THE DIRECT SUM DECOMPOSABILITY OF ${}^eM$ IN DIMENSION TWO

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Dedicated to Professor Melvin Hochster on the occasion of his sixty-fifth birthday

Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p and M a finitely generated R-module. For every $e \in \mathbb{N}$, denote by ${}^eM$ the derived R-module structure on M with scalar multiplication determined via $r \cdot x := r^{p^e}x$ for all $r \in R, x \in M$. Assuming ${}^eM$ is finite over R for all $e \in \mathbb{N}$, Hochster showed that if $\dim(M) = 1$, then ${}^eM$ can be written as a direct sum of two non-zero modules for sufficiently large e. In this paper, we study the direct sum decomposability of ${}^eM$ when $\dim(M) \geq 2$. In particular, we show that the same splitting result holds in the case of $\dim(M) = 2$ provided that, for some $P \in \min_R(M)$ such that $\dim(R/P) = 2$, there is a module-finite extension of R/P that is strongly F-regular.

0. Introduction

Throughout this paper we assume R is a Noetherian ring of prime characteristic p and M a finitely generated R-module unless specified otherwise explicitly. By $(R, \mathfrak{m}, k)$, we indicate that R is local with its maximal ideal $\mathfrak{m}$ and its residue field $k = R/\mathfrak{m}$. We always denote $q := p^e$ for varying $e \in \mathbb{N}$.

Then, for every $e \in \mathbb{N}$, there is the Frobenius map (which is a ring homomorphism) $F^e : R \to R$ defined by $F^e(r) = r^q = r^{p^e}$ for any $r \in R$. Thus, given M, there is a derived R-module structure, denoted by ${}^eM$, on the same abelian group M but with its scalar multiplication determined by $r \cdot x = r^q x = r^{p^e}x$ for $r \in R, x \in M$. It is routine to verify that $\operatorname{Ann}(M) \subseteq \operatorname{Ann}({}^eM) \subseteq \sqrt{\operatorname{Ann}(M)}$ and $\operatorname{Ass}(M) = \operatorname{Ass}({}^eM)$ for all $e \in \mathbb{N}$.

In case R is reduced, it is clear that ${}^eR$ and $R^{1/q} := \{r^{1/p^e} \mid r \in R\}$ are isomorphic as R-modules for every e. Using this terminology, a result of E. Kunz states that R is regular if and only if ${}^eR$ is flat over R for some $e \geq 1$, or equivalently, for all $e \in \mathbb{N}$ ([Ku1, Theorem 2.1]).

We say that R is *F-finite* if ${}^1R$ is finitely generated over R, or equivalently, ${}^eR$ is finitely generated over R for all $e \in \mathbb{N}$. By a result of E. Kunz in [Ku2], every F-finite ring is excellent. If R is F-finite and M is a finitely generated R-module, then it is easy to see that ${}^eM$ remains finitely generated over R for every $e \in \mathbb{N}$.

Similarly, if ${}^1M$ is finitely generated over R, then so is ${}^1(R/\operatorname{Ann}(M))$. This means that $R/\operatorname{Ann}(M)$ is an F-finite ring, i.e., ${}^e(R/\operatorname{Ann}(M))$ is finite over $R/\operatorname{Ann}(M)$ (or, equivalently, over R) for all e, which forces ${}^eM$ to be finitely generated over R for all $e \in \mathbb{N}$.

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For any $e \in \mathbb{N}$, the derived R-module ${}^eM$ can be roughly identified as the module structure of M over the subring $R^q := \{r^q = r^{p^e} \mid r \in R\}$. Thus, in general, the "size" of ${}^eM$ should increase as $e \to \infty$. Assuming ${}^eM$ is finite over R for all $e \in \mathbb{N}$, we are interested in whether it is possible that the derived R-modules ${}^eM$ remain indecomposable (i.e., can not be written as a direct sum of two non-trivial submodules) for all $e \in \mathbb{N}$. Since we can always replace R by $R/\operatorname{Ann}(M)$, we may simply assume that R is F-finite.

Here is a case where ${}^eM$ remain indecomposable for all $e \in \mathbb{N}$: Suppose R has a maximal ideal $\mathfrak{m}$ such that $k = R/\mathfrak{m}$ is a perfect field and let $M = k$. Then it is easy to see that ${}^eM \cong M = k$ and hence is indecomposable for all $e \in \mathbb{N}$. Another trivial case of non-splitting is when $M = 0$.

Hochster showed the eventual splitting of ${}^eM$ for $e \gg 0$ in many cases in [Ho]. In particular, he proved that ${}^eM$ decomposes for all $e \gg 0$ if $\dim(M) \le 1$ (except for the cases just mentioned in the last paragraph). Indeed, in case $\dim(M) = 0$, then it reduces to local case, in which we see that $\operatorname{Ann}_R(M)$ is an $\mathfrak{m}$-primary ideal of $(R, \mathfrak{m}, k)$. Then $\mathfrak{m}^{[p^{e_0}]} \subseteq \operatorname{Ann}(M)$ for some $e_0 \in \mathbb{N}$, in which $\mathfrak{m}^{[p^{e_0}]}$ denotes the ideal of R generated by $\{r^{p^{e_0}} \mid r \in \mathfrak{m}\}$. Thus the derived R-modules ${}^eM$ become vector spaces over $k = R/\mathfrak{m}$ for all $e \ge e_0$ since $\mathfrak{m} \cdot {}^eM = \mathfrak{m}^{[p^e]}M = 0$. As for the one-dimensional case, we quote what was essentially proved in [Ho, Theorem 5.16(2)].

