

[Preliminary Version]

## UNIFORM TEST EXPONENTS FOR RINGS OF FINITE F-REPRESENTATION TYPE

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ABSTRACT. Let  $R$  be a commutative Noetherian ring of prime characteristic  $p$ . Assume  $R$  (or, more generally, a finitely generated  $R$ -module  $N$  with  $\text{Supp}_R(N) = \text{Spec}(R)$ ) has finite F-representation type (abbreviated FFRT) by finitely generated  $R$ -modules. Then, for every  $c \in R^\circ$ , there is a uniform test exponent  $Q = p^E$  for  $c$  and for all  $R$ -modules.

As a consequence, we show the existence of uniform test exponents over binomial rings (in particular, affine semi-group rings).

The existence of uniform test exponents (for all modules) implies that the tight closure coincides with the finitistic tight closure, and tight closure commutes with localization for all  $R$ -modules.

### 0. INTRODUCTION

Throughout this paper we assume  $R$  is a commutative Noetherian ring with unity and of prime characteristic  $p$ . Then, for every  $e \in \mathbb{N}$ , the iterated Frobenius map  $F^e : R \rightarrow R$  defined by  $r \mapsto r^{p^e}$  is a ring homomorphism.

Let  $N$  be an  $R$ -module with the scalar multiplication denoted by  $rx$  for all  $r \in R$  and  $x \in N$ . (Thus  $N$  is naturally an  $(R, R)$ -bimodule with  $xr = rx$ .) For every  $e \in \mathbb{N}$ , we denote by  ${}^eN$  the derived  $(R, R)$ -bimodule structure on the same abelian group  $N$  with the same right module structure but with the left scalar multiplication determined by  $r \cdot x = r^{p^e}x$  for all  $r \in R$  and  $x \in N$ . (Notice that, if  $R$  is reduced, the left  $R$ -module structure of  ${}^eR$  is isomorphic to  $R^{1/q}$  for all  $q = p^e$ .) By default, the structure of  ${}^eN$  refers to its left  $R$ -module structure. When necessary, we use  ${}^l$  to indicate that the left module structure is being considered. For example,  $\text{Ann}_R^l({}^eN)$  stands for the annihilator of  ${}^eN$  as a left  $R$ -module. Then it is routine to verify that  $\text{Ann}_R(N) \subseteq \text{Ann}_R^l({}^eN) \subseteq \sqrt{\text{Ann}_R(N)}$  and  $\text{Ass}_R(N) = \text{Ass}_R^l({}^eN)$  for all  $e \in \mathbb{N}$ .

A very important concept in studying rings of characteristic  $p$  is *tight closure*, which was first introduced and developed by Hochster and Huneke; see [HH1].

**Definition 0.1** ([HH1]). Let  $M$  be an  $R$ -module. Denote  $R^\circ := R \setminus \cup_{P \in \min(R)} P$ . For every  $e \geq 0$  and  $x \in M$ , denote  $F_R^e(M) := M \otimes_R {}^eR$  and  $x_M^{p^e} := x \otimes 1 \in F_R^e(M)$ .

- (1) The *Frobenius closure* of 0 in  $M$ , denoted  $0_M^F$ , consists of  $x \in M$  such that  $x_M^{p^e} = 0$  (i.e.,  $0 = x \otimes 1 \in M \otimes_R {}^eR$ ) for some  $e \geq 0$  (or, equivalently, for all  $e \gg 0$ ).

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2000 *Mathematics Subject Classification*. Primary 13A35; Secondary 13C13, 13H10.

The author was partially supported by the National Science Foundation (DMS-0700554), and by the Research Initiation Grant of Georgia State University.

- (2) The *tight closure* of 0 in  $M$ , denoted  $0_M^*$ , is defined as follow: An element  $x \in M$  is said to be in  $0_M^*$  if there exists an element  $c \in R^\circ$  such that  $0 = x \otimes c \in M \otimes_R {}^eR$  for all  $e \gg 0$ .
- (3) The *finitistic tight closure* of 0 in  $M$ , denoted  $0_M^{*\text{fg}}$ , is defined as follows: For any  $x \in M$ , we say  $x \in 0_M^{*\text{fg}}$  if there exists a finitely generated  $R$ -submodule  $M_1$  of  $M$  such that  $x \in 0_{M_1}^*$ .

It turns out that  $0_M^F$ ,  $0_M^*$  and  $0_M^{*\text{fg}}$  are all  $R$ -submodules of  $M$ . In general, given  $R$ -modules  $L \subseteq M$ , the tight closure of  $L$  in  $M$ , denoted by  $L_M^*$ , is the (unique)  $R$ -submodule satisfying  $L \subseteq L_M^* \subseteq M$  and  $L_M^*/L = 0_{M/L}^*$ . And one can define  $L_M^F$  and  $L_M^{*\text{fg}}$  similarly. It is routine to verify that  $L_M^F \subseteq L_M^{*\text{fg}} \subseteq L_M^*$ .

From the definitions of tight closure and Frobenius closure, it is evident that the left module structure of  ${}^eR$  plays an important role in determining the membership in tight closure or Frobenius closure.

Note that, in Definition 0.1, the choice of  $c \in R^\circ$  depends on the element  $x \in 0_M^*$ . Because of this, the notion *test element* was introduced in tight closure theory, see [HH1]. Given  $c \in R$  and  $q_0 = p^{e_0}$ , we say  $c$  is a  $q_0$ -weak test element if  $c \in R^\circ$  and  $0 = x \otimes c \in M \otimes_R {}^eR$  for all  $R$ -modules  $M$ ,  $x \in 0_M^*$  and  $e \geq e_0$ . A weak test element simply means a  $q_0$ -weak test element for some  $q_0$ ; and a test element simply means a 1-weak test element. If a test element of  $R$  remains so in all the localizations of  $R$ , we call it a *locally stable test element*.

One of the most important open questions in tight closure theory was whether tight closure commutes with localization, that is, whether  $(0_M^*)_P = 0_{(M_P)}^*$  for all (finitely generated)  $R$ -module  $M$  and all  $P \in \text{Spec}(R)$ . This was answered recently: In [BM], H. Brenner and P. Monsky give an example where tight closure does not commute with localization.

However, there are many special cases where tight closure does commute with localization. It is straightforward to see that  $(0_M^*)_P \subseteq 0_{M_P}^*$ . However, the other containment, i.e.,  $(0_M^*)_P \supseteq 0_{M_P}^*$ , is not so easy to determine as the definition of tight closure involves verifying infinitely many equations. In order to study this, the notion *test exponent* was introduced in [HH3]. For simplicity (and without loss of generality), we give the definition of test exponent concerning  $0_M^*$  only.

