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MODULES WITH FINITE F-REPRESENTATION TYPE

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Abstract

Finitely generated modules with finite F-representation type (or FFRT for short) over Noetherian (local) rings of prime characteristic p are studied. If a ring R has FFRT or, more generally, if a faithful R-module has FFRT, then tight closure commutes with localizations over R. We also define F-contributors and use them to give an effective way to characterize tight closure. Then we show $\lim_{e\to\infty} \frac{\#({}^eM, M_i)}{(ap^d)^e}$ always exists under that assumption that $(R, \mathfrak{m})$ satisfies the Krull-Schmidt condition and M has FFRT by $\{M_1, M_2, \ldots, M_s\}$, in which all the M_i's are indecomposable R-modules belonging to distinct isomorphism classes and $a = [R/\mathfrak{m} : (R/\mathfrak{m})^p]$.

0. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p. Let M be an R-module. Then for any $e \geq 0$, we can derive an R-module structure on the set M with its scalar multiplication determined by $r \cdot m := r^{p^e} m$ for any $r \in R$ and $m \in M$. We denote the derived R-module by ${}^eM$.

We say that M has *finite F-representation type (FFRT)* by finitely generated R-modules $M_1, M_2, \ldots, M_s$ if, for all $e \geq 0$, the R-module ${}^eM$ are all isomorphic to finite direct sums of the R-modules $M_1, M_2, \ldots, M_s$. For each $i = 1, 2, \ldots, s$, we use $\#({}^eM, M_i)$ to denote the number of copies of M_i in the above direct sum decomposition of ${}^eM$. We say M_i is an *F-contributor* if $\lim_{e\to\infty} \frac{1}{(ap^d)^e} \#({}^eM, M_i)$ is positive or non-existent, or equivalently $\limsup_{e\to\infty} \frac{1}{(ap^d)^e} \#({}^eM, M_i) > 0$, where $d = \dim M$ and $a = [R/\mathfrak{m} : (R/\mathfrak{m})^p] < \infty$.

Rings with finite F-representation type (FFRT) were first studied by K. Smith and M. Van den Bergh in [**SVdB**]. The concept of F-contributors and the importance of R being an F-contributor can be found in recent work [**HL**] of C. Huneke and G. Leuschke.

First we show that F-contributors exist and are Cohen-Macaulay:

Theorem (See Lemma 2.1 and Lemma 2.2). *Suppose that $M \neq 0$ is a finitely generated R-module that has FFRT by $\{M_1, M_2, \ldots, M_s\}$. Then at least one of the M_i is a non-zero F-contributor and every non-zero F-contributor is Cohen-Macaulay of dimension $= \dim M$.*

There is a closure operation, called 'tight closure', defined over rings of prime characteristic p ([**HH1**]). Ever since the inception of the tight closure theory, the question whether tight closure commutes with localizations has been resistantly open although it has been proved to have positive answer in special cases. The next

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result shows that FFRT implies commutation of tight closure with localizations. It also demonstrates the importance of F-contributors in computing tight closures.

THEOREM (See Theorem 2.3, Theorem 2.5 and Remark 2.6). *Suppose R is a Noetherian ring of characteristic p.*

(i) *If there is a faithful R-module that has FFRT (e.g. R has FFRT), then tight closure commutes with localizations over R.*

(ii) *Assume that $(R, \mathfrak{m})$ is an analytically unramified, quasi-unmixed ring that has a completely stable test element (e.g. $(R, \mathfrak{m})$ is a complete domain) and that M is a faithful R-module with FFRT by $M_1, M_2, \ldots, M_s$, in which $M_1, M_2, \ldots, M_r$ are all the F-contributors. Set $N = \bigoplus_{i=1}^r M_i$. Then $K_L^* = \ker(L \to L/K \to \operatorname{Hom}_R(N, L/K \otimes_R N))$ for any finitely generated R-modules $K \subseteq L$. (In particular, $I^* = (IN :_R N) = \operatorname{Ann}_R(N/IN)$ for any ideal I of R.) This also implies that tight closure commutes with localization.*

Under the assumption that $(R, \mathfrak{m})$ is a strongly F-regular local ring and satisfies the Krull-Schmidt condition, K. Smith and M. Van den Bergh proved in [**SVdB**] that if R has FFRT by indecomposable modules $M_1, M_2, \ldots, M_s$ that belong to distinct isomorphism classes, then $\lim_{e\to\infty} \frac{\#({}^eR, M_i)}{(ap^d)^e}$ always exists for every $i = 1, 2, \ldots, s$.

We are to prove the existence of $\lim_{e\to\infty} \frac{\#({}^eM, M_i)}{(ap^d)^e}$ in a more general situation:

THEOREM (See Theorem 3.11). *Assume that $(R, \mathfrak{m})$ is a local ring that satisfies the Krull-Schmidt condition and M has FFRT by $\{M_1, M_2, \ldots, M_s\}$, in which all the M_i's are indecomposable R-modules belonging to distinct isomorphism classes. Then $\lim_{e\to\infty} \frac{\#({}^eM, M_i)}{(ap^d)^e}$ exists and is rational for every i, where $a = [R/\mathfrak{m} : (R/\mathfrak{m})^p]$.*

In Section 1, we will set up the notations carefully and review some known results. In Section 2, implications of FFRT condition and the importance of F-contributors will be studied. In Section 3, we study the existence of $\lim_{e\to\infty} \frac{\#({}^eM, M_i)}{(ap^d)^e}$.

1. *Notations and known results*

All rings are assumed to be Noetherian and have prime characteristic p unless stated otherwise explicitly. For such a ring R, there is the Frobenius homomorphism $F : R \to R$ defined by $r \mapsto r^p$ for any $r \in R$. Therefore we have iterated Frobenius homomorphism $F^e : R \to R$ defined by $r \mapsto r^{p^e}$ for any $r \in R$. Let M be an R-module. Then for any $e \geq 0$, we can derive an R-module structure on M with its scalar multiplication determined by $r \cdot m := r^{p^e} m$ for any $r \in R$ and $m \in M$. We denote the derived R-module by ${}^eM$. Notice ${}^0M = M$. It is straightforward to see that $\operatorname{Ass}_R(M) = \operatorname{Ass}_R({}^eM)$ and $\operatorname{Hom}_R(M, N) \subseteq \operatorname{Hom}_R({}^eM, {}^eN)$ for every $e \in \mathbb{N}$.

Let I be an ideal of R. Then for any $q = p^e$, we use $I^{[q]}$ to denote the ideal generated by $\{x^q \mid x \in I\}$. For any R-module M, it is easy to see that $\frac{R}{I} \otimes_R {}^eM \cong {}^eM/(I \cdot {}^eM) \cong {}^e(M/I^{[q]}M)$.

If ${}^1R$ is a finitely generated R-module (or equivalently ${}^eR$ is a finitely generated R-module for every $e \geq 0$), we say that R is F-finite. If we denote by $k(P)$ the quotient field of R/P for $P \in \operatorname{Spec}(R)$, then by [**Ku2**], Proposition 1.1, Proposition 2.3 and Theorem 2.5 (also c.f. [**Ku1**], Proposition 3.2), we know that the

F-finiteness of R implies that R has finite Krull dimension, that $[k(P) : k(P)^p] = [k(Q) : k(Q)^p]p^{\dim R_Q/PR_Q}$ for any $P, Q \in \operatorname{Spec}(R)$ such that $P \subseteq Q$, and that R is excellent.

In general, if ${}^1M$ is a finitely generated R-module, we say that M is F-finite. Notice that this implies that the ring $R/\operatorname{Ann}(M)$ is F-finite and therefore implies that ${}^eM$ is a finitely generated R-module for every $e \geq 0$.

Next we define finite F-representation type (FFRT), which will be our main interest of the following sections. Some notations are needed. For an R-module M and an integer $n > 0$, we use nM to denote the direct sum of n copies of M while we agree that $0M = 0$. For non-negative integers $n_1, n_2, \dots, n_s$ and R-modules $M_1, M_2, \dots, M_s$, we use matrix multiplication $(n_1, n_2, \dots, n_s)(M_1, M_2, \dots, M_s)^T$ to denote $n_1M_1 \oplus n_2M_2 \oplus \cdots \oplus n_sM_s = \bigoplus_{i=1}^s M_i^{\oplus n_i}$.

Rings with finite F-representation type (FFRT) were first studied by K. Smith and M. Van den Bergh in [**SVdB**].

Definition 1.1. Let R be a Noetherian ring of characteristic p and M a finitely generated R-module.

(i) We say that M has finite F-representation type (FFRT) by finitely generated R-modules $M_1, M_2, \dots, M_s$ if for every $e \geq 0$, the R-module ${}^eM$ is isomorphic to a finite direct sum of the R-modules $M_1, M_2, \dots, M_s$, i.e. there exist non-negative integers $n_{e1}, n_{e2}, \dots, n_{es}$ such that

$$ {}^eM \cong (n_{e1}, n_{e2}, \dots, n_{es})(M_1, M_2, \dots, M_s)^T = \bigoplus_{i=1}^{s} n_{ei}M_i. $$

(ii) We say $M_1, M_2, \dots, M_s$ form a FFRT system if the R-modules ${}^1M_i$ are all isomorphic to finite direct sums of the R-modules $M_1, M_2, \dots, M_s$, i.e. there exist non-negative integers a_{ij} for $1 \leq i, j \leq s$ such that

$$ {}^1M_i \cong (a_{i1}, a_{i2}, \dots, a_{is})(M_1, M_2, \dots, M_s)^T $$

for all $1 \leq i \leq s$.

(iii) We say that M has FFRT by a FFRT system $M_1, M_2, \dots, M_s$ if the R-modules $M_1, M_2, \dots, M_s$ form a FFRT system and there exists an integer $e \geq 0$ such that the R-module ${}^eM$ is isomorphic to a finite direct sum of the R-modules $M_1, M_2, \dots, M_s$, i.e. there exist non-negative integers $n_{e1}, n_{e2}, \dots, n_{es}$ such that

$$ {}^eM \cong (n_{e1}, n_{e2}, \dots, n_{es})(M_1, M_2, \dots, M_s)^T. $$

Remark 1.2. Same notations as in the Definition 1.1. Then

(i) For the sake of convenience, we allow the M_i to be zero module or $M_i \cong M_j$ for some $i \neq j$.

(ii) If M has FFRT then M is F-finite.

