OBSERVATIONS ON THE F-SIGNATURE OF LOCAL RINGS OF CHARACTERISTIC p

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Abstract. Let $(R, \mathfrak{m}, k)$ be a d-dimensional Noetherian reduced local ring of prime characteristic p such that R^{1/p^e} are finite over R for all $e \in \mathbb{N}$ (i.e. R is F-finite). Consider the sequence $\{\frac{a_e}{q^{\alpha(R)+d}}\}_{e=0}^{\infty}$, in which $\alpha(R) = \log_p[k : k^p]$, $q = p^e$, and a_e is the maximal rank of free R-modules appearing as direct summands of R-module $R^{1/q}$. Denote by $s^-(R)$ and $s^+(R)$ the liminf and limsup respectively of the above sequence as $e \to \infty$. If $s^-(R) = s^+(R)$, then the limit, denoted by $s(R)$, is called the *F-signature of* R. It turns out that the F-signature can be defined in a way that is independent of the module finite property of $R^{1/q}$ over R. We show that: (1) If $s^+(R) \geq 1 - \frac{1}{d!p^d}$, then R is regular; (2) If R is excellent such that R_P is Gorenstein for every $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$, then $s(R)$ exists; (3) If $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local flat ring homomorphism, then $s^{\pm}(R) \geq s^{\pm}(S)$ and, if furthermore $S/\mathfrak{m}S$ is Gorenstein, $s^{\pm}(S) \geq s^{\pm}(R)s(S/\mathfrak{m}S)$.

0. Introduction

Throughout this paper we assume that $(R, \mathfrak{m}, k)$ is a Noetherian local ring of prime characteristic p, where $\mathfrak{m}$ is the maximal ideal and $k = R/\mathfrak{m}$ is the residue field of R. Then there is the Frobenius homomorphism $F : R \to R$ defined by $r \mapsto r^p$ for any $r \in R$. Therefore, for any $e \in \mathbb{N}$, we have the iterated Frobenius homomorphism $F^e : R \to R$ defined by $r \mapsto r^q$ for any $r \in R$, where $q = p^e$. From now on, q will be used to denote the value p^e for various $e \in \mathbb{N}$ in the context.

Let M be an R-module. Then for any $e \geq 0$, we can derive a left R-module structure on the set M by $r \cdot m := r^{p^e} m$ for any $r \in R$ and $m \in M$. For technical reasons, we keep the original right R-module structure on M by default. We denote the derived R-R-bimodule by ${}^eM$. Thus, in ${}^eM$, we have $r \cdot m = m \cdot r^{p^e}$, which is equal to $r^q m$ in the original M. If R is reduced, then ${}^eR$, as a left R-module, is isomorphic to $R^{1/q}$. We use $\lambda^l(-), \lambda^r(-)$ to denote the left and right lengths of a bimodule. It is easy to see that $\lambda^l({}^eM) = q^{\alpha(R)}\lambda^r({}^eM) = q^{\alpha(R)}\lambda(M)$ for any finite length R-module M, in which $\alpha(R) = \log_p[k : k^p]$.

We say R is F-finite if ${}^1R$ is a finitely generated left R-module. If this is the case, it is easy to see that ${}^eM$ is a finitely generated left R-module for every $e \in \mathbb{N}$ and for every finitely generated R-modules M.

For an ideal I of R, we denote by $I^{[q]}$ the ideal generated by $\{r^q \mid r \in I\}$. Then $R/I \otimes_R {}^eM \cong {}^e(M/I^{[q]}M) \cong {}^eM \otimes_R R/I^{[q]}$ for every R-module M and every $e \in \mathbb{N}$.

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In this paper, we are going to study an invariant called 'the F-signature' of R. The notion of F-signature is first introduced and studied in [HL] by C. Huneke and G. Leuschke for F-finite rings.

Definition 0.1. Let $(R, \mathfrak{m}, k)$ be an F-finite local ring and M a finitely generated R-module. For each $e \in \mathbb{N}$, write ${}^eM \cong R^{a_e} \oplus M_e$ as left R-modules such that M_e has no non-zero free direct summand. In other words, the number a_e is the maximal rank of free direct summand of the left R-module ${}^eM$, which is independent of the particular direct sum decomposition of ${}^eM$ (since the completion $\widehat{R}$ satisfies the Krull-Schmidt condition). Denote $d := \dim R$.

(1) We may denote a_e by $\#({}^eM, R)$ and $\alpha(R) = \log_p[k : k^p] < \infty$.
(2) We denote $s^+(M) := \limsup_{e\to\infty} \frac{\#({}^eM,R)}{q^{\alpha(R)+d}}$, $s^-(M) := \liminf_{e\to\infty} \frac{\#({}^eM,R)}{q^{\alpha(R)+d}}$ and $s(M) := \lim_{e\to\infty} \frac{\#({}^eM,R)}{q^{\alpha(R)+d}}$ provided the last limit exists. In case confusion may arise, we use $s^+_R(M)$ etc. to specify the underlying ring structure.
(3) If $M = R$, we call $s(R) = \lim_{e\to\infty} \frac{\#({}^eR,R)}{q^{\alpha(R)+d}}$ the F-signature of R (see [HL]). In case $s(R)$ does not exist, we may call $s^-(R)$ and $s^+(R)$ the lower and upper F-signature of R respectively.

Remark 0.2. In the context of Definition 0.1.

(1) If R is not reduced or if M is not faithful, then $\#({}^eM, R) = 0$ for all $e > 0$.
(2) It is easy to see that $\widehat{{}^eM} \cong {}^e\left(\widehat{M}\right)$ as (left and right) $\widehat{R}$-modules for every $e \geq 0$. As a result, we may assume that R is complete without affecting the numbers a_e.

In Section 2, we observe that the definition of F-signature can be realized as

$$s^+(M) = \limsup_{e\to\infty} \frac{\lambda^r(\ker(E \otimes_R {}^eR \to E/k \otimes {}^eR))}{q^d} \quad \text{etc.},$$

where $E := E_R(k)$ is the injective hull of the residue field k and hence k is the socle of E. As it does not rely on the numbers $\#({}^eM, R)$ or the F-finite property, the notion of F-signature may be defined for any local Noetherian ring of characteristic p. Moreover, all the known results about F-signature seem to hold true in this more general setting via either direct proof or reduction to the F-finite case. Indeed, some of these results will be reviewed in Section 1 without the restriction of F-finiteness.

Like the multiplicity $e(R) = e(\mathfrak{m}, R)$ as well as the Hilbert-Kunz multiplicity $e_{HK}(R) = e_{HK}(\mathfrak{m}, R)$ of R, the F-signature $s(R)$ is an important invariant of R. But unlike $e(R)$ and $e_{HK}(R)$, the F-signature $s(R)$ and $S^{\pm}(R)$ assume their values between 0 and 1. (This follows from a simple counting of the rank of $R^{1/q}$ over R in the F-finite case.) Moreover $s^+(R) = 1 \iff R$ is regular $\iff s(R) = 1$ ([HL]) and, if R is excellent, $s^+(R) > 0 \iff R$ is strongly F-regular $\iff s^-(R) > 0$ ([AL]).

In Section 3, we prove that if $s^+(R)$ is close enough to 1 (i.e. big enough), then R is already regular.

Theorem 3.1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p with $\dim R = d$. Assume $s^+(R) > 0$ in case $\dim(R) \leq 1$, or*

$$s^+(R) \geq 1 - \frac{1}{d!p^d} \quad \textit{in case } \dim(R) \geq 2.$$

Then R is regular, which actually implies $s(R) = 1$.

Since $s^+(R)$, $s^-(R)$ and $s(R)$ are defined to be the limsup, liminf and limit of the sequence $\left\{\frac{\lambda^r(\ker(E\otimes_R {}^eR\to E/k\otimes {}^eR))}{q^d}\right\}$ as $e\to\infty$, one would naturally ask whether $s^+(R)=s^-(R)$, or equivalently the following question:

Question 0.3. Does $s(R)=\lim_{e\to\infty}\frac{\lambda^r(\ker(E\otimes_R {}^eR\to E/k\otimes {}^eR))}{q^d}$ exist?

A positive answer has been given in [HL] when $(R,\mathfrak{m})$ is Gorenstein. Another case of positive answer is proved in [SVdB] and [Yao] when R has finite F-representation type (FFRT for short, see Definition 4.5). If R is regular, then $s(M)$ exists for every finitely generated R-module M (see Corollary 2.6).

In Section 4, we show that Question 0.3 has an affirmative answer when R is Gorenstein at the punctured spectrum:

Theorem 4.3. *Let $(R,\mathfrak{m},k)$ be a Noetherian excellent local ring of prime characteristic p such that R_P is Gorenstein for every $P\in\operatorname{Spec}(R)\setminus\{\mathfrak{m}\}$. Then for any maximal Cohen-Macaulay module M, $s(M)$ exists. In particular, $s(R)$ exists.*

We also recover the result of [SVdB] and [Yao] that states: *If a finitely generated R-module M has FFRT, then $s(M)$ exists* (see Theorem 4.6).

Finally, we study the behavior of F-signature under localization and faithfully flat ring extension in Section 5.