**Definition 0.2** ([HH3]). Let  $R$  be a ring,  $c \in R$ , and  $M$  an  $R$ -module. We say  $Q = p^E$  is a test exponent for  $c$  and  $M$  if, for any  $x \in M$ , the occurrence of  $0 = x \otimes c \in M \otimes_R {}^eR = F_R^e(M)$  for one single  $e \geq E$  implies  $x \in 0_M^*$ .

If there exists a test exponent for a locally stable test element  $c \in R^\circ$  and an  $R$ -modules  $M$ , then the tight closure of 0 in  $M$  commutes with localization. This result was implicit in [McD] and was explicitly stated in [HH3, Proposition 2.3]. Moreover, Hochster and Huneke showed in [HH3] that the converse is true:

**Theorem 0.3** ([HH3]). *Let  $c \in R^\circ$  be a locally stable test element and  $M$  a finitely generated  $R$ -module. If the tight closure of 0 in  $M$  commutes with localization, then there exists a test exponent for  $c$  and  $M$ .*

In [HH3], Hochster and Huneke also asked, among other questions, whether there exists a *uniform* test exponent for a given test element and a set of modules. Over

equidimensional excellent local rings, R. Y. Sharp showed the existence of such a uniform test exponent for all modules of the form of  $R/I$  with  $I$  an ideal generated by parameters in [Sh]. (This was later also proved in [HoY1] via a different approach.) Quite generally, Hochster and this author recently have shown the existence of a uniform test exponent for all finitely generated modules of finite phantom projective dimension in [HoY2] via the approach of ‘weak embedding’.

In this paper, we study the existence of uniform test exponents for all  $R$ -modules (finitely generated or not) under the ‘finite F-representation type’ assumption. The notion *finite F-representation type* (FFRT for short) was first introduced and studied by K. Smith and M. Van den Bergh in [SVdB] over F-finite rings; also see [Yao1]. (Recall that  $R$  is said to be F-finite if  ${}^1R$  is a finitely generated  $R$ -module, which then implies that  ${}^eN$  is finitely generated over  $R$  for every  $e \in \mathbb{N}$  and for every finitely generated  $R$ -module  $N$ . By a result in [Ku], F-finite rings are excellent.) However, we do not require  $R$  to be F-finite in our definition below. (For any  $R$ -module  $T$  and any set  $\Lambda$ , we use the notation  $T^{\oplus\Lambda}$  to denote  $\bigoplus_{\lambda \in \Lambda} T_\lambda$  where  $T_\lambda = T$  for every  $\lambda \in \Lambda$ . By convention,  $T^{\oplus\Lambda} = 0$  whenever  $\Lambda = \emptyset$ .)

**Definition 0.4** (Compare with [SVdB] and [Yao1]). Let  $N$  be an  $R$ -module. We say that  $N$  has finite F-representation type (abbreviated FFRT) if there exist finitely many  $R$ -modules  $N_1, N_2, \dots, N_s$  such that, for every  $e \geq 0$ , the derived left  $R$ -module  ${}^eN$  is isomorphic to a direct sum of the  $R$ -modules  $N_1, N_2, \dots, N_s$  with multiplicities, i.e., there exist sets  $\Lambda_{e1}, \Lambda_{e2}, \dots, \Lambda_{es}$  such that

$${}^eN \cong N_1^{\oplus\Lambda_{e1}} \oplus N_2^{\oplus\Lambda_{e2}} \oplus \dots \oplus N_s^{\oplus\Lambda_{es}} = \bigoplus_{i=1}^s N_i^{\oplus\Lambda_{ei}}.$$

To be specific, we say  $N$  has FFRT by  $N_1, N_2, \dots, N_s$ .

*Remark 0.5.* (1) In the above definition, the left modules  ${}^eN$  are not required to be finitely generated over  $R$ ; the  $R$ -modules  $N_1, \dots, N_s$  are not required to be finitely generated; and the sets  $\Lambda_{e1}, \dots, \Lambda_{es}$  are allowed to be infinite. For example, over a local ring  $(R, \mathfrak{m}, k)$  with residue field  $k = R/\mathfrak{m}$ , the derived left module  ${}^e(R/\mathfrak{m})$  is a (possibly infinitely dimensional) vector space over  $k$ . So  $R/\mathfrak{m}$  has FFRT, by  $k$ , according to Definition 0.4 above. More generally, if  $N$  is an  $R$ -module such that  $\text{Ann}_R(N)$  is  $\mathfrak{m}$ -primary, say  $\mathfrak{m}^{[p^{n_1}]} \subseteq \text{Ann}_R(N)$  for some integer  $n_1 \geq 0$ , then  $N$  has FFRT by  $N, {}^1N, \dots, {}^{n_1-1}N, R/\mathfrak{m}$ .

- (2) Generally speaking, for each  $e \in \mathbb{N}$ , the decomposition of  ${}^eN$  as a direct sum of  $N_i$  is not necessarily unique. However, for each  $e$ , we may fix a direct sum decomposition of  ${}^eN$  and, from now on, refer to it as *the* direct sum decomposition of  ${}^eN$ . We say that  $N_i$  appears non-trivially in the direct sum decomposition of  ${}^eN$  if  $\Lambda_{ei} \neq \emptyset$ .
- (3) Assume  $R$  is F-finite and  $N$  is a finitely generated  $R$ -module. Then  $N$  having FFRT implies  $N$  having FFRT by finitely generated  $R$ -modules. In this case, the definition of FFRT agrees with [Yao1, Definition 1.1].

**Example 0.6.** Assume  $(R, \mathfrak{m})$  is an F-finite ring. We have the following examples of rings or modules with FFRT by finitely generated  $R$ -modules.

- (1) If  $(R, \mathfrak{m}, k)$  is regular, then  ${}^e R \cong R^{1/q} \cong R^{\oplus q^{\alpha + \dim(R)}}$  is a free  $R$ -module of rank  $q^{\alpha + \dim(R)}$ , where  $\alpha = \log_p[k : k^p] < \infty$ .
- (2) If  $(R, \mathfrak{m})$  has finite Cohen-Macaulay type and  $N$  is a finitely generated maximal Cohen-Macaulay module, then  $N$  has FFRT (since  ${}^e N$  is a maximal Cohen-Macaulay left  $R$ -module for every  $e \in \mathbb{N}$ ).
- (3) Let  $R \rightarrow S$  be a ring homomorphism such that  $S$  is module-finite over  $R$ ,  $W$  a finitely generated  $S$ -module with FFRT and  $N$  an  $R$ -submodule of  $W$  such that  $W = N \oplus N'$  as  $R$ -modules. Then  $N$  has FFRT as an  $R$ -module by [SVdB, Proposition 3.1.4] essentially. However, the Krull-Schmidt condition is not needed by the virtue of [Wi, Theorem 1.1]. In particular, if  $S$  is an F-finite regular local ring of characteristic  $p$  and  $G$  is a finite group acting on  $S$  such that  $p \nmid |G|$ , then the invariant ring  $R = S^G$  has FFRT.
- (4) Also, normal (affine) semi-group rings and rings of invariants of linearly reductive groups (over a F-finite field) have FFRT by [SVdB, Proposition 3.1.6].
- (5) Every one-dimensional complete local or graded domain with algebraically closed or finite residue field has FFRT, see [Shi] by Shibuta.
- (6) For a nice summary of examples of rings with FFRT, see [TT, Example 1.3] by Takagi and Takahashi.