(iii) Suppose that M has FFRT by indecomposable R-modules $M_1, M_2, \dots, M_s$ belonging to different isomorphism classes. If R satisfies the Krull-Schmidt condition and every M_i appears non-trivially in the direct sum decomposition of certain ${}^eM$, then M has FFRT by the FFRT system $M_1, M_2, \dots, M_s$.

(iv) Suppose that M has FFRT by the FFRT system $M_1, M_2, \dots, M_s$ as in Definition 1.1(iii) and let $A := (a_{ij})$ be the $n \times n$ matrix. Then

$$ {}^{e+n}M \cong (n_{e1}, n_{e2}, \dots, n_{es})A^n(M_1, M_2, \dots, M_s)^T $$

for all $n \geq 0$.

(v) If M has FFRT or has FFRT by a FFRT system, then for any multiplicatively closed set U in R, the localization $M_U = U^{-1}M$ also has FFRT or has FFRT by a FFRT system. The same is true for the completions of M.

(vi) If R is F-finite and has finite Cohen-Macaulay representation type, then every finitely generated Cohen-Macaulay R-module M has FFRT by the FFRT system of all distinct indecomposable Cohen-Macaulay modules.

In general, if a finitely generated R-module M has FFRT by $\{M_1, M_2, \dots, M_s\}$, the number of copies of M_i in decompositions of ${}^eM$ is not uniquely determined. But we can fix a decomposition ${}^eM \cong (n_{e1}, n_{e2}, \dots, n_{es})(M_1, M_2, \dots, M_s)^T = \bigoplus_{i=1}^s n_{ei}M_i$ of ${}^eM$ for each $e \geq 0$ in advance. So when we study an R-module M that has FFRT, we agree on the fixed decompositions as above. To make our notation more transparent, we use $\#({}^eM, M_i)$ to denote n_{ei}, the number of copies of M_i in the pre-fixed decompositions of ${}^eM$. It is in this sense that the following notion of F-contributor is defined.

The concept of F-contributors and its importance can be found in recent work [**HL**] of C. Huneke and G. Leuschke. Here we give an explicit definition:

Definition 1.3. Let M be a finitely generated R-module that has FFRT by $\{M_1, M_2, \dots, M_s\}$ and $P \in \operatorname{Spec}(R)$ be a prime ideal of R. Set $d(P) = \dim_{R_P}(M_P)$ and $a(P) = [k(P) : k(P)^p]$. We say M_i, for some $1 \leq i \leq s$, is an F-contributor of M at P if $\limsup_{e\to\infty} \frac{\#({}^eM, M_i)}{(a(P)p^{d(P)})^e} > 0$, or equivalently $\lim_{e\to\infty} \frac{\#({}^eM, M_i)}{(a(P)p^{d(P)})^e}$ is either positive or non-existent. When $(R, \mathfrak{m})$ is local, an F-contributor of M simply denotes an F-contributor of M at $\mathfrak{m}$.

Remark 1.4. Keep the notations of the above definition. Then

(i) Our definition of F-contributor depends on the pre-fixed F-representation of ${}^eM$.

(ii) If $M_P \neq 0$ for some $P \in \operatorname{Spec}(R)$, then at least one of the M_i is an F-contributor at P. See Lemma 2.1.

(iii) Let $P, Q \in \operatorname{Spec}(R)$ be two prime ideals of R such that $a(P)p^{d(P)} = a(Q)p^{d(Q)}$. Then M has the same F-contributors at P and at Q. For this reason, when $a(P)p^{d(P)}$ is constant for all $P \in \Gamma \subseteq \operatorname{Spec}(R)$, we can simply say the F-contributors of M at Γ. In particular, by [**Ku2**], we know that $a(P)p^{d(P)}$ is constant for all $P \in V(\operatorname{Ann}(M)) \subseteq \operatorname{Spec}(R)$ if $\operatorname{Spec}(R/\operatorname{Ann}(M))$ is connected and $R/\operatorname{Ann}(M)$ is locally equidimensional.

Question 1.5. Does $\lim_{e\to\infty} \frac{\#({}^eM, M_i)}{(ap^d)^e}$ always exist for every $i = 1, 2, \dots, s$?

There is a positive answer in [**SVdB**] to the above question in case R is strongly F-regular. Recall that we say a reduced Noetherian ring R of characteristic p is strongly F-regular if for any c in the complement of the union of all minimal primes of the ring R, the inclusion map $Rc^{\frac{1}{p^e}} \subset R^{\frac{1}{p^e}}$ splits for all $e \gg 0$ (or equivalently, for some $e \geq 0$).

Theorem 1.6 (K. Smith and M. Van den Bergh's results about FFRT and growth, [**SVdB**]). *Let R be a strongly F-regular ring that satisfies the Krull-*

Schmidt condition. If R has FFRT by indecomposable modules $M_1, M_2, \ldots, M_s$ that belong to distinct isomorphism classes, then $\lim_{e\to\infty} \frac{\#({}^eR, M_i)}{(ap^d)^e}$ always exists for every $i = 1, 2, \ldots, s$. And $\lim_{e\to\infty} \frac{\#({}^eR, M_i)}{(ap^d)^e} > 0$ if M_i appears non-trivially as a direct summand of ${}^eR$ for some $e \geq 0$.

Definition 1.7 ([**HH1**]). Let R be a Noetherian local ring of characteristic p and L an R-module. The tight closure of 0 in L, denoted by 0^*_L, is defined as follows: An element $x \in L$ is said to be in 0^*_L if there exists an element $c \in R^\circ$ such that $0 = x \otimes c \in L \otimes_R {}^eR$ for all $e \gg 0$, where R° is the complement of the union of all minimal primes of the ring R. Given $K \subseteq L$, the tight closure of K in L, denoted by K^*_L, is then defined as the pre-image of $0^*_{L/K}$ under the natural map $L \to L/K$.

If I is an ideal of R, then I^*_R is usually denoted by I^*. It is easy to see that an element $x \in R$ is in I^* if and only if there exists an element $c \in R^\circ$ such that $cx^{p^e} \in I^{[p^e]}$ for all $e \gg 0$.

An open question in the tight closure theory is whether tight closure commutes with localizations: Given R-modules $K \subseteq L$ and a multiplicatively closed set $U \subset R$, does $(U^{-1}K)^*_{U^{-1}L} = U^{-1}(K^*_L)$ always hold? It suffices to prove the case $K = 0$. We also mention that it is straightforward to show $(U^{-1}K)^*_{U^{-1}L} \supseteq U^{-1}(K^*_L)$.

Theorem 1.8 ([**Mo**]). *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p and $M \neq 0$ a finitely generated R module with $\dim M = d$. Then*

(i) *The limit (with $k = R/\mathfrak{m} = k(\mathfrak{m})$ and $a = [k : k^p]$)*

$$\lim_{e\to\infty} \frac{\lambda_R(M/I^{[p^e]}M)}{p^{de}} \qquad \left(= \lim_{e\to\infty} \frac{\lambda_R(\frac{R}{I} \otimes_R {}^eM)}{(ap^d)^e} \text{ if } a = [k : k^p] < \infty\right)$$

exists and is positive for every $\mathfrak{m}$-primary ideal I of R. The limit is called the Hilbert-Kunz multiplicity of M with respect to I.

(ii) *Hilbert-Kunz multiplicity is additive with respect to short exact sequence. Therefore we have the associativity formula.*

The existence of the Hilbert-Kunz multiplicity of M is generalized in [**Se1**]:

Theorem 1.9 (Seibert's results. [**Se1**], page 278). *Let $(R, \mathfrak{m})$ be an F-finite Noetherian local ring of characteristic p, $k = R/\mathfrak{m}$ and $a = [k : k^p]$. Suppose that j is an integer, that $\mathcal{C}$ is a family of finite R-modules with dimension $\leq j$, and that g is a function from $\mathcal{C}$ to $\mathbb{Z}$, such that for any short exact sequence $0 \to M' \to M \to M'' \to 0$ the following holds:*

(a) $M \in \mathcal{C}$ if and only if $M' \in \mathcal{C}$ and $M'' \in \mathcal{C}$;

(b) $g(M) \leq g(M') + g(M'')$, with equality if the sequence splits.

Then we have the following conclusions:

(i) *If $M \in \mathcal{C}$, then ${}^eM \in \mathcal{C}$ for all $e \in \mathbb{N}$;*

(ii) *For each $M \in \mathcal{C}$ there is a real number $c(M)$ such that*

$$a^{-e} g({}^eM) = c(M)p^{je} + O(p^{(j-1)e}) \qquad \text{for all} \quad n \in \mathbb{N}.$$

Furthermore $c(M)$ is an additive function of M on exact sequences.

(iii) *If g itself is additive on exact sequences, then for any $M \in \mathcal{C}$, the function*

$a^{-e}g({}^eM)$ is a polynomial in p^e of the form

$$a^{-e}g({}^eM) = b_0 + b_1p^e + b_2p^{2e} + \cdots + b_jp^{je},$$

with $b_k \in \mathbb{Q}$, for $k = 0, 1, 2, \ldots, j$.

Some examples of possible functions $g : \mathcal{C} \to Z$ may be defined by $g(M) := \lambda_S(\mathrm{Tor}_i^S(L, M))$, $\lambda_S(\mathrm{Ext}_S^i(L, M))$ or $\lambda_S(\mathrm{Ext}_S^i(M, L))$ for any $i \geq 0$, any Noetherian local ring S of characteristic p such that $R \cong S/I$ for some ideal I of S and any S-module L such that $\lambda_S(L) < \infty$.

Notation 1.10. Let $(R, \mathfrak{m})$ be an (F-finite) Noetherian local ring of prime characteristic p, L and M finitely generated R-modules with $\lambda_R(L) < \infty$ and $\dim(M) = d$.

(i) We denote $e_{HK}(L, M) := \lim_{e\to\infty} \frac{\lambda_R(L\otimes_R {}^eM)}{(ap^d)^e}$ where $a = [k : k^p]$ with $k = R/\mathfrak{m}$.

(ii) In case $L = R/I$ with I an $\mathfrak{m}$-primary ideal, we usually write $e_{HK}(L, M)$ as $e_{HK}(I, M)$, which is exactly the Hilbert-Kunz multiplicity of M with respect to I in Theorem 1.8.

(iii) Actually, the F-finite assumption can be avoided simply by considering the bimodule structure of ${}^eM$.