Theorem (Proposition 5.2, Theorem 5.4, 5.6)**.** *Let $(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a local flat ring homomorphism. We have*

(1) $s^+(R)\le s^+(R_P)$ and $s^-(R)\le s^-(R_P)$ for any $P\in\operatorname{Spec}(R)$;
(2) $s^+(R)\ge s^+(S)$ and $s^-(R)\ge s^-(S)$;
(3) If we furthermore assume that the closed fiber ring $S/\mathfrak{m}S$ is Gorenstein, then $s^+(R)s(S/\mathfrak{m}S)\le s^+(S)$ and $s^-(R)s(S/\mathfrak{m}S)\le s^-(S)$. Equalities hold if $S/\mathfrak{m}S$ is regular.

1. Review and preliminary results

This section is allocated for reviewing. Some of the displayed results will be used in the coming sections. A very important concept in studying rings of characteristic p is tight closure. Tight closure was first studied and developed by Hochster and Huneke in the 1980's. Without loss of generality, we only state the definition of the tight closure of 0 in a given R-module M.

Definition 1.1 ([HH1])**.** Let R be a Noetherian ring of characteristic p and M an R-module. The tight closure of 0 in M, denoted by 0^*_M, is defined as follow: An element $x\in M$ is said to be in 0^*_M if there exists an element $c\in R^\circ$ such that $0=x\otimes c\in M\otimes_R {}^eR$ for all $e\gg 0$, where R° is the complement of the union of all minimal primes of the ring R. The element $x\otimes 1\in M\otimes_R {}^eR$ is denoted by $x^{p^e}_M$

In general, given R-modules $N\subseteq M$, the tight closure of N in M, denoted by N^*_M, is the (unique) R-module satisfying $N\subseteq N^*_M\subseteq M$ and $N^*_M/N=0^*_{M/N}$. If R is a ring such that all of its ideals are tightly closed (in R), we say R is weakly F-regular. Moreover, if R is a ring such that every localization of R is weakly F-regular, we say R is F-regular.

Another important notion is strong F-regularity. The notion of strong F-regularity was first defined for F-finite rings in [HH2, Definition 5.1]. Then, in the following Remark 5.3 of [HH2], a more general definition of strong F-regularity for not necessarily F-finite rings is suggested. We adopt this general definition in this paper as we are concerned with rings that do not necessarily satisfy F-finite property.

Definition 1.2 ([HH2])**.** Given a local ring $(R, \mathfrak{m}, k)$ of characteristic p. We say R is strongly F-regular if for any $c \in R^\circ$, the left R-linear maps $R \to {}^eR$ defined by $1 \mapsto c$ are pure for all $e \gg 0$ (or equivalently, for some $e > 0$).

As the name suggests, strong F-regularity implies F-regularity. It is shown in [Sm, 7.1.2] that R is strongly F-regular $\iff$ $0^*_E = 0$, where $E := E(k)$ is the injective hull of the residue field $k = R/\mathfrak{m}$ (see also [LS2, Proposition 2.9]).

Next, let us list some properties of the F-signature $s(R)$. Since F-signature is going to be defined without the F-finiteness assumption, we do not assume the F-finiteness property unless stated explicitly.

Theorem 1.3 ([HL], [AL])**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic p. Then the following are true (c.f. Remark 2.4):*

(1) If $s^+(R) > 0$, then R is an F-regular, Cohen-Macaulay domain. See [HL].
(2) Actually, if R is excellent (e.g. F-finite), it is proved that $s^+(R) > 0 \iff R$ is strongly F-regular $\iff s^-(R) > 0$ in [AL].
(3) For any two $\mathfrak{m}$-primary ideals $I \subseteq J$ of R, $e_{HK}(I, R) - e_{HK}(J, R) \geq \lambda_R(J/I)s^+(R)$. See [HL]. *Therefore*
$$s^+(R) \leq \inf\{e_{HK}(I_1, R) - e_{HK}(I_2, R) \mid I_1 \subset I_2, \sqrt{I_1} = \mathfrak{m}, I_2/I_1 \cong k\}.$$
(4) Also, the inequality $(e(R) - 1)(1 - s^+(R)) \geq e_{HK}(R) - 1$ is proved in [HL]. *Hence $s^+(R) \geq 1 \implies R$ is regular $\implies s(R) = 1$.*

Remark 1.4. The value $\inf\{e_{HK}(I_1, R) - e_{HK}(I_2, R) \mid I_1 \subset I_2, \sqrt{I_1} = \mathfrak{m}, I_2/I_1 \cong k\}$ is closely related to the *minimal relative Hilbert-Kunz multiplicity for cyclic modules* of R that is defined in [WY2] by K. -i. Watanabe and K. Yoshida.

Theorem 1.5 (Kunz)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p (not necessarily F-finite) with $\dim(R) = d$. Then:*

(1) It always holds that $\lambda_R(R/\mathfrak{m}^{[p]}) \geq p^d$ while equality holds if and only if R is regular (c.f. [Ku1]*).*
(2) If R is F-finite, then R is excellent and $\alpha(R_P) = \alpha(R_Q) + \dim(R_Q/P_Q)$ for any two prime ideals $P \subseteq Q$ of R (c.f. [Ku2]*).*

Theorem 1.6. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic p (not necessarily F-finite) and M a finitely generated R-module with $\dim(R) = d$. Then (with $q = p^e$)*

(1) The limit
$$\lim_{e\to\infty} \frac{\lambda_R(M/I^{[q]}M)}{q^d} = \lim_{e\to\infty} \frac{\lambda^r_R(R/I \otimes_R {}^eM)}{q^d}$$
exists (and is positive exactly when $\dim(M) = d$) for every $\mathfrak{m}$-primary ideal I of R [Mo]. *The limit, denoted by $e_{HK}(I, M)$, is called the Hilbert-Kunz multiplicity of M with respect to I. We often write $e_{HK}(\mathfrak{m}, M)$ as $e_{HK}(M)$.*
(2) More generally, suppose that N is an R-module with $\lambda_R(N) < \infty$. Then the limit
$$\lim_{e\to\infty} \frac{\lambda^r(N \otimes_R {}^eM)}{q^d}$$

exists [Se]. *(The statement of* [Se, Page 278, Theorem] *is more general and its proof requires F-finiteness. The particular result quoted here does not need F-finiteness as one can always reduces it to the F-finite case.)*

All the remaining results in this section do not rely on characteristic p. The first is a result of S. Ding, which is used in the proof of Theorem 4.3.

Theorem 1.7 ([Di, Theorem 1.1]). *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay Noetherian local ring with a canonical module. Then the following are equivalent:*

(1) For every $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}, R_P$ is Gorenstein.
(2) There exists a positive integer n such that $R/\mathfrak{m}^n$ is not an R-linear homomorphic image of any maximal Cohen-Macaulay module without non-zero free direct summand.

A result of R. M. Guralnick is used in the proof of Theorem 4.6.

Theorem 1.8 ([Gu, Corollary 2]). *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay Noetherian local ring and M, N finitely generated R-modules. Then there exists an integer n, depending on N and M, such that M is isomorphic to a direct summand of N if and only if $M/\mathfrak{m}^n M$ is isomorphic to a direct summand of $N/\mathfrak{m}^n N$.*

The next result is used in Section 5. The exact statement of the following theorem can be found in [HH2, Theorem 7.10], which refers the readers to a more general result in [Mat, 20.F].

Theorem 1.9. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local flat ring homomorphism. If $x_1, x_2, \ldots, x_t$ form a regular sequence on $S/\mathfrak{m}S$, then they form a regular sequence on S and $R \to S/(x_1, x_2, \ldots, x_t)S$ is again a (faithfully) flat local homomorphism.*

2. An equivalent definition of the F-signature

Let $E := E_R(k)$ be the injective hull of $k = R/\mathfrak{m}$, $\phi : E \to E/k$ be the natural homomorphism, and $\psi : k \to E$ be an injective R-linear map (e.g. the inclusion map) so that $0 \to k \xrightarrow{\psi} E \xrightarrow{\phi} E/k \to 0$ is exact. Then there are induced bimodule homomorphisms $\phi \otimes_R {}^eM = \phi \otimes_R 1_{{}^eM} : E \otimes_R {}^eM \to E/k \otimes_R {}^eM$ and $\psi \otimes_R {}^eM = \psi \otimes_R 1_{{}^eM} : k \otimes_R {}^eM \to E \otimes_R {}^eM$ for any R-module M and every $e \in \mathbb{N}$.

The next lemma enables us to describe $\#({}^eM, R)$ in terms of the maps $k \xrightarrow{\psi} E \xrightarrow{\phi} E/k$. A similar formula with essentially the same effect can be found in [AE1].

Lemma 2.1. *Let $(R, \mathfrak{m}, k)$ be F-finite, M an finitely generated R-module, and let the notations be as in the context of Definition 0.1. Then, for every $e \geq 0, q = p^e$,*

$$\#({}^eM, R) = q^{\alpha(R)} \lambda^r(\ker(\phi \otimes_R 1_{{}^eM})) = q^{\alpha(R)} \lambda^r(\operatorname{image}(\psi \otimes_R 1_{{}^eM})).$$

Proof. It is enough to prove $a_e = \lambda^l(\ker(\phi \otimes_R 1_{{}^eM}))$ for any $e \in \mathbb{N}$, where ${}^eM \cong R^{a_e} \oplus M_e$ as left R-modules such that M_e has no non-zero free direct summand. Also, we may assume R is complete without loss of generality.