Theorem 0.1 ([Ho, Theorem 5.16(2)]). *Let $(R, \mathfrak{m}, k)$ be an F-finite local Noetherian ring of characteristic p and M a finitely generated R-module with $\dim(M) = 1$. Fix any $P \in \operatorname{Ass}(M)$ with $\dim(R/P) = 1$ and let $A = \overline{R/P}$ be the integral closure of R/P in its fraction field $(R/P)_P$. Then, for any $n \in \mathbb{N}$, there exists $e_0 \in \mathbb{N}$ such that ${}^eM$ has a direct summand isomorphic to A^n for all $e \ge e_0$.*

One of the main ideas in the proof of [Ho, Theorem 5.16(2)] is [Ho, Lemma 5.17], which also plays an important role in this paper. As we will need a stronger result than the the original version of [Ho, Lemma 5.17], we state the following lemma.

Lemma 0.2 (Compare with [Ho, Lemma 5.17]). *Consider the short exact sequence*

$$0 \longrightarrow D^{r+1} \oplus B \longrightarrow M \longrightarrow N \longrightarrow 0$$

of finitely generated modules over a Noetherian ring R (not necessarily of characteristic p). Assume that $\mu(E) \le r$ for all submodules $E \subseteq \operatorname{Ext}^1_R(N, D)$, where $\mu(E)$ denotes the least number of generators of E. Then M has a direct summand isomorphic to D.

Proof. This can be derived from the proof of [Ho, Lemma 5.17]. Details omitted. □

In [Ho, Fact 5.14], it was also observed that if M is a graded module over an F-finite $\mathbb{N}$-graded Noetherian ring R with R_0 a field of characteristic p and $\dim(M) \ge 1$, then for any $n \in \mathbb{N}$, there exists e such that ${}^eM$ splits as a direct sum of more than n non-zero R-modules. This splitting property was then used to prove a case of existence of small Cohen-Macaulay modules (see [Ho, Proposition 5.11]).

In this note, we study the direct sum decomposability of ${}^eM$ when $\dim(M) \ge 2$. The approach is somewhat similar to that of [Ho, Theorem 5.16(2)]. Let us state the main result, which is proved in Section 1.

Main Theorem (See Theorem 1.8). *Let $(R, \mathfrak{m}, k)$ be an F-finite Noetherian local ring of characteristic p and M a finitely generated R-module with $\dim(M) = 2$. Let*

A be the integral closure of R/P in some finite algebraic extension field of $(R/P)_P$ for some $P \in \operatorname{Ass}(M)$ with $\dim(R/P) = 2$. If A is strongly F-regular, then, given any $n \in \mathbb{N}$, A^n is isomorphic to a direct summand of ${}^eM$ for every $e \gg 0$.

Recall that an F-finite ring R is said to be *strongly F-regular* (cf. [HH2, Definition 5.1]) if, for any $c \in R^\circ := R \backslash \cup_{P \in \min(R)} P$, the R-linear map $R \to {}^eR$ defined by $1 \mapsto c$ splits for some $e > 0$ (or equivalently, for all $e \gg 0$). Strong F-regularity can be equivalently defined in terms of *tight closure* (cf. [HH1]): $(R, \mathfrak{m}, k)$ is strongly F-regular if and only if 0 is *tightly closed* in the injective hull of k. For example, if $(R, \mathfrak{m}, k)$ is an F-finite regular local ring, then ${}^eR$ is free over R for all e by [Ku1, Theorem 2.1]. Thus, for any $c \neq 0 \in R$, the R-linear map $R \to {}^eR$ sending 1 to c splits as long as e is large enough that $c \notin \mathfrak{m}^{[p^e]} = \mathfrak{m} \cdot {}^eR$. This shows that every F-finite regular ring is strongly F-regular.

In Hochster's result (i.e., Theorem 0.1), as R/P is a domain with $\dim(R/P) \leq 1$, its integral closure $A = \overline{R/P}$ is regular (and hence strongly F-regular) automatically. However, when $\dim(R/P) = 2$, its integral closure may not be regular. Nevertheless, Theorem 1.8(1) states that if there is a module-finite domain extension of R/P that is strongly F-regular, then the same splitting result for ${}^eM$ still holds. In this sense, Theorem 1.8(1) may be regarded as a generalization of Theorem 0.1.

1. The eventual splitting of ${}^eM$ in dimension two

We would like to begin this section with an easy remark.

Remark 1.1. Let R be a ring and $M_1 \to M \to M_2$ be an exact sequence. Then

$$\sup\{\mu(E) \mid E \subseteq M\} \leq \sup\{\mu(E_1) \mid E_1 \subseteq M_1\} + \sup\{\mu(E_2) \mid E_2 \subseteq M_2\}.$$

Throughout this paper, $\mu(E)$ denotes the minimal number of generators for any R-module E.

Let us next recall a familiar and useful fact about 1-dimensional R-modules. We use $\lambda_R(-)$ to denote the length of an R-module.