Theorem 1.11 ([**HH1**], Theorem 8.17). *Let $(R, \mathfrak{m})$ be a local Noetherian ring, M and $K \subseteq L$ R-modules such that $\dim(M) = \dim(R)$ and $\lambda(L) < \infty$, and $I \subseteq J$ $\mathfrak{m}$-primary ideals of R.*

(i) *If $K \subseteq 0_L^*$, then $e_{HK}(L, M) = e_{HK}(L/K, M)$. In particular, if $J \subseteq I^*$, then $e_{HK}(I, M) = e_{HK}(J, M)$.*

(ii) *Conversely, if R is an analytically unramified, quasi-unmixed ring with a completely stable test element (e.g. $(R, \mathfrak{m})$ is a complete domain), then $e_{HK}(L, R) = e_{HK}(L/K, R)$ implies $K \subseteq 0_L^*$. In particular, $e_{HK}(I, R) = e_{HK}(J, R)$ implies $J \subseteq I^*$.*

Actually in [**HH1**], Theorem 8.17, more general results are proved.

2. *F-contributors and tight closures.*

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p and $M \neq 0$ a finitely generated R-module that has FFRT by $\{M_1, M_2, \ldots, M_s\}$. Set $a = [k(\mathfrak{m}) : k(\mathfrak{m})^p]$ and $d = \dim(M)$. Then the sequence $\left\{\frac{\#({}^eM, M_i)}{(ap^d)^e}\right\}_{e=0}^{\infty}$ is bounded for every $i = 1, 2, \ldots, s$ such that $M_i \neq 0$ and at least one of the M_i is a non-zero F-contributor.*

Proof. Without loss of generality, assume that $M_i \neq 0$ for all $i = 1, 2, \ldots, s$. Then by [**Mo**],

$$\lim_{e\to\infty} \frac{\lambda_R(M/\mathfrak{m}^{[q]}M)}{p^{de}} = \lim_{e\to\infty} \sum_{i=1}^{s} \frac{\#({}^eM, M_i)}{(ap^d)^e} \lambda_R(M_i/\mathfrak{m}M_i)$$

exists and is equal to $e_{HK}(\mathfrak{m}, M) > 0$. The existence of the limit and the fact that $\lambda_R(M_i/\mathfrak{m}M_i) > 0$ for all $i = 1, 2, \ldots, s$ prove the boundedness while the fact that $e_{HK}(\mathfrak{m}, M) > 0$ proves the existence of at least one F-contributor. $\square$

LEMMA 2.2. *Let (R, m) be local and $M \neq 0$ a finitely generated R-module that has FFRT by $\{M_1, M_2, \ldots, M_s\}$. Set $a = [k(\mathfrak{m}) : k(\mathfrak{m})^p]$ and $d = \dim(M)$. For any $i_0 = 1, 2, \ldots, s$, if $M_{i_0} \neq 0$ and $\liminf_{e\to\infty} \frac{\#({}^eM, M_{i_0})}{(ap^c)^e} > 0$, then $\operatorname{depth} M_{i_0} \geq c$. In particular, every non-zero F-contributor of M is Cohen-Macaulay.*

Proof. Without loss of generality, we assume that M is a faithful R-module. Let $\underline{x} := x_1, x_2, \ldots, x_d$ be a system of parameters of R. Then $\underline{x}^q := x_1^q, x_2^q, \ldots, x_d^q$ is also a system of parameters of R for every $q = p^e$. Let $\mathrm{H}^j_R(\underline{x}^q, M)$ be the j-th Koszul cohomology. Then we have $\lim_{e\to\infty} \frac{\lambda_R(\mathrm{H}^j_R(\underline{x}^{p^e}, M))}{p^{ce}} = 0$ for all $j = 0, 1, \ldots, c-1$ by a result which is implicit in [**Ro**] and explicitly stated in [**HH2**, Theorem 6.2]. On the other hand we have

$$\begin{aligned}\lim_{e\to\infty} \frac{\lambda_R(\mathrm{H}^j_R(\underline{x}^{p^e}, M))}{p^{ce}} &= \lim_{e\to\infty} \frac{\lambda_R(\mathrm{H}^j_R(\underline{x}, {}^eM))}{(ap^c)^e} \\ &= \lim_{e\to\infty} \sum_{i=1}^{s} \frac{\#({}^eM, M_i)}{(ap^c)^e} \lambda_R(\mathrm{H}^j_R(\underline{x}, M_i)).\end{aligned}$$

Therefore $\lambda_R(\mathrm{H}^j_R(\underline{x}, M_{i_0})) = 0$ for all $j = 0, 1, \ldots, c-1$ by our assumption on M_{i_0}. Hence $\operatorname{depth} M_{i_0} \geq c$. In particular, every non-zero F-contributor of M is Cohen-Macaulay. $\square$

Next we study the localization problem under the assumption of FFRT. One way to attack the question of whether tight closure commutes with localizations is to study, for a given $I \subset R$, the finiteness of $\cup_{e\geq 0} \operatorname{Ass}(\frac{R}{I^{[p^e]}})$ and the annihilators of $\mathrm{H}^0_{\mathfrak{m}}(\frac{R}{I^{[p^e]}})$ (see [**HH1**] and [**Ka**], also see [**Hu**] and [**Vr**] for results along this line) while another is to study the 'linear growth' property of the primary decompositions of $I^{[p^e]}$ in R (see [**SS**] or [**SN**]). Our next theorem shows that rings with FFRT satisfy nice properties one would want and consequently tight closure commutes with localizations whenever R has FFRT. The proof of Theorem 2.3(ii) below is similar to that of [**SN**, Theorem 7.6(ii)] and of [**AHH**, Theorem 3.7].

THEOREM 2.3. *Let R and S be Noetherian rings of prime characteristic p and M a finitely generated R-module with FFRT by $\{M_1, M_2, \ldots, M_s\}$.*

(i) *For any finitely generated R-module L, the set $\cup_{e\in\mathbb{N}} \operatorname{Ass}(L \otimes_R {}^eM)$ is finite and there exists an integer $k \in \mathbb{N}$ such that (a) and (b) are satisfied:*
 (a) *For every $e \in \mathbb{N}$, there exists a primary decomposition*
 $$0 = Q_{e1} \cap Q_{e2} \cap \cdots \cap Q_{es_e} \quad \text{of } 0 \text{ in } L \otimes_R {}^eM,$$
 where $\operatorname{Ass}(L \otimes {}^eM) = \{P_{ej} \mid 1 \leq j \leq s_e\}$ and Q_{ej} are P_{ej}-primary components of $0 \subset L \otimes_R {}^eM$ satisfying $P^k_{ej}(L \otimes_R {}^eM) \subseteq Q_{ej}$ for all $1 \leq j \leq s_e$;
 (b) *For all $J \subset R$ and for all $q = p^e$, we have*
 $$J^k(0 :_{L\otimes_R {}^eM} J^\infty) = 0, \text{ i.e., } J^k\, \mathrm{H}^0_J(L \otimes_R {}^eM) = 0.$$

(ii) *Consequently, tight closure commutes with localization if $\operatorname{Ann}_R(M) \subseteq \sqrt{(0)}$, the nilradical of R (e.g. M is faithful over R or $M = R$).*

(iii) *More generally, tight closure commutes with localizations over S provided that $S/\sqrt{(0)} \cong R/\sqrt{\operatorname{Ann}_R(M)}$ as rings.*

Proof. (i): For each $i = 1, 2, \ldots, s$, write down a primary decomposition of 0 in

$L \otimes_R M_i$ (ignore the M_i such that $L \otimes_R M_i = 0$) as following

$$0 = Q'_{i1} \cap Q'_{i2} \cap \cdots \cap Q'_{it_i},$$

where Q'_{ij} is a P'_{ij}-primary component of $0 \subset L \otimes_R M_i$. Naturally we get an induced primary decomposition of $0 \subset L \otimes_R {}^eM$ for every e since ${}^eM$ is a direct sum of the M_i. Choose $k \in \mathbb{N}$ so that ${P'_{ij}}^k(L \otimes_R M_i) \subseteq Q'_{ij}$ for all $i = 1, 2, \ldots, s$ and all $j = 1, 2, \ldots, t_i$. Then (a) is evidently true. And we also have $J^k(0 :_{L \otimes_R M_i} J^\infty) = 0$ for all i and all $J \subset R$. Thus $J^k(0 :_{L \otimes_R {}^eM} J^\infty) = 0$ for all $J \subset R$, $e \in \mathbb{N}$.

(ii): Let L be any finitely generated R-module and U any multiplicatively closed subset of R. We need to show $0^*_{U^{-1}L} \subseteq U^{-1}(0^*_L)$. We know $\cup_{e \in \mathbb{N}} \operatorname{Ass}(L \otimes_R {}^eM)$ is finite by part (i), say $\cup_{e \in \mathbb{N}} \operatorname{Ass}(L \otimes_R {}^eM) = \{P_1, P_2, \ldots, P_t\}$. Without loss of generality, we assume that, for some $1 \le r \le t$, $P_i \cap U = \emptyset$ and $P_j \cap U \neq \emptyset$ for all $1 \le i \le r, r+1 \le j \le t$. Then there exists $u \in U$ such that $u \in \cap_{j=r+1}^t P_j$. To prove $0^*_{U^{-1}L} \subseteq U^{-1}(0^*_L)$, it suffices to show that if $\frac{x}{1} \in 0^*_{U^{-1}L}$ with $x \in L$, then $x \in U^{-1}(0^*_L)$. The assumption that $\frac{x}{1} \in 0^*_{U^{-1}L}$ implies that there exist $c \in R^\circ$ and $u_e \in U$ such that $0 = u_e x \otimes c \in L \otimes_R {}^eR$ for all $e \gg 0$ (see [**AHH**, Lemma 3.3]). This implies that $0 = u_e x \otimes cm \in L \otimes_R {}^eM$ for all $m \in M$ and all $e \gg 0$ (since the R-linear map $R \to M$ defined by $1 \mapsto m \in M$ induces an R-linear map ${}^eR \to {}^eM$). Since part (i)(a) holds for M, we adopt the notations there. In particular, for every $m \in M$ and $e \gg 0$,

$$u_e(x \otimes cm) = u_e x \otimes cm = 0 \in Q_{e1} \cap Q_{e2} \cap \cdots \cap Q_{es_e} \subseteq L \otimes_R {}^eM$$

as in (i)(a). Then, for each $e \gg 0$ and $1 \le j \le s_e$, we have $x \otimes cm \in Q_{ej}$ if $P_{ej} \cap U = \emptyset$ while $u^k x \otimes cm \in P_{ej}^k L \otimes_R {}^eM \subseteq Q_{ej}$ if $P_{ej} \cap U \neq \emptyset$. All in all, we have

$$u^k x \otimes cm \in \bigcap_{j=1}^{s_e} Q_{ej} = 0 \subseteq L \otimes_R {}^eM \quad \text{for all } e \gg 0 \text{ and all } m \in M.$$