Therefore, for the rest of this proof, we simply regard ${}^eM$ as a module over commutative ring R determined by $r \cdot m = m \cdot r = r^{p^e} m$ where $r \in R$ and $m \in M$ and prove $a_e = \lambda(\ker(\phi \otimes_R 1_{{}^eM}))$. Let $-^\vee := \operatorname{Hom}_R(-, E)$ denote the Matlis duality of any R-module. Then we have isomorphisms $E^\vee \cong R$ and $(E/k)^\vee \cong \mathfrak{m}$, under which $\phi^\vee : (E/k)^\vee \to E^\vee$ corresponds to the inclusion map $\mathfrak{m} \to R$. Since M_e has no non-trivial free direct summand, every R-linear map $h \in \operatorname{Hom}_R(M_e, R)$ satisfies $h(M_e) \subseteq \mathfrak{m}$. In other words, the induced map $\operatorname{Hom}_R(M_e, \phi^\vee) : \operatorname{Hom}_R(M_e, (E/k)^\vee) \to \operatorname{Hom}_R(M_e, E^\vee)$ is an isomorphism. Thus,

$$\begin{aligned}\lambda\left(\ker(\phi \otimes_R {}^eM)\right) &= \lambda\left(\operatorname{coker}\left((\phi \otimes_R {}^eM)^\vee\right)\right) = \lambda\left(\operatorname{coker}\left(\operatorname{Hom}_R\left({}^eM, \phi^\vee\right)\right)\right) \\ &= \lambda\left(\operatorname{coker}\left(\operatorname{Hom}_R\left(R^{a_e}, \phi^\vee\right)\right)\right) + \lambda\left(\operatorname{coker}\left(\operatorname{Hom}_R\left(M_e, \phi^\vee\right)\right)\right) \\ &= a_e + 0 = a_e,\end{aligned}$$

which is what we want. □

As the expression $\frac{\lambda^r(\ker(\phi \otimes_R 1_{{}^eM}))}{q^{\dim(R)}}$ does not rely on the F-finiteness of R, the notion of the F-signature may be defined for all Noetherian local rings of prime characteristic p which is equivalent to Definition 0.1 when R is F-finite.

Definition 2.2. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p with $\dim(R) = d$ and M a finitely generated R-module. Keep E, ψ and ϕ as above.

(1) Denote $\#({}^eM) := \lambda^r(\ker(\phi \otimes_R 1_{{}^eM})) = \lambda^r(\operatorname{image}(\psi \otimes_R 1_{{}^eM}))$ for all $e \in \mathbb{N}$. In case confusion may arise, we use $\#_R({}^eM)$ to specify the underlying ring structure.

(2) We define $s^-(M)$ and $s^+(M)$ to be, respectively, the liminf and limsup of the sequence $\left\{\frac{\#({}^eM)}{q^d}\right\}_{e=0}^\infty$ as $e \to \infty$. If $s^-(M) = s^+(M)$, the limit is denoted by $s(M)$. Once again, we may use $s_R^-(M)$, $s_R^+(M)$ and $s_R(M)$ to clarify the underlying ring structure.

(3) In the case of $M = R$, we call $s^-(R), s^+(R)$ and $s(R)$ the lower F-signature, upper F-signature and F-signature of R respectively.

Remark 2.3. Keep the notations as in Definition 2.2.

(1) As a right R-submodule of $E \otimes_R {}^eM$, $\operatorname{image}(\psi \otimes_R 1_{{}^eM})$ has length no larger than $\lambda_R(M/\mathfrak{m}^{[q]}M)$. Hence the sequence $\left\{\frac{\#({}^eM)}{q^d}\right\}_{e=0}^\infty$ is bounded. In case $M = R$, the right R-submodule $\operatorname{image}(\psi \otimes_R 1_{{}^eR}) \subseteq E \otimes_R {}^eR =: F^e(E)$ is generated by the element $u \otimes 1 \in E \otimes_R {}^eR$ for any $0 \neq u \in k \subseteq E$. Recall that the element $u \otimes 1 \in E \otimes_R {}^eR$ is denoted by $u_E^{p^e}$ in the context of defining tight closure of submodules (c.f. [HH1] or Definition 1.1). Therefore, we have $\#({}^eR) = \lambda_R(R/\operatorname{Ann}_R^r(u^{p^e}))$ for every $e \in \mathbb{N}$.

(2) Let e be any fixed integer. Then there exists a finite length R-submodule of $E' \subseteq E$ such that $\#({}^eM) = \lambda^r(\ker(\phi \otimes_R 1_{{}^eM})) = \lambda^r(\ker(\phi' \otimes_R 1_{{}^eM}))$ with $\phi' : E' \to E'/k$ being the natural R-homomorphism. Alternatively, let $\{\mathfrak{a}_n\}_{n=1}^\infty$ be a sequence of $\mathfrak{m}$-primary ideals cofinal with $\{\mathfrak{m}^n\}_{n=1}^\infty$ and denote $E_n := (0 :_E \mathfrak{a}_n)$ for every $n \in \mathbb{N}$. Then $\#({}^eM) = \lambda^r(\ker(\phi \otimes_R 1_{{}^eM})) = \lambda^r(\ker(\phi_n \otimes_R 1_{{}^eM}))$ for all $n \gg 0$, where $\phi_n : E_n \to E_n/k$ are the natural homomorphisms. This fact has been observed and used in [AL].

(3) Suppose that $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a flat local homomorphism of rings of characteristic p such that $\mathfrak{m}S = \mathfrak{n}$. Let $\mathfrak{a}_n, E_n$ be as in the above part (2). Then $E_n \otimes_R S \cong (0 :_{E_S(l)} \mathfrak{a}_nS)$ for every n as they both have one-dimensional socle with the same annihilator as S-modules. Hence by the remark made in part (2) above, it is straightforward to see that $\#({}^eM) = \#({}^e(M \otimes_R S))$. (For a more general statement, see Theorem 5.6.) Thus, as far as the F-signature over R is concerned, we may assume that R is complete (by $R \to \widehat{R}$), R has a infinite residue field (by $R \to R[T]_{\mathfrak{m}[T]}$) or R is F-finite (by $R \to \widehat{R} \to \widehat{R} \otimes_{k[[X_1,\ldots,X_n]]} k^\infty[[X_1, \ldots, X_n]]$,

in which $k[[X_1, \ldots, X_n]]$ is such that there is a ring homomorphism from $k[[X_1, \ldots, X_n]]$ onto $\widehat{R}$ and k^∞ is the perfect closure of $k = R/\mathfrak{m}$).

(4) The value $s^-(R)$ is the same as the invariant called the *minimal relative Hilbert-Kunz multiplicity* of R in [WY2] by K. -i. Watanabe and K. Yoshida.

Remark 2.4. The known results (as well as the main themes of their original proofs) about the F-signature seem to hold true without the assumption of F-finiteness, although sometimes R needs to be excellent. We remark on some of the results of [HL] and [AL] that are quoted in Theorem 1.3.

(1) It is easy to see that $s^+(R) > 0$ implies the weakly F-regularity of R (for example, by part (3) below). Then, it follows from Proposition 5.2 that every localization of R remains weakly F-regular. Hence Theorem 1.3(1).
(2) The proof in [AL] for the implications that $s^+(R) > 0 \iff R$ is strongly F-regular $\iff s^-(R) > 0$ is valid for all excellent rings R. Actually, with the new formulation of $s^+(M)$, a standard argument as in the proof of [HH1, Theorem 8.17] readily shows that $s^+(M) > 0 \implies 0^*_E = 0$, the latter of which is equivalent to the strongly F-regularity of R. Indeed, if $0^*_E \neq 0$ on the contrary, then $u \in 0^*_E$ for any nonzero $u \in k \subseteq E$. That is, there exists an element $c \in R \backslash \cup_{P \in \min(R)} P$ such that $0 = u^q_E \cdot c = u \otimes c \in E \otimes_R {}^e R$ for all $e \gg 0$. Hence $\lambda^r(\ker(\phi \otimes_R 1_{{}^e M})) \leq \lambda\left(M/(\mathfrak{m}^{[q]}, c)M\right) = o(q^d)$ as $e \to \infty$ since $\dim(M/cM) < d = \dim(R)$, which contradicts the assumption $s^+(M) > 0$. (This explains Theorem 1.3(2).)
(3) Theorem 1.3(3) reduces itself to the F-finite case (c.f. Remark 2.3(3)), which is verified in [HL]. It is also a special case of the next Lemma 2.5(2).
(4) The proof for the inequality $(e(R)-1)(1-s^+(R)) \geq e_{HK}(R)-1$ in [HL] can be used verbatim to prove the general case. Alternatively, we may argue that it reduces to the F-finite case.

Lemma 2.5. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p with $\dim(R) = d$ and M a finitely generated R-module. Given (not necessarily finitely generated) R-modules L and D and an R-homomorphism $\psi' : L \to D$ such that $\lambda_R(\mathrm{image}(\psi')) = \lambda_R(\psi'(L)) < \infty$. Then (recall that $q = p^e$)*

(1) $\lambda(\psi'(L))\#({}^e M) \leq \lambda^r(\mathrm{image}(\psi' \otimes {}^e M)) \leq \lambda(\psi'(L))\lambda(M/\mathfrak{m}^{[q]}M)$ for every $e \in \mathbb{N}$, and hence,
(2) $\lambda(\psi'(L))s^+(M) \leq \limsup_{e\to\infty} \frac{\lambda^r(\mathrm{image}(\psi' \otimes {}^e M))}{q^d} \leq \lambda(\psi'(L))e_{HK}(M)$ and $\lambda(\psi'(L))s^-(M) \leq \liminf_{e\to\infty} \frac{\lambda^r(\mathrm{image}(\psi' \otimes {}^e M))}{q^d} \leq \lambda(\psi'(L))e_{HK}(M)$.
(3) $\#({}^e R)\#({}^{e'} M) \leq \#({}^{e+e'} M)$ for every $e, e' \in \mathbb{N}$. As a result,
  (a) R is regular $\iff \#({}^e R) = q^d$ for some (or for all) $e > 0$; and
  (b) R is not regular $\iff \#({}^e R) \leq q^d - 1$ for some (or for all) $e > 0$.