Generally speaking, for any  $R$ -module  $N$ , the “size” of  ${}^e N$  (as a left  $R$ -module) increases as  $e$  increases. However, when  $N$  has FFRT (say by  $N_1, N_2, \dots, N_s$ ), the left module structures of  ${}^e N$  for all  $e \geq 0$  are ‘captured’ in these of  $N_1, N_2, \dots, N_s$ .

**Theorem 0.7.** *Assume  $R$  is F-finite and assume there exists a finitely generated  $R$ -module  $N$  with  $\text{Supp}_R(N) = \text{Spec}(R)$  (e.g.,  $N = R$ ) that has FFRT. Then*

- (1) *Tight closure commutes with localization for all finitely generated  $R$ -modules; see [Yao1, Theorem 2.3]. (Also see Corollary 3.5.)*
- (2)  *$R$  is F-regular if and only if  $R$  is strongly F-regular (cf. [Yao1, Remark 4.3]).*
- (3) *The F-signature of  $N$  exists and is a rational number; see [Yao2, Theorem 4.6] and also [Yao1, Theorem 3.11 and Corollary 3.12].*<sup>1</sup>

In light of Theorem 0.3 and Theorem 0.7 (1), one could ask whether there exists a *uniform* test exponent for a given  $c \in R^\circ$  and for all (finitely generated)  $R$ -modules. This is answered positively in the main theorem of the paper:

**Theorem** (See Theorem 3.1). *If there exists a finitely generated  $R$ -module  $N$  with  $\text{Ann}_R(N) \subseteq \sqrt{0}$  such that  $N$  has FFRT by finitely generated  $R$ -modules, then, for every  $c \in R^\circ$ , there is a (uniform) test exponent for  $c$  and for all modules over  $R$  and over all its localizations.*

As a corollary, we show the existence of uniform test exponents over binomial rings (in particular, affine semi-group rings). Moreover, the existence of test exponents implies the tight closure is the same as the finitistic tight closure, and, as mentioned above, tight closure commutes with localization for all  $R$ -modules.

<sup>1</sup>In fact, the F-signature of  $N$  exists as long as  $N$  is a finitely generated  $R$ -module that has FFRT as defined in this paper (without assuming  $R$  is F-finite). This could be proved via an approach similar to that in the proof of [Yao2, Theorem 4.6], for example.

## 1. DEFINING THE FROBENIUS CLOSURE AND TIGHT CLOSURE VIA A MODULE

First, we make some definitions that are closely related to the Frobenius closure and tight closure. They will be used in the sequel. (For any  $R$ -modules  $M$ ,  $L \subseteq N$  and  $x \in M$ , we use  $0 = x \otimes L \subseteq M \otimes_R N$  to express the meaning that  $0 = x \otimes y \in M \otimes_R N$  for all  $y \in L$ .)

**Definition 1.1.** Let  $R$  be a ring of characteristic  $p$  and  $M, N$  be  $R$ -modules.

- (1) Define  $0_M^{F,N}$ , which may be called the  $N$ -Frobenius closure of 0 in  $M$ , as

$$0_M^{F,N} := \{x \in M \mid 0 = x \otimes {}^e N \subseteq M \otimes_R {}^e N \text{ for all } e \gg 0\}.$$

- (2) Define  $0_M^{*N}$ , which may be called the  $N$ -tight closure of 0 in  $M$ , as follows: An element  $x \in M$  is in  $0_M^{*N}$  if there exists  $c \in R \setminus \cup_{P \in \min(R/\text{Ann}_R(N))} P$  such that

$$0 = x \otimes {}^e(cN) \subseteq M \otimes_R {}^e N \quad \text{for all } e \gg 0.$$

For all  $R$ -modules  $M$  and  $N$ , it is routine to verify that  $0_M^{F,N}$  and  $0_M^{*N}$  are both  $R$ -submodules of  $M$ . Moreover, it is straightforward to see

$$0_M^{F,N} \subseteq 0_M^{*N}, \quad 0_M^F = 0_M^{F,R} \subseteq 0_M^{F,N} \quad \text{and} \quad 0_M^* = 0_M^{*R},$$

because of the natural isomorphisms  ${}^e N \cong {}^e R \otimes_R N$  for all  $e \geq 0$ . Moreover, when persistence of tight closure holds (e.g., when  $R$  is excellent), the inclusion  $0_M^{*R} \subseteq 0_M^{*N}$  holds; see Lemma 1.3 below. It turns out that  $0_M^{*N}$  is closely related to the ordinary tight closure, as shown in the following Lemma 1.3.

*Remark 1.2.* We want to clarify some notations, which will be used throughout the paper: Let  $M$  and  $N$  be  $R$ -modules. Then, for every  $e \in \mathbb{N}$ , there is a canonical isomorphism  $F_R^e(M) \otimes_R N = M \otimes_R {}^e R \otimes_R N \cong M \otimes_R {}^e N$ . Under this isomorphism,  $(x \otimes c) \otimes N \subseteq F_R^e(M) \otimes_R N$  corresponds to  $x \otimes {}^e(cN) \subseteq M \otimes_R {}^e N$  for all  $x \in M$  and  $c \in R$ . Also note that  ${}^e(cN)$  is not equal to  $c({}^e N)$  in general, as  $c({}^e N) = {}^e(c^p N)$ . However, we do have  ${}^e(cN) = {}^e(Nc) = ({}^e N)c$  (or simply  ${}^e Nc$ ) in light of the  $(R, R)$ -bimodule structure of  ${}^e N$  (and of  $N$ ).