Now, the assumption that $\operatorname{Ann}_R(M) \subseteq \sqrt{(0)}$ implies that there is an R-linear map $h : M \to R/\sqrt{(0)}$ such that $h(m_0) \in \left(R/\sqrt{(0)}\right)^\circ$ for some $m_0 \in M$. Applying h, we get $0 = u^k x \otimes ch(m_0) \in L \otimes_R {}^e\left(R/\sqrt{(0)}\right)$ for all $e \gg 0$. Notice that $h(m)$ can be lifted back to some $d \in R^\circ$ under the natural ring homomorphism $R \to R/\sqrt{(0)}$. Also observe that, for any given $q_0 = p^{e_0}$, the Frobenius mapping $r \mapsto r^{p^{e_0}}$ defines an R-linear map $F^{e_0} : {}^eR \to {}^{e+e_0}R$ for all e. Choose q_0 large enough so that $\sqrt{(0)}^{[q_0]} = 0$. Then F^{e_0} factors through ${}^e\left(R/\sqrt{(0)}\right)$, which means there exists an R-linear map $G^{e_0} : {}^e\left(R/\sqrt{(0)}\right) \to {}^{e+e_0}R$ such that $G^{e_0}(h(m_0)) = d^{q_0} \in {}^{e+e_0}R$ for all e. Now apply G^{e_0} to the equation $0 = u^k x \otimes ch(m_0) \in L \otimes_R {}^e\left(R/\sqrt{(0)}\right)$ to get $0 = u^k x \otimes (cd)^{q_0} \in L \otimes_R {}^{e+e_0}R$ for all $e \gg 0$, which implies that $u^k x \in 0^*_L$ or, equivalently, $x \in U^{-1}(0^*_L)$.

(iii): This follows from part (ii) as, for a general ring T of characteristic p, tight closure commutes with localization over T if and only if it is true over $T/\sqrt{(0)}$. $\square$

Next we see the usefulness of F-contributors in the tight closure theory.

Proposition 2.4. *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring of characteristic p, M a finitely generated R-module with $\dim(M) = \dim(R)$. Assume that M*

has FFRT by $\{M_1, M_2, \ldots, M_s\}$ and that $\{M_1, M_2, \ldots, M_r\}$ is the set of all F-contributors for some $r \leq s$. Set $N = \bigoplus_{i=1}^r M_i$.

(i) *For any finitely generated R-modules $K \subseteq L$, K_L^* is contained in the kernel of $L \to L/K \to \mathrm{Hom}_R(N, L/K \otimes_R N)$, the composition of the natural and the evaluation R-homomorphisms.*

(ii) *If, furthermore, R is analytically unramified, quasi-unmixed with a completely stable test element (e.g. $(R, \mathfrak{m})$ is a complete domain) and M is faithful over R, then $K_L^* = \ker(L \to L/K \to \mathrm{Hom}_R(N, L/K \otimes_R N))$.*

Proof. Without loss of generality, we assume $K = 0$. Since $0_L^* \subseteq \bigcap_{n>0} (\mathfrak{m}^n L)_L^*$ and equality holds if there is a test element (by [**HH1**], Proposition 8.13(b)) and $\ker(L \to \mathrm{Hom}_R(N, L \otimes N)) = \bigcap_{n>0} \ker\big(\frac{L}{\mathfrak{m}^n L} \to \mathrm{Hom}_R(N, \frac{L}{m^n L} \otimes N)\big)$, we assume $\lambda_R(L) < \infty$, still, without loss of generality. Let D be an arbitrary R-submodule of L and denote $L' := L/D$. Set $a = [k : k^p]$, $d = \dim(R) = \dim(M)$. Then we have

$$\begin{aligned} e_{HK}(L, M) - e_{HK}(L', M) &= \lim_{e\to\infty} \frac{\lambda_R(L \otimes_R {}^eM)}{(ap^d)^e} - \lim_{e\to\infty} \frac{\lambda_R(L' \otimes_R {}^eM)}{(ap^d)^e} \\ &= \lim_{e\to\infty} \sum_{i=1}^s \frac{\#({}^eM, M_i)}{(ap^d)^e} \lambda_R(L \otimes_R M_i) - \lim_{e\to\infty} \sum_{i=1}^s \frac{\#({}^eM, M_i)}{(ap^d)^e} \lambda_R(L' \otimes_R M_i) \\ &= \lim_{e\to\infty} \sum_{i=1}^s \frac{\#({}^eM, M_i)}{(ap^d)^e} \big(\lambda_R(L \otimes_R M_i) - \lambda_R(L' \otimes_R M_i)\big) \\ &= \lim_{e\to\infty} \sum_{i=1}^r \frac{\#({}^eM, M_i)}{(ap^d)^e} \big(\lambda_R(L \otimes_R M_i) - \lambda_R(L' \otimes_R M_i)\big), \end{aligned}$$

which implies that $e_{HK}(L, M) = e_{HK}(L', M) \iff \lambda_R(L \otimes_R M_i) = \lambda_R(L' \otimes_R M_i)$ for all $i = 1, 2, \ldots, r \iff \lambda_R(L \otimes_R N) = \lambda_R(L' \otimes_R N) \iff D \subseteq \{x \in L \,|\, 0 = x \otimes y \in L \otimes_R N, \forall y \in N\} = \ker(L \to \mathrm{Hom}_R(N, L \otimes_R N))$.

(i): Since $e_{HK}(L, M) = e_{HK}(L/0_L^*, M)$ by Theorem 1.11, we have, by the above argument, $0_L^* \subseteq \ker(L \to \mathrm{Hom}_R(N, L \otimes_R N))$.

(ii): Let $D' = \ker(L \to \mathrm{Hom}_R(N, L \otimes_R N))$ and $L'' = L/D'$. Then, by the above argument again, $e_{HK}(L, M) = e_{HK}(L'', M)$. This implies that $e_{HK}(L, R/P) = e_{HK}(L'', R/P)$ for every $P \in \min(M) = \min(R)$ by the associativity formula, the fact that R is equidimensional and the fact that, a priori, $e_{HK}(L, R/P) \geq e_{HK}(L'', R/P)$ for each minimal prime P. Hence $e_{HK}(L, R) = e_{HK}(L'', R)$, by the associativity formula again, which implies $D' \subseteq 0_L^*$ by Theorem 1.11. Combined with the result in (i), this gives $0_L^* = \ker(L \to \mathrm{Hom}_R(N, L \otimes_R N))$. □

The next theorem is a global version of the above Proposition 2.4. Notice that Theorem 2.5(iii) is just a special case of Theorem 2.3(ii) but is proved differently. Recall that persistence of tight closure holds if R is essentially of finite type over an excellent local ring or if $R/\sqrt{(0)}$ is F-finite by [**HH3**, Theorem 6.24].

THEOREM 2.5. *Let R be a Noetherian ring of characteristic p and M a finitely generated R-module with FFRT by $\{M_1, M_2, \ldots, M_s\}$. Consider the following conditions:*

(1) *$[k(\mathfrak{m}) : k(\mathfrak{m})^p] p^{\dim(M_\mathfrak{m})}$ is constant for all maximal ideals $\mathfrak{m}$ of R. Under this condition, we set $N = \bigoplus_{i=1}^r M_i$ be a direct sum of all the F-contributors at all maximal ideals $\mathfrak{m}$ of R (see Remark 1.4(iii)).*

(2) *Either* (a) *persistence of tight closure holds; or* (b) $\dim(M_{\mathfrak{m}}) = \dim(R_{\mathfrak{m}})$ *for all maximal ideals* $\mathfrak{m}$ *of* R*.*

(3) M *is faithful,* R *has a test element, and, for every maximal ideal* $\mathfrak{m}$ *of* R*,* $R_{\mathfrak{m}}$ *is an analytically unramified, quasi-unmixed and with a completely stable test element.*

Then:

(i) *Assume* (1) *and* (2)*. Then* $K^*_L \subseteq \ker(L \to L/K \to \operatorname{Hom}_R(N, L/K \otimes_R N))$ *for any finitely generated* R*-modules* $K \subseteq L$*.*

(ii) *Assume* (1) *and* (3)*. Then* $K^*_L = \ker(L \to L/K \to \operatorname{Hom}_R(N, L/K \otimes_R N))$ *for any finitely generated* R*-modules* $K \subseteq L$*.*

(iii) *Assume* (3)*. Then tight closure commutes with localization over* R*, that is,* $(U^{-1}K)^*_{U^{-1}L} = U^{-1}(K^*_L)$ *for any finitely generated* R*-modules* $K \subseteq L$ *and for any multiplicatively closed set* $U \subset R$*.*

Proof. Without loss of generality, we assume $K = 0$. Notice that condition (3) implies condition (2)(b).

(i): If condition (2)(a) is satisfied, then it is enough to prove the desired result over $R/\operatorname{Ann}(M)$ via the natural map $R \to R/\operatorname{Ann}(M)$. But notice that M is faithful over $R/\operatorname{Ann}(M)$ hence (2)(b) is satisfied. So we assume (2)(b) without loss of generality. For every maximal ideal $\mathfrak{m}$ of R, we have $(0^*_L)_{\mathfrak{m}} \subseteq 0^*_{L_{\mathfrak{m}}}$. We then apply Proposition 2.4(i) to local ring $R_{\mathfrak{m}}$ and get $0^*_{L_{\mathfrak{m}}} \subseteq (\ker(L \to \operatorname{Hom}_R(N, L \otimes_R N)))_{\mathfrak{m}}$. Hence $0^*_L \subseteq \ker(L \to \operatorname{Hom}_R(N, L \otimes_R N))$.