Proof. (1): We may simply assume that $\psi' : L \to D$ is a monomorphism (hence $\lambda(L) = \lambda(\psi'(L)) < \infty$). Then, by induction on $\lambda(L)$, it is enough to prove the case where $L = k$. Since E is an injective R-module, the map $\psi : k \to E$ (as in Definition 2.2) factors through the injective map ψ'. Consequently $\#({}^e M) = \lambda^r(\mathrm{image}(\psi \otimes 1_{{}^e M})) \leq \lambda^r(\mathrm{image}(\psi' \otimes 1_{{}^e M}))$ for every $e \in \mathbb{N}$, the desired result. The inequality $\lambda^r(\mathrm{image}(\psi' \otimes {}^e M)) \leq \lambda(\psi'(L))\lambda(M/\mathfrak{m}^{[q]}M)$ is well-known and also obvious in this context.

(2): Divide the inequalities in (1) by $q^d = p^{ed}$ and then take the limit as $e \to \infty$.

(3): Let $\psi : k \to E$ be as in Definition 2.2. Then, by part (1), $\#({}^eR)\#({}^{e'}M) \leq \lambda^r(\text{image}((\psi \otimes_R {}^eR) \otimes_R {}^{e'}M)) = \#({}^{e+e'}M)$ for every $e, e' \in \mathbb{N}$. To finish the rest of the proof for (3), we simply observe that $\#({}^eR) \geq p^{ed}$ for some $e > 0 \implies \#({}^{ne}R) \geq p^{ned}$ for all $n \in \mathbb{N} \implies s^+(R) \geq 1 \implies R$ is regular $\implies \#({}^eR) = (p^e)^d$ for all $e \in \mathbb{N}$. $\square$

Corollary 2.6. *If $(R, \mathfrak{m}, k)$ is regular and M is a finitely generated R-module, then $s(M)$ exists.*

Proof. Say $\dim(R) = d$. By Lemma 2.5(3), we have $p^d\#({}^eM) = \#({}^1R)\#({}^eM) \leq \#({}^{e+1}M)$ for every $e \in \mathbb{N}$. Thus the sequence $\left\{\frac{\#({}^eM)}{p^{ed}}\right\}_{e=0}^{\infty}$ is non-decreasing and hence has a limit. $\square$

3. Rings with big enough F-signature are regular

If R is not regular, then $s^+(R) < 1$. We show that, for non-regular rings R of fixed dimension, the F-signature $s^+(R)$ can not be arbitrarily close to 1.

Theorem 3.1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p with $\dim R = d$. Assume $s^+(R) > 0$ in case $\dim(R) \leq 1$, or*

$$s^+(R) \geq 1 - \frac{1}{d!p^d} \qquad \textit{in case } \dim(R) \geq 2.$$

Then R is regular, which actually implies $s(R) = 1$.

Proof. If $\dim R \leq 1$ and $s^+(R) > 0$, then R is normal and hence regular. So we assume $\dim R \geq 2$. Suppose, on the contrary, that R is not regular. Then $e(R) > 1$, $e_{HK}(R) > 1$ (c.f. [WY1] or [HY]) and $\#({}^1R) \leq p^d - 1$ (c.f. Lemma 2.5(3)).

Firstly, we have $(e(R) - 1)(1 - s^+(R)) \geq e_{HK}(R) - 1$ by [HL], which implies

$$(*) \qquad s^+(R) \leq 1 - \frac{e_{HK}(R) - 1}{e(R) - 1} < 1 - \frac{e_{HK}(R) - 1}{d!e_{HK}(R) - 1} = \frac{d!e_{HK}(R) - e_{HK}(R)}{d!e_{HK}(R) - 1},$$

as we have $1 \leq e_{HK}(R) \leq e(R) < d!e_{HK}(R)$. (Note that the strict inequality $e(R) < d!e_{HK}(R)$ when $\dim(R) \geq 2$ is a recent result of D. Hanes in [Ha].)

Secondly, let $\psi : k \to E$ be an injective R-linear map as in Definition 2.2 and hence an induced bimodule map $\psi' := \psi \otimes_R {}^1R : k \otimes_R {}^1R \to E \otimes_R {}^1R$. For every $e \in \mathbb{N}$, it is easy to see that $\lambda^r(\text{image}(\psi' \otimes_R {}^eR)) = \lambda^r(\text{image}(\psi \otimes_R {}^{e+1}R)) = \#({}^{e+1}R)$ and hence $\limsup_{e\to\infty} \frac{\lambda^r(\text{image}(\psi' \otimes {}^eR))}{p^{ed}} = \limsup_{e\to\infty} \frac{\#({}^{e+1}R)}{p^{ed}} = p^d s^+(R)$ by the definition of the F-signature. We also have $\limsup_{e\to\infty} \frac{\lambda^r(\text{image}(\psi' \otimes {}^eR))}{p^{ed}} \leq \lambda^r(\text{image}(\psi'))e_{HK}(R) = \#({}^1R)e_{HK}(R) \leq (p^d - 1)e_{HK}(R)$ by Lemma 2.5(2-3). Hence

$$(**) \qquad p^d s^+(R) \leq (p^d - 1)e_{HK}(R) \implies s^+(R) \leq \frac{(p^d - 1)e_{HK}(R)}{p^d}.$$

Define functions $f(x) = \frac{d!x - x}{d!x - 1} = \frac{d! - 1}{d!} + \frac{d! - 1}{d!(d!x - 1)}$ and $g(x) = \frac{(p^d - 1)x}{p^d}$ over the open interval $(1, \infty)$. It is easy to see that $f(x)$ is a strictly decreasing function and $g(x)$ is strictly increasing over $(1, \infty)$.

If $e_{HK}(R) \geq \frac{d!p^d - 1}{d!(p^d - 1)}$, then $s^+(R) < \frac{d!e_{HK}(R) - e_{HK}(R)}{d!e_{HK}(R) - 1} \leq f\left(\frac{d!p^d - 1}{d!(p^d - 1)}\right) = 1 - \frac{1}{d!p^d}$ by $(*)$, a contradiction. If, otherwise, $1 < e_{HK}(R) < \frac{d!p^d - 1}{d!(p^d - 1)}$, then we get $s^+(R) \leq \frac{(p^d - 1)e_{HK}(R)}{p^d} < g\left(\frac{d!p^d - 1}{d!(p^d - 1)}\right) = 1 - \frac{1}{d!p^d}$ by $(**)$, still a contradiction.

Therefore the assumption $s^+(R) \geq 1 - \frac{1}{d!p^d}$ implies that R is regular. $\square$

Remark 3.2. M. Blickle and F. Enescu showed the following result in [BE]: *Let $(R, \mathfrak{m})$ be a Noetherian unmixed local ring of characteristic p with $\dim(R) = d$. If $e_{HK}(R) \leq 1 + \max\{\frac{1}{d!p^d}, \frac{1}{p^d e(R)}\}$, then R is regular.* Theorem 3.1 is inspired by the result of [BE] and has a similar effect.

4. Some cases where $s(M) = \lim_{e\to\infty} \frac{\#({}^eM)}{q^{\dim(R)}}$ exists

Proposition 4.1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p and M a finitely generated R-module. Keep the notations as in Definition 2.2. Suppose that there exists a finitely generated R-submodule $E' \subseteq E$ such that $\#({}^eM) = \lambda^r(\ker(\phi' \otimes_R 1_{{}^eM}))$ for all (sufficiently large) $e \in \mathbb{N}$, where $\phi' : E' \to E'/k$ is the naturally induced R-homomorphism. Then (with $q = p^e$)*

1. *$s(M) = \lim_{e\to\infty} \frac{\#({}^eM)}{q^d}$ exists.*
2. *$s(M) = \inf\{e_{HK}(I_1, M) - e_{HK}(I_2, M) \,|\, I_1 \subset I_2, \sqrt{I_1} = \mathfrak{m}, I_2/I_1 \cong k\}$ and the value is attained at certain such ideals of R.*
3. *Suppose R is excellent and M is faithful over R. Then*
$$R \text{ is weakly } F\text{-regular} \iff s(M) > 0 \iff R \text{ is strongly } F\text{-regular.}$$

Proof. (1): Indeed, as $\lambda(E') < \infty$, the limit

$$\begin{aligned} s(M) = \lim_{e\to\infty} \frac{\#({}^eM)}{q^d} &= \lim_{e\to\infty} \frac{\lambda^r(\ker(\phi' \otimes_R 1_{{}^eM}))}{q^d} \\ &= \lim_{e\to\infty} \frac{\lambda^r(E' \otimes_R {}^eM)}{q^d} - \lim_{e\to\infty} \frac{\lambda^r((E'/k) \otimes_R {}^eM)}{q^d} \end{aligned}$$

exists by a result of G. Seibert (c.f. Theorem 1.6).