Lemma 1.2. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, k)$ (not necessarily of characteristic p) with $\dim(M) \leq 1$. Then*

$$\sup\{\mu(E) \mid E \subseteq M\} \leq \lambda(\mathrm{H}^0_{\mathfrak{m}}(M)) + e(M) < \infty,$$

in which $\mathrm{H}^0_{\mathfrak{m}}(M) := \cup_{n \in \mathbb{N}}(0 :_M \mathfrak{m}^n)$ and $e(M) := \lim_{n\to\infty} \frac{\lambda(M/\mathfrak{m}^n M)}{n}$, the Hilbert multiplicity of M (as a module of dimension one).

Proof. We sketch a proof. Let E be an arbitrary submodule of M. Then $\mathrm{H}^0_{\mathfrak{m}}(E) = \mathrm{H}^0_{\mathfrak{m}}(M) \cap E$ and hence there exists an exact sequence $0 \to E/\mathrm{H}^0_{\mathfrak{m}}(E) \to M/\mathrm{H}^0_{\mathfrak{m}}(M)$. Notice that $E/\mathrm{H}^0_{\mathfrak{m}}(E)$ is either 0 or Cohen-Macaulay of dimension 1. Thus, by Remark 1.1, etc., we have

$$\begin{aligned}\mu(E) &\leq \mu(\mathrm{H}^0_{\mathfrak{m}}(E)) + \mu(E/\mathrm{H}^0_{\mathfrak{m}}(E)) \leq \lambda(\mathrm{H}^0_{\mathfrak{m}}(E)) + e(E/\mathrm{H}^0_{\mathfrak{m}}(E)) \\ &\leq \lambda(\mathrm{H}^0_{\mathfrak{m}}(M)) + e(M/\mathrm{H}^0_{\mathfrak{m}}(M)) = \lambda(\mathrm{H}^0_{\mathfrak{m}}(M)) + e(M).\end{aligned}$$

(In the above, we used the fact that $\mu(E/\mathrm{H}^0_{\mathfrak{m}}(E)) \leq e(E/\mathrm{H}^0_{\mathfrak{m}}(E))$, which holds since $N := E/\mathrm{H}^0_{\mathfrak{m}}(E)$ is a one-dimensional Cohen-Macaulay R-module. To prove this, we assume $\dim(R) = 1$ and $|k| = \infty$ without loss of generality. Then there exists $x \in \mathfrak{m}$ such that x is N-regular and xR is a reduction of $\mathfrak{m}$. Consequently, we have $e(N) = \lim_{n\to\infty} \frac{\lambda(N/\mathfrak{m}^n N)}{n} = \lim_{n\to\infty} \frac{\lambda(N/x^n N)}{n} = \lambda_R(N/xN) \geq \mu(N)$.) $\square$

The next result plays an important role in the proof of the main theorem of this paper. Before stating the result, we first explain some more notations and terminologies that we are going to use.

Notation 1.3. Let $L, D \neq 0$ be finitely generated modules over an F-finite ring R.

(1) We denote by $\#_R(L, D)$, or $\#(L, D)$ if the ring R is clearly understood, the maximal integer n such that $L \cong D^n \oplus N$ for some R-module N.
(2) When $(R, \mathfrak{m}, k)$ is local, we denote $\alpha(R) = \log_p[k : k^p]$ (i.e., $p^{\alpha(R)}$ is the rank of ${}^1k$ as a k-vector space).
(3) Assuming $(R, \mathfrak{m}, k)$ is local, we say that D is an F-contributor of L if $\limsup_{e\to\infty} \frac{\#({}^eL, D)}{q^{\alpha(R)+\dim(L)}} > 0$, in which $q = p^e$. (See [Yao1] for some of the properties of F-contributors.)
(4) Also, given functions $f, g : \mathbb{N} \to \mathbb{N}$, we say $f(e) = O(g(e))$ if there exists $a \in \mathbb{N}$ such that $f(e) \leq ag(e)$ for all $e \in \mathbb{N}$.

Recall that, for any $P \in \operatorname{Spec}(R)$ and $e \in \mathbb{N}$, the derived module ${}^e(R/P)$ has torsion-free rank $q^{\alpha(R)+\dim(R/P)}$ over R/P ([Ku2, Proposition 2.3]). The next lemma gives a criterion as to when the eventual splitting of ${}^eM$ occurs.

Lemma 1.4. *Let $(R, \mathfrak{m}, k)$ be an F-finite local Noetherian ring of prime characteristic p and M a finitely generated R-module. Suppose, for some $e_0 \geq 0$, there exists a short exact sequence*

$$0 \longrightarrow L \longrightarrow {}^{e_0}M \longrightarrow N \longrightarrow 0,$$

such that $\dim(N) = d \leq 1$ *and* $\limsup_{e\to\infty} \frac{\#({}^eL, D)}{q^{\alpha(R)+d}} = \infty$ *for some finitely generated R-module $D \neq 0$ (e.g., $\dim(L) > d$ and D is a F-contributor of L). Then, for any $n \in \mathbb{N}$, there exists $e \in \mathbb{N}$ such that ${}^eM$ has a direct summand isomorphic to D^n.*

Proof. As the assumption also implies that $\limsup_{e\to\infty} \frac{\#({}^eL, D^n)}{q^{\alpha(R)+d}} = \infty$ for any $n \in \mathbb{N}$, we may simply prove the lemma in the case of $n = 1$. Also, as ${}^e({}^{e_0}M) = {}^{e+e_0}M$ for all $e \in \mathbb{N}$, we may relabel ${}^{e_0}M$ with M and, thus, assume $e_0 = 0$ without loss of generality.

