\alpha M}{I^{\alpha+m}M}, \frac{(0:_N J^{\beta+n})}{(0:_N J^\beta)}\right)$ are (naturally) Artinian B-module for all $(\alpha, \beta) \in \mathbb{Z}^s \times \mathbb{Z}^t$ and $(m, n) \in \mathbb{N}^s \times \mathbb{N}^t$. By Lemma 11.2, it suffices to prove that the families $\boldsymbol{\mathcal{E}}^{(\alpha,\beta)}$ satisfy the linear growth property for secondary representation over B with a uniform slope. Note that B is a complete semi-local ring.

Next, we apply the Matlis duality functor $\mathrm{D}(-)$ (over B) to the modules in the families $\boldsymbol{\mathcal{E}}^{(\alpha,\beta)}$. By Observation 8.4, we only need to prove the linear growth property of primary decomposition, with a uniform slope, for the following families over B:

$$\mathrm{D}\,\boldsymbol{\mathcal{E}}^{(\alpha,\beta)} = \left\{ \operatorname{Tor}_c^R \left(\frac{I^\alpha M}{I^{\alpha+m} M}, \frac{J^\beta\,\mathrm{D}(N)}{J^{\beta+n}\,\mathrm{D}(N)} \right) \;\Big|\; (m,n) \in \mathbb{N}^s \times \mathbb{N}^t = \mathbb{N}^{s+t} \right\}.$$

Note that $\mathrm{D}(N)$ is Noetherian over B, while $B = \prod_{i-1}^r \widehat{R_{\mathfrak{m}_i}}$ is a Noetherian ring that is flat over R.

Now, by Theorem 5.2, all the families $\mathrm{D}\,\boldsymbol{\mathcal{E}}^{(\alpha,\beta)}$ satisfy the linear growth property of primary decomposition over B with a uniform slope. The proof is complete. □

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