(2): To prove this, we may assume that R is complete without loss of generality. If R is weakly F-regular, then R is reduced and hence approximately Gorenstein. Therefore there exists a $\mathfrak{m}$-primary ideal I of R such that $E' \subseteq (0 :_E I) \cong R/I$. Choose $I_2 = (I_1 :_R \mathfrak{m})$ to get $\lambda(I_1/I_2) = 1$ and $s(M) = e_{HK}(I_1, M) - e_{HK}(I_2, M)$. If R is not weakly F-regular, then choose $I_1 \subset I_2$ to be any $\mathfrak{m}$-primary ideals such that $I_2 \subseteq I_1^*$ and $\lambda(I_1/I_2) = 1$ to get $e_{HK}(I_1, M) - e_{HK}(I_2, M) = 0 = s(M)$.

(3): We have $s(M) = e_{HK}(I_1, M) - e_{HK}(I_2, M)$ for $\mathfrak{m}$-primary ideals $I_1 \subset I_2$ such that $\lambda(I_1/I_2) = 1$ by (2) above. Suppose R is weakly F-regular. Then, since R is excellent, $\widehat{R}$ is also weakly F-regular, which in turn implies that $\widehat{R}$ is a domain. Therefore we can apply [HH1, Theorem 8.17] to get $e_{HK}(I_1, R) - e_{HK}(I_2, R) > 0$, which, as M is faithful, forces $s(M) = e_{HK}(I_1, M) - e_{HK}(I_2, M) > 0$. Hence R is strongly F-regular. The rest implications are clear. $\square$

Lemma 4.2. *Let $(R, \mathfrak{m}, k)$ be an F-finite Noetherian local ring of characteristic p and keep the notations as in Definition 0.1. Then the following are equivalent:*

1. *There exists a finite-length R-submodule $E_1 \subseteq E$ such that $\#({}^eM) = \lambda^r_R(\ker(\phi_1 \otimes_R 1_{{}^eM}))$ for all (sufficiently large) $e \in \mathbb{N}$, where $\phi_1 : E_1 \to E_1/k$ is the natural R-homomorphism.*
2. *There exists an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R such that $R/\mathfrak{a}$ is not an R-linear homomorphic image of left R-module M_e for any (sufficiently large) $e \in \mathbb{N}$.*

Proof. By Matlis duality functor $-^\vee := \operatorname{Hom}_R(-, E)$, there is a one-one correspondence from the family of all finite-length R-modules to itself. In particular, we have $E_1 \leftrightarrow R/\operatorname{Ann}_R(E_1)$, $E_1/k \leftrightarrow \mathfrak{m}/\operatorname{Ann}_R(E_1)$ and $\phi_1 \leftrightarrow i$ where $\phi_1 : E_1 \to E_1/k$ and $i : \mathfrak{m}/\operatorname{Ann}_R(E_1) \to R/\operatorname{Ann}_R(E_1)$ are the natural surjection and inclusion maps respectively.

As in the proof of Lemma 2.1, we regard ${}^eM$ as an R-module with its scalar multiplication defined by $r \cdot m = r^{p^e} m = m \cdot r$ for any $r \in R, m \in M$. Then

$$\begin{aligned}
(1) &\iff \lambda_R(\ker(\phi_1 \otimes_R {}^eM)) = a_e \quad \text{for all } e \gg 0 \\
&\iff \lambda_R\left(\operatorname{coker}\left((\phi_1 \otimes_R {}^eM)^\vee\right)\right) = a_e \quad \text{for all } e \gg 0 \\
&\iff \lambda_R\left(\operatorname{coker}\left(\operatorname{Hom}_R({}^eM, \phi_1^\vee)\right)\right) = a_e \quad \text{for all } e \gg 0 \\
&\iff \lambda_R\left(\operatorname{coker}\left(\operatorname{Hom}_R(R^{a_e} \oplus M_e, \phi_1^\vee)\right)\right) = a_e \quad \text{for all } e \gg 0 \\
&\iff a_e + \lambda\left(\operatorname{coker}\left(\operatorname{Hom}_R\left(M_e, \phi_1^\vee\right)\right)\right) = a_e \quad \text{for all } e \gg 0 \\
&\iff \lambda\left(\operatorname{coker}\left(\operatorname{Hom}_R\left(M_e, \phi_1^\vee\right)\right)\right) = 0 \quad \text{for all } e \gg 0 \\
&\iff (2),
\end{aligned}$$

which finishes the proof. □

Theorem 4.3. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic p such that $\widehat{R}_P$ is Gorenstein for every $P \in \operatorname{Spec}(\widehat{R}) \setminus \{\mathfrak{m}\widehat{R}\}$ (e.g. R is an excellent local ring such that R_P is Gorenstein for every $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$). Then for any maximal Cohen-Macaulay module M, the results (1), (2) and (3) listed in Proposition 4.1 hold. In particular, (A) $s(R) = \lim_{e\to\infty} \frac{\#({}^eR)}{q^d}$ exists; (B) the value $\inf\{e_{HK}(I_1, R) - e_{HK}(I_2, R) \mid I_1 \subset I_2, \sqrt{I_1} = \mathfrak{m}, I_2/I_1 \cong k\}$ is attained and is equal to $s(R)$; and (C) Assuming R is excellent, we have R is weakly F-regular if and only if R is strongly F-regular.*

Proof. It is enough to prove the case where R is complete, weakly F-regular and hence Cohen-Macaulay (otherwise the statements are all trivially true), and also F-finite (by extending its coefficient field to its perfect closure as described in Remark 2.3(3) and the fact that the extension ring remains Gorenstein at the punctured spectrum). Hence canonical module exists over R. Since M is maximal Cohen-Macaulay, so are ${}^eM$ and hence M_e for every $e \geq 0$. By the result of Ding in [Di] quoted as Theorem 1.7, there exists an integer $n \in \mathbb{N}$ such that $R/\mathfrak{m}^n$ is not an R-linear homomorphic image of any maximal Cohen-Macaulay module without non-zero free direct summand. Hence, because of Lemma 4.2, Proposition 4.1 applies to M and the proof is complete. □

Remark 4.4.
1. Aberbach and Enescu recently proved the existence of $s(R)$ under a weaker condition that R_P is $\mathbb{Q}$-Gorenstein for every $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$, or R is $\mathbb{N}$-graded (see [AE2]). Their proof also shows that these rings satisfy the assumption of Proposition 4.1. Also, Singh has recently proved that the F-signature of an affine semigroup ring always exists in [Si].
2. Whether or when weak F-regularity, F-regularity and strong F-regularity are equivalent is an open question. B. MacCrimmon proved in [Mac] that weak F-regularity is equivalent to strong F-regularity if R_P is $\mathbb{Q}$-Gorenstein for every $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. There is also a proof of the above statement provided by I. Aberbach in [Ab2]. The equivalence also holds in case R is $\mathbb{N}$-graded, which is proved by Lyubeznik and Smith [LS1].

Before proving the next result, let us recall the definition of R-modules with finite F-representation type (FFRT for short).

Definition 4.5. Given a finitely generated R-module M. We say that M has finite F-representation type (FFRT) if there exist finitely generated R-modules $M_1, M_2, \ldots, M_s$ such that for every $e \geq 0$, the R-module ${}^eM$ is isomorphic to a finite direct sum of the R-modules $M_1, M_2, \ldots, M_s$, i.e. there exist non-negative integers $n_{e1}, n_{e2}, \ldots, n_{es}$ such that

$$^{e}M \cong M_1^{n_{e1}} \oplus M_2^{n_{e2}} \oplus \cdots \oplus M_s^{n_{es}}.$$

Examples of FFRT include: (1) If R has finite (maximal) Cohen-Macaulay type, then every maximal Cohen-Macaulay module has FFRT. (2) Let $R \to S$ be a ring homomorphism such that S is module-finite over R, W a finite S-module with FFRT and M an R-submodule of W such that $W = M \oplus N$ as R-modules. Then M has FFRT as an R-module by [SVdB, Proposition 3.1.4], in which the Krull-Schmidt condition is not needed by virtue of [Wi, Theorem 1.1].

If R has FFRT, K. Smith and M. Van den Bergh proved that $\lim_{e\to\infty} \frac{\#({}^eR,R)}{q^{\alpha(R)+d}}$ exists in [SVdB]. In general, if a finitely generated R-module M has FFRT, then $\lim_{e\to\infty} \frac{\#({}^eM,R)}{q^{\alpha(R)+d}}$ exists and is rational (see [Yao]). The result (about the existence of the limit $s(R)$) is recovered in the next theorem.