We can filter N by finitely many submodules such that successive quotients are isomorphic to either $k = R/\mathfrak{m}$ or R/P with $P \in \operatorname{Spec}(R)$ and $\dim(R/P) = 1$. For each such P, denote by $\overline{R/P}$ the integral closure of R/P in its fraction field. Then $\overline{R/P}$ is regular and finitely generated over R/P since R is excellent. Hence there exists an exact sequence $0 \to \overline{R/P} \to R/P \to U \to 0$ with $\lambda_R(U) < \infty$. This shows that N may be filtered by finitely many submodules with successive quotients isomorphic to either $k = R/\mathfrak{m}$ or $\overline{R/P}$ with $P \in \operatorname{Spec}(R)$ and $\dim(R/P) = 1$. Fix such a filtration, say

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_r = N,$$

together with $\Lambda_0 \subseteq \{1, 2, \ldots, r\}$ and $\Lambda_1 = \{1, 2, \ldots, r\} \setminus \Lambda_0$ such that $N_i/N_{i-1} \cong k = R/\mathfrak{m}$ when $i \in \Lambda_0$ and $N_i/N_{i-1} \cong \overline{R/P_i}$ with $P_i \in \operatorname{Spec}(R)$ and $\dim(R/P_i) = 1$ when $i \in \Lambda_1$. As the integral closure $\overline{R/P_i}$ is a 1-dimensional regular semi-local domain for each $i \in \Lambda_1$, we have ${}^e(\overline{R/P_i}) \cong \overline{R/P_i}^{q^{\alpha(R)+1}}$ (cf. [Ku2, Proposition 2.3] and Lemma 1.10). Therefore, for any $e \in \mathbb{N}$, the derived R-module ${}^eN$ may be filtered correspondingly as follows

$$0 = {}^eN_0 \subsetneq {}^eN_1 \subsetneq \cdots \subsetneq {}^eN_r = {}^eN,$$

in which ${}^eN_i/{}^eN_{i-1} \cong {}^ek \cong k^{q^{\alpha(R)}}$ if $i \in \Lambda_0$ and ${}^eN_i/{}^eN_{i-1} \cong {}^e\big(\overline{R/P_i}\big) \cong \overline{R/P_i}^{\,q^{\alpha(R)+1}}$ if $i \in \Lambda_1$. Thus, by induction on r (details omitted) and by Remark 1.1 repeatedly, we have, for all $e \in \mathbb{N}$,

$$\begin{aligned}\sup\{\mu(E) \,|\, E \subseteq \mathrm{Ext}^1_R({}^eN, D)\} &\le \sum_{i=1}^{r} \sup\{\mu(E) \,|\, E \subseteq \mathrm{Ext}^1_R({}^eN_i/{}^eN_{i-1}, D)\} \\ &= q^{\alpha(R)} \sum_{i\in\Lambda_0} \sup\{\mu(E) \,|\, E \subseteq \mathrm{Ext}^1_R(k, D)\} \\ &\quad + q^{\alpha(R)+1} \sum_{i\in\Lambda_1} \sup\{\mu(E) \,|\, E \subseteq \mathrm{Ext}^1_R(\overline{R/P_i}, D)\}.\end{aligned}$$

Therefore, we conclude that $\sup\{\mu(E) \,|\, E \subseteq \mathrm{Ext}^1_R({}^eN, D)\} = O(q^{\alpha(R)+d})$. (Notice that, in case $\dim(N) = d = 0$, we have $\Lambda_1 = \emptyset$.)

Denote $\mu(e) = \sup\{\mu(E) \,|\, E \subseteq \mathrm{Ext}^1_R({}^eN, D)\}$ for every $e \in \mathbb{N}$. As $\mu(e) = O(q^{\alpha(R)+d})$ and $\limsup_{e\to\infty} \frac{\#({}^eL,D)}{q^{\alpha(R)+d}} = \infty$, there exists a large enough e such that $\#({}^eL, D) \ge \mu(e) + 1$. That is, ${}^eL \cong D^{\mu(e)+1} \oplus B$ for some R-module B and hence we have an exact sequence

$$0 \longrightarrow D^{\mu(e)+1} \oplus B \longrightarrow {}^eM \longrightarrow {}^eN \longrightarrow 0.$$

By Lemma 0.2, we see that D is isomorphic to a direct summand of ${}^eM$. $\square$