**Lemma 1.3.** *Let  $M$  and  $N$  be  $R$ -modules. Assume  $N \neq 0$  and  $N$  is finitely generated over  $R$ .*

- (1) *If  $\text{Ann}_R(N) \subseteq \sqrt{0}$  (i.e.,  $\text{Supp}_R(N) = \text{Spec}(R)$ ), then  $0_M^{*N} = 0_M^*$ .*  
 (2) *In general,  $0_M^{*N}$  is the preimage of  $0_{(M/\text{Ann}_R(N)M)}^*$  (computed over the ring  $R/\text{Ann}_R(N)$ ) under the natural homomorphism  $M \rightarrow M/\text{Ann}_R(N)M$ .*

*Proof.* (1) It suffices to show  $0_M^{*N} \subseteq 0_M^*$ . Let  $x \in 0_M^{*N}$ . Denote  $\bar{x} := x + \sqrt{0}M \in \bar{M} = M/\sqrt{0}M$  and  $\bar{N} := N/\sqrt{0}N$ . It is clear that  $\bar{x} \in 0_{\bar{M}}^{*\bar{N}}$  over  $R/\sqrt{0}$ . On the other hand, we have that  $x \in 0_M^*$  over  $R$  if and only if  $\bar{x} \in 0_{\bar{M}}^*$  over  $R/\sqrt{0}$  (cf. [HH1]). Thus we may assume  $R$  is reduced and, hence,  $\text{Ann}_R(N) = 0$  without loss of generality. Consequently, there exists  $h \in \text{Hom}_R(N, R)$  such that  $h(N) \cap R^\circ \neq \emptyset$ . (To see this, just invert all elements in  $R^\circ$ .) Say  $h(z) = d \in R^\circ$  for some  $z \in N$ . Then  $h(cz) = cd \in R^\circ$ . Then, as  $h \in \text{Hom}_R(N, R) \subseteq \text{Hom}_R^l({}^e N, {}^e R)$ , we apply the homomorphism

$\text{id}_M \otimes h$  to  $0 = x \otimes {}^e(cN) \subseteq M \otimes_R {}^eN$ , for all  $e \gg 0$ , to see that

$$0 = x \otimes h({}^e(cN)) \subseteq M \otimes_R {}^eR,$$

which implies  $0 = x \otimes h(cz) \in M \otimes_R {}^eR$  for all  $e \gg 0$ ,

in other words,  $0 = x \otimes cd \in M \otimes_R {}^eR$  for all  $e \gg 0$ ,

which implies  $x \in 0_M^*$ . Thus  $0_M^{*N} \subseteq 0_M^*$ , the desired inclusion. (This was also shown implicitly in the proof of [Yao1, Theorem 2.3] with a slightly different argument.)

(2) This follows from (1) above and Definition 1.1, in light of the natural isomorphism  $M \otimes_R {}^eN \cong (M/\text{Ann}_R(N)M) \otimes_{R/\text{Ann}_R(N)} {}^eN$ .  $\square$

The reader might want to compare the following result with the proof of [HH3, Proposition 2.6]. This will be used in the proof of the main theorem.

**Proposition 1.4.** *Let  $c \in R$  and  $N$  an  $R$ -module. For every  $R$ -module  $M$  and every integer  $e \geq 0$ , define*

$$M_e := \{x \in M \mid x \otimes c \in 0_{F_R^e(M)}^{F,N} \subseteq F_R^e(M) = M \otimes_R {}^eR\}.$$

Then  $M_0 \supseteq M_1 \supseteq \cdots \supseteq M_e \supseteq M_{e+1} \supseteq \cdots$ , that is,  $\{M_e\}_{e=0}^\infty$  is a descending chain of  $R$ -submodules of  $M$ .

*Proof.* For any  $e \geq 0$  and any  $x \in M$ , we see that

$$\begin{aligned} x \in M_{e+1} &\iff x \otimes c \in 0_{F_R^{e+1}(M)}^{F,N} \subseteq F_R^{e+1}(M) = M \otimes_R {}^{e+1}R \\ &\iff 0 = x \otimes c \otimes {}^nN \subseteq M \otimes_R {}^{e+1}R \otimes_R {}^nN \text{ for all } n \gg 0 \\ &\implies 0 = x \otimes c^p \otimes {}^nN \subseteq M \otimes_R {}^{e+1}R \otimes_R {}^nN \text{ for all } n \gg 0 \\ &\iff 0 = x \otimes c \otimes {}^{n+1}N \subseteq M \otimes_R {}^eR \otimes_R {}^{n+1}N \text{ for all } n \gg 0 \\ &\iff x \otimes c \in 0_{F_R^e(M)}^{F,N} \subseteq F_R^e(M) = M \otimes_R {}^eR \iff x \in M_e, \end{aligned}$$

which proves  $M_e \supseteq M_{e+1}$ . As it is routine to verify that all  $M_e$  are  $R$ -submodules of  $M$ , the proof is complete.  $\square$

Given an  $R$ -module  $N$  that has FFRT, we agree to use the following notations throughout the remainder of the paper:

*Notation 1.5.* Let  $N$  be an  $R$ -module that has FFRT. Then we agree that  $N$  has FFRT by  $R$ -modules  $N_1, \dots, N_r, N_{r+1}, \dots, N_s$ , in which  $N_1, \dots, N_r$  are exactly the ones appearing in the direct sum decompositions of  ${}^eN$  non-trivially for infinitely many  $e$ . Thus there is an integer  $n_1 \geq 0$  such that  $N_1, \dots, N_r$  are the only ones that could appear non-trivially in the direct sum decompositions of  ${}^eN$  for all  $e \geq n_1$ . Then there is another integer  $n_2 \geq n_1$  such that each of  $N_1, \dots, N_r$  appears non-trivially in at least one of the direct sum decompositions of  ${}^{n_1}N, {}^{n_1+1}N, \dots, {}^{n_2}N$ . Let  $T := \bigoplus_{i=1}^r N_i$ . Also denote by  $K$  the  $R$ -module determined by the left  $R$ -module structure of  $\bigoplus_{e=n_1}^{n_2} {}^eN$ .

*Observation 1.6.* Let  $N, r$  and  $K$  be as in Notation 1.5. Then

- (1) As an  $R$ -module,  $K$  has FFRT by  $N_1, \dots, N_r$ .

- (2) Each of  $N_1, \dots, N_r$  appears non-trivially in the direct sum decomposition of  $K = {}^0K$ .
- (3) Each of  $N_1, \dots, N_r$  appears non-trivially in the direct sum decompositions of  ${}^eK$  for infinitely many  $e$ .
- (4)  $\text{Ann}_R(K) = \sqrt{\text{Ann}_R(K)} = \sqrt{\text{Ann}_R(N)}$ . See Lemma 1.7 (3).
- (5) For any  $R$ -module  $M$ , we have  $0_M^{F,K} = 0_M^{F,N}$ . (This follows from the fact that  ${}^nK = \bigoplus_{e=n_1}^{n_2} {}^{n+e}N$  as left  $R$ -modules for all integers  $n \geq 0$ .)
- (6) If  $R$  is F-finite and  $N$  is finitely generated over  $R$ , then so is  $K$ .