Theorem 4.6. *Let $(R, \mathfrak{m}, k)$ be a F-finite Noetherian local ring of prime characteristic p and M a finitely generated R-module. If M has FFRT, the results (1), (2) and (3) listed in Proposition 4.1 hold. In particular, $s(M)$ exists.*

Proof. Without loss of generality of the definition of FFRT, there are finitely generated R-modules $N_1, N_2, \ldots, N_s$, none of which has non-zero free direct summand, such that for every $e \geq 0$, ${}^eM \cong R^{a_e} \oplus N_1^{n_{e1}} \oplus N_2^{n_{e2}} \oplus \cdots \oplus N_s^{n_{es}}$ for some non-negative integers $a_e, n_{e1}, n_{e2}, \ldots, n_{es}$. By the result in [Gu] quoted in Theorem 1.8, there exists an integer $n \in \mathbb{N}$ such that $R/\mathfrak{m}^n$ is not a homomorphic image of N_i for any $i = 1, 2, \ldots, s$. Hence $R/\mathfrak{m}^n$ is not a homomorphic image of $M_e \cong \oplus_{i=1}^s N_i^{n_{ei}}$ for any $e \geq 0$ and the desired results follow from Proposition 4.1. □

Remark 4.7. Let R be a subring of an F-finite regular local ring S of characteristic p such that S is module finite over R and the inclusion $R \to S$ splits over R. Denote the rank of S over R by $\mathrm{rank}_R(S)$. (This is the case if R is the ring of invariants of S under a finite group G of order prime to the characteristic, i.e. $p \nmid |G|$. See [HL, Corollary 20] and notice that $\mathrm{rank}_R(S) = |G|$.) Hence ${}^eR$ is direct summand of ${}^eS$ as an R-module. On the other hand, ${}^eS \cong S^{\alpha(S)+\dim(S)}$ as S-modules (hence as R-modules), which implies that S has FFRT as an R-module. Say $S \cong R^f \oplus M$ such that R is not a direct summand of M. Then, considered as R-modules, ${}^eS \cong R^{f(\alpha(S)+\dim(S))} \oplus M^{\alpha(S)+\dim(S)}$ for all $e \geq 0$, which implies that $s_R(S) = f$ (as $\alpha(R) = \alpha(S)$ and $\dim(R) = \dim(S)$). Moreover, as ${}^eR$ is a direct summand of ${}^eS$ for every $e \geq 0$, R has FFRT by [Wi, Theorem 1.1] or, under the Krull-Schmidt assumption, by [SVdB, Proposition 3.1.4]. In [HL, Corollary 20] (where R is an invariant subring of S), it is proved that

$$s(R) = \frac{f}{\mathrm{rank}_R(S)},$$

under the assumption that R is Gorenstein. Now that we have Theorem 4.6, the Gorenstein assumption turns out to be superfluous. Indeed, since both S and R

have FFRT as R-modules, we can choose $\mathfrak{m}$-primary ideals $I_1 \subset I_2$ of R such that $\lambda_R(I_2/I_1) = 1$, $s(R) = e_{HK}(I_1) - e_{HK}(I_2)$ and $s_R(S) = e_{HK}(I_1, S) - e_{HK}(I_2, S)$ as in Proposition 4.1. Therefore $s_R(S) = \operatorname{rank}_R(S)s(R)$, which gives $s(R) = \frac{f}{\operatorname{rank}_R(S)}$.

5. The F-signature under local flat extensions

Given a local ring homomorphism $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$, a finitely generated module M over R and $P \in \operatorname{Spec}(R)$, we get an S-module $N := M \otimes_R S$ by scalar extension and an R_P-module M_P by localization. To avoid the cumbersome subscripts, we sometimes simply write $s(M \otimes_R S)$, $s(S/\mathfrak{m}S)$ and $s(M_P)$ etc. instead of $s_S(M \otimes_R S)$, $s_{S/\mathfrak{m}S}(S/\mathfrak{m}S)$ and $s_{S_P}(M_P)$ etc. respectively. As always, ψ is a fixed injective map (e.g. the inclusion map) from k to $E = E_R(k)$ and hence the induced S-linear map $\psi \otimes_R S : k \otimes_R S \to E \otimes_R S$. Finally, we denote by $\bar{S}$ the closed fiber ring $S/\mathfrak{m}S$.

We are to study the behavior of the F-signature under local flat (i.e. faithfully flat) homomorphisms. Sometimes we make our statements more general so that they apply to some cases of local pure homomorphisms. A homomorphism $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is pure $\iff 0 \neq \operatorname{image}(\psi \otimes_R S) \subseteq E \otimes_R S$ (see [HR, Proposition 6.11]). We start with a special case of pure local extension where $0 \neq \lambda_S(\operatorname{image}(\psi \otimes_R S)) < \infty$ (e.g. $\bar{S}$ is 0-dimensional).

Lemma 5.1. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a pure local ring homomorphism such that $\lambda_S(\operatorname{image}(\psi \otimes_R S)) < \infty$, and M a finitely generated R-module. We have*

(1) Set $I := \operatorname{Ann}_{\bar{S}}(\operatorname{image}(\psi \otimes_R S)) \subset \bar{S}$. Then (with $q = p^e$)
  - *(a) $\#({}^eM) \geq \frac{\lambda_S(\operatorname{image}(\psi \otimes_R S))}{\lambda_{\bar{S}}(\bar{S}/I^{[q]})} \#({}^e(M \otimes_R S))$ for every $e \in \mathbb{N}$; and*
  - *(b) $s^{\pm}(M) \geq \frac{\lambda_S(\operatorname{image}(\psi \otimes_R S))}{e_{HK}(I, \bar{S})} s^{\pm}(M \otimes_R S)$, if $\dim(S) = \dim(R) + \dim(\bar{S})$.*

(2) In particular, if $\bar{S} = S/\mathfrak{m}S$ is 0-dimensional, then
  - *(a) $\#({}^eM) \geq \frac{\lambda_S(\operatorname{image}(\psi \otimes_R S))}{\lambda_S(S/\mathfrak{m}S)} \#({}^e(M \otimes_R S))$ for every $e \in \mathbb{N}$; and*
  - *(b) $s^{\pm}(M) \geq \frac{\lambda_S(\operatorname{image}(\psi \otimes_R S))}{\lambda_S(S/\mathfrak{m}S)} s^{\pm}(M \otimes_R S)$.*

(3) If the ring homomorphism $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is flat, then
  - *(a) $\#({}^eM) \geq \#({}^e(M \otimes_R S))$ for every $e \in \mathbb{N}$; and therefore*
  - *(b) $s^+(M) \geq s^+(M \otimes_R S)$ and $s^-(M) \geq s^-(M \otimes_R S)$.*

Proof. (1)(a): For every $e \in \mathbb{N}$, we have a composition of natural isomorphisms

$$\begin{aligned}(E \otimes_R S) \otimes_S {}^e(M \otimes_R S) &\cong E \otimes_R (S \otimes_S {}^e(M \otimes_R S)) \\ &\cong E \otimes_R {}^e(M \otimes_R S) \cong (E \otimes_R {}^eM) \otimes_R S,\end{aligned}$$

under which $\operatorname{image}((\psi \otimes_R S) \otimes_S {}^e(M \otimes_R S)) \cong \operatorname{image}((\psi \otimes_R {}^eM) \otimes_R S)$. Hence we get $\operatorname{Ann}_S(\operatorname{image}((\psi \otimes_R {}^eM) \otimes_R S)) = \operatorname{Ann}^r_S(\operatorname{image}((\psi \otimes_R S) \otimes_S {}^e(M \otimes_R S))) \supseteq (\operatorname{Ann}_S(\operatorname{image}(\psi \otimes_R S)))^{[q]}$ for every $e \in \mathbb{N}$ and $q = p^e$, which implies that

$$\begin{aligned}\lambda^r_S(\operatorname{image}((\psi \otimes_R S) \otimes_S {}^e(M \otimes_R S))) &= \lambda_S(\operatorname{image}((\psi \otimes_R {}^eM) \otimes_R S)) \\ &\leq \lambda_R(\operatorname{image}(\psi \otimes_R {}^eM))\lambda_{\bar{S}}(\bar{S}/I^{[q]}) = \#({}^eM)\lambda_{\bar{S}}(\bar{S}/I^{[q]})\end{aligned}$$

for every $e \in \mathbb{N}$. On the other hand, we have

$$\lambda_S(\operatorname{image}(\psi \otimes_R S)\#({}^e(M \otimes_R S)) \leq \lambda^r_S(\operatorname{image}((\psi \otimes_R S) \otimes_S {}^e(M \otimes_R S)))$$

by Lemma 2.5(1). Combining the two inequalities together, we get

$$\lambda_{\bar{S}}(\bar{S}/I^{[q]})\#({}^eM) \geq \lambda_S(\operatorname{image}(\psi \otimes_R S)\#({}^e(M \otimes_R S))$$

for every $e \in \mathbb{N}$, which gives the desired result of (1)(a).

(1)(b): Divide (1)(a) by $q^{\dim(R)}$ and take the limits as $e \to \infty$.

(2): This follows from (1) since $\lambda(S/\mathfrak{m}S) \geq \lambda(\bar{S}/I^{[q]})$ and $\dim(S) = \dim(R)$. Indeed, $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is pure $\implies \mathrm{H}^{\dim(R)}_{\mathfrak{n}}(S) \cong H^{\dim(R)}_{\mathfrak{m}}(R) \otimes_R S \neq 0 \implies \dim(S) \geq \dim(R) \implies \dim(S) = \dim(R)$.

To prove (3), we observe that the extra assumption on the flatness of S over R implies $\mathrm{image}(\psi \otimes_R S) \cong S/\mathfrak{m}S$. Hence (3) is a special case of (2). $\square$

Next we prove that the F-signature is non-decreasing upon further localization, which, in the F-finite case, is obvious from Definition 0.1.