(ii): We have $0^*_L = \bigcap_{\mathfrak{m}} \bigcap_{n \geq 0} (\mathfrak{m}^n L)^*_L$ (by [**HH1**], Proposition 8.13(b)), where $\mathfrak{m}$ runs over all maximal ideals of R. For each maximal ideal $\mathfrak{m}$ of R, let $\phi_{\mathfrak{m}}$ denote the natural ring homomorphism $R \to R_{\mathfrak{m}}$. By [**HH1**, Proposition 8.9], we have

$$\begin{aligned}
(\mathfrak{m}^n L)^*_L &= \phi_{\mathfrak{m}}^{-1}\big(((\mathfrak{m}^n L)_{\mathfrak{m}})^*_{L_{\mathfrak{m}}}\big) \\
&= \phi_{\mathfrak{m}}^{-1}\big(\ker(L_{\mathfrak{m}} \to \Big(\frac{L}{\mathfrak{m}^n L}\Big)_{\mathfrak{m}} \to \operatorname{Hom}(N_{\mathfrak{m}}, \Big(\frac{L}{\mathfrak{m}^n L}\Big)_{\mathfrak{m}} \otimes N_{\mathfrak{m}}))\big) \\
&= \phi_{\mathfrak{m}}^{-1}\big((\ker(L \to \frac{L}{\mathfrak{m}^n L} \to \operatorname{Hom}(N, \frac{L}{\mathfrak{m}^n L} \otimes N)))_{\mathfrak{m}}\big) \\
&= \ker\big(L \to \frac{L}{\mathfrak{m}^n L} \to \operatorname{Hom}(N, \frac{L}{\mathfrak{m}^n L} \otimes N)\big).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
0^*_L &= \bigcap_{\mathfrak{m}} \bigcap_{n \geq 0} (\mathfrak{m}^n L)^*_L = \bigcap_{\mathfrak{m}} \bigcap_{n \geq 0} \ker\big(L \to \frac{L}{\mathfrak{m}^n L} \to \operatorname{Hom}(N, \frac{L}{\mathfrak{m}^n L} \otimes N)\big) \\
&= \ker\big(L \to \operatorname{Hom}(N, L \otimes N)\big).
\end{aligned}$$

(iii): If $\operatorname{Spec}(R)$ is disconnected, i.e. $R = R_1 \times R_2$, then both R_1 and R_2 satisfy the conditions of the theorem. Also to show that tight closure commutes with localization for R, it is enough to show the same results for both R_1 and R_2.

Hence we may assume that $\operatorname{Spec}(R)$ is connected so that $[k(P) : k(P)^p]p^{\dim R_P} = [k(Q) : k(Q)^p]p^{\dim R_Q}$ for any two prime ideals P and Q of R. Therefore condition (1) is satisfied by any localization of R and hence result in part (i) applies.

To prove tight closure commutes with localization, it is enough to show, for any multiplicatively closed set $U \subset R$, $U^{-1}(0^*_L) = 0^*_{U^{-1}L}$. Applying result in part (i) to $U^{-1}R$, we have $0^*_{U^{-1}L} \subseteq \ker(U^{-1}L \to \operatorname{Hom}(U^{-1}N, U^{-1}L \otimes U^{-1}N)) = U^{-1}(\ker(L \to \operatorname{Hom}(N, L \otimes N)))$. But we have $0^*_L = \ker(L \to \operatorname{Hom}(N, L \otimes N))$

by (ii) above. Hence $0^*_{U^{-1}L} \subseteq U^{-1}(0^*_L)$. And we conclude that $U^{-1}(0^*_L) = 0^*_{U^{-1}L}$ as $U^{-1}(0^*_L) \subseteq 0^*_{U^{-1}L}$ is automatic. $\square$

Remark 2.6. We might be interested in the ideals cases of Theorem 2.3 and Theorem 2.5. It is straightforward to get the results by letting $L = R/I$.

(i) Theorem 2.3(i) says the set $\cup_{e\in\mathbb{N}} \operatorname{Ass}(\frac{R}{I} \otimes_R {}^eM) = \cup_{e\in\mathbb{N}} \operatorname{Ass}(\frac{M}{I^{[q]}M})$ is finite and $J^k \cdot \mathrm{H}^0_J(\frac{^eM}{I\cdot{}^eM}) = 0$, for all $J \subset R$ and for all $q = p^e$, which implies

$$J^{(k+\mu(J))q}(I^{[q]}M :_M J^\infty) \subseteq I^{[q]}M, \text{ i.e., } J^{(k+\mu(J))q}\,\mathrm{H}^0_J(\frac{M}{I^{[q]}M}) = 0,$$

where $\mu(J)$ is the least number of generators of the ideal J.

(ii) Theorem 2.5(ii) simply says $I^* = (IN :_R N) = \operatorname{Ann}_R(N/IN)$.

Remark 2.7. Let R be a Noetherian ring of characteristic p that has FFRT by $\{M_1, M_2, \ldots, M_s\}$. Say that $\{M_1, M_2, \ldots, M_t\}$ is the set of all modules that appear in the decompositions of ${}^eR$ non-trivially for infinitely many e. Let $N' = \bigoplus_{i=1}^t M_i$. Then the Frobenius closure of 0 in an R-module L, denoted by 0^F_L, is determined by $0^F_L = \ker(L \to \operatorname{Hom}_R(N', L \otimes_R N'))$. In particular, the Frobenius closure of an ideal I in R, denoted by I^F, is characterized by $I^F = (IN' :_R N')$. The proof is similar to the one of Proposition 2.4 but more direct.

Discussion 2.8. Let R be as in Theorem 2.5(ii) and adopt the notations there. We furthermore assume $\#({}^{e_0}R, M_i) > 0$ for some e_0 and for all $i = 1, 2, \ldots, r$. Let $q_0 = p^{e_0}$. Then $N = \bigoplus_{i=1}^r M_i$ may be realized as a direct summand of R^{1/q_0} since ${}^eR \cong R^{1/p^e}$ as R-modules for every e. Say that $N = \bigoplus_{i=1}^r M_i$ is generated by $c_1^{1/q_0}, c_2^{1/q_0}, \ldots, c_t^{1/q_0}$ as an R-submodule of R^{1/q_0}. Let $\tau_0 = (c_1, c_2, \ldots, c_t)$ be the ideal of R generated by $c_1, c_2, \ldots, c_t$. Then for any ideal I of R and an element $x \in R$, we have $x \in I^*$ if and only if $\tau_0 x^{q_0} \subseteq I^{[q_0]}$. Indeed, $x \in I^*$ if and only if $xN \subseteq IN$, i.e. $x(c_1^{1/q_0}, c_2^{1/q_0}, \ldots, c_t^{1/q_0}) \subseteq I(c_1^{1/q_0}, c_2^{1/q_0}, \ldots, c_t^{1/q_0})$ if and only if $x(c_1^{1/q_0}, c_2^{1/q_0}, \ldots, c_t^{1/q_0}) \subseteq IR^{1/q_0}$ if and only if $\tau_0 x^{q_0} \subseteq I^{[q_0]}$. Here the second 'if and only if' follows from the fact that N is a direct summand of R^{1/q_0} while the third 'if and only if' follows by taking the q_0-th Frobenius power or the q_0-th root. More generally, we mention that, for any finitely generated R-modules $K \subseteq L$ and any element $x \in L$, we have $x \in K^*_L$ if and only if $\tau_0 x^{q_0}_L \subseteq K^{[q_0]}_L$. (See [**HH1**, Discussion 8.1] for the meaning of $x^{q_0}_L$ and $K^{[q_0]}_L$.) The proof, which we omit, is similar to the one for the ideal case above. Once again, we deduce that tight closure commutes with localization in this case.

Remark 2.9. Of course we can talk about F-contributors for any F-finite R-module M without the assumption of FFRT. If $[k(\mathfrak{m}) : k(\mathfrak{m})^p]p^{\dim(M_\mathfrak{m})}$ is constant for all maximal ideals $\mathfrak{m}$ of R and N is a non-zero F-contributor of M at all maximal ideals $\mathfrak{m}$ of R, then we always have:

(i) Suppose that $\dim M = \dim R$. Then for any finitely generated R-module L, we have $0^*_L \subseteq \ker(L \to \operatorname{Hom}_R(N, L \otimes_R N))$.

(ii) N is necessarily a Cohen-Macaulay module if R is local. More generally, results similar to Lemma 2.2 can be proved.

3. *The sequence* $\left\{\frac{\#({}^eM,M_i)}{(ap^d)^e}\right\}_{e=0}^{\infty}$

In this section we study the growth of $\#({}^eM, M_i)$ as $e \to \infty$. We restrict ourselves to the case where $(R, \mathfrak{m})$ is local and $M \neq 0$ is a finitely generated R-module with FFRT by a FFRT system $M_1, M_2, \dots, M_s$. Without loss of generality, we may simply assume that $M \cong XY$ and ${}^eY \cong A^eY$ for all $e \geq 0$, where $X = (n_1, n_2, \dots, n_s)$ is a $1 \times s$ matrix, $A := (a_{ij})$ is an $s \times s$ matrix with non-negative integer entries and $Y = (M_1, M_2, \dots, M_s)^T$. Consequently ${}^eM \cong XA^eY$ for all $e \geq 0$. For each $i = 1, 2, \dots, s$, let $E_i = (0, \dots, 0, 1, 0, \dots, 0)^T$. Then we can easily see that $\#({}^eM, M_i) = XA^eE_i$. Then $\frac{\#({}^eM,M_i)}{(ap^d)^e} = XB^eE_i$, where $B = \frac{1}{ap^d}A$. We use E to denote the identity matrix of various sizes and use $Z = (z_1, z_2, \dots, z_s)^T \in \mathbb{C}^s$ to denote an arbitrarily chosen and then fixed $s \times 1$ matrix with entries in $\mathbb{C}$. Similarly $X = (n_1, n_2, \dots, n_s)$ is used to denote an arbitrarily chosen and then fixed vector. But we may insist that the entries of X be non-negative integers in order to maintain the realization that $XB^eE_i = \frac{\#({}^eM,M_i)}{(ap^d)^e}$ where $M = \bigoplus_{i=1}^s n_iM_i$.

We also assume that $\{M_1, M_2, \dots, M_r\}$ is the set of all F-contributors of $\oplus_{i=1}^s M_i$ so that, for any R-module $M \cong XY$, the set of F-contributors of M is contained in $\{M_1, M_2, \dots, M_r\}$. We call $M_1, M_2, \dots, M_r$ the general F-contributors of the FFRT system $\{M_1, M_2, \dots, M_s\}$. Also we set $Y' = (M_1, M_2, \dots, M_r, 0, \dots, 0)^T$, $a = [k(\mathfrak{m}) : k(\mathfrak{m})^p]$ and $d = \dim M$.