**Lemma 1.7.** *Let  $R$  be a ring,  $c \in R$ , and  $N$  an  $R$ -module with FFRT. Adopt the notations in Notation 1.5. Then*

- (1) For any  $R$ -module  $M$ ,  $0_M^{F,N}$  can be characterized as

$$\begin{aligned}
 0_M^{F,N} &= 0_M^{F,K} = \{x \in M \mid 0 = x \otimes T \subseteq M \otimes_R T\} \\
 &= \{x \in M \mid 0 = x \otimes {}^nK \subseteq M \otimes_R {}^nK \text{ for all } n \geq 0\} \\
 &= \{x \in M \mid 0 = x \otimes K \subseteq M \otimes_R K\} \\
 &= \{x \in M \mid 0 = x \otimes {}^nN \subseteq M \otimes_R {}^nN \text{ for all } n = n_1, \dots, n_2\} \\
 &= \{x \in M \mid 0 = x \otimes {}^nN \subseteq M \otimes_R {}^nN \text{ for all } n \geq n_1\} \\
 &\subseteq \{x \in M \mid 0 = x \otimes {}^{n_1}N \subseteq M \otimes_R {}^{n_1}N\}.
 \end{aligned}$$

- (2) In particular, for every  $e \geq 0$ ,  $M_e$  (as defined in Proposition 1.4) satisfies

$$M_e \subseteq \{x \in M \mid 0 = x \otimes {}^{e+n_1}(c^{p^{n_1}}N) \subseteq M \otimes_R {}^{e+n_1}N\}.$$

- (3)  $\text{Ann}_R(T) = \text{Ann}_R(K) = \sqrt{\text{Ann}_R(K)} = \sqrt{\text{Ann}_R(N)}$ .

*Proof.* (1) This simply follows from the definition of  $0_M^{F,N}$  and of  $0_M^{F,K}$  as well as the direct sum decompositions of  $T$ ,  ${}^nK$  and of  ${}^nN$ .

(2) This simply follows from part (1) above applied to  $F_R^e(M) = M \otimes_R {}^eR$ . Also see Remark 1.2.

(3) Since both  $T$  and  $K$  are direct sums of  $N_1, \dots, N_r$  with each  $N_i$  ( $i = 1, \dots, r$ ) involved non-trivially, we see  $\text{Ann}_R(K) = \text{Ann}_R(T)$ .

Next, as  $\text{Ann}_R^l({}^eN) \subseteq \sqrt{\text{Ann}_R(N)}$  for all  $e \geq 0$ , we see  $\text{Ann}_R(K) \subseteq \text{Ann}_R^l({}^{n_1}N) \subseteq \sqrt{\text{Ann}_R(N)}$ . (Recall that  $\text{Ann}_R^l({}^eN)$  stands for the annihilator of  ${}^eN$  as a left  $R$ -module.) Also, for every  $1 \leq i \leq r$ , there exists an integer  $e_i$  such that  $N_i$  appears non-trivially in the direct sum decomposition of  ${}^{e_i}N$ ; and by taking  $e_i$  large enough, we may further assume  $\sqrt{\text{Ann}_R(N)}^{[p^{e_i}]} \subseteq \text{Ann}_R(N)$ . Thus

$$\text{Ann}_R(N_i) \supseteq \text{Ann}_R^l({}^{e_i}N) = \sqrt{\text{Ann}_R(N)}.$$

Consequently, we have

$$\text{Ann}_R(K) = \text{Ann}_R(T) = \bigcap_{i=1}^r \text{Ann}_R(N_i) \supseteq \sqrt{\text{Ann}_R(N)}.$$

Thus  $\text{Ann}_R(K) = \sqrt{\text{Ann}_R(N)}$ . Now it is clear that  $\text{Ann}_R(K) = \sqrt{\text{Ann}_R(K)}$ .  $\square$

*Remark 1.8.* Suppose  $N = R$  has FFRT and adopt the notations in Notation 1.5. For any  $R$ -module  $M$ , we see that (compare with Lemma 1.7 (1))

$$\begin{aligned} 0_M^F &= \{x \in M \mid 0 = x \otimes {}^n R \subseteq M \otimes_R {}^n R \text{ for all } n \geq n_1\} \quad (\text{by Lemma 1.7 (1)}) \\ &= \{x \in M \mid 0 = x \otimes {}^{n_1} R \subseteq M \otimes_R {}^{n_1} R\} = \{x \in M \mid 0 = x \otimes 1 \in M \otimes_R {}^{n_1} R\}. \end{aligned}$$

In other words,  $p^{n_1}$  is a *uniform Frobenius exponent* for all  $R$ -modules. This was noted in a discussion with Mel Hochster, and was also implicit in [Yao1, Remark 2.7].

## 2. SOME PRELIMINARIES ON TEST EXPONENTS

In this section, we list some properties and implications related to the existence of uniform test exponents. The first two lemmas, Lemma 2.1 and Lemma 2.2, are from [HoY2]. We include their proofs for completeness only.

**Lemma 2.1** ([HoY2]). *Let  $R$  be a Noetherian ring of characteristic  $p$ . Say the set of minimal primes of  $R$  is  $\min(R) = \{P_1, P_2, \dots, P_t\}$  so that  $\sqrt{0} = \bigcap_{i=1}^t P_i$ . For any  $c \in R$  and any  $R$ -module  $M$ , the following statements hold.*

- (1) *If  $Q = p^E$  is a test exponent for  $c + P_i \in R/P_i$  and  $M/P_i M$  over  $R/P_i$  for all  $i = 1, 2, \dots, t$ , then  $Q$  is a test exponent for  $c$  and  $M$  over  $R$ .*
- (2) *If  $Q$  is a test exponent for  $c + \sqrt{0} \in R/\sqrt{0}$  and  $M/\sqrt{0}M$  over  $R/\sqrt{0}$ , then  $Q$  is a test exponent for  $c$  and  $M$  over  $R$ .*

*Proof.* (1) For any  $x \in M$ , suppose  $0 = x \otimes c \in F_R^e(M)$  for some  $e \geq E$ . Then,  $0 = (x + P_i M) \otimes (c + P_i) \in F_{R/P_i}^e(M/P_i M)$ , which implies  $x + P_i M \in 0_{M/P_i M}^*$  over  $R/P_i$  for all  $i = 1, 2, \dots, t$ . This forces  $x \in 0_M^*$  (see [HH1]).

(2) This follows similarly. □

The next lemma deals with integral and pure ring extensions. In particular, the lemma applies to any reduced excellent ring with its integral closure in its total quotient ring, and to any complete local ring (with any of its  $\Gamma$ -constructions, see [HH2]). Some parts of the following lemma can also be found in [HoY1].