Proposition 5.2. *Let R be a Noetherian ring of characteristic p, M a finitely generated R-module and $P_1 \supseteq P_2$ two prime ideals of R. Then (with $q = p^e$)*

(1) $\#({}^eM_{P_1}) \leq q^{\dim((R/P_2)_{P_1})}\#({}^eM_{P_2})$ for every $e \in \mathbb{N}$, and therefore,

(2) $s^+(M_{P_1}) \leq s^+(M_{P_2})$ and $s^-(M_{P_1}) \leq s^-(M_{P_2})$.

Proof. Without loss of generality, we may simply assume $(R, \mathfrak{m})$ is local with $P_1 = \mathfrak{m}$ and $P_2 = P \in \mathrm{Spec}(R)$. Fix a flat local ring homomorphism $R \to \widehat{R} \to \widehat{R} \otimes_{k[[X_1,\dots,X_n]]} k^\infty[[X_1, \dots, X_n]] =: S$, in which $k[[X_1, \dots, X_n]]$ is such that there is a ring homomorphism from $k[[X_1, \dots, X_n]]$ onto $\widehat{R}$ and k^∞ is the perfect closure of $k = R/\mathfrak{m}$ (c.f. Remark 2.3(3)). Denote by N the right and left S-module $M \otimes_R S$. Choose $Q \in \mathrm{Spec}(S)$ such that $PS \subseteq Q$ and $\dim(S/Q) = \dim(R/P)$. Hence $\dim(R_P) = \dim(S_Q)$ and $\#({}^eM_P) \geq \#({}^eN_Q)$ by Lemma 5.1(2). Since S is F-finite, we have $\#({}^eN) = \#({}^eN, S) \leq \#({}^eN_Q, S_Q) = q^{\dim(S/Q)}\#({}^eN_Q)$ by the meaning of $\#({}^eN, S)$ and $\#({}^eN_Q, S_Q)$ in Definition 0.1. Therefore, we have $\#({}^eM) = \#({}^eN) \leq q^{\dim(S/Q)}\#({}^eN_Q) \leq q^{\dim(R/P)}\#({}^eM_P)$, the result of (1).

To see that (2) follows from (1), we notice the non-trivial case is when $s^+(M) > 0$, which implies $\widehat{R}$ is Cohen-Macaulay $\implies \dim(R/P) + \dim(R_P) = \dim(R)$. $\square$

Remark 5.3. $\#({}^eR) > q^{\dim(R)} - q$ for some $e > 0 \implies \#({}^eR_P) \geq \frac{\#({}^eR)}{q^{\dim(R/P)}} > q^{\dim(R)-\dim(R/P)} - q^{1-\dim(R/P)} \geq q^{\dim(R_P)} - 1$ for every $P \in \mathrm{Spec}(R) \setminus \{\mathfrak{m}\} \implies \#({}^eR_P) \geq q^{\dim(R_P)}$ for every $P \in \mathrm{Spec}(R) \setminus \{\mathfrak{m}\} \implies R_P$ is regular for every $P \in \mathrm{Spec}(R) \setminus \{\mathfrak{m}\} \implies s(R)$ exists by Theorem 4.3.

Theorem 5.4. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a pure local ring homomorphism and M a finitely generated R-module. Then there exists $Q \in \mathrm{Spec}(S)$ such that $0 \neq \lambda_S(\mathrm{image}(\psi \otimes_R S_Q)) < \infty$. For every such Q, we have (with $q = p^e$)*

(1) Set $I := \mathrm{Ann}_{\bar{S}}(\mathrm{image}(\psi \otimes_R S)) \subset \bar{S} = S/\mathfrak{m}S$. Then

(a) $q^{\dim(S/Q)}\#({}^eM) \geq \frac{\lambda_{S_Q}(\mathrm{image}(\psi \otimes_R S_Q))}{\lambda_{\bar{S}_Q}(\bar{S}_Q/I_Q^{[q]})}\#({}^e(M \otimes_R S))$ for every $e \in \mathbb{N}$;

(b) $s^\pm(M) \geq \frac{\lambda_{S_Q}(\mathrm{image}(\psi \otimes_R S_Q))}{e_{HK}(I_Q, \bar{S}_Q)} s^\pm(M \otimes_R S)$ if $\dim(S) = \dim(R) + \dim(\bar{S})$.

(2) In particular, if $\bar{S}_Q = S_Q/\mathfrak{m}S_Q$ is 0-dimensional, then

(a) $q^{\dim(S/Q)}\#({}^eM) \geq \frac{\lambda_{S_Q}(\mathrm{image}(\psi \otimes_R S_Q))}{\lambda_{S_Q}(S_Q/\mathfrak{m}S_Q)}\#({}^e(M \otimes_R S))$ for every $e \in \mathbb{N}$;

(b) $s^\pm(M) \geq \frac{\lambda_{S_Q}(\mathrm{image}(\psi \otimes_R S_Q))}{\lambda_{S_Q}(S_Q/\mathfrak{m}S_Q)} s^\pm(M \otimes_R S)$.

(3) If the local ring homomorphism $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is flat, then

(a) $q^{\dim(S/Q)}\#({}^eM) \geq \#({}^e(M \otimes_R S))$ for every $e \in \mathbb{N}$; hence,

(b) $s^+(M) \geq s^+(M \otimes_R S)$ and $s^-(M) \geq s^-(M \otimes_R S)$.

Proof. Indeed, Q may be any minimal prime over $\operatorname{Ann}_S(\operatorname{image}(\psi \otimes_R S)) \subsetneq S$. For every such $Q \in \operatorname{Spec}(S)$, Lemma 5.1 and Proposition 5.2 may be applied to the pure local ring homomorphism $R \to S_Q$ and the localization of S at Q respectively. (In proving (1)(b), notice that the non-trivial case is when $s^{\pm}(S) > 0$, which implies that $\dim(S_Q) = \dim(R) + \dim(\bar{S}_Q)$ under the assumption.) □

Remark 5.5. If a local ring homomorphism $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a pure, then, by definition (see [HH2, Theorem 5.5]), the strong F-regularity of S implies the strong F-regularity of R, which amounts to "$s^{\pm}(S) > 0 \implies s^{\pm}(R) > 0$" in terms of F-signature. Theorem 5.4(1)(b) above reveals a relation between $s^{\pm}(S)$ and $s^{\pm}(R)$, which refines the implication "$s^{\pm}(S) > 0 \implies s^{\pm}(R) > 0$" provided that the condition $\dim(S) = \dim(R) + \dim(S/\mathfrak{m}S)$ holds (e.g. the homomorphism is flat).

Theorem 5.6. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local flat ring homomorphism with the closed fiber $\bar{S} := S/\mathfrak{m}S$ being Gorenstein and M a finitely generated R-module. Then*

(1) $\#_R({}^eM)\#_{\bar{S}}({}^e\bar{S}) \le \#_S({}^e(M \otimes_R S))$, for every $e \in \mathbb{N}$, and therefore,
(2) $s^+(M)s(S/\mathfrak{m}S) \le s^+(M \otimes_R S)$ and $s^-(M)s(S/\mathfrak{m}S) \le s^-(M \otimes_R S)$.

Equalities hold in (1) and (2) if $S/\mathfrak{m}S$ is regular.

Proof. It is enough to prove the inequalities as the equalities would then be forced by the above Theorem 5.4 in case of regular closed fiber. Nevertheless, everything (including the case of regular closed fiber) is proved from scratch.

We may assume both R and S to be complete (c.f. Remark 2.3 (3)) and hence excellent. As the only interesting case is when R is reduced (otherwise $\#({}^eM) = 0$ for all $e > 0$), we may assume that R is approximately Gorenstein. For notational convenience, we denote the resulted left and right S-module $M \otimes_R S$ by N and $S/\mathfrak{m}S$ by $\bar{S}$. For the same reason, we treat R as a subring of S.

Let $E_R(k), E_{\bar{S}}(l)$ and $E_S(l)$ be the injective hulls of the residue fields over the respective rings. Recall that (see Definition 2.2)

$$\#_R({}^eM) = \lambda_R^r\left(\ker\left(E_R(k) \otimes_R {}^eM \to \frac{E_R(k)}{k} \otimes_R {}^eM\right)\right),$$
$$\#_{\bar{S}}({}^e\bar{S}) = \lambda_{\bar{S}}^r\left(\ker\left(E_{\bar{S}}(l) \otimes_{\bar{S}} {}^e\bar{S} \to \frac{E_{\bar{S}}(l)}{l} \otimes_{\bar{S}} {}^e\bar{S}\right)\right) \quad \text{and}$$
$$\#_S({}^eN) = \lambda_S^r\left(\ker\left(E_S(l) \otimes_S {}^eN \to \frac{E_S(l)}{l} \otimes_S {}^eN\right)\right)$$

for every $e \in \mathbb{N}$.

It is enough to prove (1), i.e.

$$\#_R({}^eM)\#_{\bar{S}}({}^e\bar{S}) \le \#_S({}^eN)$$

(equality in case of $\bar{S}$ being regular), which will give the desired result of (2) since $\dim(S) = \dim(R) + \dim(\bar{S})$ and $s(\bar{S})$ exists (c.f. Definition 2.2).