Remark 1.5. We may sketch another proof of the above Lemma 1.4: Again, it suffices to prove the case where $n = 1$. The assumption $\limsup_{e\to\infty} \frac{\#({}^eL,D)}{q^{\alpha(R)+d}} = \infty$ implies that D has depth at least $d + 1 = \dim(N) + 1$ (see the proof of [Yao1, Lemma 2.2]). Thus there exists $x \in \mathrm{Ann}(N) \subseteq \mathrm{Ann}({}^eN)$ for all $e \in \mathbb{N}$ such that x is D-regular. Let $\overline{R} = R/\mathrm{Ann}_R(N)$ and $\overline{D} = D/xD$. Hence, for all $e \in \mathbb{N}$, $\mathrm{Ext}^1_R({}^eN, D) \cong \mathrm{Hom}_R({}^eN, D/xD) \subseteq \mathrm{Hom}_R(\overline{R}^{\mu({}^eN)}, \overline{D}) \cong \mathrm{Hom}_R(\overline{R}, \overline{D})^{\mu({}^eN)}$. As $\mathrm{Hom}_R(\overline{R}, \overline{D})$ has dimension at most one, Remark 1.1 and Lemma 1.2 imply that

$$\begin{aligned}\sup\{\mu(E) \,|\, E \subseteq \mathrm{Ext}^1_R({}^eN, D)\} &\le \sup\left\{\mu(E) \,|\, E \subseteq \mathrm{Hom}_R(\overline{R}, \overline{D})^{\mu({}^eN)}\right\} \\ &\le \mu({}^eN) \sup\{\mu(E) \,|\, E \subseteq \mathrm{Hom}_R(\overline{R}, \overline{D})\} = O(\mu({}^eN)).\end{aligned}$$

On the other hand, $\mu({}^eN) = \lambda({}^eN/\mathfrak{m} \cdot {}^eN) = q^{\alpha(R)}\lambda(N/\mathfrak{m}^{[q]}N) = O(q^{\alpha(R)+d})$ by the existence of Hilbert-Kunz multiplicity (see [Mo]), where $d = \dim(N)$. Hence $\sup\{\mu(E) \,|\, E \subseteq \mathrm{Ext}^1_R({}^eN, D)\} = O(q^{\alpha(R)+d})$ and from here the proof goes just as in the original proof of Lemma 1.4.

Remark 1.6. From the proof of Lemma 1.4 we see that if $\lim_{e\to\infty} \frac{\#({}^eL,D)}{q^{\alpha(R)+d}} = \infty$ then, for any given $n \in \mathbb{N}$, there exists $e_1 \in \mathbb{N}$ such that ${}^eM$ has a direct summand isomorphic to D^n for all $e \ge e_1$.

Next, we use the above criterion (i.e., Lemma 1.4) to produce a situation where ${}^eM$ splits for $e \gg 0$. For any finitely generated R-module M, set

$$\mathrm{Assh}(M) = \{P \in \mathrm{Ass}(M) \mid \dim(R/P) = \dim(M)\},$$

which is the same as $\{P \in \min(M) \mid \dim(R/P) = \dim(M)\}$. We point out that some of the arguments in the proof of the next proposition are similar to the ones outlined in the proof of [Ho, Theorem 5.16(2)].

Proposition 1.7. *Let $(R, \mathfrak{m}, k)$ be an F-finite local Noetherian ring of characteristic p and M, L, D finitely generated non-zero R-modules such that $\dim(M) = 2$, $\mathrm{Ass}(L) \subseteq \mathrm{Assh}(M)$ (so that $\dim(L) = 2$) and $\limsup_{e\to\infty} \frac{\#({}^eL, D)}{q^{\alpha(R)+1}} = \infty$ (e.g., D is an F-contributor of L). Then, for any $n \in \mathbb{N}$, there exists $e \in \mathbb{N}$ such that D^n is isomorphic to a direct summand of ${}^eM$.*

Proof. Choose a primary decomposition of 0 in M, say

$$0 = Q_1 \cap Q_2 \cap \cdots \cap Q_s$$

such that $\mathrm{Ass}(M/Q_i) = \{P_i\}$. Assume the primary decomposition is minimal so that $\mathrm{Ass}(M) = \{P_1, P_2, \ldots, P_s\}$. Say that $\mathrm{Assh}(M) = \{P_1, P_2, \ldots, P_r\}$ for some $1 \le r \le s$. Let $S = R \setminus \cup_{i=1}^r P_i$. Then over the localization ring $S^{-1}R$, we get a primary decomposition of 0 in $S^{-1}M$

$$0 = S^{-1}Q_1 \cap S^{-1}Q_2 \cap \cdots \cap S^{-1}Q_r,$$

which shows that $S^{-1}(\oplus_{i=1}^r M/Q_i) \cong S^{-1}M$ by the Chinese Remainder Theorem. Lifting the isomorphism back to R, we get a short exact sequence

$$0 \longrightarrow \oplus_{i=1}^r M/Q_i \longrightarrow M \longrightarrow N \longrightarrow 0$$

for some finitely generated R-module N with $\dim(N) \le 1$. Then, for every $e \in \mathbb{N}$, there is a short exact sequence

$$0 \longrightarrow \oplus_{i=1}^r {}^e(M/Q_i) \longrightarrow {}^eM \longrightarrow {}^eN \longrightarrow 0 \tag{1.7.1}$$