We will keep these notations throughout this section.

Therefore Question 1.5 can be restated as: Does $\lim_{e\to\infty} XB^eE_i$ exist for every $i = 1, 2, \dots, s$? Or equivalently, does $\lim_{e\to\infty} XB^e$ exist? Or, still equivalently, does $\lim_{e\to\infty} XB^eZ$ exist for every $Z \in \mathbb{C}^s$?

A slightly stronger question would be:

Question 3.1. Does the limit $\lim_{e\to\infty} XB^eE_i$ exist for every $X \in \mathbb{N}^s$ and every $i = 1, 2, \dots, s$? Or equivalently, does $\lim_{e\to\infty} B^e$ exist? Or, still equivalently, does $\lim_{e\to\infty} XB^eZ$ exist for every $X \in \mathbb{N}^s$ and every $Z \in \mathbb{C}^s$?

Example 3.2. Actually we should not expect a positive answer to the above question in general. There might be relations among $M_1, M_2, \dots, M_s$ in terms of direct sums. Indeed, let $R = k$ be a field of characteristic $p = 2$ such that $[k : k^2] = 2$ and let $M = M_1 = M_2 = k$. Then M has FFRT by a FFRT system M_1, M_2 and we may pre-fix the direct sum decompositions of ${}^eM$ so that $X = (1, 0)$ and $A = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$. But it is easy to see that $\lim_{e\to\infty} \frac{\#({}^eM,M_i)}{2^e}$ do not exist for $i = 1, 2$. Or even simpler, let $R = k = M = M_1 = M_2$ where k is a perfect field and $X = (1, 0)$ so that $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

By a result of K. Smith and Van den Bergh, quoted as Theorem 1.6, the limit always exists and is always positive for $M = R$ where R is strongly F-regular ring with FFRT by finitely many indecomposable modules which satisfies the Krull-Schmidt condition. Notice that in this case R does have FFRT by a FFRT system.

In this section, we first study the properties of the matrix B in the general situations of FFRT by a FFRT system. Then, in Theorem 3.11, we will give a positive answer to Question 3.1 under the assumption that R satisfies the Krull-

Schmidt condition and that $M_1, M_2, \ldots, M_s$ are all indecomposable, non-zero and belong to different isomorphism classes.

LEMMA 3.3. *All of the eigenvalues of B have absolute values ≤ 1.*

Proof. It follows from Lemma 2.1: Suppose, on the contrary, that there exist an $\lambda \in \mathbb{C}$ and a complex vector $V = (v_1, v_2, \ldots, v_s)^T \neq 0$ such that $|\lambda| > 1$ and $BV = \lambda V$. Then $B^e V = \lambda^e V$. By choosing a proper $X \in \mathbb{N}^s$ such that $XV \neq 0$, we have that $|XB^e V| = |\lambda^e XV| = |\lambda|^e |XV| \to \infty$ as $e \to \infty$. But by Lemma 2.1 applied to $M = X(M_1, M_2, \ldots, M_s)^T$, $|XB^e V| \leq \sum_{i=1}^s |v_i| \frac{\#({}^eM, M_i)}{(ap^d)^e}$ defines a bounded sequence. Contradiction. $\square$

Let $\lambda_1, \lambda_2, \ldots, \lambda_l$ be the distinct eigenvalues of B such that $|\lambda_i| = 1$ for $i = 1, 2, \ldots, k$ and $|\lambda_i| < 1$ for $i = k+1, k+2, \ldots, l$. We can think of B as a $\mathbb{C}$-linear transformation of $\mathbb{C}^s$. Now, by the primary decomposition theorem (or Jordan Canonical Form theorem), we can write $\mathbb{C}^s$ as $\mathbb{C}^s = \bigoplus_{j=1}^l \mathcal{Z}_j$, where $\mathcal{Z}_j = \ker((\lambda_j E - B)^s) = \ker((\lambda_j E - B)^n)$ for sufficiently large n. Then every $Z \in \mathbb{C}^s$ can be written as $Z = \sum_{i=1}^l Z_i$ where $Z_i \in \mathcal{Z}_i$ for every $i = 1, 2, \ldots, l$. In particular, $N_i^s Z_i = 0$ for every $i = 1, 2, \ldots, l$, where B_i is the restriction of B to $\mathcal{Z}_i$ and $N_i := B_i - \lambda_i E$ for each $i = 1, 2, \ldots, l$.

Then we have $XB^e Z = \sum_{i=1}^l XB^e Z_i$. For all $e \geq s$, we have $XB^e Z_i = X(\lambda_i E + N_i)^e Z_i = X(\sum_{j=0}^s \binom{e}{j} \lambda_i^{e-j} N_i^j) Z_i = \sum_{j=0}^s \binom{e}{j} \lambda_i^{e-j} X N_i^j Z_i$, which can be realized as $\lambda_i^e \sum_{j=1}^s c_{ij} \binom{e}{j} = \lambda_i^e P_i(e)$, where $c_{ij} = X(\frac{1}{\lambda_i} N_i)^j Z_i$ and $P_i(e)$ is the value of the polynomial $P_i(W) = \sum_{j=1}^s c_{ij} \binom{W}{j} \in \mathbb{C}[W]$ at $W = e$ for each $1 \leq i \leq l$. (Here we assume that all the eigenvalues of B are non-zero. If 0 is an eigenvalue of B, we can treat the part corresponding to 0 separately to get a similar result.) Therefore we have $XB^e Z = \sum_{i=1}^l \lambda_i^e P_i(e)$.

Alternatively we can derive the above result in the following (essentially the same) way by means of matrices: By the primary decomposition theorem, there exists an invertible $s \times s$ matrix T with complex entries such that

$$T^{-1}BT = \begin{pmatrix} B_1 & 0 & \ldots & 0 \\ 0 & B_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & & B_l \end{pmatrix},$$

where, for each $i = 1, 2, \ldots, l$, B_i is an $s_i \times s_i$ matrix such that $N_i' = B_i - \lambda_i E$ is nilpotent for each $i = 1, 2, \ldots, l$. In particular, $(N_i')^s = 0$.

Let $U = XT$ and $V = T^{-1}Z$. Corresponding to the partition of $T^{-1}BT$, we write $U = (U_1, U_2, \ldots, U_l)$ and $V^T = (V_1^T, V_2^T, \ldots, V_l^T)$ so that U_i and V_i^T are both $1 \times s_i$ complex matrices. Then we have $XB^e Z = \sum_{i=1}^l U_i B_i^e V_i$. For all $e \geq s$, we have $U_i B_i^e V_i = U_i(\lambda_i E + N_i')^e V_i = U_i(\sum_{j=0}^s \binom{e}{j} \lambda_i^{e-j} (N_i')^j) V_i = \sum_{j=0}^s \binom{e}{j} \lambda_i^{e-j} U_i (N_i')^j V_i$, which can be realized as $\lambda_i^e \sum_{j=1}^s c_{ij} \binom{e}{j} = \lambda_i^e P_i(e)$, where $c_{ij} = U_i(\frac{1}{\lambda_i} N_i')^j V_i$ and $P_i(e)$ is the value of the polynomial $P_i(W) = \sum_{j=1}^s c_{ij} \binom{W}{j} \in \mathbb{C}[W]$ at $W = e$ for each $1 \leq i \leq l$. (Here we assume that all the eigenvalues of B are non-zero. If 0 is an eigenvalue of B, we can treat the part corresponding to 0 separately to get a similar result.) Therefore we have $XB^e Z = \sum_{i=1}^l \lambda_i^e P_i(e)$.

LEMMA 3.4. *Keep the notations as above. Then:*

(i) *The value 1 is an eigenvalue of B.*

(ii) $P_i(W) = c_{i0} = XZ_i$ *are constant polynomials for all* $i = 1, 2, \ldots, k$.

(iii) *For some fixed* X *and* $Z = \sum_{i=1}^{l} Z_i$ *where* $Z_i \in \mathcal{Z}_i$ *for every* $i = 1, 2, \ldots, l$, *we have* $\lim_{e\to\infty} XB^eZ$ *exists if and only if* $P_i(W) = c_{i0} = XZ_i = 0$ *for every* $i = 1, 2, \ldots, k$ *such that* $\lambda_i \neq 1$.

The proof follows from a lemma in [**Se2**], either directly or indirectly. Also we need to use the fact that the set $\{\binom{W}{j} \mid j = 1, 2, \ldots, s\}$, considered as a subset of the $\mathbb{C}$-vector space $\mathbb{C}[W]$, is linearly independent over $\mathbb{C}$. First we state the lemma.

Lemma 3.5 ([**Se2**], Lemma 2.3). *Given* $\gamma_1, \ldots, \gamma_t \in \mathbb{C}\setminus\{0\}$ *and* $P_1(W), P_2(W), \ldots, P_t(W) \in \mathbb{C}[W] \setminus \{0\}$ *for some* $t \in \mathbb{N}$. *Assume that* $\gamma_1, \gamma_2, \ldots, \gamma_t$ *are distinct. Set* $f(e) := \sum_{i=1}^{t} \gamma_i^e P_i(e)$ *for all* $e \in \mathbb{N}$. *Then we have:*

(i) *The following are equivalent:*
 (a) $\lim_{e\to\infty} f(e) = 0$;
 (b) $|\gamma_i| < 1$ *for all* $i = 1, 2, \ldots, t$.

(ii) *For any* $c \in \mathbb{C} \setminus \{0\}$, *the following are equivalent:*
 (a) $\lim_{e\to\infty} f(e) = c$;
 (b) *There is an* $i_0 \in \mathbb{N}$ *with* $1 \leq i_0 \leq t$ *such that* $\gamma_{i_0} = 1$, $P_{i_0} = c$ *and* $|\gamma_i| < 1$ *for all* $1 \leq i \leq t$ *with* $i \neq i_0$.

Proof of Lemma 3.4. (i): This is basically proved in [**Se2**]. We include a proof for completeness.