Choose a sequence of irreducible $\mathfrak{m}$-primary ideals $\{\mathfrak{a}_n\}$ (so that $R/\mathfrak{a}_n \cong (0 :_{E_R(k)} \mathfrak{a}_n)$ for all $n > 0$) satisfying $\mathfrak{a}_n \subseteq \mathfrak{m}^n$. Choose elements $x_1, x_2, \ldots, x_t \in S$ such that their images form a full system of parameters for $\bar{S}$ and denote $I_n = (x_1^n, x_2^n, \ldots, x_t^n)S$ for all $n > 0$. (In case $\bar{S}$ is regular, make sure that the images of $x_1, x_2, \ldots, x_t \in S$ form a regular system of parameters for $\bar{S}$.) For each

n, choose $u_n \in R, v_n \in S$ such that $u_n + \mathfrak{a}_n$ generates $(0 :_{R/\mathfrak{a}_n} \mathfrak{m})$, the socle of $R/\mathfrak{a}_n$, and $v_n + I_n\bar{S}$ generates the socle of $\bar{S}/I_n\bar{S}$. (In case $\bar{S}$ is regular, choose $v_n = (x_1x_2\cdots x_t)^{n-1}$.) Recall that $S/I_n^{[q]}$ is flat over R for every n and every $q = p^e$ by Theorem 1.9. (In case $\bar{S}$ is regular, $S/(I_n, v_n)^{[q]}S = S/(x_1^{nq}, x_2^{nq}\ldots, x_t^{nq}, (x_1x_2\cdots x_t)^{(n-1)q})S$ is also flat over R for every n and every q since it has a filtration by modules of the form $S/(x_1, x_2, \ldots, x_t)S$.) Then the element $u_nv_n + \mathfrak{a}_nS + I_n$ generates the socle of $S/(\mathfrak{a}_nS + I_n)$ for every n and hence $S/(\mathfrak{a}_nS + I_n)$ is a 0-dimensional Gorenstein ring for every $n > 0$. Notice that $\mathfrak{a}_nS + I_n \subseteq \mathfrak{n}^n$ for all n.

Let $e \in \mathbb{N}$ be any fixed integer. Then by Remark 2.3(2) and our choice of $\mathfrak{a}_n, u_n, I_n$ and v_n, we have (with $q = p^e$)

$$\begin{aligned}
\lambda_R^r\left(\ker\left(E_R(k)\otimes_R {}^eM \to \frac{E_R(k)}{k}\otimes_R {}^eM\right)\right) &= \lambda_R\left(\frac{(\mathfrak{a}_n, u_n)^{[q]}M}{\mathfrak{a}_n^{[q]}M}\right),\\
\lambda_{\bar{S}}^r\left(\ker\left(E_{\bar{S}}(l)\otimes_{\bar{S}} {}^e\bar{S} \to \frac{E_{\bar{S}}(l)}{l}\otimes_{\bar{S}} {}^e\bar{S}\right)\right) &= \lambda_{\bar{S}}\left(\frac{(I_n, v_n)^{[q]}\bar{S}}{I_n^{[q]}\bar{S}}\right) \quad \text{and}\\
\lambda_S^r\left(\ker\left(E_S(l)\otimes_S {}^eN \to \frac{E_S(l)}{l}\otimes_S {}^eN\right)\right) &= \lambda_S\left(\frac{(\mathfrak{a}_nS, I_n, u_nv_n)^{[q]}N}{(\mathfrak{a}_nS, I_n)^{[q]}N}\right)
\end{aligned}$$

for all $n \gg 0$, while the second equality holds for all $n > 0$. But we have

$$\begin{aligned}
&\lambda_S\left(\frac{(\mathfrak{a}_nS, I_n, u_nv_n)^{[q]}N}{(\mathfrak{a}_nS, I_n)^{[q]}N}\right) = \lambda_S\left(\frac{(\mathfrak{a}_nS, I_n, u_n)^{[q]}N}{(\mathfrak{a}_nS, I_n)^{[q]}N}\right) - \lambda_S\left(\frac{(\mathfrak{a}_nS, I_n, u_n)^{[q]}N}{(\mathfrak{a}_nS, I_n, u_nv_n)^{[q]}N}\right)\\
&= \lambda_S\left(\frac{(\mathfrak{a}_n, u_n)^{[q]}M}{\mathfrak{a}_n^{[q]}M}\otimes_R \frac{S}{I_n^{[q]}}\right) - \lambda_S\left(\frac{N}{((\mathfrak{a}_nS, I_n, u_nv_n)^{[q]}N :_N u_n^q)}\right)\\
&= \lambda_R\left(\frac{(\mathfrak{a}_n, u_n)^{[q]}M}{\mathfrak{a}_n^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{I_n^{[q]}\bar{S}}\right) - \lambda_S\left(\frac{N}{((\mathfrak{a}_nS, I_n)^{[q]}N :_N u_n^q) + v_n^qN}\right)\\
&= \lambda_R\left(\frac{(\mathfrak{a}_n, u_n)^{[q]}M}{\mathfrak{a}_n^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{I_n^{[q]}\bar{S}}\right) - \lambda_S\left(\frac{N}{\left(\mathfrak{a}_n^{[q]}N :_N u_n^q\right) + I_n^{[q]}N + v_n^qN}\right)\\
&= \lambda_R\left(\frac{(\mathfrak{a}_n, u_n)^{[q]}M}{\mathfrak{a}_n^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{I_n^{[q]}\bar{S}}\right) - \lambda_S\left(\frac{N}{\left(\mathfrak{a}_n^{[q]}N :_N u_n^q\right)}\otimes_S \frac{S}{(I_n, v_n)^{[q]}S}\right)\\
&= \lambda_R\left(\frac{(\mathfrak{a}_n, u_n)^{[q]}M}{\mathfrak{a}_n^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{I_n^{[q]}\bar{S}}\right) - \lambda_S\left(\frac{M}{\left(\mathfrak{a}_n^{[q]}M :_M u_n^q\right)}\otimes_R \frac{S}{(I_n, v_n)^{[q]}S}\right)\\
&\geq \lambda_R\left(\frac{(\mathfrak{a}_n, u_n)^{[q]}M}{\mathfrak{a}_n^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{I_n^{[q]}\bar{S}}\right) - \lambda_R\left(\frac{M}{\left(\mathfrak{a}_n^{[q]}M :_M u_n^q\right)}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{(I_n, v_n)^{[q]}\bar{S}}\right)\\
&= \lambda_R\left(\frac{(\mathfrak{a}_n, u_n)^{[q]}M}{\mathfrak{a}_n^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{I_n^{[q]}\bar{S}}\right) - \lambda_R\left(\frac{(\mathfrak{a}_n, u_n)^{[q]}M}{\mathfrak{a}_n^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{(I_n, v_n)^{[q]}\bar{S}}\right)\\
&= \lambda_R\left(\frac{(\mathfrak{a}_n, u_n)^{[q]}M}{\mathfrak{a}_n^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{(I_n, v_n)^{[q]}\bar{S}}{I_n^{[q]}\bar{S}}\right)
\end{aligned}$$

for every $n \in \mathbb{N}$. (In case $\bar{S} = S/\mathfrak{m}S$ is regular, equality holds throughout because of the flatness of $\frac{S}{(I_n, v_n)^{[q]}S}$ over R and $\lambda_{\bar{S}}\left(\frac{(I_n, v_n)^{[q]}\bar{S}}{I_n^{[q]}\bar{S}}\right) = q^{\dim(\bar{S})}$.) Hence the proof is complete. □

As a corollary, we state a result of Ian Aberbach in [Ab1], which may now be easily understood in terms of F-signature in light of Theorem 5.6 together with the main result of [AL] applied to excellent rings.

Theorem 5.7 ([Ab1, Theorem 3.6])**.** *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local flat ring homomorphism with $S/\mathfrak{m}S$ being Gorenstein. Assume that R and $S/\mathfrak{m}S$ are both excellent. Then the strong F-regularity of R and of $S/\mathfrak{m}S$ implies the strong F-regularity of S.*

Proof. The strong F-regularity of R and $S/\mathfrak{m}S \implies s^+(R)s(S/\mathfrak{m}S) > 0 \implies s^+(S) > 0 \implies$ the strong F-regularity of S. □

Remark 5.8. In [AL], I. Aberbach and G. Leuschke define the s-dimension of $(R, \mathfrak{m})$, denoted by $\operatorname{sdim}(R)$, to be the largest integer i such that $\limsup_{e\to\infty} \frac{\#({}^eR, R)}{q^{\alpha(R)+i}} > 0$ in case R is F-finite. Recently, I. Aberbach and F. Enescu showed results concerning $\operatorname{sdim}(R)$ in [AE1]. We would like to remark that the notion may just as well be defined as the largest integer i such that $\limsup_{e\to\infty} \frac{\#({}^eR)}{q^i} > 0$ for any Noetherian local ring of characteristic p. The results in this section may be used to analyze the behavior of s-dimension under localization and flat local extension. In particular, we have $\operatorname{sdim}(R) \le \operatorname{sdim}(R_P) + \dim(R/P)$ by Proposition 5.2. Similarly, if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local flat ring homomorphism, then $\operatorname{sdim}(S) \le \operatorname{sdim}(R) + \dim(S/\mathfrak{m}S)$ by Theorem 5.4. If we further assume that $S/\mathfrak{m}S$ is Gorenstein, then Theorem 5.6 shows that $\operatorname{sdim}(S) \ge \operatorname{sdim}(R) + \operatorname{sdim}(S/\mathfrak{m}S)$ while equality holds if $S/\mathfrak{m}S$ is strongly F-regular.

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