with $\dim({}^eN) = \dim(N) \le 1$. Since $\mathrm{Ass}(M/Q_i) = \{P_i\}$, we see that ${}^e(M/Q_i) \neq 0$ are finitely generated torsion-free R/P_i-modules for all $e \gg 0$. (Indeed, as $\sqrt{\mathrm{Ann}_R(M/Q_i)} = P_i$, there exists $e_0 \in \mathbb{N}$ such that $(\mathrm{Ann}_R(M/Q_i))^{[p^{e_0}]} \subseteq P_i$, which implies that ${}^e(M/Q_i)$ is annihilated by P_i for every $e \ge e_0$. Moreover, for any $x \in R \setminus P_i$, as x is a non-zero-divisor on M/Q_i, it remains so on ${}^e(M/Q_i)$ for every $e \ge 0$.) For any $e \ge e_0$ and any $i = 1, \ldots, r$, let $n(e,i)$ denote the torsion-free rank of ${}^e(M/Q_i)$ over R/P_i. Then $n(e,i) > 0$ and there exists a short exact sequence

$$0 \longrightarrow (R/P_i)^{n(e,i)} \longrightarrow {}^e(M/Q_i) \longrightarrow N_{(e,i)} \longrightarrow 0 \tag{1.7.2}$$

so that $N_{(e,i)}$ is finitely generated over R/P_i with $\dim(N_{(e,i)}) < \dim(M/Q_i) = 2$ for every $i = 1, \ldots, r$. Putting (1.7.1) and (1.7.2) together, we get a short exact sequence

$$0 \longrightarrow \oplus_{i=1}^r (R/P_i)^{n(e,i)} \longrightarrow {}^eM \longrightarrow N_e \longrightarrow 0 \tag{$*_e$}$$

for each $e \ge e_0$, in which N_e is a finitely generated R-module with $\dim(N_e) \le 1$. Notice that $n(e+1,i) = p^{\alpha(R)+2} n(e,i)$ for each $e \ge e_0$ and each $i \in \{1, 2, \ldots, r\}$ (cf. [Ku2, Proposition 2.3]). In particular, we see that $n(e,i) \to \infty$ as $e \to \infty$.

Now we carry out a similar procedure on L: Say $\mathrm{Ass}(L) = \{P_1, P_2, \ldots, P_t\}$ for some $1 \le t \le r$. Fix a primary decomposition of 0 in L, say $0 = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_t$ with $\mathrm{Ass}(L/Q'_i) = \{P_i\}$. Let $U = R \setminus \cup_{i=1}^t P_i$. Then we see that $U^{-1}(\oplus_{i=1}^t L/Q'_i) \cong U^{-1}L$, which gives a short exact sequence

$$0 \longrightarrow L \longrightarrow \oplus_{i=1}^t L/Q'_i \longrightarrow N' \longrightarrow 0 \tag{1.7.3}$$

for some finitely generated R-module N' with $\dim(N') \leq 1$. Similarly, we can find a large enough $e' \in \mathbb{N}$ such that, for each $i = 1, 2, \ldots, t$, ${}^{e'}(L/Q_i')$ is torsion-free over R/P_i (say with torsion-free rank $n(i)$) and, hence, there exists a short exact sequence

$$(1.7.4) \qquad 0 \longrightarrow {}^{e'}(L/Q_i') \longrightarrow (R/P_i)^{n(i)} \longrightarrow N_i' \longrightarrow 0$$

for some finitely generated R/P_i-module N_i' with $\dim(N_i') \leq 1$. Together, (1.7.3) and (1.7.4) produce a short exact sequence

$$(**) \qquad 0 \longrightarrow {}^{e'}L \longrightarrow \oplus_{i=1}^{t}(R/P_i)^{n(i)} \longrightarrow N'' \longrightarrow 0$$

for some finitely generated R-module N'' with $\dim(N'') \leq 1$.

Now fix a sufficiently large $e_1 \in \mathbb{N}$ such that $n(e_1, i) \geq n(i)$ for all $i = 1, \ldots, t$. Then, the exact sequences $(*_{e_1})$ and $(**)$ generate a short exact sequence

$$0 \longrightarrow {}^{e'}L \oplus B \longrightarrow {}^{e_1}M \longrightarrow N''' \longrightarrow 0,$$

in which $B = \big(\oplus_{i=1}^{t} (R/P_i)^{n(e_1,i)-n(i)} \big) \bigoplus \big(\oplus_{i=t+1}^{r} (R/P_i)^{n(e_1,i)} \big)$ and N''' is a finitely generated R-module with $\dim(N''') \leq 1$. Moreover, it is clear that

$$\limsup_{e\to\infty} \frac{\#({}^eL, D)}{q^{\alpha(R)+1}} = \infty \quad \text{implies} \quad \limsup_{e\to\infty} \frac{\#({}^e({}^{e'}L \oplus B), D)}{q^{\alpha(R)+1}} = \infty.$$

Now the desired result follows from Lemma 1.4. $\square$

The above proposition can be applied to the following case, which proves the main theorem of this paper. First, recall that in [AL], I. Aberbach and G. Leuschke proved the following result concerning strongly F-regularity: *An F-finite local ring $(R, \mathfrak{m})$ is strongly F-regular if and only if* $\liminf_{e\to\infty} \frac{\#({}^eR,R)}{q^{\alpha(R)+\dim(R)}} > 0$. Also, we refer the readers to [SVdB] and [Yao1] for the definition and properties of modules of *finite F-representation type* (abbreviated *FFRT*).