**Lemma 2.2.** *Let  $R \subseteq S$  be an extension of rings such that at least one of the following holds: (A)  $S$  is module-finite over  $R$ , or (B)  $S$  is integral over  $R$  and they share a common weak test element  $d \in R^\circ$ , or (C)  $S$  is a pure extension of  $R$  and they share a common weak test element  $d \in R^\circ$ .*

*For any  $c \in R$  and any  $R$ -module  $M$ , if  $Q = p^E$  is a test exponent for  $c$  and  $M \otimes_R S$  over  $S$ , then  $Q$  is a test exponent for  $c$  and  $M$  over  $R$ .*

*Proof.* Suppose  $0 = x \otimes c \in M \otimes_R {}^e R = F_R^e(M)$  for some  $x \in M$  and  $p^e \geq Q$ . Then  $0 = (x \otimes 1) \otimes c \in (M \otimes_R S) \otimes_S {}^e S = F_S^e(M \otimes_R S)$  and hence  $x \otimes 1 \in 0_{(M \otimes_R S)}^*$  over  $S$ .

In case of (A) or (C), it is immediate that  $x \in 0_M^*$ . (Case (A) reduces to the domain case and then [HH2, Lemma 6.25] (or Lemma 1.3) applies.)

Case (B): Say  $d \in R^\circ$  is a common  $q_0$ -weak test element of  $R$  and  $S$  where  $q_0 = p^{e_0}$ . Then  $0 = x \otimes d \in M \otimes_R {}^e S$  for all  $e \geq e_0$ . Thus, for each  $e \geq e_0$ , there is a module-finite ring extension  $S_e$  such that  $R \subseteq S_e \subseteq S$  and satisfying

$$0 = x \otimes d \in M \otimes_R {}^e (S_e), \quad \text{which implies } x \otimes d \in 0_{M \otimes_R {}^e R}^* \subseteq M \otimes_R {}^e R$$

for the same reason as in the last paragraph ([HH2, Lemma 6.25] or Lemma 1.3). Then, as  $d$  is a  $q_0$ -weak test element, we see

$$0 = x \otimes d \otimes d \in M \otimes_R {}^e R \otimes_R {}^{e_0} R, \quad \text{that is,} \quad 0 = x \otimes d^{q_0+1} \in M \otimes_R {}^{e+e_0} R$$

for all  $e \geq e_0$ . Now, as  $d^{q_0+1} \in R^\circ$ , this shows  $x \in 0_M^*$ .  $\square$

As an easy corollary, we show that uniform test exponents exist if  $\dim(R) \leq 1$ .

**Corollary 2.3.** *Let  $R$  be an excellent ring of prime characteristic  $p$  that is either F-finite or semi-local. Assume  $\dim(R) \leq 1$ .*

*Then, for every  $c \in R^\circ$ , there is a test exponent for  $c$  and all  $R$ -modules.*

*Proof.* In case  $R$  is semi-local, then we may assume  $R$  is F-finite by passing to its completion and then a proper  $\Gamma$ -construction (see [HH2] and Lemma 2.2).

So we may simply assume  $R$  is F-finite. By Lemma 2.1, we may further assume  $R$  is a domain. Then  $\bar{R}$ , the integral closure of  $R$  in its fraction field, is regular. Thus, we may as well assume  $R$  is F-finite and regular by Lemma 2.2. In particular,  $R$  is strongly F-regular. Therefore, for every  $c \in R^\circ$ , there exists  $Q = p^E$  such that the left  $R$ -submodule of  ${}^e R$  spanned by  $c$  is a free direct summand of  ${}^e R$  for all  $e \geq E$ . Thus, for every  $R$ -module  $M$ ,  $x \in M$  and  $e \geq E$ , the equation  $0 = x \otimes c \in M \otimes_R {}^e R$  implies  $x = 0$ , which is in  $0_M^*$ .<sup>2</sup>  $\square$

Quite generally, the existence of a uniform test exponent implies that tight closure agrees with finitistic tight closure.

**Lemma 2.4.** *Let  $R$  be a ring such that at least one of the following holds:*

- (1) *For every  $c \in R^\circ$ , there is a test exponent for  $c$  and all  $R$ -modules.*
- (2) *For some weak test element  $c \in R^\circ$  (for all  $R$ -modules), there is a test exponent for  $c$  and all  $R$ -modules.*

*Then tight closure coincides with the finitistic tight closure (cf. Definition 0.1 (2–3)).*

*Proof.* Without loss of generality, we show  $0_M^{*\text{fg}} = 0_M^*$  for an arbitrary  $R$ -module  $M$ . Let  $x \in 0_M^*$ , so that there exists  $c \in R^\circ$  such that  $0 = x \otimes c \in M \otimes_R {}^e R$  for all  $e \gg 0$ . (In case (2) holds, let  $c$  be the weak test element.) Let  $Q = p^E$  be a test exponent for  $c$  and all  $R$ -modules. Then, obviously, there exists  $e_0 \geq E$  such that  $0 = x \otimes c \in M \otimes_R {}^{e_0} R$ . By a property of tensor product, there exists a finitely generated  $R$ -submodule  $M_1 \subseteq M$  such that  $x \in M_1$  and

$$0 = x \otimes c \in M_1 \otimes_R {}^{e_0} R \quad \text{for the same } e_0 \geq E.$$

Thus  $x \in 0_{M_1}^*$ , since  $Q = p^E$  is a uniform test exponent. This shows  $x \in 0_M^{*\text{fg}}$  by the definition of  $0_M^{*\text{fg}}$ . Hence  $0_M^* \subseteq 0_M^{*\text{fg}}$ , which is enough to complete the proof.  $\square$

**Lemma 2.5.** *Let  $R$  be a ring and  $c \in R$ . If  $Q = p^E$  is a test exponent for  $c$  and all finitely generated  $R$ -modules, then  $Q$  is a test exponent for  $c$  and all  $R$ -modules.*

<sup>2</sup>As a by-product, we see that if  $R$  is strongly F-regular and  $c \in R^\circ$ , there is a uniform test exponent for  $c$  and all  $R$ -modules  $M$ .