Let $Z = (\lambda_R(M_1/\mathfrak{m}M_1), \lambda_R(M_2/\mathfrak{m}M_2), \ldots, \lambda_R(M_s/\mathfrak{m}M_s))^T$. Then

$$\lim_{e\to\infty} \sum_{i=1}^{l} \lambda_i^e P_i(e) = \lim_{e\to\infty} \frac{\lambda_R({}^eM/\mathfrak{m}\cdot {}^eM)}{(ap^d)^e} = \lim_{e\to\infty} \frac{\lambda_R(M/\mathfrak{m}^{[p^e]}M)}{p^{de}} = e_{HK}(\mathfrak{m}, M)$$

and the fact that $e_{HK}(\mathfrak{m}, M) > 0$ implies that $\lambda_{i_0} = 1$ for some $1 \leq i_0 \leq l$ by Lemma 3.5 (ii).

(ii): For each $i = 1, 2, \ldots, l$, set $P_i'(W) = \frac{P_i(W)-P_i(0)}{W} \in \mathbb{C}[W]$. Since $\{XB^eZ = \sum_{i=1}^{l} \lambda_i^e P_i(e)\}_{e=0}^{\infty}$ is bounded, we have

$$0 = \lim_{e\to\infty} \frac{XB^eZ}{e} = \lim_{e\to\infty} \sum_{i=1}^{l} \lambda_i^e \frac{P_i(e)}{e} = \lim_{e\to\infty} \sum_{i=1}^{l} \lambda_i^e P_i'(e),$$

which forces $P_i'(W) = 0$ for all $i = 1, 2, \ldots, k$, which implies that $P_i(W) = c_{i0} = XZ_i$ are constant polynomials for all $i = 1, 2, \ldots, k$.

(iii): Follows directly from part (ii) and Lemma 3.5 (ii). □

Lemma 3.6. *Keep the above notations. Then*

(i) $\mathcal{Z}_i = \ker(B - \lambda_i E) = \ker(N_i)$ *is the eigen-space of* λ_i *(or, in terms of matrix,* $B_i = \lambda_i E$, *i.e.* $N_i' = 0$*) for all* $i = 1, 2, \ldots, k$.

(ii) *Let* $M = XY$ *be a fixed* R*-module. Also we assume that* $\lambda_k = 1$ *without loss of generality. Then* $\lim_{e\to\infty} \frac{\#({}^eM, M_i)}{(ap^d)^e}$ *exists for every* $i = 1, 2, \ldots, s$ *if and only if* $XZ = 0$ *for every* $Z \in \bigoplus_{j=1}^{k-1} \mathcal{Z}_j$.

(iii) *We assume that* $\lambda_k = 1$ *without loss of generality. Let* $Z = \sum_{i=1}^{l} Z_i$ *where* $Z_i \in \mathcal{Z}_i$ *for every* $i = 1, 2, \ldots, l$. *Then* $\lim_{e\to\infty} XB^eZ$ *exists for every* X *if and only if* $Z_i = 0$ *for* $i = 1, 2, \ldots, k-1$.

(iv) *The limit* $\lim_{e\to\infty} \frac{\#({}^eM, M_i)}{(ap^d)^e}$ *exists for every module* $M = XY$ *and for every*

$i = 1, 2, \ldots, s$ if and only if $k = 1$, i.e. $\lambda_1 = 1$ is the only eigenvalue of B with absolute value equal to 1.

Proof. (i). By the above Lemma, we know that $\sum_{j=1}^{s} c_{ij}\binom{W}{j} = P_i(W) = c_i$. Since the set $\{\binom{W}{j} \mid j = 1, 2, \ldots, r\}$, considered as a subset of the $\mathbb{C}$-vector space $\mathbb{C}[W]$, is linearly independent over $\mathbb{C}$, we have $c_{ij} = 0$ for all $j = 1, 2, \ldots, s$. In particular $c_{i1} = 0$. But $c_{i1} = \frac{1}{\lambda_i} X N_i Z_i$. Therefore $X N_i Z_i = 0$. By running X over all possible choices and running Z over all vectors in $\mathbb{C}^s$ (actually it is enough to run Z over all vectors in $\mathcal{Z}_1$), we deduce that $N_1 Z_1 = 0$ for all $Z_1 \in \mathcal{Z}_1$, which proves (i).

(ii) and (iii) immediately follow from the above lemma.

(4) immediately follows from (ii) or (iii). Alternatively it can be proved directly. □

Discussion 3.7. For any $X \in \mathbb{N}^s$, let $\mathcal{V}_X$ to be the set of all $s \times 1$ matrices $V \in \mathbb{C}^s$ with complex entries such that $\lim_{e\to\infty} XB^eV$ exists. It is easy to show that $\mathcal{V}_X$ is a B-subspace of $\mathbb{C}^s$ and that $\lim_{e\to\infty} XB^e$ exists if and only if $\mathcal{V}_X = \mathbb{C}^s$. By the definition of F-contributors, we know that $E_i \in \mathcal{V}_X$, for all $i = r+1, r+2, \ldots, s$ if $M_1, M_2, \ldots, M_r$ are all the F-contributors of $M = XY$.

Similarly, we define $\mathcal{V}$ to be the set of all $s \times 1$ matrices $V \in \mathbb{C}^s$ with complex entries such that $\lim_{e\to\infty} B^eV$ exists. It is easy to show that $\mathcal{V}$ is a B-subspace of $\mathcal{V}_X \subseteq \mathbb{C}^s$ for any $X \in \mathbb{N}^s$ and that $\lim_{e\to\infty} B^e$ exists if and only if $\mathcal{V} = \mathbb{C}^s$. By the definition of the general F-contributors, we know that $E_i \in \mathcal{V}$, for all $i = r+1, r+2, \ldots, s$ since $\{M_1, M_2, \ldots, M_r\}$ contains all the F-contributors of $M = XY$ for all possible X.

Let L be an R-module such that $\lambda_R(L) < \infty$ and $M \cong XY$ so that M has FFRT by $M_1, M_2, \ldots, M_s$. By [**Se1**], we know that

$$\begin{aligned}\lim_{e\to\infty} XB^e\lambda_R(\mathrm{Hom}_R(Y', L)) &= \lim_{e\to\infty} XB^e\lambda_R(\mathrm{Hom}_R(Y, L)) \\ &= \lim_{e\to\infty} \frac{\lambda_R(\mathrm{Hom}_R({}^eM, L))}{(ap^d)^e}\end{aligned}$$

exists. Hence $\{\lambda_R(\mathrm{Hom}_R(Y', L)) \mid \lambda_R(L) < \infty\}$ and $\{\lambda_R(\mathrm{Hom}_R(Y, L)) \mid \lambda_R(L) < \infty\}$ are all contained in $\mathcal{V}$. Hence a sufficient condition for a positive answer to Question 3.1 would be that the $\{\lambda_R(\mathrm{Hom}_R(Y, L)) \mid \lambda_R(L) < \infty\}$ spans $\mathbb{Q}^s$ or that $\{\lambda_R(\mathrm{Hom}_R(Y', L)) \mid \lambda_R(L) < \infty\}$ spans $\mathbb{Q}^r$.

In the remaining part of this section we assume the R-modules $M_1, M_2, \ldots, M_r$ satisfy the following unique condition:

$$\sum_{i=1}^{r} n_i M_i \cong \sum_{i=1}^{r} m_i M_i \quad \text{if and only if} \quad m_i = n_i \text{ for all } 1 \le i \le r. \tag{3.1}$$

This condition is satisfied if, for example, R satisfies the Krull-Schmidt condition and $M_1, M_2, \ldots, M_r$ are all indecomposable, non-zero and belong to different isomorphism classes. Indeed, under the uniqueness condition (3.1), we can show that $\lim_{e\to\infty} \frac{\#({}^eM, M_i)}{(ap^d)^e} = \lim_{e\to\infty} XBE_i$ exists for every $i = 1, 2, \ldots, s$ and every $X \in \mathbb{N}^s$. Its proof uses the following theorems of Robert M. Guralnick [**Gu**] and M. Auslander [**Au**]. We only quote a special version of each of the theorems. See the original papers for their general versions and proofs.

THEOREM 3.8 ([**Gu**], Corollary 1). *Let $(R, \mathfrak{m})$ be a Noetherian local ring, not necessarily of characteristic p and M and N are finite R-modules. If $M/\mathfrak{m}^n M \cong N/\mathfrak{m}^n N$ for a sufficiently large $n \in \mathbb{N}$, then $M \cong N$.*

The next theorem of Auslander can be found in [**Au**] and [**AR**]. A simple and direct proof of the result is provided in [**Bo**] by Klaus Bongartz.

THEOREM 3.9 ([**Au**], [**AR**] and [**Bo**]). *Let R be a Artinian ring, not necessarily of characteristic p and M and N are finite R-modules. Then $M \cong N$ if and only if $\lambda_R(\mathrm{Hom}_R(M, L)) = \lambda_R(\mathrm{Hom}_R(N, L))$ for all finite R-modules L, which is also equivalent to that $\lambda_R(M \otimes L) = \lambda_R(N \otimes L)$ for all finite R-modules L.*

Actually it is the following corollary of the above two theorems that is used in the proof of Theorem 3.11:

COROLLARY 3.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, not necessarily of characteristic p and M and N are finite R-modules. Then $M \cong N$ if and only if $\lambda_R(\mathrm{Hom}_R(M, L)) = \lambda_R(\mathrm{Hom}_R(N, L))$ for all finite R-modules L such that $\lambda_R(L) < \infty$ if and only if $\lambda_R(M \otimes L) = \lambda_R(N \otimes L)$ for all finite R-modules L such that $\lambda_R(L) < \infty$.*

Proof. For any $n \in \mathbb{N}$ and for any finitely generated $R/\mathfrak{m}^n$-module L, we have $\lambda_R(\mathrm{Hom}_R(M, L)) = \lambda_R(\mathrm{Hom}_R(N, L))$ by assumption. That is the same as to say that $\lambda_{R/\mathfrak{m}^n}(\mathrm{Hom}_{R/\mathfrak{m}^n}(M/\mathfrak{m}^n M, L)) = \lambda_{R/\mathfrak{m}^n}(\mathrm{Hom}_{R/\mathfrak{m}^n}(N/\mathfrak{m}^n N, L))$ for any finitely generated $R/\mathfrak{m}^n$-module L. Hence by Theorem 3.9, $M/\mathfrak{m}^n M \cong N/\mathfrak{m}^n N$ as $R/\mathfrak{m}^n$-modules (and as R-modules) for any $n \in \mathbb{N}$. Then Theorem 3.8 gives the desired result that $M \cong N$ as R-modules. $\square$

THEOREM 3.11. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of characteristic p and M a finitely generated R-module with FFRT by a FFRT system $\{M_1, M_2, \ldots, M_s\}$, of which $M_1, M_2, \ldots, M_r$ are the general F-contributors which satisfy the uniqueness condition* (3.1). *Then $\lim_{e\to\infty} \frac{\#({}^eM, M_i)}{(ap^d)^e} = \lim_{e\to\infty} XB^eE_i$ exists and is rational for every $i = 1, 2, \ldots, s$ and every $X \in \mathbb{N}^s$, where $M \cong XY$. Or equivalently, the matrix B has exactly one eigenvalue, i.e.* 1, *with absolute value equal to* 1.