Theorem 1.8. *Let $(R, \mathfrak{m}, k)$ be an F-finite local Noetherian ring of characteristic p and M a finitely generated R-module with* $\dim(M) = 2$. *Let A be a domain that is a module-finite extension of R/P for some* $P \in \mathrm{Assh}(M)$.

1. *(1) If A is strongly F-regular, then, for any $n \in \mathbb{N}$, there exists $e_1 \in \mathbb{N}$ such that ${}^eM$ has a direct summand isomorphic to A^n for all $e \geq e_1$.*
2. *(2) If there is a finitely generated torsion-free A-module $L \neq 0$ that has FFRT, then there exists an R-module $D \neq 0$ such that for any $n \in \mathbb{N}$, there is $e_1 \in \mathbb{N}$ such that ${}^{e_1}M$ has a direct summand isomorphic to D^n.*

Proof. (1): Evidently $\dim(A) = 2$, $\mathrm{Ass}_R(A) = \{P\}$ and A is a semi-local F-finite ring, say with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_c$. Also notice that $\dim(A_{\mathfrak{m}_i}) = \dim(A) = 2$ and $\alpha(A_{\mathfrak{m}_i}) = \alpha(R)$ for each $i = 1, 2, \ldots, c$. (Indeed, the equation $\alpha(A_{\mathfrak{m}_i}) = \alpha(R)$ holds because $A/\mathfrak{m}_i$ is a finite field extension of $R/\mathfrak{m}$. Then $\dim(A_{\mathfrak{m}_i}) = \dim(R)$ follows from [Ku2, Proposition 2.3]. It also follows from Ratliff's dimension formula (cf. [Ma, Theorem 15.6]).)

We are to show $\liminf_{e\to\infty} \frac{\#_R({}^eA,A)}{q^{\alpha(R)+\dim(A)}} > 0$ in order to apply Proposition 1.7 and Remark 1.6. Thus it suffices to show $\liminf_{e\to\infty} \frac{\#_A({}^eA,A)}{q^{\alpha(R)+\dim(A)}} > 0$ by considering A as an A-module. Since A is strongly F-regular, so is $A_{\mathfrak{m}_i}$ for each $i = 1, 2, \ldots, c$.

Therefore $\liminf_{e\to\infty} \frac{\#_{A_{\mathfrak{m}_i}}({}^eA_{\mathfrak{m}_i}, A_{\mathfrak{m}_i})}{q^{\alpha(A_{\mathfrak{m}_i})+\dim(A_{\mathfrak{m}_i})}} > 0$ for each $i = 1, 2, \dots, c$ (see [AL]). Finally, by Lemma 1.10 below, we have

$$\liminf_{e\to\infty} \frac{\#_A({}^eA, A)}{q^{\alpha(R)+\dim(A)}} = \min\left\{ \liminf_{e\to\infty} \frac{\#_{A_{\mathfrak{m}_i}}({}^eA_{\mathfrak{m}_i}, A_{\mathfrak{m}_i})}{q^{\alpha(R)+\dim(A)}} \mid 1 \le i \le c\right\} > 0,$$

which finishes the proof.

(2): Clearly, we have $\mathrm{Ass}_R(L) = \{P\} \subseteq \mathrm{Assh}_R(M)$ so that $\dim(L) = 2$. Also, the assumption that L has FFRT as an A-module implies that L has FFRT as an R-module (from definition). By [Yao1, Lemma 2.1], there is an non-zero F-contributor D of L over R. Now Proposition 1.7 applies. $\square$

Remark 1.9. As every strongly F-regular ring A is normal, we see that if A is a module-finite extension of R/P (as in Theorem 1.8), then A is the integral closure of R/P in some finite field extension of $(R/P)_P$.

In general, for any (F-finite) local ring $(R, \mathfrak{m}, k)$ of prime characteristic p, the invariant $s(R) = \lim_{e\to\infty} \frac{\#({}^eR, R)}{q^{\alpha(R)+\dim(R)}}$, if it exists, is called the F-signature of R. It was first defined and studied in [HL]. For some other related work on the F-signature, see [AE1], [AE2], [AL], [Si], [SVdB], [Yao1] and [Yao2], etcetera.

Although Lemma 1.10 might be well-known, we state and prove it for the completeness of the proof of Theorem 1.8. (Lemma 1.10 was also referred to in the proof of Lemma 1.4.) Before stating the lemma, we observe that for any Noetherian local ring $(R, \mathfrak{m})$ and any finitely generated R-modules $N, D \neq 0$, we have

$$(\dagger) \qquad \#_R(N \oplus D^n, D) = \#_R(N, D) + n$$

for every $n \in \mathbb{N}$, which follows from the fact that $\#_{\widehat{R}}(\widehat{M}, \widehat{D}) = \#_R(M, D)$ and the Krull-Schmidt property of $\widehat{R}$, the $\mathfrak{m}$-adic completion of R.