*Proof.* Let  $M$  be an arbitrary  $R$ -module. For any  $x \in M$ , suppose  $0 = x \otimes c \in F_R^e(M)$  for some  $e \geq E$ . By a property of tensor product, there exists a finitely generated  $R$ -submodule  $M_1 \subseteq M$  such that  $x \in M_1$  and  $0 = x \otimes c \in M_1 \otimes_R {}^eR$ , which implies  $x \in 0_{M_1}^*$ . Thus  $x \in 0_M^*$ .  $\square$

**Lemma 2.6** (Compare with [HH3, Proposition 2.3]). *Let  $R$  be a ring and  $M$  an  $R$ -module such that at least one of the following holds:*

- (1) *For every  $c \in R^\circ$ , there is a test exponent  $p^E$  for  $c$  and  $M$ .*
- (2) *For some locally stable weak test element  $c \in R^\circ$  (for  $M$ ), there is a test exponent  $p^E$  for  $c$  and  $M$ . (This case was covered in [HH3, Proposition 2.3].)*

*Then the tight closure of 0 in  $M$  commutes with localization.*

*Proof.* The proof is the same as that of [HH3, Proposition 2.3], in spirit. It suffices to prove  $0_{U^{-1}M}^* \subseteq U^{-1}(0_M^*)$  for any multiplicatively closed subset  $U$  of  $R$ . For  $x \in M$  and  $u \in U$ , suppose  $\frac{x}{u} \in 0_{U^{-1}M}^*$  over  $U^{-1}R$ . Then there exists  $c \in R^\circ$  such that  $0 = \frac{x}{u} \otimes \frac{c}{1} \in U^{-1}M \otimes_{U^{-1}R} {}^e(U^{-1}R)$  for all  $e \gg 0$  (see [AHH, Lemma 3.3]). (In case (2) holds, let  $c$  be the locally stable weak test element.) Thus, there exist  $e \geq E$  and  $v = v_e \in U$  such that  $0 = vx \otimes c \in M \otimes_R {}^eR$ . This implies  $vx \in 0_M^*$ , which shows that  $\frac{x}{u} = \frac{vx}{vu} \in U^{-1}(0_M^*)$ .  $\square$

### 3. THE EXISTENCE OF UNIFORM TEST EXPONENTS

In this section, we show the existence of uniform test exponents under a FFRT assumption. When an  $R$ -module  $N$  has FFRT, we adopt the notations in Notation 1.5 (in particular, the meaning of  $N_i$ ,  $r$ ,  $s$ ,  $T$ ,  $n_1$ ,  $n_2$ , and  $K$ ) without further comments. Note that  $T$  is finitely generated exactly when  $N_1, \dots, N_r$  are all finitely generated.

**Theorem 3.1.** *Let  $R$  be a ring (Noetherian, of prime characteristic  $p$ ). Assume there exists a finitely generated  $R$ -module  $N$  with  $\text{Supp}_R(N) = \text{Spec}(R)$  (i.e.,  $\text{Ann}_R(N) \subseteq \sqrt{0}$ ) such that  $N$  has FFRT by  $N_1, \dots, N_r, \dots, N_s$ . Further assume that  $T = \bigoplus_{i=1}^r N_i$  is finitely generated over  $R$  (which is automatic if  $R$  is  $F$ -finite).*

*Then, for every  $c \in R^\circ$ , there is a (uniform) test exponent  $Q = p^E$  for  $c$  and for all  $R$ -modules. Moreover, this very same  $Q$  is a (uniform) test element for  $c/1 \in U^{-1}R$  and for all  $U^{-1}R$ -modules over every localization  $U^{-1}R$  of  $R$ , in which  $U$  is a multiplicatively closed subset of  $R$ .*

*Proof.* Fix an arbitrary  $c \in R^\circ$ . For all integers  $n \geq 0$  and  $e \geq 0$ , denote

$$\begin{aligned} \text{Hom}_R^l(n+n_1N, T)(n+n_1(c^{p^{n_1}}N)) &:= \langle \{h(c^{p^{n_1}}z) \mid h \in \text{Hom}_R^l(n+n_1N, T), z \in N\} \rangle \\ &= \langle \bigcup_{h \in \text{Hom}_R^l(n+n_1N, T)} h(n+n_1(c^{p^{n_1}}N)) \rangle \subseteq T, \\ T_e &:= \sum_{n=0}^e [\text{Hom}_R^l(n+n_1N, T)(n+n_1(c^{p^{n_1}}N))] \subseteq T. \end{aligned}$$

(Here  $\text{Hom}_R^l(n+n_1N, T)$  stands for the set of left  $R$ -module homomorphisms from  $n+n_1N$  to  $T$ . For any subset  $S$  of  $T$ , we use  $\langle S \rangle$  to denote the  $R$ -submodule of  $T$

generated by  $S$ .) From the construction of  $T_e$ , we clearly get an ascending chain of  $R$ -submodules of  $T$  as follows

$$T_0 \subseteq T_1 \subseteq \cdots \subseteq T_e \subseteq T_{e+1} \subseteq \cdots .$$

As  $T$  is Noetherian (by assumption), there exists a non-negative integer  $E$  such that  $T_e = T_E$  for all  $e \geq E$ . Let  $Q := p^E$ .

We are going to show that  $Q = p^E$  is a uniform test exponent for  $c$  and for all  $R$ -modules. To this end, let  $M$  be a (not necessarily finitely generated)  $R$ -module and let  $x \in M$  such that  $0 = x \otimes c \in M \otimes_R {}^{e_0}R$  for some  $e_0 \geq E$ . It remains to show  $x \in 0_M^*$  in order to finish the proof.

Since  $0 = x \otimes c \in M \otimes_R {}^{e_0}R = F_R^{e_0}(M)$ , it is obvious that  $x \otimes c \in 0_{F_R^{e_0}(M)}^{F, N} \subseteq F_R^{e_0}(M)$  (cf. Definition 1.1 (1)). This means  $x \in M_{e_0}$ . (Note that  $M_e$  is defined in Proposition 1.4.) As  $\{M_e\}_{e=0}^\infty$  is a descending chain by Proposition 1.4, we see

$$x \in M_{e_0} \subseteq \cdots \subseteq M_0.$$

That is,  $x \in M_n$  for all  $n = 0, \dots, e_0$ . By Lemma 1.7 (2), this implies

$$0 = x \otimes {}^{n+n_1}(c^{p^{n_1}} N) \subseteq M \otimes_R {}^{n+n_1}N \quad \text{for all } n = 0, \dots, e_0.$$

Therefore  $0 = x \otimes h({}^{n+n_1}(c^{p^{n_1}} N)) \subseteq M \otimes_R T$  for all  $n \in \{0, \dots, e_0\}$  and all  $h \in \text{Hom}_R^l({}^{n+n_1}N, T)$ . Denote

$$\text{Ann}_T(x \in M) := \{t \in T \mid 0 = x \otimes t \in M \otimes_R T\},$$

which is an  $R$ -submodule of  $T$ . Then the last statement simply says  $h({}^{n+n_1}(c^{p^{n_1}} N)) \subseteq \text{Ann}_T(x \in M)$  for all  $n \in \{0, \dots, e_0\}$  and all  $h \in \text{Hom}_R^l({}^{n+n_1}N, T)$ . Consequently, by the construction of  $T_e$ , we see  $T_{e_0} \subseteq \text{Ann}_T(x \in M)$ ; and hence

$$T_e \subseteq \text{Ann}_T(x \in M) \quad \text{for all } e \geq E$$

by our choice of  $E$  and the assumption that  $e_0 \geq E$ .