Proof. We first arbitrarily choose and then fix an $X \in \mathbb{N}^s$ and set $M \cong XY$. By discussion 3.7, it suffices to show that the set of vectors $\{\lambda_R(\mathrm{Hom}_R(Y', L)) = (\lambda_R(\mathrm{Hom}_R(M_1, L)), \lambda_R(\mathrm{Hom}_R(M_2, L)), \ldots, \lambda_R(\mathrm{Hom}_R(M_r, L))) \in \mathbb{Q}^r \mid \lambda(L) < \infty\}$ spans $\mathbb{Q}^r$. Suppose not. Then there are integers $c_1, c_2, \ldots, c_r$, not all zero, such that $(c_1, c_2, \ldots, c_r)\lambda_R(\mathrm{Hom}_R(Y', L)) = 0$, i.e.

$$c_1\lambda_R(\mathrm{Hom}_R(M_1, L)) + c_2\lambda_R(\mathrm{Hom}_R(M_2, L)) + \cdots + c_r\lambda_R(\mathrm{Hom}_R(M_r, L)) = 0$$

for all R-modules L such that $\lambda_R(L) < \infty$. Without loss of generality, we may assume that $c_i \geq 0$ for $i = 1, 2, \ldots, t$ and $c_j = -b_j < 0$ for $j = t+1, t+2, \ldots, r$. Let $N' = \bigoplus_{i=1}^t c_i M_i$ and $N'' = \bigoplus_{j=t+1}^r b_j M_j$. Then $(c_1, c_2, \ldots, c_r)\lambda_R(\mathrm{Hom}_R(Y', L)) = 0$ means that $\lambda_R(\mathrm{Hom}_R(N', L)) = \lambda_R(\mathrm{Hom}_R(N'', L))$ for all R-modules L such that $\lambda_R(L) < \infty$, which implies that $N' \cong N''$ from the above Corollary 3.10. But this is impossible as $M_1, M_2, \ldots, M_r$ satisfy the uniqueness condition (3.1).

It remains to show that $\lim_{e\to\infty}\frac{\#({}^eM,M_i)}{(ap^d)^e}=\lim_{e\to\infty}XBE_i$ is rational for every $i=1,2,\dots,s$ and every $X\in\mathbb{N}^s$. This follows directly from a lemma of Seibert [**Se2**], Lemma 2.4. We include a proof for completeness. Indeed, since we know that the only uni-modular eigenvalue of B is 1 and the zero space of $B-E$ is the same as the zero space of $(B-E)^n$ for all $n\in\mathbb{N}$, there exists an invertible matrix $T\in M_{s\times s}(\mathbb{Q})$ such that

$$T^{-1}BT=\begin{pmatrix}E_{s_1\times s_1} & 0\\ 0 & B_{s_2\times s_2}\end{pmatrix},$$

where $E_{s_1\times s_1}$ is the $s_i\times s_i$ identity matrix and $B_{s_2\times s_2}$ is an $s_2\times s_2$ matrix with all its eigenvalues having absolute values strictly less than 1. In particular, $\lim_{n\to\infty}B^n_{s_2\times s_2}=0$.

Write $XT^{-1}=(X',X'')$ and $TE_i=(E_i',E_i'')^T$, where X',X'',E_i' and E_i'' are $1\times s_1, 1\times s_2, s_1\times 1$ and $s_2\times 1$ matrices respectively with rational entries. Then

$$\lim_{n\to\infty}XB^nE_i=\lim_{n\to\infty}(X'E_i'+X''B^n_{s_2\times s_2}E_i'')=X'E_i',$$

which is rational. □

Corollary 3.12. *Let $(R,\mathfrak{m})$ be a local Noetherian ring of characteristic p (not necessarily satisfying the Krull-Schmidt condition) and M be a finitely generated R-module with FFRT. If we use $\#({}^eM,R)$ to denote the maximal number of copies of R appearing as a direct summand of ${}^eM$, then $\lim_{e\to\infty}\frac{\#({}^eM,R)}{(ap^d)^e}$ exists.*

Proof. We may assume that R is complete since $\#({}^eM,R)=\#({}^e\widehat{M},\widehat{R})$. Then the existence of the limit follows immediately from Theorem 3.11 as complete rings satisfy the Krull-Schmidt condition. □

Remark 3.13. The limit $\lim_{e\to\infty}\frac{\#({}^eR,R)}{(ap^d)^e}$ is studied in [**HL**] by C. Huneke and G. Leuschke and is called "the F-signature of R" there.

Question 3.14. Now let us return to the general situation as at the beginning of the section, i.e. we do not assume that R satisfies the Krull-Schmidt condition or that $M_1,M_2,\dots,M_s$ are all indecomposable belonging to distinct isomorphism classes. Let $P(W)\in\mathbb{Q}[W]$ be the characteristic polynomial of B. Suppose $\lambda\in\mathbb{C}$ is a root of $P(W)$ and $|\lambda|=1$. Then is λ an n-th root of 1?

Does Theorem 3.11 help with anything in this direction as we can complete the ring R without loss of generality? If the answer to the above question is positive, then we can show that the sequence $\{\frac{\#({}^eM,M_i)}{(ap^d)^e}\}_{e=0}^{\infty}$ is 'periodically convergent', i.e. there exists an integer $k>0$ such that for every $i=1,2,\dots,k$, $\lim_{n\to\infty}\frac{\#({}^{nk+i}M,M_i)}{(ap^d)^{nk+i}}$ exists.

4. *About* $\bigcap_L \mathrm{Ann}_R(\ker(L\to\mathrm{Hom}_R(N,L\otimes N)))$

Let us return to the situation of Proposition 2.4(ii) and Theorem 2.5(ii) and keep the notations. Both results claim $K_L^*=\ker(L\to L/K\to\mathrm{Hom}_R(N,L/K\otimes N))$ for any finitely generated R-modules $K\subseteq L$, in which N is the direct sum of all F-contributors. Thus the test ideal of R is $\tau=\bigcap_{K\subseteq L}(K:_R(\ker(L\to L/K\to$

$\mathrm{Hom}_R(N, L/K \otimes N))))$, where $K \subseteq L$ run over all finitely generated R-modules. As $K^*_L/K = 0^*_{L/K}$, we may always assume that $K = 0$ to get $\tau = \bigcap_L \mathrm{Ann}_R(\ker(L \to \mathrm{Hom}_R(N, L \otimes N)))$ and it is easy to see that $\ker(L \to \mathrm{Hom}_R(N, L \otimes N))$ consists of $x \in L$ such that $x \otimes N$ is zero in $L \otimes_R N$. In the case of R being approximately Gorenstein, the test ideal can be simplified as $\tau = \bigcap_{I \subset R}(I :_R I^*) = \bigcap_{I \subset R}(I :_R (IN :_R N))$. Our next definition is inspired by this observation.

Definition 4.1. Let R be a Noetherian ring, not necessarily of characteristic p. For any R-module N, we define $\tau(N) = \bigcap_L \mathrm{Ann}_R(\ker(L \to \mathrm{Hom}_R(N, L \otimes N)))$ with L running over all finitely generated R-modules.

Lemma 4.2. *Let R be a Noetherian ring, not necessarily of characteristic p, N be a finitely generated R-module and U a multiplicatively closed subset of R. Then $\tau(N) \cap U \neq \emptyset$ if and only if there exists $n \in \mathbb{N}$ such that $nN_U = N_U \oplus \cdots \oplus N_U$ has a direct summand isomorphic to R_U ($n = 1$ if $R_U = U^{-1}R$ is local).*

Proof. First, we assume that nN_U has a direct summand isomorphic to R_U for some positive integer n. Since $\tau(nN) = \tau(N)$, we may assume $n = 1$. Therefore there exists an element $c \in U$ such that R_c is a homomorphic image of N_c. That is the same as to say that there is R-homomorphism $f : N \to R$ such that $c^i \in f(N)$ for some i. We may as well assume that $i = 1$. Then for any finitely generated R-module L and for any $x \in \ker(L \to \mathrm{Hom}_R(N, L \otimes N))$, we have $x \otimes N = 0$ in $L \otimes_R N$. Applying $1_L \otimes f$ on $L \otimes_R N$, we get $cx = 0 \in L \cong L \otimes R$, which in turn implies that $c \in \mathrm{Ann}_R(\ker(L \to \mathrm{Hom}_R(N, L \otimes N)))$. Hence $c \in \tau(N)$, which gives $\tau(N) \cap U \neq \emptyset$, the desired result.

For the converse implication, we assume that $\tau(N) \cap U \neq \emptyset$. By relabeling R_U and N_U with R and N respectively, we may simply assume that $\tau(N) = R$ and prove nN has a direct summand isomorphic to R for some $n \in \mathbb{N}$. Say N is generated by $x_1, x_2, \ldots, x_n$. Define an R-linear map $\phi : R \to nN$ by $r \mapsto (rx_1, rx_2, \ldots, rx_n)$. The assumption that $\tau(N) = R$ says exactly that the induced map $1_L \otimes \phi : L \otimes_R R \to L \otimes_R nN$ is injective for any finitely generated (hence any) R-module L, i.e. ϕ is pure. Since nN is Noetherian, we get that $\phi : R \to nN$ is a split injection and hence nN has a direct summand isomorphic to R. $\square$

Remark 4.3. Let us again return to Proposition 2.4(ii) and Theorem 2.5(ii) with M being a FFRT faithful R-module. Then R is weakly F-regular if and only if $\tau(N) = R$ if and only if R is an F-contributor of M (by the above Lemma 4.2) if and only if R is strongly F-regular (by a recent result of [**AL**]).

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