Lemma 1.10. *Let A be a semi-local Noetherian ring (not necessarily with prime characteristic p) with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_c$ exactly and $M, D \neq 0$ finitely generated A-modules. Then $\#_A(M, D) = \min\{\#_{A_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}) \mid 1 \le i \le c\}$.*

Proof. One could prove this lemma by using the fact that $\#_A(M, D) = \#_{\widehat{A}}(\widehat{M}, \widehat{D})$, in which $\widehat{\ } = \widehat{\ }^{\mathfrak{m}}$ denotes the completion with respect to the $\mathfrak{m}$-adic topology where $\mathfrak{m} = \cap_{i=1}^c \mathfrak{m}_i$ is the Jacobson radical, and the fact that $\widehat{A} = \prod_{i=1}^c \widehat{A_{\mathfrak{m}_i}}^{\mathfrak{m}_i}$, in which $\widehat{\ }^{\mathfrak{m}_i}$ stands for the $(\mathfrak{m}_i A_{\mathfrak{m}_i})$-adic completion.

For an alternative proof, let $\#_A(M, D) = n$ and say $M \cong N \oplus D^n$. Then $\#_A(N, D) = 0$ and, by (†) above, we have the equation

$$\min\{\#_{A_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}) \mid 1 \le i \le c\} = n + \min\{\#_{A_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}) \mid 1 \le i \le c\}.$$

Now it suffices to prove $\min\{\#_{A_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}) \mid 1 \le i \le c\} = 0$. Suppose, on the contrary, that $\min\{\#_{A_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}) \mid 1 \le i \le c\} > 0$. Then, for each $1 \le i \le c$, there exist homomorphisms

$$\begin{aligned} & \phi_i/s_i \in (\mathrm{Hom}_A(N, D))_{\mathfrak{m}_i} = \mathrm{Hom}_{A_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}) \\ \text{and} \quad & \psi_i/s_i \in (\mathrm{Hom}_A(D, N))_{\mathfrak{m}_i} = \mathrm{Hom}_{A_{\mathfrak{m}_i}}(D_{\mathfrak{m}_i}, N_{\mathfrak{m}_i}) \\ \text{such that} \quad & (\phi_i/s_i) \circ (\psi_i/s_i) = 1_{D_{\mathfrak{m}_i}}, \end{aligned}$$

in which $\phi_i \in \mathrm{Hom}_A(N, D)$, $\psi_i \in \mathrm{Hom}_A(D, N)$ and $s_i \in A \setminus \mathfrak{m}_i$. Choose $r_i \in \cap_{j\neq i}\mathfrak{m}_j \setminus \mathfrak{m}_i$ for each $i = 1, 2, \ldots, c$. Then it is routine to verify that

$$\Big(\sum_{i=1}^{c} r_i\phi_i\Big) \circ \Big(\sum_{i=1}^{c} r_i\psi_i\Big) \in \mathrm{Hom}_A(D, D)$$

is surjective (by Nakayama's Lemma) and hence is an isomorphism, which implies that N has a direct summand isomorphic to D, a contradiction. □

It is noted that Proposition 1.7 and Theorem 1.8 apply to 2-dimensional cases only. In case the dimension is higher, we have the following result, which was obtained during a discussion with Melvin Hochster.

Theorem 1.11. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of prime characteristic p with $\dim(R) \geq 2$ such that R_P is integrally closed in its fraction field for all $P \in \mathrm{Spec}(R)$ with $\dim(R/P) \geq 2$. Let $A := \overline{R}$ be the integral closure of R in its fraction field. If A is strongly F-regular, then for any finitely generated faithful R-module M and any $n \in \mathbb{N}$, there exists e_1 such that ${}^eM$ has a direct summand isomorphic to A^n (as an R-module) for all $e \geq e_1$.*

Proof. Let $\mathfrak{C} = \{r \in R \,|\, rA \subseteq R\}$ be the conductor, which is the largest common ideal of R and A. Then we see $\dim(R/\mathfrak{C}) \leq 1$ by the assumption. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{C}M \longrightarrow M \longrightarrow M/\mathfrak{C}M \longrightarrow 0,$$

in which $\dim(M/\mathfrak{C}M) \leq 1$ and $\mathfrak{C}M$ is a finitely generated faithful A-module. Thus there exist $h \in \mathrm{Hom}_A(\mathfrak{C}M, A)$ and $x \in \mathfrak{C}M$ such that $h(x) = c \in A^\circ$. Then, as A is strongly F-regular, there are $e_0 \in \mathbb{N}$ and $g \in \mathrm{Hom}_A({}^{e_0}A, A)$ such that $g(c) = 1$. Consequently, we get an A-linear homomorphism $g \circ h : {}^{e_0}(\mathfrak{C}M) \to {}^{e_0}A \to A$ that maps x to 1, showing that A is a direct summand of ${}^{e_0}(\mathfrak{C}M)$ as an A-module.

The strong F-regularity of A also implies $\liminf_{e\to\infty} \frac{\#_A({}^eA, A)}{q^{\alpha(R)+\dim(A)}} > 0$ (cf. [AL] and the proof of Theorem 1.8(1) above). Thus, we have

$$\liminf_{e\to\infty} \frac{\#_R({}^e(\mathfrak{C}M), A)}{q^{\alpha(R)+\dim(R)}} \geq \liminf_{e\to\infty} \frac{\#_A({}^e(\mathfrak{C}M), A)}{q^{\alpha(R)+\dim(A)}} > 0 \quad \text{(cf. last paragraph).}$$

Now the claim follows from Lemma 1.4 and Remark 1.6. □

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