We are to prove that  $0 = x \otimes {}^{e+n_1}(c^{p^{n_1}} N) \subseteq M \otimes_R {}^{e+n_1}N$  for all  $e \geq E$ . Indeed, as  ${}^{e+n_1}N$  is (isomorphic to) a direct sum of  $N_1, \dots, N_r$  and  $T = \bigoplus_{i=1}^r N_i$ , we see that  ${}^{e+n_1}N$  (as a left  $R$ -module) is a direct summand of  $T^{\oplus \Lambda}$  for some index set  $\Lambda$ . (Recall that  $T^{\oplus \Lambda} = \bigoplus_{\lambda \in \Lambda} T_\lambda$  in which  $T_\lambda = T$  for every  $\lambda \in \Lambda$ .) Therefore, there exist  $\phi_n \in \text{Hom}_R^l({}^{e+n_1}N, T^{\oplus \Lambda})$  and  $\psi_n \in \text{Hom}_R^l(T^{\oplus \Lambda}, {}^{e+n_1}N)$  such that  $\psi_n \circ \phi_n = \text{id}_{({}^{e+n_1}N)}$ , the identity map on  ${}^{e+n_1}N$ . For each  $\lambda \in \Lambda$ , let  $\pi_\lambda : T^{\oplus \Lambda} \rightarrow T$  be the projection to  $T_\lambda = T$ . Now, since  $T_e \subseteq \text{Ann}_T(x \in M)$  for all  $e \geq E$  (cf. the end of last paragraph), we see (for all  $e \geq E$ )

$$\begin{aligned} 0 = x \otimes T_e &\subseteq M \otimes_R T, && \text{which implies} \\ 0 = x \otimes [\pi_\lambda \circ \phi_n({}^{e+n_1}(c^{p^{n_1}} N))] &\subseteq M \otimes_R T \text{ for all } \lambda \in \Lambda, && \text{which implies} \\ 0 = x \otimes \phi_n({}^{e+n_1}(c^{p^{n_1}} N)) &\subseteq M \otimes_R T^{\oplus \Lambda}, && \text{which implies} \\ 0 = x \otimes [\psi_n \circ \phi_n({}^{e+n_1}(c^{p^{n_1}} N))] &\subseteq M \otimes_R {}^{e+n_1}N, && \text{that is} \\ 0 = x \otimes {}^{e+n_1}(c^{p^{n_1}} N) &\subseteq M \otimes_R {}^{e+n_1}N && \text{for all } e \geq E. \end{aligned}$$

Thus  $x \in 0_M^{*N}$  (cf. Definition 1.1 (2)), as  $c^{p^{n_1}} \in R^\circ$  and  $\min(R/\text{Ann}_R(N)) = \min(R)$ . So we conclude  $x \in 0_M^{*N} = 0_M^*$  by Lemma 1.3, as  $N$  is a finitely generated  $R$ -module

with  $\text{Ann}_R(N) \subseteq \sqrt{0}$ . Thus  $Q = p^E$  is a (uniform) test exponent for  $c$  and for the arbitrary  $R$ -module  $M$ .

Finally, let  $U^{-1}R$  be any localization of  $R$ , in which  $U$  is a multiplicatively closed subset of  $R$ . For every  $(U^{-1}R)$ -module (and hence an  $R$ -module)  $M$  and  $x \in M$ , suppose  $0 = x \otimes \frac{c}{1} \in M \otimes_{U^{-1}R} {}^e(U^{-1}R)$  for some  $e \geq E$ . Then, as  ${}^e(U^{-1}R) = U^{-1}({}^eR)$ , the above can actually be rewritten as  $0 = x \otimes c \in M \otimes_R {}^eR$ . Since  $Q = p^E$  is a test exponent for  $c$  and  $M$  (as an  $R$ -module), we see that  $x \in 0_M^*$  over  $R$ , which implies  $x \in 0_M^*$  over  $U^{-1}R$ .

Now the proof of Theorem 3.1 is complete.  $\square$

*Remark 3.2.* Suppose  $R$  is F-finite in the context of Theorem 3.1. Then, as noted in Observation 1.6,  $K$  is a finitely generated  $R$ -module having FFRT with  $\text{Ann}_R(K) = \sqrt{0}$ . Consequently, by replacing  $N$  with  $K$ , we may further assume that  $N$  has FFRT such that  $n_1 = 0$  (cf. Notation 1.5). This should make the proof of Theorem 3.1 a bit simpler in appearance.

Now we list some corollaries resulted from the existence of uniform test exponents over rings of FFRT. The first corollary covers the case of binomial rings, which, in particular, include affine semi-group rings.

**Corollary 3.3.** *Let  $R$  be a binomial ring over a field  $k$  (i.e.,  $R$  is a quotient ring of a polynomial ring over  $k$  modulo an ideal generated by binomials). Then, for every  $c \in R^\circ$ , there exists a uniform test exponent for  $c$  and all  $R$ -modules.*

*Proof.* The proof goes very much like [Sm, proof of the last corollary]. Without loss of generality, we assume  $k$  is algebraically closed (cf. Lemma 2.2). Then, modulo each of its minimal primes and taking the integral closure, we may assume  $R$  is a normal affine semi-group ring. Now the claim follows as  $R$  has FFRT (see Example 0.6 (4)). For more details, see [Sm, proof of the last corollary].  $\square$

**Corollary 3.4.** *Let  $R$  be as in Corollary 2.3, Theorem 3.1, or Corollary 3.3. Then the finitistic tight closure coincides with the tight closure for all  $R$ -modules.*

*Proof.* This follows from the existence of a uniform test exponent, see Lemma 2.4.  $\square$

The existence of test exponents implies tight closure commuting with localizations.

**Corollary 3.5.** *Let  $R$  be as in Corollary 2.3, Theorem 3.1, or Corollary 3.3. Then tight closure commutes with localization for all (not necessarily finitely generated)  $R$ -modules.*

*Proof.* This follows from the existence of a uniform test exponent, see Lemma 2.6. (This was already well-known when  $\dim(R) \leq 1$ .)  $\square$

The last corollary generalizes the result in [Yao1, Theorem 2.3] to all  $R$ -modules, and provides an alternative proof of K. Smith's result in [Sm] that tight closure commutes with localization over binomial rings.

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